PROGRESSIVE PARTICIPATION

By

Dirk Bergemann and Philipp Strack

August 2019

COWLES FOUNDATION DISCUSSION PAPER NO. 2189

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.yale.edu/

Progressive Participation*

Dirk Bergemann† Philipp Strack‡

August 14, 2019

Abstract

A single seller faces a sequence of buyers with unit demand. The buyers are forward-looking and long-lived but vanish (and are replaced) at a constant rate. The arrival time *and* the valuation is private information of each buyer and unobservable to the seller. Any incentive compatible mechanism has to induce truth-telling about the arrival time and the evolution of the valuation.

We derive the optimal stationary mechanism, characterize its qualitative structure and derive a closed form solution. As the arrival time is private information, the agent can choose the time at which he reports his arrival. The truth-telling constraint regarding the arrival time can be represented as an optimal stopping problem. The stopping time determines the time at which the agent decides to participate in the mechanism. The resulting value function of each agent can not be too convex and has to be continuously differentiable everywhere, reflecting the option value of delaying participation. The optimal mechanism thus induces progressive participation by each agent: he participates either immediately or at a future random time.

KEYWORDS: Dynamic Mechanism Design, Observable Arrival, Unobservable Arrival, Repeated Sales, Interim Incentive Constraints, Interim Participation Constraints, Stopping Problem, Option Value, Progressive Participation.

JEL CLASSIFICATION: D44, D82, D83.

*The first author acknowledges financial support through NSF Grant SES 1459899. We thank seminar audiences at the Econometric Society North America Winter Meetings 2017 and University of Chile Dynamic Mechanism Design Workshop 2017 for many productive comments. We thank Rahul Deb, Daniel Garrett, Al Klevorick, and Maher Said for many helpful conversations.

†Department of Economics, Yale University, dirk.bergemann@yale.edu;

‡Department of Economics, Yale University, philipp.strack@gmail.com

1 Introduction

1.1 Motivation

We consider a classic mechanism design problem in a dynamic and stationary environment. The designer wants to repeatedly sell a good (or service) to buyers with randomly evolving valuation. There is a representative buyer with a unit demand for a good in every period (or equivalently a continuum of buyers). The willingness to pay of each buyer is private information of the buyer and evolves randomly over time. We assume a stationary environment in which each buyer is replaced at random, and with a constant rate, by a new buyer whose willingness-to-pay is randomly drawn from a given distribution. The objective of the seller is to find a stationary revenue maximizing policy in this dynamic environment. The choice of policy is unrestricted and may consist of leasing contracts, sale contracts, or any other form of dynamic contract.

We depart from the earlier analysis of dynamic mechanisms in our treatment of the participation decision of the buyer. We allow the buyer (he), once he has arrived in the economy, to choose the time at which he enters into a contract with the seller (she). Thus, while he can sign a contract with the seller immediately upon arrival, he has the option to postpone the participation decision until a future date. The buyer therefore has the option to wait and sign any contract only after he has received additional information about his willingness to pay. In particular, he can time the acceptance of a contract until he has a sufficiently high willingness to pay. Thus, both the incentive constraints that are in place *after* the agent has signed the contract, and the participation constraints that are in place *before* the agent has signed the contract are fully responsive to the arrival of new information, and thus are represented as sequential constraints. In particular, the buyer can enter the contract upon arrival or at any later time. His participation is therefore determined progressively as he receives additional information. For brevity, we sometimes refer to the current setting with interim participation and interim incentive constraints as *progressive mechanism design*.

We can contrast this with the received perspective in *dynamic mechanism design*. There, the seller knows the arrival time of the agent in the economy, and can commit herself to make a single and once-and-for-all offer to the agent at the moment of arrival. In particular, the seller can commit herself to *never* ever make another offer to the agent. In our view these two features: (i) the ability of the seller to time the offer to the arrival time of the buyer and (ii) the ability to refrain from any future offers seem unlikely to be present in many economic environments of interest. For example, the consumer clearly has a choice when to sign up for a mobile phone contract, a gym membership, or a service contract for a kitchen appliance. Importantly, as the consumer waits, he may receive more information about his willingness to pay for the product. Thus relative to the specific assumption in the earlier literature, we allow the arrival time and the identity of the agent to be private information to the agent. Thus, the contract or the menu of contracts cannot be timed to the arrival of the agent and the contract (or lack of contract) offer cannot be tied to the identity of the agent. In a stationary environment in which agents arrive and depart to maintain a balanced population, we show that the resulting dynamic mechanism will also be stationary, and time independent.

