Supplemental Material to
FUNCTIONAL COEFFICIENT PANEL MODELING WITH COMMUNAL SMOOTHING COVARIATES

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May 2019

COWLES FOUNDATION DISCUSSION PAPER NO. 2193S

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
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Functional Coefficient Panel Modeling with Communal Smoothing Covariates: the case without intercept*

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May 27, 2019

Abstract

This document provides supplementary material for Phillips and Wang (2019). In particular, we deal here with the case where a functional intercept term is not included in the model. Two types of estimators, one based on cross-section averaged data and the other on the full panel, are presented. Limit properties of these estimators are derived covering cases where the cross-section and time series sample sizes diverge at various rates. An illustrative simulation is provided to examine the performance of the proposed estimators.

JEL classification: C14, C23

Keywords: Communal covariates, Fixed effects, Functional coefficients, Panel data.

*Research support from the Kelly Fund and Marsden Grant 16-UOA-239 at the University of Auckland and an LKC Fellowship at Singapore Management University is gratefully acknowledged. Peter C. B. Phillips email: peter.phillips@yale.edu; Ying Wang email: ying.w@auckland.ac.nz

1 Model and estimation procedures

We consider the model

$$y_{it} = \alpha_i + x_{it}'\beta(z_t) + u_{it}, \tag{1.1}$$

where the regressor x_{it} $(p \times 1)$ does not include a constant term. So the only difference with Phillips and Wang (2019) is that there is no functional intercept term $\beta_0(z)$ in (1.1).

As in Phillips and Wang (2019), we provide two different approaches to estimation. The first method estimates the cross-section averaged version of (1.1), whereas the second uses the full panel. We adopt the profile method of Su and Ullah (2011) by profiling out the individual effects α_i or $\bar{\alpha} = N^{-1}\sum_{i=1}^{N}\alpha_i$ first. The two different estimation approaches are presented in the next two subsections. Section 2 develops the asymptotic theory, Section 3 reports simulations concerning finite sample performance of the estimates, and proofs are given in the Appendix.

1.1 Average Profile Local Constant (APLC) estimation

Taking averages over i in (1.1), we get the aggregate system

$$\bar{y}_{At} = \bar{\alpha} + \bar{x}_{At}'\beta(z_t) + \bar{u}_{At}, t = 1, \cdots, T. \tag{1.2}$$

Denote $\bar{Y}_A = (\bar{y}_{A1}, \cdots, \bar{y}_{AT})'$, $\bar{X}_A = (\bar{x}_{A1}, \cdots, \bar{x}_{AT})'$ and $\bar{U}_A = (\bar{u}_{A1}, \cdots, \bar{u}_{AT})'$. Suppose $\bar{\alpha}$ is known, then the local constant estimate of $\beta(z)$ is given as

$$\hat{\beta}_{APLC}^{oracle}(z) = [\bar{X}_A' K_z \bar{X}_A]^{-1}\bar{X}_A' K_z(\bar{Y}_A - \bar{\alpha}1_{T\times 1}), \tag{1.3}$$

where $K_z = diag(K(H^{-1}(z_1 - z)), \cdots, K(H^{-1}(z_T - z)))$ is the $T \times T$ diagonal kernel weighting matrix . We call $\hat{\beta}_{APLC}^{oracle}(z)$ the oracle estimator as it assumes $\bar{\alpha}$ is known.

Let $s(z) = [\bar{X}_A' K_z \bar{X}_A]^{-1}\bar{X}_A' K_z$, then $\hat{\beta}_{APLC}^{oracle}(z) = s(z)(\bar{Y}_A - \bar{\alpha}1_{T\times 1})$. Using this oracle estimator in (1.2) gives

$$\bar{y}_{At} = \bar{\alpha} + \bar{x}_{At}' s(z_t)(\bar{Y}_A - \bar{\alpha}1_{T\times 1}) + \tilde{u}_t, \tag{1.4}$$

where $\tilde{u}_t = \bar{y}_{At} - \bar{\alpha} - \bar{x}_{At}'\hat{\beta}_{APLC}^{oracle}(z_t) = \bar{x}_{At}'[\beta(z_t) - \hat{\beta}_{APLC}^{oracle}(z_t)] + \bar{u}_{At}$. Rewrite the above equation as

$$\bar{y}_{At} - \bar{x}_{At}' s(z_t)\bar{Y}_A = (1 - \bar{x}_{At}' s(z_t)1_{T\times 1})\bar{\alpha} + \tilde{u}_t, t = 1, \cdots, T. \tag{1.5}$$

Denote $y^*_{T,t} = \bar{y}_{At} - \bar{x}'_{At}s(z_t)\bar{Y}_A$ and $x^*_{T,t} = 1 - \bar{x}'_{At}s(z_t)1_{T\times 1}$. Equation (1.5) can be expressed as

$$y^*_{T,t} = x^*_{T,t}\bar{\alpha} + \tilde{u}_t, t = 1, \cdots, T. \tag{1.6}$$

Then $\bar{\alpha}$ can be estimated by least squares (OLS) as

$$\hat{\bar{\alpha}} = \left(\sum_t x^{*2}_{T,t}\right)^{-1} \sum_t x^*_{T,t} y^*_{T,t}. \tag{1.7}$$

Using this estimate in the oracle estimator $\hat{\beta}^{oracle}_{APLC}(z)$ gives the following feasible estimator

$$\hat{\beta}_{APLC}(z) = [\bar{X}'_A K_z \bar{X}_A]^{-1} \bar{X}'_A K_z (\bar{Y}_A - \hat{\bar{\alpha}} 1_{T\times 1}). \tag{1.8}$$

We refer to this estimator as the feasible average profile local constant (APLC) estimator.

1.2 Profile Local Constant (PLC) estimation

Based on model (1.1), if $\alpha = (\alpha_1, \cdots, \alpha_N)'$ is treated as known, the local level estimator for $\beta(z)$ is

$$\hat{\beta}^{oracle}_{PLC}(z) = [X'K(z)X]^{-1}X'K(z)(Y - D\alpha), \tag{1.9}$$

where X is the $n \times p$ matrix formed by stacking the $1 \times p$ vector x'_{it}, $K(z) = I_N \otimes K_z$ is an $n \times n$ diagonal matrix, $K_z = diag(K_{1H}, \cdots, K_{TH})$, $K_{tH} = K(H^{-1}(z_t - z))$, $Y = (y_{11}, \cdots, y_{1T}, \cdots, y_{N1}, \cdots, y_{NT})'$, $D = I_N \otimes 1_T$, $U = (u_{11}, \cdots, u_{1T}, \cdots, u_{N1}, \cdots, u_{NT})'$, and $n := NT$.

Let $w(z) := [X'K(z)X]^{-1}X'K(z)$, which is $p \times n$, then $\hat{\beta}^{oracle}_{PLC}(z) = w(z)(Y - D\alpha)$. Plugging this oracle estimator into model (1.1), we get

$$y_{it} = \alpha_i + x'_{it}w(z_t)(Y - D\alpha) + v_{it}, \tag{1.10}$$

where $v_{it} = y_{it} - \alpha_i - x'_{it}\hat{\beta}^{oracle}_{PLC}(z_t) = x'_{it}[\beta(z_t) - \hat{\beta}^{oracle}_{PLC}(z_t)] + u_{it}$. Collecting terms involving α in (1.10), rewrite the equation as

$$\begin{aligned} y_{it} &= \alpha_i + x'_{it}w(z_t)(Y - D\alpha) + v_{it}, \\ &= \alpha_i + \sum_{d=1}^{p} x_{it,d}e'_d w(z_t)(Y - D\alpha) + v_{it} \\ &= \alpha_i + \sum_{d=1}^{p} x_{it,d}w_d(z_t)(Y - D\alpha) + v_{it}, \end{aligned} \tag{1.11}$$

where e_d is a $p \times 1$ vector with 1 in the d-th element and 0 elsewhere, and $w_d(z_t) = e_d' w(z_t)$ is $1 \times n$. Writing (1.11) in vector form, we have

$$\begin{aligned} Y - D\alpha &= \sum_{d=1}^{p} \mathbf{x}_d \odot [1_N \otimes w_d(Z)(Y - D\alpha)] + V \\ &= \left[\sum_{d=1}^{p} (\mathbf{x}_d \otimes 1_n') \odot (1_N \otimes w_d(Z)) \right] (Y - D\alpha) + V \\ &=: Q_1(Y - D\alpha) + V, \end{aligned}$$

where $\mathbf{x}_d = (x_{11,d}, \cdots, x_{1T,d}, \cdots, x_{N1,d}, \cdots, x_{NT,d})'$, $w_d(Z) = (w_d(z_1)', \cdots, w_d(z_T)')'$ is $T \times n$, $\odot$ denotes the Hadamard product, the definition of Q_1 follows from the context. Consequently, we have

$$(I_{NT} - Q_1)Y = (I_{NT} - Q_1)D\alpha + V. \tag{1.12}$$

Let $M_1 = I_{NT} - Q_1$, $M_2 = M_1 D$, and then we have

$$\hat{\alpha}_{PLC} = [M_2' M_2]^{-1} M_2' M_1 Y, \tag{1.13}$$

where the affix 'PLC' stands for 'Profile Local Constant'. Using $\hat{\alpha}_{PLC}$ in $\hat{\beta}_{PLC}^{oracle}(z)$, we get the feasible estimator

$$\hat{\beta}_{PLC}(z) = w(z)(Y - D\hat{\alpha}_{PLC}) = [X'K(z)X]^{-1} X'K(z)(Y - D\hat{\alpha}_{PLC}). \tag{1.14}$$

2 Asymptotics

We make the following assumptions and use $\int$ for $\int_{-1}^{1}$ in what follows unless otherwise stated.

Assumption 1. *(a) The kernel function $k(\cdot)$ is a symmetric bounded probability function with support $[-1, 1]$, $\int k(w)dw = 1$, and $\int wk(w)dw = 0$. Denote $\int w^2 k(w)dw = \mu_2$, $\int k^2(w)dw = \nu_0$ and $\int w^2 k^2(w)dw = \nu_2$;*

(b) The product kernel $K(v) = \prod_{j=1}^{q} k(v_j)$, with $v = (v_1, \cdots, v_q)'$, $\int vv'K(v)dv = \mu_2 I_q$, $\int K^2(v)dv = \nu_0^q$, and $\int vv'K^2(v)dv = \nu_2 I_q$.

Assumption 2. *(a) $\{(x_i, u_i), i \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) variates over i, where $x_i = (x_{it}, t \geq 1)$ and $u_i = (u_{it}, t \geq 1)$. Further,* *for each $i \geq 1$, $\{(x_{it}, z_t, u_{it}), t \geq 1\}$ is stationary and α-mixing with mixing coefficients α_k satisfying $\alpha_k = O(k^{-\tau})$, where $\tau > \frac{\lambda+2}{\lambda}$ for some $\lambda > 0$, as in (b) and (c) below. Furthermore, u_{it} is independent of x_{it} and z_t for all i and t;*

(b) *Let $\mathbb{E}x_{it} = \eta$, $\mathbb{E}(x_{it}x'_{it}) = V_{xx}$ is positive definite, and $\gamma_x(j) = \mathbb{E}x_{it}x'_{it+j}$. Denote $\Sigma_{xx} = Var(x_{it}) = V_{xx} - \eta\eta'$. Furthermore, $\mathbb{E}(||x_{it}||^{2(2+\lambda)}) < \infty$, where $||\cdot||$ is Euclidean distance. In addition, the first and second order conditional moments of x_{it} given $z_t = z$ are independent of z;*

(c) *The error process $\{u_{it}\}$ satisfies $\mathbb{E}u_{it} = 0$, $\mathbb{E}u_{it}^2 = \sigma_u^2 < \infty$, $\mathbb{E}|u_{it}|^{2+\lambda} < \infty$, $\sum_{j=0}^{\infty} |\gamma_u(j)| < \infty$ where $\gamma_u(j) = \mathbb{E}(u_{1t}u_{1,t+j})$ and $\gamma_u^2 = \sum_{j=-\infty}^{\infty} \gamma_u(j)$ is the common long-run variance of u_{it};*

(d) *z_t has probability density $f_z(z)$. $f_z(\cdot)$ and $\beta_*(\cdot)$ have continuous derivatives up to the second order.*

Assumption 3. *Define $H = diag(h_1, \cdots, h_q)$, $||H|| = \sqrt{\sum_{j=1}^{q} h_j^2} \to 0$ and $|H| = h_1 \cdots h_q$. As $T \to \infty$, $||H|| \to 0$ and $T|H| \to \infty$.*

Discussion of these assumptions is given in Phillips and Wang (2019).

2.1 Asymptotics of the APLC estimators

The following results give the limit distributions of $\hat{\beta}_{APLC}^{oracle}(z)$, $\hat{\bar{\alpha}}$ and $\hat{\beta}_{APLC}(z)$ under various conditions on the cross-section and time series sample sizes (N, T) and the two cases $\eta = 0, \eta \neq 0$.

Theorem 2.1. *Under Assumptions 1-3, as $T \to \infty$, $|H| \to 0$, and $T|H| \to \infty$, the following results hold.*

(a) *For the oracle estimator $\hat{\beta}_{APLC}^{oracle}(z)$, we have:*

(a1) *if N is fixed (η can be zero or nonzero),*

$$\sqrt{T|H|}(\hat{\beta}_{APLC}^{oracle}(z) - \beta(z) - B_1(z)) \Rightarrow N\left(0, f_z^{-1}(z)\sigma_u^2\nu_0^q N^{-1}V_{xx,N}^{-1}\right), \tag{2.1}$$

where $V_{xx,N} = \frac{1}{N}V_{xx} + (1 - \frac{1}{N})\eta\eta' = \frac{1}{N}\Sigma_{xx} + \eta\eta'$;

(a2) *if $N \to \infty$ and $\eta = 0$,*

$$\sqrt{T|H|}(\hat{\beta}_{APLC}^{oracle}(z) - \beta(z) - B_1(z)) \Rightarrow N\left(0, f_z^{-1}(z)\sigma_u^2\nu_0^q\Sigma_{xx}^{-1}\right); \tag{2.2}$$

(a3) if $N \to \infty$, $\eta \neq 0$, and $p = 1$,

$$\sqrt{NT|H|}(\hat{\beta}^{oracle}_{APLC}(z) - \beta(z) - B_1(z)) \Rightarrow N\left(0, f_z^{-1}(z)\sigma_u^2 \nu_0^q \eta^{-2}\right); \tag{2.3}$$

(a4) if $N \to \infty$, $\eta \neq 0$, and $p > 1$,

$$\sqrt{T|H|}(\hat{\beta}^{oracle}_{APLC}(z) - \beta(z) - B_1(z)) \Rightarrow N\left(0, f_z^{-1}(z)\sigma_u^2 \nu_0^q \left(V_{xx}^{-1} - \frac{V_{xx}^{-1}\eta\eta' V_{xx}^{-1}}{\eta' V_{xx}^{-1}\eta}\right)\right), \tag{2.4}$$

which is a degenerate distribution;

where the bias is

$$B_1(z) = f_z^{-1}(z)\mu_2 \sum_{s=1}^{q} h_s^2 \left[\frac{\partial f_z(z)}{\partial z_s}\frac{\partial \beta(z)}{\partial z_s} + \frac{1}{2}\frac{\partial^2 \beta(z)}{\partial^2 z_s} f_z(z)\right]. \tag{2.5}$$

(b) For the estimator $\hat{\bar{\alpha}}$ given in (1.7), we have:

(b1) if N is fixed,

$$\sqrt{T}(\hat{\bar{\alpha}} - \bar{\alpha}) \Rightarrow N\left(0, \sigma^2_{\bar{\alpha},N}\right), \tag{2.6}$$

where the asymptotic variance $\sigma^2_{\bar{\alpha},N}$ is given in (A.12). If we further have $\{u_{it}\}$ iid across t, we have

$$\sigma^2_{\bar{\alpha},N} = \sigma_u^2\left(\frac{1}{N} + \eta' \Sigma_{xx}^{-1}\eta\right); \tag{2.7}$$

(b2) if $N \to \infty$ and $\eta = 0$, $\hat{\bar{\alpha}} - \bar{\alpha} = O_p(1/\sqrt{NT})$.

(c) The feasible estimator $\hat{\beta}_{APLC}(z)$ given in (1.8) is asymptotically equivalent to the oracle estimator $\hat{\beta}^{oracle}_{APLC}(z)$ when N is finite or N goes to infinity and $\eta = 0$.

Remark 2.1. ***(Comments regarding case (a4) in Theorem 2.1)*** *When $N \to \infty$, $V_{xx,N} \to \eta\eta'$, which is a singular matrix when $p > 1$. This gives rise to the degenerate distribution in Theorem 2.1 (a4). The result in (2.4) is obtained by first taking the inverse and then letting $N \to \infty$. Clearly, the quadratic form $\eta'\left(V_{xx}^{-1} - \frac{V_{xx}^{-1}\eta\eta' V_{xx}^{-1}}{\eta' V_{xx}^{-1}\eta}\right)\eta = 0$. Hence, the linear combination $\sqrt{T|H|}\eta'(\hat{\beta}^{oracle}_{APLC}(z) - \beta(z)) \to_p 0$ and the rate of convergence of the estimator is higher in this direction.*

To develop a complete asymptotic theory in this case, we can use the rotation methods of (Park and Phillips, 1988) and transform the model (1.2) as follows

$$\bar{y}_{At} - \bar{\alpha} = \omega_t'\theta(z_t) + \bar{u}_{At}, t = 1, \cdots, T, \tag{2.8}$$

where $\omega_t = W'\bar{x}_{At}$, $W = (\xi, W_2)$ is an orthogonal matrix of order p, $\xi = (\eta'\eta)^{-1/2}\eta$, and $\theta(z_t) = W'\beta(z_t)$ is a linear transform of $\beta(z_t)$. Since W is orthogonal, we have $W_2'\eta = 0$. Based on model (2.8), we get the following oracle estimator of $\theta(z)$ (assuming $\bar{\alpha}$ is known)

$$\hat{\theta}^{oracle}_{APLC}(z) = \left[\sum_t \omega_t K(H^{-1}(z_t - z))\omega_t'\right]^{-1} \sum_t \omega_t K(H^{-1}(z_t - z))\bar{u}_{At}. \tag{2.9}$$

The asymptotics of $\hat{\theta}^{oracle}_{APLC}(z)$ can be obtained analogously with Theorem 2.1 (a). The key step is to notice that as $T \to \infty$, we have

$$\frac{1}{T|H|}\sum_t \omega_t K(H^{-1}(z_t - z))\omega_t' \xrightarrow{p} f_z(z)\mathbb{E}\omega_t\omega_t', \tag{2.10}$$

where

$$\begin{aligned}\mathbb{E}\omega_t\omega_t' &= W'\mathbb{E}(\bar{x}_{At}\bar{x}_{At}')W = W'V_{xx,N}W \\ &= \begin{pmatrix} (\eta'\eta)^{-1/2}\eta' \\ W_2' \end{pmatrix} \left(\frac{1}{N}\Sigma_{xx} + \eta\eta'\right) \begin{pmatrix} (\eta'\eta)^{-1/2}\eta & W_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{N}(\eta'\eta)^{-1}\eta'\Sigma_{xx}\eta + \eta'\eta & \frac{1}{N}(\eta'\eta)^{-1/2}\eta'\Sigma_{xx}W_2 \\ \frac{1}{N}(\eta'\eta)^{-1/2}W_2'\Sigma_{xx}\eta & \frac{1}{N}W_2'\Sigma_{xx}W_2 \end{pmatrix}. \end{aligned} \tag{2.11}$$

Now if $N \to \infty$, it is not hard to see that, with $D_N = diag(1, \sqrt{N}I_{p-1})$,

$$D_N\mathbb{E}\omega_t\omega_t'D_N = \begin{pmatrix} \frac{1}{N}(\eta'\eta)^{-1}\eta'\Sigma_{xx}\eta + \eta'\eta & \frac{1}{\sqrt{N}}(\eta'\eta)^{-1/2}\eta'\Sigma_{xx}W_2 \\ \frac{1}{\sqrt{N}}(\eta'\eta)^{-1/2}W_2'\Sigma_{xx}\eta & W_2'\Sigma_{xx}W_2 \end{pmatrix} \to \begin{pmatrix} \eta'\eta & 0 \\ 0 & W_2'\Sigma_{xx}W_2 \end{pmatrix} \equiv V_{xx,\eta}, \tag{2.12}$$

which is non-singular. Therefore, as T and N go to infinity simultaneously, we have

$$\frac{1}{T|H|}D_N\sum_t \omega_t K(H^{-1}(z_t - z))\omega_t'D_N \xrightarrow{p} f_z(z)V_{xx,\eta}. \tag{2.13}$$

Then, with these transformations we get the following limit theory

$$\sqrt{T|H|}\begin{pmatrix} \sqrt{N} & \\ & I_{p-1} \end{pmatrix} (\hat{\theta}^{oracle}_{APLC}(z) - \theta(z) - B_2(z)) \Rightarrow N(0, f_z^{-1}(z)\sigma_u^2\nu_0^q V_{xx,\eta}^{-1}), \tag{2.14}$$

where $B_2(z) = f_z^{-1}(z)\mu_2 \sum_{s=1}^q h_s^2 \left[\frac{\partial f_z(z)}{\partial z_s}\frac{\partial \theta(z)}{\partial z_s} + \frac{1}{2}\frac{\partial^2\theta(z)}{\partial^2 z_s} f_z(z)\right]$. *Notice that* $\theta(z_t) = W'\beta(z_t) = \begin{pmatrix}(\eta'\eta)^{-1/2}\eta' \\ W_2'\end{pmatrix}\beta(z_t)$. *From* (2.14), *it is apparent that the linear combination* $\eta'\hat{\beta}_{APLC}^{oracle}(z)$ *in direction* η *converges at rate* $\sqrt{NT|H|}$, *whereas linear combinations in other directions converge at rate* $\sqrt{T|H|}$.

Remark 2.2. ***(Comments on cases (b) and (c) in Theorem 2.1)*** *In these two cases we only provide results here for finite* N *or for* $N \to \infty$ *when* $\eta = 0$. *The case where* $N \to \infty$ *with* $\eta \neq 0$ *is complicated by degeneracy in the first order asymptotics of* $\hat{\hat{\alpha}}$. *More details are given in the appendix, following the proof of (b2). Full consideration of this case involves detailed evaluation of higher order terms and is beyond the scope of the supplement. Analysis of this case is therefore left for future work.*

2.2 Asymptotics of the PLC estimators

Theorem 2.2. *Under Assumptions 1-3, as* $T \to \infty$, $|H| \to 0$, *and* $T|H| \to \infty$, *the following results hold.*

(i) For the oracle estimator $\hat{\beta}_{PLC}^{oracle}(z)$ *defined in* (1.9), *we have (*N *can be fixed or goes to infinity):*

$$\sqrt{NT|H|}(\hat{\beta}_{PLC}^{oracle}(z) - \beta(z) - B_1(z)) \Rightarrow N(0, f_z^{-1}(z)\nu_0^q\sigma_u^2 V_{xx}^{-1}). \tag{2.15}$$

(ii) For a finite N, *and the estimator* $\hat{\alpha}_{PLC}$ *defined in* (1.13), *we have* $\hat{\alpha}_{PLC} = \alpha + O_p(1/\sqrt{T})$. *Further, if* $\{u_{it}\}$ *is i.i.d. over* t, *then*

$$\sqrt{T}(\hat{\alpha}_{PLC} - \alpha) \Rightarrow N\left(0, \sigma_u^2\left[I_N - \frac{1}{N}\eta' V_{xx}^{-1}\eta(1_N \otimes 1_N')\right]^{-1}\right).$$

(iii) For a finite N *or* N *goes to infinity, the feasible estimator* $\hat{\beta}_{PLC}(z)$ *defined in* (1.14) *is asymptotically equivalent to the oracle estimator* $\hat{\beta}_{PLC}^{oracle}(z)$.

3 Numerical studies

3.1 Finite sample performance of the oracle and feasible APLC estimators

This section investigates the finite sample performance of the oracle estimator given in (1.3), the OLS estimator given in (1.7) and the feasible APLC estimator given in (1.8). We use the following DGP:

$$\begin{aligned} y_{it} &= \alpha_i + x_{it}\beta(z_t) + u_{it}, \\ x_{it} &= \eta + x_{it}^0, x_{it}^0 = \rho_1 x_{it-1}^0 + \xi_{it}, \\ \alpha_i &= c_0 x_{iA}^0 + v_i, \end{aligned}$$

where (u_{it}, ξ_{it}, v_i) is i.i.d. $N(\mathbf{0}, diag(1, 1 + \eta^2, 1))$, z_t is i.i.d. $U(-1, 1)$. The functional coefficient is $\beta(z) = 1 + z^2$. The parameter c_0 controls the correlation between α_i and $x_{iA}^0 = T^{-1}\sum_{t=1}^T x_{it}^0$. We use the values $c_0 = 1$ (fixed effect) and $\rho_1 = 0.5$. The DGP of ξ_{it} ensures that the variance of ξ_{it} is of order $|\eta|^2$, and so is the variance of x_{it}. This is mainly to avoid the peculiar situation where there is not enough variation in x_{it} when η is large. The bandwidth is set as $h = \hat{\sigma}_z T^{-1/5}$, where $\hat{\sigma}_z$ denotes the sample standard deviation of $\{z_t\}_{t=1}^T$.