We view these two restrictions as necessary steps to bring the design of dynamic revenue maximizing mechanism closer to many interesting economic applications. To the extent that these restrictions impose additional constraints on the seller, they necessarily weaken the power of dynamic mechanism design. We therefore investigate what the impact of these additional constraints is on the ability of the seller to raise revenues from the buyers using dynamic contracts. The additional constraints for the seller are reflected in a larger set of reporting strategies for the buyers. A buyer can misreport both his willingness to pay as well as his arrival time. This creates an option value for the buyer as instead of choosing a contract immediately he can wait and enter into a contract with the seller when it is most favorable for him to do so. Given the menu of contracts offered by the seller, the buyer thus solves an optimal stopping problem to determine when to enter into a contractual relationship with the seller. From the point of view of the buyer, the choice of an optimal contract from the menu therefore has an option element. Subject to the (random) evolution of his type, his willingness to pay, he can choose when to enter into an agreement with the seller. This suggests that the buyer will receive a larger information rent than in the standard dynamic mechanism design framework where the buyer has to sign a contract with the seller immediately.

We develop our analysis in a continuous time setting where the willingness to pay of the agent follows a geometric Brownian motion. The prior distribution of the willingness to pay upon arrival is given exogenously and together with the renewal rate in the population generates an ergodic distribution which forms the stationary environment. The revenue maximizing static mechanism, i.e. contract which does not condition on a buyers history, is a leasing contract which offers the good in every period for the posted price that is optimal given the ergodic distribution of the valuations of the buyers.

In the absence of the sequential participation constraint, the revenue maximizing dynamic mechanism would sell the object with probability one and forever at fixed price (see Bergemann and Strack, 2015). Thus, the object would be sold rather than leased to all buyers who have an initial willingness to pay above a certain threshold. Conversely, all buyers whose initial value is below this threshold would not buy the object, neither at the beginning of time, nor anytime thereafter. In a first pass, we then restrict attention to a sale price policy, which is optimal in the absence of sequential participation constraints and determine the optimal price in such a policy in the presence of sequential participation constraints. Here, the comparison of thresholds and prices between dynamic and progressive mechanism design are instructive. We find that the threshold for the willingness to pay at which a buyer purchases the object is strictly higher in the progressive model than in the dynamic model without progressive participation constraint. By contrast, the price at which the buyer can acquire the object can be either below or above the price charged in the dynamic setting.

We can gain some initial insight by considering how a buyer would react to the option to buy at fixed price that would exactly extract the expected surplus from owning the object. This would be the threshold type in the dynamic setting and would leave the agent with zero expected net surplus. In the progressive setting, he could and clearly should delay the purchase until his willingness to pay is sufficiently above the threshold level to guarantee himself a positive net surplus. Thus, at any threshold level, the seller will be able to extract less surplus from the buyer than he could in the presence of a static participation constraint. In response to the weakened ability to extract surplus, the seller has to adjust her policy along the price *and* the quantity margin at the same time. We show that the seller will generally choose to implement a higher threshold for the willingness-to pay. Thus, there will be fewer initial sales relative to the static participation constraint. But the seller also adjusts along the dimension of the price and will ask for a price below the price at which the threshold type would have received zero expected net surplus. Interestingly, the price with sequential participation constraints may either be below or above the price charged under the static participation constraint. Most importantly, a gap now arises between the price paid to receive the object, and the expected value assigned to the object by the threshold type.