Note that x_{it}^0 has zero-mean, so η is the mean of x_{it}. We tried $\eta \in (0, 1, 5)$ under three different values of T (50, 100 and 200). N is fixed at $N = 5$. For each η, we replicate the estimation procedure B=400 times and obtain 400 estimates, denoted $\{\hat{\beta}^{(l)}(z)\}_{l=1}^B$ and $\{\hat{\bar{\alpha}}^{(l)}\}_{l=1}^B$.

To evaluate estimation accuracy we use the following squared error criteria:

$$AMSE(\hat{\beta}(z)) = \frac{1}{B}\sum_{l=1}^{B}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\hat{\beta}^{(l)}(z_t) - \beta(z_t)\right)^2\right]; \tag{3.1}$$

$$ASE(\hat{\bar{\alpha}}) = \frac{1}{B}\sum_{l=1}^{B}\left(\hat{\bar{\alpha}}^{(l)} - \bar{\alpha}^{(l)}\right)^2. \tag{3.2}$$

Results are summarized in Table 1. The oracle estimator is evidently slightly better in overall performance than the feasible estimator. As T increase, all estimators have improved performance. We also observe that the performance of the estimator $\hat{\bar{\alpha}}$ deteriorates as η increases, which leads to greater discrepancy between the oracle and feasible estimators in such case. It is possible to study asymptotic behavior as $|\eta| \to \infty$ but investigation of that case is left for future study.

3.2 Finite sample performance of the oracle and feasible PLC estimators

This section considers the PLC estimators, including the oracle estimator given in (1.9), the feasible estimator given in (1.14). The DGP is the same with the previous subsection. The bandwidth h is determined by $h = \hat{\sigma}_z(NT)^{-1/5}$, where $\hat{\sigma}_z$ is the sample standard deviation of $\{z_t\}_{t=1}^T$. Upon replication, the PLC estimation procedure returns B=400 estimates $\{\hat{\alpha}_{i,PLC}^{(l)}\}_{l=1}^B$, for $i = 1, \cdots, N$. We use the following criterion to assess performance

$$AMSE(\alpha) = \frac{1}{B}\sum_{l=1}^{B}(\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_{i,PLC}^{(l)} - \alpha_i^{(l)})^2). \quad (3.3)$$

The results are summarized in Table 2. Evidently, the oracle estimator has overall performance that is slightly better than the feasible estimator. The performance of $\hat{\alpha}_{PLC}$ also deteriorates as η increases, but at a very slow rate. Comparison with Table 1 reveals that the PLC method is more efficient than the APLC method, as expected.

Appendix

We start with some notation. We use 'smaller order' to represent terms that are of smaller order than the terms stated explicitly. C is a constant that could take different values at different places. For a column vector ξ, denote $|\xi|$ as its L_2 norm, namely $|\xi| = (\xi'\xi)^{1/2}$. For a diagonal matrix $A = diag(a_1, \cdots, a_p)$, denote $|A| = a_1 \cdots a_p$. The notation $VecDiag(B)$ means take the diagonal elements of the square matrix B and stack them as a column vector. According to the context, we use := and =: to signify definitional equality. For random vectors ξ and η, we use $\xi \sim_a \eta$ to indicate that they share the same asymptotic distribution. $1_{T\times 1}$ denotes a $T \times 1$ vector with unity in each position.

A Proof of Theorem 2.1

(a) Proof for the oracle estimator $\hat{\beta}^{oracle}_{APLC}$

Note that model (1.2) is a standard time series functional-coefficient model if we assume $\bar{\alpha}$ is known. The limit theory for $\hat{\beta}^{oracle}_{APLC}(z)$ therefore follows standard results and we only provide a sketch of the proof here.

First, the d-th element of $\beta(z_t)$, $\beta_d(z_t)$, $d = 1, \cdots, p$, admits the following second order Taylor expansion at a given point $z = (z_1, \cdots, z_q)'$:

$$\begin{aligned}
\beta_d(z_t) &= \beta_d(z) + (z_t - z)'\beta_d^{(1)}(z) + \frac{1}{2}(z_t - z)'\beta_d^{(2)}(z)(z_t - z) + o_p(|z_t - z|^2) \\
&= \beta_d(z) + [H^{-1}(z_t - z)]'H\beta_d^{(1)}(z) + \frac{1}{2}[H^{-1}(z_t - z)]'H\beta_d^{(2)}(z)H[H^{-1}(z_t - z)] + o_p(|z_t - z|^2) \\
&=: \beta_d(z) + \dot{\beta}_d(z, z_t) + \frac{1}{2}\ddot{\beta}_d(z, z_t) + o_p(|z_t - z|^2),
\end{aligned}$$

where $H = diag(h_1, \cdots, h_q)$, $|z_t - z|^2 = \sum_{j=1}^{q}(z_{t,j} - z_j)^2$. The definition of $\dot{\beta}_d(z, z_t)$ and $\ddot{\beta}_d(z, z_t)$ follow from the context and they are scalars. Then for the $p \times 1$ vector $\beta(z_t)$, we can write

$$\beta(z_t) = \begin{pmatrix} \beta_1(z_t) \\ \vdots \\ \beta_p(z_t) \end{pmatrix} = \beta(z) + \dot{\beta}(z, z_t) + \frac{1}{2}\ddot{\beta}(z, z_t) + o_p(|z_t - z|^2)1_{p\times 1}, \tag{A.1}$$

where $\dot{\beta}(z, z_t)$ is a $p \times 1$ vector with d-th element $\dot{\beta}_d(z, z_t)$ and $\ddot{\beta}(z, z_t)$ is $p \times 1$ with d-th element being $\ddot{\beta}_d(z, z_t)$. Now we plug the Taylor expansion of $\beta(z_t)$ in (A.1) into model

(1.2) and get

$$\bar{y}_{At} = \bar{\alpha} + \bar{x}'_{At}\beta(z) + \bar{x}'_{At}\dot{\beta}(z, z_t) + \frac{1}{2}\bar{x}'_{At}\ddot{\beta}(z, z_t) + \bar{u}_{At} + \textit{smaller order}, \tag{A.2}$$

or in vector form

$$\bar{Y}_A = \bar{\alpha}1_{T\times 1} + \bar{X}_A\beta(z) + \bar{U}_A + VecDiag(\bar{X}_A\dot{\beta}(z)) + \frac{1}{2}VecDiag(\bar{X}_A\ddot{\beta}(z)) + \textit{smaller order}, \tag{A.3}$$

where $\bar{X}_A$ is $T \times p$ by stacking $\bar{x}'_{At}$, $\dot{\beta}(z)$ is $p \times T$ with the t-th column being the $p \times 1$ vector $\dot{\beta}(z, z_t)$, and $\ddot{\beta}(z)$ is similarly defined. Then we have

$$\begin{aligned}
\hat{\beta}^{oracle}_{APLC}(z) - \beta(z) =&[\bar{X}'_A K_z \bar{X}_A]^{-1}\bar{X}'_A K_z(\bar{Y}_A - \bar{\alpha}1_{T\times 1}) - \beta(z) \\
=&[\bar{X}'_A K_z \bar{X}_A]^{-1}\bar{X}'_A K_z \Big(\bar{X}_A\beta(z) + \bar{U}_A + VecDiag(\bar{X}_A\dot{\beta}(z)) \\
&+1/2VecDiag(\bar{X}_A\ddot{\beta}(z)) + \textit{smaller order}\Big) - \beta(z) \\
=&[\bar{X}'_A K_z \bar{X}_A]^{-1}\bar{X}'_A K_z \bar{U}_A + [\bar{X}'_A K_z \bar{X}_A]^{-1}\bar{X}'_A K_z VecDiag(\bar{X}_A\dot{\beta}(z)) \\
&+ 1/2[\bar{X}'_A K_z \bar{X}_A]^{-1}\bar{X}'_A K_z VecDiag(\bar{X}_A\ddot{\beta}(z)) + \textit{smaller order}.
\end{aligned} \tag{A.4}$$

Consider the first term in (A.4). Combining Lemma C.1 and C.2, we have the following results as $T \to \infty$. First, when N is fixed,

$$\sqrt{T|H|}[\bar{X}'_A K_z \bar{X}_A]^{-1}\bar{X}'_A K_z \bar{U}_A \Rightarrow N(0, f_z^{-1}(z)\nu_0^q\sigma_u^2 N^{-1}V_{xx,N}^{-1}); \tag{A.5}$$

when $N \to \infty$ and $\eta = 0$,

$$\sqrt{T|H|}[\bar{X}'_A K_z \bar{X}_A]^{-1}\bar{X}'_A K_z \bar{U}_A \Rightarrow N(0, f_z^{-1}(z)\nu_0^q V_{xx}^{-1}\sigma_u^2);$$

when $N \to \infty$, $\eta \neq 0$ and $p = 1$,

$$\sqrt{NT|H|}[\bar{X}'_A K_z \bar{X}_A]^{-1}\bar{X}'_A K_z \bar{U}_A \Rightarrow N(0, f_z^{-1}(z)\nu_0^q \eta^{-2}\sigma_u^2);$$

when $N \to \infty$, $\eta \neq 0$ and $p > 1$,

$$\sqrt{T|H|}[\bar{X}'_A K_z \bar{X}_A]^{-1}\bar{X}'_A K_z \bar{U}_A \Rightarrow N(0, f_z^{-1}(z)\nu_0^q\sigma_u^2(V_{xx}^{-1} - \frac{V_{xx}^{-1}\eta\eta' V_{xx}^{-1}}{\eta' V_{xx}^{-1}\eta})),$$

which is a degenerate limit distribution.

Now we look at the bias term, namely the second and third terms in (A.4). In view of Lemma C.1 and C.3, we have when N is fixed,

$$[\bar{X}_A' K_z \bar{X}_A]^{-1} \bar{X}_A' K_z diag(\bar{X}_A \dot{\beta}(z)) \xrightarrow{p} f_z^{-1}(z)\mu_2 \sum_s h_s^2 \frac{\partial f_z(z)}{\partial z_s} \frac{\partial \beta(z)}{\partial z_s}, \tag{A.6}$$

and

$$[\bar{X}_A' K_z \bar{X}_A]^{-1} \bar{X}_A' K_z VecDiag(\bar{X}_A \ddot{\beta}(z)) \xrightarrow{p} \mu_2 \sum_s h_s^2 \frac{\partial^2 \beta(z)}{\partial^2 z_s}. \tag{A.7}$$

When $N \to \infty$ (A.6) and (A.7) continue to hold. The analyses are similar and are omitted. So the asymptotic bias is

$$B_1(z) = f_z^{-1}(z)\mu_2 \sum_{s=1}^{q} h_s^2 [\frac{\partial f_z(z)}{\partial z_s} \frac{\partial \beta(z)}{\partial z_s} + \frac{1}{2} \frac{\partial^2 \beta(z)}{\partial^2 z_s} f_z(z)].$$

Finally, we obtain the following results. When N is fixed,

$$\sqrt{T|H|}(\hat{\beta}_{APLC}^{oracle}(z) - \beta(z) - B_1(z)) \Rightarrow N(0, f_z^{-1}(z)\nu_0^q \sigma_u^2 N^{-1} V_{xx,N}^{-1});$$

when $N \to \infty$ and $\eta = 0$,

$$\sqrt{T|H|}(\hat{\beta}_{APLC}^{oracle}(z) - \beta(z) - B_1(z)) \Rightarrow N(0, f_z^{-1}(z)\nu_0^q V_{xx}^{-1} \sigma_u^2);$$

when $N \to \infty$, $\eta \neq 0$ and $p = 1$,

$$\sqrt{NT|H|}(\hat{\beta}_{APLC}^{oracle}(z) - \beta(z) - B_1(z)) \Rightarrow N(0, f_z^{-1}(z)\nu_0^q \eta^{-2} \sigma_u^2);$$

when $N \to \infty$, $\eta \neq 0$ and $p > 1$,

$$\sqrt{T|H|}(\hat{\beta}_{APLC}^{oracle}(z) - \beta(z) - B_1(z)) \Rightarrow N(0, f_z^{-1}(z)\nu_0^q \sigma_u^2 (V_{xx}^{-1} - \frac{V_{xx}^{-1}\eta\eta' V_{xx}^{-1}}{\eta' V_{xx}^{-1} \eta})),$$

which is a degenerate distribution. ■

(b) Proof for the estimator $\hat{\bar{\alpha}}$

Based on (1.7), we have

$$\begin{aligned}
\hat{\bar{\alpha}} - \bar{\alpha} =& (\sum_t x_{T,t}^{*2})^{-1} \sum_t x_{T,t}^* \tilde{u}_t \\
=& (\sum_t x_{T,t}^{*2})^{-1} \sum_t x_{T,t}^* \bar{x}_{At}' [\beta(z_t) - \hat{\beta}_{APLC}^{oracle}(z_t)] + (\sum_t x_{T,t}^{*2})^{-1} \sum_t x_{T,t}^* \bar{u}_{At}.
\end{aligned} \tag{A.8}$$

To analyze the two terms in (A.8), we first look at $x^*_{T,t}$. Note that

$$\begin{aligned} x^*_{T,t} &= 1 - \bar{x}'_{At} s(z_t) 1_{T\times 1} \\ &= 1 - \bar{x}'_{At}\left[\sum_s \bar{x}_{As}\bar{x}'_{As} K(H^{-1}(z_s - z_t))\right]^{-1}\left[\sum_s \bar{x}_{As} K(H^{-1}(z_s - z_t))\right] \\ &= 1 - \bar{x}'_{At}\left[\frac{1}{T|H|}\sum_s \bar{x}_{As}\bar{x}'_{As} K(H^{-1}(z_s - z_t))\right]^{-1}\left[\frac{1}{T|H|}\sum_s \bar{x}_{As} K(H^{-1}(z_s - z_t))\right], \end{aligned} \tag{A.9}$$

which involves averages over the time dimension. To simplify the analysis, we replace those averages with their probability limits.

(b1) First consider the case where $T \to \infty$ and N is fixed. As $T \to \infty$, we have $\frac{1}{T|H|}\sum_s \bar{x}_{As} K(H^{-1}(z_s - z_t)) \xrightarrow{p} \mathbb{E}(\bar{x}_{As}) f_z(z_t) = \eta f_z(z_t)$, and $\frac{1}{T|H|}\sum_s \bar{x}_{As}\bar{x}'_{As} K(H^{-1}(z_s - z_t)) \xrightarrow{p} \mathbb{E}(\bar{x}_{As}\bar{x}'_{As}) f_z(z_t) = V^{-1}_{xx,N} f_z(z_t)$ (Lemma C.1). Then we have

$$x^*_{T,t} \xrightarrow{p} 1 - \bar{x}'_{At} V^{-1}_{xx,N}\eta. \tag{A.10}$$

Similarly, it can be shown that as $T \to \infty$,

$$\frac{1}{T}\sum_t x^{*2}_{T,t} \xrightarrow{p} \mathbb{E}[1 - \bar{x}'_{At} V^{-1}_{xx,N}\eta]^2 = 1 - \eta' V^{-1}_{xx,N}\eta.$$

Then consider $\sum_t x^*_{T,t}\bar{u}_{At}$. Evidently, $\mathbb{E}[\sum_t x^*_{T,t}\bar{u}_{At}] = \sum_t \mathbb{E}(x^*_{T,t}\bar{u}_{At}) = 0$ because $\{u_{it}\}$ is zero-mean and independent with $\{z_t, x_{it}\}$. We compute its long run variance by calculating

$$\begin{aligned} \mathbb{E}\left(\frac{1}{\sqrt{T}}\sum_t x^*_{T,t}\bar{u}_{At}\right)^2 &= \frac{1}{T}\sum_t\sum_s \mathbb{E}x^*_{T,t}\bar{u}_{At}x^*_{T,s}\bar{u}_{As} \\ &= \frac{1}{T}\left(\sum_t \mathbb{E}x^{*2}_{T,t}\mathbb{E}\bar{u}^2_{At} + \sum_{t\neq s}\mathbb{E}x^*_{T,t}x^*_{T,s}\mathbb{E}\bar{u}_{At}\bar{u}_{As}\right) \\ &= \mathbb{E}x^{*2}_{T,t}\mathbb{E}\bar{u}^2_{At} + 2\sum_{\ell=1}^{T-1}(1-\ell/T)\gamma_{x^*_T}(\ell)\gamma_{\bar{u}}(\ell), \end{aligned}$$

where $\mathbb{E}\bar{u}^2_{At} = \frac{1}{N}\sigma^2_u$, $\gamma_{\bar{u}}(\ell) = \frac{1}{N}\gamma_u(\ell)$ since u_{it} is i.i.d. across i with zero-mean, and

$$\begin{aligned} \gamma_{x^*_T}(\ell) = \mathbb{E}x^*_{T,t}x^*_{T,t+\ell} &\to \mathbb{E}[1 - \bar{x}'_{At}V^{-1}_{xx,N}\eta][1 - \bar{x}'_{At+\ell}V^{-1}_{xx,N}\eta] \\ &= 1 - 2\eta' V^{-1}_{xx,N}\eta + \eta' V^{-1}_{xx,N}\mathbb{E}[\bar{x}_{At}\bar{x}'_{At+\ell}]V^{-1}_{xx,N}\eta, \end{aligned}$$

where

$$\mathbb{E}[\bar{x}_{At}\bar{x}'_{At+\ell}] = \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} x_{it}][\frac{1}{N}\sum_{j=1}^{N} x'_{jt+\ell}] = \frac{1}{N^2}\sum_{i}\sum_{j}\mathbb{E}x_{it}x'_{jt+\ell}$$
$$= \frac{1}{N^2}(\sum_{i}\mathbb{E}x_{it}x'_{it+\ell} + \sum_{i\neq j}\mathbb{E}x_{it}x'_{jt+\ell})$$
$$= \frac{1}{N}\gamma_x(\ell) + (1-\frac{1}{N})\eta\eta'.$$

Therefore we have

$$\gamma_{x^*_T}(\ell) \to 1 - 2\eta' V_{xx,N}^{-1}\eta + \eta' V_{xx,N}^{-1}\left(\frac{1}{N}(\gamma_x(\ell) - \eta\eta') + \eta\eta'\right) V_{xx,N}^{-1}\eta$$
$$= 1 - 2\eta' V_{xx,N}^{-1}\eta + (1-\frac{1}{N})(\eta' V_{xx,N}^{-1}\eta)^2 + \frac{1}{N}\eta' V_{xx,N}^{-1}\gamma_x(\ell)V_{xx,N}^{-1}\eta$$
$$=: \gamma_{x^*}(\ell).$$

Thus, as $T\to\infty$

$$\mathbb{E}\left(\frac{1}{\sqrt{T}}\sum_t x^*_{T,t}\bar{u}_{At}\right)^2 \to (1-\eta' V_{xx,N}^{-1}\eta)\frac{1}{N}\sigma_u^2 + 2\sum_{\ell=1}^{\infty}\gamma_{x^*}(\ell)\frac{1}{N}\gamma_u(\ell).$$

Therefore, for the second term in (A.8), we have

$$\sqrt{T}(\sum_t x^{*2}_{T,t})^{-1}\sum_t x^*_{T,t}\bar{u}_{At} \Rightarrow N(0, \sigma^2_{\bar{\alpha},N}), \tag{A.11}$$

where

$$\sigma^2_{\bar{\alpha},N} =(1-\eta' V_{xx,N}^{-1}\eta)^{-2}\left[(1-\eta' V_{xx,N}^{-1}\eta)\frac{1}{N}\sigma_u^2 + 2\sum_{\ell=1}^{\infty}\gamma_{x^*}(\ell)\frac{1}{N}\gamma_u(\ell)\right]$$
$$=\frac{\sigma_u^2}{N}(1-\eta' V_{xx,N}^{-1}\eta)^{-1} + \frac{2}{N}(1-\eta' V_{xx,N}^{-1}\eta)^{-2}\sum_{\ell=1}^{\infty}\gamma_{x^*}(\ell)\gamma_u(\ell). \tag{A.12}$$

If we further assume that $\{u_{it}\}$ is i.i.d. across t, then $\gamma_u(\ell)=0$ for all $\ell>0$ and we have

$$\sigma^2_{\bar{\alpha},N} = (1-\eta' V_{xx,N}^{-1}\eta)^{-1}\frac{1}{N}\sigma_u^2 = \sigma_u^2(\frac{1}{N} + \eta'\Sigma_{xx}^{-1}\eta). \tag{A.13}$$

Now consider the first term in (A.8). In view of (A.9), we have

$$x^*_{T,t}\bar{x}'_{At} \xrightarrow{p} (1-\bar{x}'_{At}V_{xx,N}^{-1}\eta)\bar{x}'_{At}.$$

Note that $\mathbb{E}(1-\bar{x}'_{At}V_{xx,N}^{-1}\eta)\bar{x}'_{At}=\mathbb{E}(\bar{x}'_{At})-\eta'V_{xx,N}^{-1}\mathbb{E}\bar{x}_{At}\bar{x}'_{At}=0$ and, by uniform convergence of the kernel regression estimator, $\sup_{t\le T}|\beta(z_t)-\hat{\beta}_{APLC}^{oracle}(z_t)|=o_p(1)$, so we have $T^{-1/2}\sum_t x_{T,t}^*\bar{x}'_{At}[\beta(z_t)-\hat{\beta}_{APLC}^{oracle}(z_t)]=o_p(1)$. Thus the first term in (A.8) is of order $o_p(1/\sqrt{T})$, which is smaller than that of the second term. So the asymptotic distribution is determined by the second term, which gives $\sqrt{T}(\hat{\bar{\alpha}}-\bar{\alpha})\Rightarrow N(0,\sigma^2_{\bar{\alpha},N})$.

(b2) Then we let $N\to\infty$ and $\eta=0$. Here, $\sqrt{N}\bar{x}_{At}=O_p(1)$ and $\frac{1}{T|H|}\sum_s\bar{x}_{As}K(H^{-1}(z_s-z_t))=O_p(1/\sqrt{NT|H|})$. From Lemma C.1, we know $\frac{1}{T|H|}\sum_s\bar{x}_{As}\bar{x}'_{As}K(H^{-1}(z_s-z_t))=O_p(1/N)$. Consequently, the second term in (A.9) is of order $O_p(1/\sqrt{T|H|})$. Therefore we have $x_{T,t}^*\xrightarrow{p}1$. Similarly, it can be shown that $\frac{1}{T}\sum_t x_{T,t}^{*2}\xrightarrow{p}1$. Thus we have $\sum_t x_{T,t}^*\bar{u}_{At}\sim_a\sum_t\bar{u}_{At}$, where $\frac{\sqrt{N}}{\sqrt{T}}\sum_t\bar{u}_{At}=O_p(1)$. So the second term of (A.8) is of order $O_p(1/\sqrt{NT})$. Similarly, one can show that the first term of (A.8) is of order $o_p(1/\sqrt{NT})$. Then we have $\hat{\bar{\alpha}}-\bar{\alpha}=O_p(1/\sqrt{NT})$.

In the case where $N\to\infty$ and $\eta\neq0$, the above approach fails. To see this, when $p=1$ observe that $x_{T,t}^*\xrightarrow{p}1-\eta'(\eta\eta')^{-1}\eta=0$ and $T^{-1}\sum_t x_{T,t}^{*2}\xrightarrow{p}0$. If $p>1$, the probability limit, $\eta\eta'f_z(z_t)$, of $\frac{1}{T|H|}\sum_s\bar{x}_{As}\bar{x}_{As}K(H^{-1}(z_s-z_t))$ is singular. In view of the zero/singular limit, the convergence rates in this case are different and analysis involves the limit behavior of higher order terms. The derivations are more complex and go beyond the scope of this paper and supplement. We therefore leave the investigation of this degenerate case to future work.

■

(c) Proof for the feasible estimator $\hat{\beta}_{APLC}(z)$

First we have

$$\begin{aligned}\hat{\beta}_{APLC}(z)-\beta(z)&=[\bar{X}'_AK_z\bar{X}_A]^{-1}\bar{X}'_AK_z(\bar{Y}_A-\hat{\bar{\alpha}}1_{T\times1})-\beta(z)\\&=[\bar{X}'_AK_z\bar{X}_A]^{-1}\bar{X}'_AK_z[\bar{Y}_A-\bar{\alpha}1_{T\times1}+(\bar{\alpha}-\hat{\bar{\alpha}})1_{T\times1}]-\beta(z)\\&=\hat{\beta}_{APLC}^{oracle}(z)-\beta(z)-[\bar{X}'_AK_z\bar{X}_A]^{-1}\bar{X}'_AK_z(\hat{\bar{\alpha}}-\bar{\alpha})1_{T\times1}.\end{aligned}\tag{A.14}$$

We see that the asymptotic distribution of the feasible estimator depends on that of the oracle estimator and that of $\hat{\bar{\alpha}}$.

First consider the case where N is fixed. Since $\frac{1}{T|H|}\bar{X}'_AK_z\bar{X}_A\xrightarrow{p}f_z(z)V_{xx,N}$ and $\frac{1}{T|H|}\bar{X}'_AK_z1_{T\times1}\xrightarrow{p}f_z(z)\eta$, we will have

$$\hat{\beta}_{APLC}(z)-\beta(z)-B_1(z)\sim_a\hat{\beta}_{APLC}^{oracle}(z)-\beta(z)-B_1(z)-V_{xx,N}^{-1}\eta(\hat{\bar{\alpha}}-\bar{\alpha}).\tag{A.15}$$

Rewrite (A.15) as

$$\sqrt{T|H|}(\hat{\beta}_{APLC}(z) - \beta(z) - B_1(z)) \sim_a \sqrt{T|H|}(\hat{\beta}^{oracle}_{APLC}(z) - \beta(z) - B_1(z)) - \sqrt{|H|}V^{-1}_{xx,N}\eta\sqrt{T}(\hat{\bar{\alpha}} - \bar{\alpha}), \tag{A.16}$$

We already know $\sqrt{T|H|}(\hat{\beta}^{oracle}_{APLC}(z) - \beta(z) - B_1(z))$ is of order $O_p(1)$, and $\sqrt{T}(\hat{\bar{\alpha}} - \bar{\alpha})$ is of order $O_p(1)$. Since $|H| \to 0$, the asymptotic distribution is dominated by that of $\hat{\beta}^{oracle}_{APLC}(z)$. Then the asymptotic distribution of $\hat{\beta}_{APLC}(z)$ is the same as (2.1).

If $N \to \infty$ and $\eta = 0$, we have $\frac{1}{T|H|}\bar{X}'_A K_z \bar{X}_A = O_p(1/N)$ and $\frac{1}{T|H|}\bar{X}'_A K_z 1_{T\times 1} = O_p(1/\sqrt{NT|H|})$,and $\hat{\bar{\alpha}} - \bar{\alpha} = O_p(1/\sqrt{NT})$. Therefore, the second term of (A.14) is of order $O_p(1/T\sqrt{|H|})$, which is smaller than that of the first term. Thus the asymptotic distribution of the feasible estimator is dominated by that of the oracle estimator. ■

B Proof of Theorem 2.2

(i) Proof for the oracle estimator $\hat{\beta}^{oracle}_{PLC}(z)$

First we have the following second order Taylor expansion for $\beta(z)$ (see (A.1) for details)

$$\beta(z_t) = \beta(z) + \dot{\beta}(z, z_t) + \frac{1}{2}\ddot{\beta}(z, z_t) + o_p(|z_t - z|^2)1_{p\times 1}.$$

Using this Taylor expansion in model (1.1) we get

$$y_{it} = \alpha_i + x'_{it}\beta(z) + x'_{it}\dot{\beta}(z, z_t) + x'_{it}\frac{1}{2}\ddot{\beta}(z, z_t) + u_{it} + smaller\ order,$$

or in vector form

$$Y = D\alpha + X\beta(z) + VecDiag(X[1_{1\times N} \otimes \dot{\beta}(z)]) + \frac{1}{2}VecDiag(X[1_{1\times N} \otimes \ddot{\beta}(z)]) + U + smaller\ order,$$

where $\dot{\beta}(z)$ is $p \times T$ with the t-th column being the $p \times 1$ vector $\dot{\beta}(z, z_t)$, and where $\ddot{\beta}(z)$ is defined in a similar way. Then we have

$$\begin{aligned}\hat{\beta}^{oracle}_{PLC}(z) - \beta(z) =& [X'K(z)X]^{-1}X'K(z)U + [X'K(z)X]^{-1}X'K(z)VecDiag(X[1_{1\times N} \otimes \dot{\beta}(z)]) \\ &+ [X'K(z)X]^{-1}X'K(z)\frac{1}{2}VecDiag(X[1_{1\times N} \otimes \ddot{\beta}(z)]) + smaller\ order.\end{aligned} \tag{B.1}$$

Consider the first term in (B.1). In view of Lemma C.4 and C.5, we have $\frac{1}{NT|H|}X'K(z)X \xrightarrow{p} f_z(z)V_{xx}$ and $\frac{1}{\sqrt{NT|H|}}X'K(z)U \Rightarrow N(0, f_z(z)\nu_0^q\sigma_u^2 V_{xx})$. So for the first term in (B.1), we have

$$\sqrt{NT|H|}[X'K(z)X]^{-1}X'K(z)U \quad \Rightarrow \quad N(0, f_z^{-1}(z)\nu_0^q\sigma_u^2 V_{xx}^{-1}). \tag{B.2}$$

Next consider the third term in (B.1). In view of Lemma C.6, we have

$$\frac{1}{NT|H|}X'K(z)VecDiag[X(1_{1\times N}\otimes\ddot{\beta}(z))] \xrightarrow{p} f_z(z)\mu_2 V_{xx}\sum_{s=1}^{q} h_s^2 \frac{\partial^2\beta(z)}{\partial^2 z_s}.$$

Then

$$[X'K(z)X]^{-1}X'K(z)VecDiag(X[1_{1\times N}\otimes\ddot{\beta}(z)]) \xrightarrow{p} \mu_2\sum_{s=1}^{q} h_s^2 \frac{\partial^2\beta(z)}{\partial^2 z_s}. \tag{B.3}$$

A similar technique can be used to analyze the second term in (B.1), leading to

$$\frac{1}{NT|H|}X'K(z)VecDiag(X[1_{1\times N}\otimes\dot{\beta}(z)]) \xrightarrow{p} \mu_2 V_{xx}\sum_{s=1}^{q} h_s^2 \frac{\partial\beta(z)}{\partial z_s}\frac{\partial f_z(z)}{\partial z_s}.$$

Thus

$$[X'K(z)X]^{-1}X'K(z)VecDiag(X[1_{1\times N}\otimes\dot{\beta}(z)]) \xrightarrow{p} \mu_2 f_z^{-1}(z)\sum_{s=1}^{q} h_s^2 \frac{\partial\beta(z)}{\partial z_s}\frac{\partial f_z(z)}{\partial z_s}. \tag{B.4}$$

Combining (B.1), (B.2), (B.3) and (B.4), we have

$$\sqrt{NT|H|}(\hat{\beta}_{PLC}^{oracle}(z) - \beta(z) - B_1(z)) \Rightarrow N(0, f_z^{-1}(z)\nu_0^q\sigma_u^2 V_{xx}^{-1}), \tag{B.5}$$

where the bias is $B_1(z) = \mu_2 f_z^{-1}(z)\sum_{s=1}^{q} h_s^2\left[\frac{\partial\beta(z)}{\partial z_s}\frac{\partial f_z(z)}{\partial z_s} + \frac{1}{2}\frac{\partial^2\beta(z)}{\partial^2 z_s}f_z(z)\right]$.

(ii) Proof for $\hat{\alpha}_{PLC}$

For the estimator $\hat{\alpha}_{PLC}$, based on (1.12) and (1.13) we have

$$\hat{\alpha}_{PLC} - \alpha = [M_2'M_2]^{-1}M_2'V, \tag{B.6}$$

where $V = VecDiag(X\{1_N'\otimes[\beta(Z) - \hat{\beta}_{PLC}^{oracle}(Z)]\}) + U$, with $v_{it} = x_{it}'[\beta(z_t) - \hat{\beta}_{PLC}^{oracle}(z_t)] + u_{it}$, where $\beta(Z)$ is $p\times T$ with t-th column $\beta(z_t)$, and $\hat{\beta}_{PLC}^{oracle}(Z)$ is similarly defined. Then we have

$$\hat{\alpha}_{PLC} - \alpha = [M_2'M_2]^{-1}M_2'VecDiag(X\{1_N'\otimes[\beta(Z) - \hat{\beta}_{PLC}^{oracle}(Z)]\}) + [M_2'M_2]^{-1}M_2'U. \tag{B.7}$$

We now analyze the term $[M_2'M_2]^{-1}$, noting that $M_2 = (I_{NT} - Q_1)D$. First consider the matrix Q_1 which has the form

$$
\begin{aligned}
Q_1 &= \sum_{d=1}^{p} (\mathbf{x}_d \otimes 1_n') \odot (1_N \otimes w_d(Z)) \\
&= \begin{pmatrix} \sum_d x_{11,d} w_d(z_1) \\ \vdots \\ \sum_d x_{1T,d} w_d(z_T) \\ \vdots \\ \sum_d x_{N1,d} w_d(z_1) \\ \vdots \\ \sum_d x_{NT,d} w_d(z_T) \end{pmatrix} = \begin{pmatrix} x_{11}' w(z_1) \\ \vdots \\ x_{1T}' w(z_T) \\ \vdots \\ x_{N1}' w(z_1) \\ \vdots \\ x_{NT}' w(z_T) \end{pmatrix},
\end{aligned}
$$

where $w(z_t) = [X'K(z_t)X]^{-1}X'K(z_t)$ is $p \times n$. Then each row $x_{it}'w(z_t)$ is $1 \times n$. For the matrix Q_1D, the typical row at the j-th column is $x_{it}'[X'K(z_t)X]^{-1}\sum_{s=1}^{T} x_{js}K(H^{-1}(z_s - z_t))$, for $j = 1, ..., N$. We already know $\frac{1}{NT|H|}X'K(z)X \xrightarrow{p} f_z(z)V_{xx}$ from Lemma C.4. Then we have the following asymptotic approximation as $T \to \infty$

$$
Q_1 D \sim_a \frac{1}{N} \begin{pmatrix} x_{11}' V_{xx}^{-1} \eta_1 & \cdots & x_{11}' V_{xx}^{-1} \eta_N \\ \vdots & & \vdots \\ x_{1T}' V_{xx}^{-1} \eta_1 & \cdots & x_{1T}' V_{xx}^{-1} \eta_N \\ \vdots & & \vdots \\ x_{N1}' V_{xx}^{-1} \eta_1 & \cdots & x_{N1}' V_{xx}^{-1} \eta_N \\ \vdots & & \vdots \\ x_{NT}' V_{xx}^{-1} \eta_1 & \cdots & x_{NT}' V_{xx}^{-1} \eta_N \end{pmatrix} = \frac{1}{N} X V_{xx}^{-1} (\eta_1, \cdots, \eta_N).
$$

Thus

$$
M_2 = D - Q_1 D \sim_a \begin{pmatrix} 1_T - \frac{1}{N}\begin{pmatrix} x_{11}' \\ \vdots \\ x_{1T}' \end{pmatrix} V_{xx}^{-1}\eta_1 & -\frac{1}{N}\begin{pmatrix} x_{11}' \\ \vdots \\ x_{1T}' \end{pmatrix} V_{xx}^{-1}\eta_2 & \cdots & -\frac{1}{N}\begin{pmatrix} x_{11}' \\ \vdots \\ x_{1T}' \end{pmatrix} V_{xx}^{-1}\eta_N \\ \vdots & & & \vdots \\ -\frac{1}{N}\begin{pmatrix} x_{N1}' \\ \vdots \\ x_{NT}' \end{pmatrix} V_{xx}^{-1}\eta_1 & & \cdots & 1_T - \frac{1}{N}\begin{pmatrix} x_{N1}' \\ \vdots \\ x_{NT}' \end{pmatrix} V_{xx}^{-1}\eta_N \end{pmatrix},
$$

and so as $T \to \infty$

$$T^{-1}M_2'M_2 \xrightarrow{p} \begin{pmatrix} 1-\frac{1}{N}\eta_1'V_{xx}^{-1}\eta_1 & -\frac{1}{N}\eta_1'V_{xx}^{-1}\eta_2 & \cdots & -\frac{1}{N}\eta_1'V_{xx}^{-1}\eta_N \\ \vdots & & & \vdots \\ -\frac{1}{N}\eta_N'V_{xx}^{-1}\eta_1 & & \cdots & 1-\frac{1}{N}\eta_N'V_{xx}^{-1}\eta_N \end{pmatrix}$$
$$= I_N - \frac{1}{N}\begin{pmatrix}\eta_1' \\ \vdots \\ \eta_N'\end{pmatrix} V_{xx}^{-1}(\eta_1, \cdots, \eta_N).$$

Since $\eta_1 = \cdots = \eta_N = \eta$, we have

$$T^{-1}M_2'M_2 \xrightarrow{p} I_N - \frac{1}{N}\eta' V_{xx}^{-1}\eta(1_N \otimes 1_N'). \tag{B.8}$$

Now consider the first term of (B.7). A typical row of $VecDiag(X\{1_N' \otimes [\beta(Z) - \hat{\beta}_{PLC}^{oracle}(Z)]\})$ is $x_{it}'[\beta(z_t) - \hat{\beta}_{PLC}^{oracle}(z_t)]$. The j-th row of $M_2'VecDiag(X\{1_N' \otimes [\beta(Z) - \hat{\beta}_{PLC}^{oracle}(Z)]\})$ is $\sum_{t=1}^T x_{jt}'[\beta(z_t) - \hat{\beta}_{PLC}^{oracle}(z_t)] - \eta_j' V_{xx}^{-1}\frac{1}{N}\sum_{i=1}^N\sum_{t=1}^T x_{it}x_{it}'[\beta(z_t) - \hat{\beta}_{PLC}^{oracle}(z_t)] = \sum_{t=1}^T[x_{jt}' - \eta_j'V_{xx}^{-1}\frac{1}{N}\sum_{i=1}^N x_{it}x_{it}'][\beta(z_t) - \hat{\beta}_{PLC}^{oracle}(z_t)]$. Then the j'th row of

$$T^{-1}M_2'VecDiag(X\{1_N' \otimes [\beta(Z) - \hat{\beta}_{PLC}^{oracle}(Z)]\})$$

is

$$\frac{1}{T}\sum_{t=1}^T[x_{jt}' - \eta_j'V_{xx}^{-1}\frac{1}{N}\sum_{i=1}^N x_{it}x_{it}'][\beta(z_t) - \hat{\beta}_{PLC}^{oracle}(z_t)]$$
$$\sim_a \quad \frac{1}{T}\sum_{t=1}^T[x_{jt}' - \eta_j'V_{xx}^{-1}\frac{1}{N}\sum_{i=1}^N x_{it}x_{it}'] \times o_p(1) = o_p(1), \tag{B.9}$$

since $\sup_{t\le T}|\beta(z_t) - \hat{\beta}_{PLC}^{oracle}(z_t)| = o_p(1)$ by uniform convergence of the oracle estimator, $\frac{1}{T}\sum_{t=1}^T x_{jt}' \xrightarrow{p} \eta_j = \eta$, and (B.8). Thus, the first term of (B.7) is $o_p(1/\sqrt{T})$. Below we will show the second term of (B.7) is of order $O_p(1/\sqrt{T})$. Thus the first term of (B.7) is negligible.

Now consider the second term of (B.7). We have

$$M_2'U = \begin{pmatrix} \sum_{t=1}^T(1-\frac{1}{N}\eta_1'V_{xx}^{-1}x_{1t})u_{1t} - \frac{1}{N}\eta_1'V_{xx}^{-1}\sum_{i\neq 1}\sum_t x_{it}u_{it} \\ \sum_{t=1}^T(1-\frac{1}{N}\eta_2'V_{xx}^{-1}x_{2t})u_{2t} - \frac{1}{N}\eta_2'V_{xx}^{-1}\sum_{i\neq 2}\sum_t x_{it}u_{it} \\ \vdots \\ \sum_{t=1}^T(1-\frac{1}{N}\eta_N'V_{xx}^{-1}x_{Nt})u_{Nt} - \frac{1}{N}\eta_N'V_{xx}^{-1}\sum_{i\neq N}\sum_t x_{it}u_{it} \end{pmatrix}.$$

To simplify computations, we now assume u_{it} is i.i.d. over t. The proof here is concerned with establishing the order of $\hat{\alpha}_{PLC}$, while the form of its asymptotic distribution is less important and it is sufficient to give the derivation when u_{it} is i.i.d. over t. In this case, we have, for $j = 1, \cdots, N$,

$$T^{-1/2}\sum_{t=1}^{T}\left(1-\frac{1}{N}\eta_j' V_{xx}^{-1}x_{jt}\right)u_{jt} \Rightarrow N\left(0, \sigma_u^2\left[1-\frac{2}{N}\eta_j' V_{xx}^{-1}\eta_j+\frac{1}{N^2}\eta_j' V_{xx}^{-1}\eta_j\right]\right),$$

and

$$-T^{-1/2}\frac{1}{N}\eta_j' V_{xx}^{-1}\sum_{i\neq j}\sum_{t}x_{it}u_{it} \Rightarrow N(0, \sigma_u^2\frac{N-1}{N^2}\eta_j' V_{xx}^{-1}\eta_j).$$

Since u_{it} is also iid across i, then the above two terms in the j-th row of $M_2'U$ are independent. Thus we have, for the j-th row of $M_2'U$,

$$T^{-1/2}\left[\sum_{t=1}^{T}(1-\frac{1}{N}\eta_j' V_{xx}^{-1}x_{jt})u_{jt}-\frac{1}{N}\eta_j' V_{xx}^{-1}\sum_{i\neq j}\sum_{t}x_{it}u_{it}\right] \Rightarrow N(0, \sigma_u^2[1-\frac{1}{N}\eta_j' V_{xx}^{-1}\eta_j]).$$

To compute the covariance between the j-th row and the k-th row of $M_2'U$, we rewrite the j-th row as $\sum_{t=1}^{T}(1-\frac{1}{N}\eta_j' V_{xx}^{-1}x_{jt})u_{jt}-\frac{1}{N}\eta_j' V_{xx}^{-1}\sum_t x_{kt}u_{kt}-\frac{1}{N}\eta_j' V_{xx}^{-1}\sum_{i\neq j,k}\sum_t x_{it}u_{it}$, and the k-th row as $\sum_{t=1}^{T}(1-\frac{1}{N}\eta_k' V_{xx}^{-1}x_{kt})u_{kt}-\frac{1}{N}\eta_k' V_{xx}^{-1}\sum_t x_{jt}u_{jt}-\frac{1}{N}\eta_k' V_{xx}^{-1}\sum_{i\neq j,k}\sum_t x_{it}u_{it}$. Then the covariance is found to be is $-\sigma_u^2\frac{1}{N}\eta_j' V_{xx}^{-1}\eta_k$. Therefore we have

$$\begin{aligned}T^{-1/2}M_2'U &\Rightarrow N\left(0, \sigma_u^2\begin{pmatrix}1-\frac{1}{N}\eta_1' V_{xx}^{-1}\eta_1 & -\frac{1}{N}\eta_1' V_{xx}^{-1}\eta_2 & \cdots & -\frac{1}{N}\eta_1' V_{xx}^{-1}\eta_N\\ -\frac{1}{N}\eta_2' V_{xx}^{-1}\eta_1 & & \cdots & \vdots\\ \vdots & & & \\ -\frac{1}{N}\eta_N' V_{xx}^{-1}\eta_1 & & \cdots & 1-\frac{1}{N}\eta_N' V_{xx}^{-1}\eta_N\end{pmatrix}\right)\\ &= N\left(0, \sigma_u^2\left[I_N-\frac{1}{N}\begin{pmatrix}\eta_1'\\ \vdots\\ \eta_N'\end{pmatrix}V_{xx}^{-1}(\eta_1, \cdots, \eta_N)\right]\right).\end{aligned}$$

Hence,

$$(T^{-1}M_2'M_2)^{-1}T^{-1/2}M_2'U \Rightarrow N\left(0, \sigma_u^2\left[I_N-\frac{1}{N}\begin{pmatrix}\eta_1'\\ \vdots\\ \eta_N'\end{pmatrix}V_{xx}^{-1}(\eta_1, \cdots, \eta_N)\right]^{-1}\right). \qquad \text{(B.10)}$$

Combining (B.7), (B.9) and (B.10), we have

$$\sqrt{T}(\hat{\alpha}_{PLC}-\alpha) \Rightarrow N\left(0, \sigma_u^2\left[I_N - \frac{1}{N}\begin{pmatrix}\eta_1'\\ \vdots \\ \eta_N'\end{pmatrix} V_{xx}^{-1}(\eta_1,\cdots,\eta_N)\right]^{-1}\right).$$

Using the fact that $\eta_1=\eta_2=\cdots=\eta_N=\eta$, we have

$$\begin{aligned}\sqrt{T}(\hat{\alpha}_{PLC}-\alpha) \Rightarrow & N\left(0, \sigma_u^2\left[I_N - \frac{1}{N}\eta' V_{xx}^{-1}\eta(1_N\otimes 1_N')\right]^{-1}\right)\\ =& N\left(0, \sigma_u^2\left[I_N + \frac{\eta' V_{xx}^{-1}\eta}{N(1-\eta' V_{xx}^{-1}\eta)}1_{N\times 1}\otimes 1_{1\times N}\right]\right)\end{aligned} \tag{B.11}$$

under the additional assumption that $\{u_{it}\}$ is iid across t. ■

(iii) Proof for the feasible estimator $\hat{\beta}_{PLC}(z)$

Following (1.14) and (1.9), we have

$$\begin{aligned}\hat{\beta}_{PLC}(z)-\beta(z) &= [X'K(z)X]^{-1}X'K(z)(Y-D\hat{\alpha}_{PLC})-\beta(z)\\ &= [X'K(z)X]^{-1}X'K(z)(Y-D\alpha)-\beta(z)+[X'K(z)X]^{-1}X'K(z)D(\alpha-\hat{\alpha}_{PLC})\\ &= \hat{\beta}_{PLC}^{oracle}(z)-\beta(z)+[X'K(z)X]^{-1}X'K(z)D(\alpha-\hat{\alpha}_{PLC}).\end{aligned}$$

Hence, the limit theory for the feasible estimator is jointly determined by that of the oracle estimator and that of $\hat{\alpha}_{PLC}$.

We already have $\frac{1}{NT|H|}X'K(z)X \xrightarrow{p} f_z(z)V_{xx}$ from Lemma C.4, and it is not hard to verify that $(T|H|)^{-1}X'K(z)D \xrightarrow{p} f_z(z)\eta\otimes 1_{1\times N}$. Then we can write

$$\begin{aligned}&\sqrt{NT|H|}(\hat{\beta}_{PLC}(z)-\beta(z)-B_1(z))\\ &= \sqrt{NT|H|}(\hat{\beta}_{PLC}^{oracle}(z)-\beta(z)-B_1(z)) - \sqrt{|H|}V_{xx}^{-1}\sqrt{T/N}\eta\otimes 1_{1\times N}(\hat{\alpha}_{PLC}-\alpha).\end{aligned} \tag{B.12}$$

Clearly, if $\eta=0$, the feasible estimator is asymptotically equivalent to the oracle estimator. When $\eta\neq 0$, we need to investigate the order of $\eta\otimes 1_{1\times N}(\hat{\alpha}_{PLC}-\alpha)=\eta 1_{1\times N}(\hat{\alpha}_{PLC}-\alpha)$. In view of (B.11), it is evident that $AsymVar(1_{1\times N}(\hat{\alpha}_{PLC}-\alpha))=O(N/T)$. Therefore, $\sqrt{T/N}\eta\otimes 1_{1\times N}(\hat{\alpha}_{PLC}-\alpha)=O_p(1)$. Since $|H|\to 0$ as $T\to\infty$, the second term in (B.12) is diminishing. So the feasible estimator $\hat{\beta}_{PLC}(z)$ is asymptotically equivalent as $T\to\infty$ to the oracle estimator $\hat{\beta}_{PLC}^{oracle}(z)$, which gives

$$\sqrt{NT|H|}(\hat{\beta}_{PLC}(z)-\beta(z)-B_1(z)) \Rightarrow N(0, f_z^{-1}(z)\nu_0^q\sigma_u^2 V_{xx}^{-1}).$$

■

C Useful Lemmas

Lemma C.1. *Under Assumptions 1-3, as $T \to \infty$, we have:*

1. *when N is fixed,*

$$\frac{1}{T|H|}\bar{X}'_A K_z \bar{X}_A \xrightarrow{p} f_z(z)V_{xx,N};$$

2. *when $N \to \infty$ and $\eta \neq 0$,*

$$\frac{1}{T|H|}\bar{X}'_A K_z \bar{X}_A \xrightarrow{p} f_z(z)\eta\eta';$$

3. *when $N \to \infty$ and $\eta = 0$,*

$$\frac{N}{T|H|}\bar{X}'_A K_z \bar{X}_A \xrightarrow{p} f_z(z)\Sigma_{xx} = f_z(z)V_{xx}.$$

Proof When N is fixed,

$$\frac{1}{T|H|}\bar{X}'_A K_z \bar{X}_A = \frac{1}{T|H|}\sum_t \bar{x}_{At}K_{tH}\bar{x}'_{At} \xrightarrow{p} f_z(z)\mathbb{E}\bar{x}_{At}\bar{x}'_{At},$$

where

$$\begin{aligned}
\mathbb{E}(\bar{x}_{At}\bar{x}'_{At}) &= \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N} x_{it}\right)\left(\frac{1}{N}\sum_{j=1}^{N} x'_{jt}\right) = \frac{1}{N^2}\sum_i\sum_j \mathbb{E}x_{it}x'_{jt} \\
&= \frac{1}{N^2}(\sum_i \mathbb{E}x_{it}x'_{it} + \sum_{i\neq j}\mathbb{E}x_{it}x'_{jt}) = \frac{1}{N^2}(NV_{xx} + (N^2-N)\eta\eta') \\
&= \frac{1}{N}V_{xx} + \left(1-\frac{1}{N}\right)\eta\eta' =: V_{xx,N}.
\end{aligned}$$

Thus

$$\frac{1}{T|H|}\bar{X}'_A K_z \bar{X}_A \xrightarrow{p} f_z(z)V_{xx,N}.$$

When $N \to \infty$, the derivations are similar and are omitted. ■

Lemma C.2. *Under Assumptions 1-3, as $T \to \infty$, we have:*

1. *when N is fixed,*

$$\frac{1}{\sqrt{T|H|}}\bar{X}_A' K_z \bar{U}_A = \frac{1}{\sqrt{T|H|}}\sum_t \bar{x}_{At} K_{tH} \bar{u}_{At} \Rightarrow N(0, f_z(z)\nu_0^q \sigma_u^2 N^{-1} V_{xx,N});$$

2. *when $N \to \infty$ and $\eta \neq 0$,*

$$\frac{\sqrt{N}}{\sqrt{T|H|}}\bar{X}_A' K_z \bar{U}_A = \frac{\sqrt{N}}{\sqrt{T|H|}}\sum_t \bar{x}_{At} K_{tH} \bar{u}_{At} \Rightarrow N(0, f_z(z)\nu_0^q \eta\eta' \sigma_u^2);$$

3. *when $N \to \infty$ and $\eta = 0$,*

$$\frac{N}{\sqrt{T|H|}}\bar{X}_A' K_z \bar{U}_A = \frac{N}{\sqrt{T|H|}}\sum_t \bar{x}_{At} K_{tH} \bar{u}_{At} \Rightarrow N(0, f_z(z)\nu_0^q V_{xx} \sigma_u^2).$$

Proof Omitted.

Lemma C.3. *Under Assumptions 1-3, as $T \to \infty$, we have:*

1. *when N is fixed,*

$$\frac{1}{T|H|}\bar{X}_A' K_z VecDiag(\bar{X}_A \dot{\beta}(z)) \xrightarrow{p} \mu_2 \left[\frac{1}{N}V_{xx} + (1-\frac{1}{N})\eta\eta'\right] \sum_s h_s^2 \frac{\partial f_z(z)}{\partial z_s}\frac{\partial \beta(z)}{\partial z_s}, \tag{C.1}$$

and

$$\frac{1}{T|H|}\bar{X}_A' K_z VecDiag(\bar{X}_A \ddot{\beta}(z)) \xrightarrow{p} \mu_2 f_z(z) \left[\frac{1}{N}V_{xx} + (1-\frac{1}{N})\eta\eta'\right] \sum_s h_s^2 \frac{\partial^2 \beta(z)}{\partial^2 z_s}; \tag{C.2}$$

2. *when $N \to \infty$ and $\eta \neq 0$,*

$$\frac{1}{T|H|}\bar{X}_A' K_z VecDiag(\bar{X}_A \dot{\beta}(z)) \xrightarrow{p} \mu_2 \eta\eta' \sum_s h_s^2 \frac{\partial f_z(z)}{\partial z_s}\frac{\partial \beta(z)}{\partial z_s},$$

and

$$\frac{1}{T|H|}\bar{X}_A' K_z VecDiag(\bar{X}_A \ddot{\beta}(z)) \xrightarrow{p} \mu_2 f_z(z) \eta\eta' \sum_s h_s^2 \frac{\partial^2 \beta(z)}{\partial^2 z_s};$$

3. when $N \to \infty$ and $\eta = 0$,

$$\frac{N}{T|H|}\bar{X}_A' K_z VecDiag(\bar{X}_A \dot{\beta}(z)) \xrightarrow{p} \mu_2 V_{xx} \sum_s h_s^2 \frac{\partial f_z(z)}{\partial z_s}\frac{\partial \beta(z)}{\partial z_s},$$

and

$$\frac{N}{T|H|}\bar{X}_A' K_z VecDiag(\bar{X}_A \ddot{\beta}(z)) \xrightarrow{p} \mu_2 f_z(z) V_{xx} \sum_s h_s^2 \frac{\partial^2 \beta(z)}{\partial^2 z_s}.$$

Proof We provide the proof only for the case where N is fixed. We first prove (C.1). Note that

$$\begin{aligned}\bar{X}_A' K_z VecDiag(\bar{X}_A \dot{\beta}(z)) &= \sum_t \bar{x}_{At} K(H^{-1}(z_t - z))\bar{x}_{At}' \dot{\beta}(z, z_t)\\ &= \sum_t \bar{x}_{At} K(H^{-1}(z_t - z)) \sum_{j=1}^p \bar{x}_{At,j} \dot{\beta}_j(z, z_t).\end{aligned}$$

Then the d-th element of $\bar{X}_A' K_z VecDiag(\bar{X}_A \dot{\beta}(z))$ is

$$\sum_t \bar{x}_{At,d} K(H^{-1}(z_t - z)) \sum_{j=1}^p \bar{x}_{At,j} \dot{\beta}_j(z, z_t) = \sum_t \sum_{j=1}^p \bar{x}_{At,d} K(H^{-1}(z_t - z)) \bar{x}_{At,j} \dot{\beta}_j(z, z_t).$$

We can show that

$$\begin{aligned}&\mathbb{E}[\bar{x}_{At,d} K(H^{-1}(z_t - z)) \bar{x}_{At,j} \dot{\beta}_j(z, z_t)]\\ =&\frac{1}{N^2}\sum_i \sum_\ell \mathbb{E} x_{it,d} x_{\ell t,j} \int K(H^{-1}(z_t - z))[H^{-1}(z_t - z)]' H \beta_j^{(1)}(z) f_z(z_t) dz_t\\ =&\frac{1}{N^2}(\sum_i \mathbb{E} x_{it,d} x_{it,j} + \sum_{i \neq \ell} x_{it,d} x_{\ell t,j}) \int K(u) u' H \beta_j^{(1)}(z) f_z(z + Hu) d(z + Hu)\\ =&\left[\frac{1}{N} V_{xx,(d,j)} + (1 - \frac{1}{N})\eta_d \eta_j\right] \int K(u) u' H \beta_j^{(1)}(z)[f_z(z) + (f_z^{(1)}(z))' Hu + \mathit{smaller\ order}]|H| du\\ =&\left[\frac{1}{N} V_{xx,(d,j)} + (1 - \frac{1}{N})\eta_d \eta_j\right] \int K(u) u' H \beta_j^{(1)}(z)(f_z^{(1)}(z))' Hu |H| du + \mathit{smaller\ order}\\ =&\left[\frac{1}{N} V_{xx,(d,j)} + (1 - \frac{1}{N})\eta_d \eta_j\right] \int K(u) \sum_s u_s^2 h_s^2 \frac{\partial \beta_j(z)}{\partial z_s}\frac{\partial f_z(z)}{\partial z_s} du |H|\\ =&\left[\frac{1}{N} V_{xx,(d,j)} + (1 - \frac{1}{N})\eta_d \eta_j\right] \mu_2 \sum_s h_s^2 \frac{\partial \beta_j(z)}{\partial z_s}\frac{\partial f_z(z)}{\partial z_s}|H|,\end{aligned}$$

where $\eta = (\eta_1, \cdots, \eta_p)'$ (note that we use a symmetric kernel and have $\int K(u)u_s^2 du = \int k(u_1)\cdots k(u_q)u_s^2 du_1 \cdots du_q = \mu_2$).

Then for the d-th element of $\bar{X}_A' K_z VecDiag(\bar{X}_A \dot{\beta}(z))$, we have

$$
\begin{aligned}
&\frac{1}{T|H|}\sum_t \bar{x}_{At,d} K(H^{-1}(z_t - z)) \sum_{j=1}^p \bar{x}_{At,j}\dot{\beta}_j(z, z_t) \\
\xrightarrow{p}\ & \mu_2 \sum_{j=1}^p \left[\frac{1}{N}V_{xx,(d,j)} + (1-\frac{1}{N})\eta_d\eta_j\right] \sum_s h_s^2 \frac{\partial \beta_j(z)}{\partial z_s}\frac{\partial f_z(z)}{\partial z_s} \\
=\ & \mu_2 \sum_s h_s^2 \frac{\partial f_z(z)}{\partial z_s} \sum_{j=1}^p \left[\frac{1}{N}V_{xx,(d,j)} + (1-\frac{1}{N})\eta_d\eta_j\right] \frac{\partial \beta_j(z)}{\partial z_s}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\frac{1}{T|H|}\bar{X}_A' K_z VecDiag(\bar{X}_A \dot{\beta}(z)) \xrightarrow{p} & \mu_2 \sum_s h_s^2 \frac{\partial f_z(z)}{\partial z_s} \sum_{j=1}^p \left[\frac{1}{N}\begin{pmatrix} V_{xx,(1,j)} \\ \vdots \\ V_{xx,(p,j)} \end{pmatrix} + (1-\frac{1}{N})\eta\eta_j\right] \frac{\partial \beta_j(z)}{\partial z_s} \\
=& \mu_2 \sum_s h_s^2 \frac{\partial f_z(z)}{\partial z_s} \left[\frac{1}{N}V_{xx} + (1-\frac{1}{N})\eta\eta'\right] \frac{\partial \beta(z)}{\partial z_s} \\
=& \mu_2 \left[\frac{1}{N}V_{xx} + (1-\frac{1}{N})\eta\eta'\right] \sum_s h_s^2 \frac{\partial f_z(z)}{\partial z_s}\frac{\partial \beta(z)}{\partial z_s}.
\end{aligned}
$$

Next we prove (C.2). We have

$$
\begin{aligned}
\bar{X}_A' K_z VecDiag(\bar{X}_A \ddot{\beta}(z)) &= \sum_t \bar{x}_{At} K(H^{-1}(z_t - z)) \bar{x}_{At}' \ddot{\beta}(z, z_t) \\
&= \sum_t \bar{x}_{At} K(H^{-1}(z_t - z)) \sum_{j=1}^p \bar{x}_{At,j} \ddot{\beta}_j(z, z_t).
\end{aligned}
$$

The d-th element of $\bar{X}_A' K_z VecDiag(\bar{X}_A \ddot{\beta}(z))$ is

$$
\sum_t \bar{x}_{At,d} K(H^{-1}(z_t - z)) \sum_{j=1}^p \bar{x}_{At,j}\ddot{\beta}_j(z, z_t) = \sum_t \sum_{j=1}^p \bar{x}_{At,d} K(H^{-1}(z_t - z)) \bar{x}_{At,j}\ddot{\beta}_j(z, z_t),
$$

where

$$
\begin{aligned}
&\mathbb{E}[\bar{x}_{At,d}K(H^{-1}(z_t-z))\bar{x}_{At,j}\ddot{\beta}_j(z,z_t)] \\
=&\frac{1}{N^2}\sum_i\sum_\ell \mathbb{E}x_{it,d}x_{\ell t,j}\int K(H^{-1}(z_t-z))[H^{-1}(z_t-z)]'H\beta_j^{(2)}(z)H[H^{-1}(z_t-z)]f_z(z_t)dz_t \\
=&\frac{1}{N^2}(\sum_i \mathbb{E}x_{it,d}x_{it,j}+\sum_{i\neq\ell}x_{it,d}x_{\ell t,j})\int K(u)u'H\beta_j^{(2)}(z)Huf_z(z+Hu)d(z+Hu) \\
=&\left[\frac{1}{N}V_{xx,(d,j)}+(1-\frac{1}{N})\eta_d\eta_j\right]\int K(u)u'H\beta_j^{(2)}(z)Hu[f_z(z)+(f_z^{(1)}(z))'Hu+\textit{smaller order}]|H|du \\
=&\left[\frac{1}{N}V_{xx,(d,j)}+(1-\frac{1}{N})\eta_d\eta_j\right]\int K(u)u'H\beta_j^{(2)}(z)Huf_z(z)|H|du+\textit{smaller order} \\
=&\left[\frac{1}{N}V_{xx,(d,j)}+(1-\frac{1}{N})\eta_d\eta_j\right]\int K(u)\sum_s u_s^2h_s^2\frac{\partial^2\beta_j(z)}{\partial^2 z_s}duf_z(z)|H|+\textit{smaller order} \\
=&\left[\frac{1}{N}V_{xx,(d,j)}+(1-\frac{1}{N})\eta_d\eta_j\right]\mu_2\sum_s h_s^2\frac{\partial^2\beta_j(z)}{\partial^2 z_s}|H|f_z(z)+\textit{smaller order.}
\end{aligned}
$$

For the d-th element of $\bar{X}_A'K_zVecDiag(\bar{X}_A\ddot{\beta}(z))$, we have

$$
\begin{aligned}
&\frac{1}{T|H|}\sum_t \bar{x}_{At,d}K(H^{-1}(z_t-z))\sum_{j=1}^p \bar{x}_{At,j}\ddot{\beta}_j(z,z_t) \\
\xrightarrow{p}\ &\mu_2 f_z(z)\sum_{j=1}^p\left[\frac{1}{N}V_{xx,(d,j)}+(1-\frac{1}{N})\eta_d\eta_j\right]\sum_s h_s^2\frac{\partial^2\beta_j(z)}{\partial^2 z_s} \\
=\ &\mu_2 f_z(z)\sum_s h_s^2\sum_{j=1}^p\left[\frac{1}{N}V_{xx,(d,j)}+(1-\frac{1}{N})\eta_d\eta_j\right]\frac{\partial^2\beta_j(z)}{\partial^2 z_s}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{1}{T|H|}\bar{X}_A'K_zVecDiag(\bar{X}_A\ddot{\beta}(z)) &\xrightarrow{p} \mu_2 f_z(z)\sum_s h_s^2\sum_{j=1}^p\left[\frac{1}{N}\begin{pmatrix}V_{xx,(1,j)}\\ \vdots \\ V_{xx,(p,j)}\end{pmatrix}+(1-\frac{1}{N})\eta\eta_j\right]\frac{\partial^2\beta_j(z)}{\partial^2 z_s} \\
&=\mu_2 f_z(z)\sum_s h_s^2\left[\frac{1}{N}V_{xx}+(1-\frac{1}{N})\eta\eta'\right]\frac{\partial^2\beta(z)}{\partial^2 z_s} \\
&=\mu_2 f_z(z)\left[\frac{1}{N}V_{xx}+(1-\frac{1}{N})\eta\eta'\right]\sum_s h_s^2\frac{\partial^2\beta(z)}{\partial^2 z_s}.
\end{aligned}
$$

■

Lemma C.4. *Under Assumptions 1-3, as $T \to \infty$, we have*

$$\frac{1}{NT|H|} X'K(z)X = \frac{1}{NT|H|} \sum_{i=1}^{N} \sum_{t=1}^{T} x_{it} K_{tH} x_{it}' \xrightarrow{p} f_z(z)\mathbb{E}[x_{it}x_{it}'|z_t = z] = f_z(z)V_{xx}, \quad \text{(C.3)}$$

where $K_{tH} = K(H^{-1}(z_t - z)) = k((z_{t,1} - z_1)/h_1) \cdots k((z_{t,q} - z_q)/h_q)$ is a scalar.

Proof First consider the mean of the LHS of (C.3).

$$\frac{1}{NT|H|}\mathbb{E}[X'K(z)X] = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{T|H|}\sum_{t=1}^{T}\mathbb{E}\left[x_{it}K_{tH}x_{it}'\right] \to f_z(z)\mathbb{E}[x_{it}x_{it}'] = f_z(z)V_{xx}.$$

Then consider the variance of the LHS of (C.3).

$$\begin{aligned}
&Var\left(\frac{1}{NT|H|}X'K(z)X\right)\\
=&\frac{1}{N^2T^2|H|^2}Var\left(\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}K_{tH}x_{it}'\right)\\
=&\frac{1}{N^2}\sum_{i=1}^{N}Var\left(\frac{1}{T|H|}\sum_{t=1}^{T}x_{it}K_{tH}x_{it}'\right)\\
=&\frac{1}{N}\left[\frac{1}{T^2|H|^2}\sum_{t=1}^{T}Var(x_{1t}K_{tH}x_{1t}') + \frac{2}{T|H|^2}\sum_{\ell=1}^{T-1}\left(1-\frac{\ell}{T}\right)Cov(x_{11}K_{1H}x_{11}', x_{1,\ell+1}K_{\ell+1,H}x_{1,\ell+1}')\right]\\
=&\frac{1}{NT|H|^2}Var(x_{1t}K_{tH}x_{1t}') + \frac{2}{NT|H|^2}\sum_{\ell=1}^{T-1}\left(1-\frac{\ell}{T}\right)Cov(x_{11}K_{1H}x_{11}', x_{1,\ell+1}K_{\ell+1,H}x_{1,\ell+1}')\\
=&\Pi_1 + \Pi_2,
\end{aligned}$$

since we have assumed *iid* across i and stationarity over t. By Assumption 1 and 2 (b), we have $Var(x_{it}K_{tH}x_{it}') = O(|H|)$. So $\Pi_1 = O(1/(NT|H|))$.

Next consider the second term Π_2. Based on the Davydov's Inequality (Davydov, 1968), we have, for $\delta = \lambda + 2 > 2$, and $\ell \geq 1$,

$$\begin{aligned}
|Cov(x_{11}K_{1H}x_{11}', x_{1,\ell+1}K_{\ell+1,H}x_{1,\ell+1}')| \leq & C\alpha_\ell^{1-2/\delta}[\mathbb{E}|x_{11}K_{1H}x_{11}'|^\delta]^{1/\delta}[\mathbb{E}|x_{1,\ell+1}K_{\ell+1,H}x_{1,\ell+1}'|^\delta]^{1/\delta}\\
\leq & C\alpha_\ell^{1-2/\delta}|H|^2.
\end{aligned}$$

So we have

$$\begin{aligned}
\Pi_2 \leq&\frac{2}{NT|H|^2}\sum_{\ell=1}^{T-1}|Cov(x_{11}K_{1H}x_{11}', x_{1,\ell+1}K_{\ell+1,H}x_{1,\ell+1}')|\\
\leq&\frac{C}{NT}\sum_{\ell=1}^{\infty}\alpha_\ell^{1-2/\delta} = \frac{C}{NT}\sum_{\ell=1}^{\infty}\ell^{-\tau(1-2/\delta)} = O(1/(NT)) = o(1/(NT|H|)),
\end{aligned}$$

because $\tau(1-2/\delta) > 1$.

We have therefore shown that $Var\left(\frac{1}{NT|H|}X'K(z)X\right) = o(1)$. This completes the proof of (C.3). ■

Lemma C.5. *Under Assumptions 1-3, as $T \to \infty$, we have*

$$\frac{1}{\sqrt{NT|H|}}X'K(z)U \Rightarrow N(0, f_z(z)\nu_0^q\sigma_u^2 V_{xx}). \tag{C.4}$$

Proof First note that $X'K(z)U = \sum_i \sum_t x_{it}K_{tH}u_{it}$ and $\mathbb{E}(x_{it}K_{tH}u_{it}) = 0$. The variance of $\frac{1}{\sqrt{NT|H|}}X'K(z)U$ is

$$\frac{1}{NT|H|}Var(\sum_i \sum_t x_{it}K_{tH}u_{it}) = f_z(z)\nu_0^q\sigma_u^2 V_{xx} + o(1),$$

where $\nu_0^q = \int K^2(u)du$ is a scalar.

Denote $\{\xi_j, j = 1, \cdots, NT\} = \{(x_{11}, u_{11}), \cdots, (x_{1T}, u_{1T}), \cdots, (x_{N1}, u_{N1}), \cdots, (x_{NT}, u_{NT})\}$. Then $\{\xi_j\}$ is stationary and α-mixing with mixing coefficient

$$\alpha_k^* = \begin{cases} \alpha_k \ or \ 0, & k < T, \\ 0, & k \geq T, \end{cases}$$

where α_k is the mixing coefficient of $\{(x_{it}, z_t, u_{it})\}$. So by Theorem 2.2 of Peligrad and Utev (1997), we have $\frac{1}{\sqrt{NT|H|}}X'K(z)U \Rightarrow N(0, f_z(z)\nu_0^q\sigma_u^2 V_{xx})$. ■

Lemma C.6. *Under Assumptions 1-3, as $T \to \infty$, we have*

$$\frac{1}{NT|H|}X'K(z)VecDiag[X(1_{1\times N} \otimes \ddot{\beta}(z))] \xrightarrow{p} f_z(z)\mu_2 V_{xx}\sum_{s=1}^{q} h_s^2 \frac{\partial^2\beta(z)}{\partial^2 z_s}.$$

Proof First we have

$$\frac{1}{NT|H|}X'K(z)VecDiag[X(1_{1\times N} \otimes \ddot{\beta}(z))] = \frac{1}{NT|H|}\sum_i \sum_t x_{it}K(H^{-1}(z_t - z))x_{it}'\ddot{\beta}(z, z_t). \tag{C.5}$$

Note that the d-th element of $\frac{1}{NT|H|}X'K(z)VecDiag[X(1_{1\times N} \otimes \ddot{\beta}(z))]$ is

$$\frac{1}{NT|H|}\sum_i \sum_t x_{it,d}K_{tH}(\sum_{j=1}^{p} x_{it,j}\ddot{\beta}_j(z, z_t)) = \frac{1}{NT|H|}\sum_i \sum_t \sum_{j=1}^{p} x_{it,d}K_{tH}x_{it,j}\ddot{\beta}_j(z, z_t),$$

where

$$\begin{aligned}\mathbb{E}x_{it,d}K_{tH}x_{it,j}\ddot{\beta}_j(z,z_t) &= V_{xx,(d,j)}\int K(H^{-1}(z_t-z))[H^{-1}(z_t-z)]'H\beta_j^{(2)}(z)H[H^{-1}(z_t-z)]f_z(z_t)dz_t\\ &= V_{xx,(d,j)}\int K(u)u'H\beta_j^{(2)}(z)Huf_z(z+Hu)|H|du\\ &= V_{xx,(d,j)}|H|f_z(z)\mu_2\sum_{s=1}^{q}h_s^2\frac{\partial^2\beta_j(z)}{\partial^2 z_s}+\textit{smaller order}.\end{aligned}$$

Then for the d-th element of $\frac{1}{NT|H|}X'K(z)VecDiag[X(1_{1\times N}\otimes\ddot{\beta}(z))]$, we have

$$\begin{aligned}\frac{1}{NT|H|}\sum_i\sum_t\sum_{j=1}^{p}x_{it,d}K_{tH}x_{it,j}\ddot{\beta}_j(z,z_t) &\xrightarrow{p} f_z(z)\mu_2\sum_{j=1}^{p}V_{xx,(d,j)}\sum_{s=1}^{q}h_s^2\frac{\partial^2\beta_j(z)}{\partial^2 z_s}\\ &= f_z(z)\mu_2\sum_{s=1}^{q}h_s^2\sum_{j=1}^{p}V_{xx,(d,j)}\frac{\partial^2\beta_j(z)}{\partial^2 z_s}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{NT|H|}X'K(z)VecDiag[X(1_{1\times N}\otimes\ddot{\beta}(z))] &\xrightarrow{p} f_z(z)\mu_2\sum_{s=1}^{q}h_s^2\begin{pmatrix}\sum_{j=1}^{p}V_{xx,(1,j)}\frac{\partial^2\beta_j(z)}{\partial^2 z_s}\\ \vdots\\ \sum_{j=1}^{p}V_{xx,(p,j)}\frac{\partial^2\beta_j(z)}{\partial^2 z_s}\end{pmatrix}\\ &= f_z(z)\mu_2\sum_{s=1}^{q}h_s^2V_{xx}\frac{\partial^2\beta(z)}{\partial^2 z_s}\\ &= f_z(z)\mu_2V_{xx}\sum_{s=1}^{q}h_s^2\frac{\partial^2\beta(z)}{\partial^2 z_s}.\end{aligned}$$

■

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Table 1: Oracle and feasible APLC methods (all figures have been multiplied by 10^3)

	η	AMSE($\hat{\beta}^{oracle}_{APLC}(z)$)	AMSE($\hat{\beta}_{APLC}(z)$)	ASE($\hat{\hat{\alpha}}$)
$T = 50$	0	103.5	105.3	4.4
	1	21.1	37.7	13.7
	5	3.9	4.7	24.6
$T = 100$	0	52.6	53.1	2.1
	1	11.2	18.2	5.7
	5	2.3	2.7	11.7
$T = 200$	0	26.5	26.7	1.1
	1	6.3	9.9	3.1
	5	1.4	1.6	4.8

Table 2: Oracle and feasible PLC methods (all figures have been multiplied by 10^3)

	η	AMSE($\hat{\beta}^{oracle}_{PLC}(z)$)	AMSE($\hat{\beta}_{PLC}(z)$)	AMSE($\hat{\alpha}_{PLC}$)
$T = 50$	0	18.2	18.5	19.0
	1	8.6	9.2	22.9
	5	3.3	3.3	22.3
$T = 100$	0	11.0	11.1	10.5
	1	4.6	4.9	11.8
	5	2.0	2.0	12.1
$T = 200$	0	5.9	6.0	5.6
	1	3.0	3.1	5.7
	5	1.3	1.3	6.1