Following the analysis of the optimal price policy under sequential participation constraint, we then show that indeed a single sale price is an optimal progressive mechanism in the class of all possible mechanisms. In other words, a single sale price as a specific and simple indirect implementation of a direct mechanism achieves the revenue maximizing optimum. The main challenge for establishing this result is that it is unclear how to handle the progressive participation constraint. As our example with the threshold type illustrates this constraint will always bind for some type and thus cannot be ignored. This constraint is non-standard as it states that the value function of the agent must be the solution to an optimal stopping problem which itself involves the value function. We relax this problem by restricting the agent to a small set of deviations, namely cut-off strategies which are indexed by the cut-off. This relaxation has the advantage that the agent's participation strategies can be mapped into $\mathbb{R}$ which allows us to reduce the problem into a static mechanism design problem. This static problem is a variant of the classical setup by Mussa and Rosen (1978) with the non-standard feature that each agent can (deterministically) increase his type at the cost of multiplicatively decreasing his interim utility. This additional constraint leads to a failure of the first-order approach. We show that the resulting mathematical program can be expressed as a Pontryagin control problem with contact constraints and we develop a verification result for such problems which might be of independent interest. We highlight the implications that the option to wait has for the effectiveness of dynamic mechanism in three different ways. First, we compare the revenue of the optimal dynamic mechanism under the progressive participation constraint with the revenue of the static mechanism. We show that the dynamic mechanism may display no revenue advantage at all against a purely static mechanism choice. Second, we consider the ratio between the revenue difference between dynamic and static mechanism under progressive participation and the revenue difference between dynamic and static mechanism without progressive participation constraint. Interestingly, the revenue difference remains much larger, and in particular positive without progressive participation constraint. Third, we consider the counterfactual, and ask how much revenue can the seller loose by choosing an optimal mechanism in the absence of progressive participation constraints when the buyer has indeed the option to postpone any purchase, and hence the realized revenue is subject to the progressive participation constraint.

1.2 Related Literature

The analysis of revenue-maximizing mechanism in an environment where the agent's private information changes over time started with Baron and Besanko (1984) and Besanko (1985). Since these early contributions, the literature has developed considerably in recent years with notable contributions by Courty and Li (2000), Battaglini (2005), Eső and Szentes

(2007) and Pavan et al. (2014).[1] These paper derive in increasing generality the dynamic revenue maximizing mechanism. The analysis in these contributions have in common the same set of constraints on the choice of mechanism. The principal has to satisfy all of the sequential incentive constraints, but only a single ex-ante participation constraint. In earlier work, Bergemann and Strack (2015), we considered the same set of constraints in a continuous-time setting where the stochastic process that describes the evolution of the flow utility was governed by a Brownian motion. The continuous-time setting allowed us to obtain additional and explicit results regarding the nature of the optimal allocation policy, which are unavailable in the discrete-time setting. In the present paper, we will use the continuous-time setting again for very similar reasons.

While most of the literature on dynamic mechanism design assumes that the arrival of the agents is known to the principal and that the principal can make a single, take-it-or-leave-it offer at the moment of the agent's arrival, Garrett (2016) offers a notable exception. He allows for the random arrival of buyers in a stationary environment and suggests that it may offer an explanation for temporary price reduction. His model is also a continuous-time model but the set of possible values of the buyer is binary. More precisely, the value of the buyer changes between low and high values according to a Markov process. In the choice of mechanism, he restricts attention to deterministic sale price policies only. Garrett (2016) observes that an optimal policy in the class of all dynamic direct mechanisms, one that does not restrict attention to deterministic sale price path (and implied restrictions on reporting types), may lead to very different results and implications.

By contrast, we consider an environment with a continuum of values whose evolution is governed by a geometric Brownian motion. In this environment, we establish that a deterministic and time-invariant sale price indeed constitutes a revenue maximizing mechanism in the class of all mechanisms. Interestingly, in the binary value environment of Garrett (2016), a time-invariant price path is not even optimal within the class of deterministic price path, and he obtains conditions on the binary values under which a deterministic price cycle prevails in the optimal contract.

The importance of a privately observed arrival time is also investigated in Deb (2014) and Garrett (2017). In contrast to the present work, these papers do not investigate a stationary environment. Instead, while the mechanism starts at time $t = 0$, the agent may arrive at a later time. The main concern therefore is how to encourage the early arrivals to contract

[1] Bergemann and Valimaki (2019) provide a survey into the recent developments of dynamic mechanism design.

early. In a setting with either a durable good or a non-durable good, respectively, these authors find that the optimal mechanism treats early arriving participants more favorably than late arriving participants. The late arriving participants face less favorable prices and purchase lower quantities than the early arrivals.

There are related concerns with the emphasis on the ex-ante participation constraints in the literature on dynamic mechanism design that pursue different directions from the one presented here. Lobel and Paes Leme (2019) question the unlimited ability of the seller to commit to make only a single offer to the buyer. They suggest that while the seller may have "positive commitment" power, she may lack in "negative commitment" power. That is, he can commit to any contractual promise, but may not be able to commit never to make any further offer in the future. They show that in a finite horizon model with a sequence of perishable goods, the equilibrium is long-term efficient and that the seller's revenue is a function of the buyer's ex ante utility under a no commitment model. Skreta (2006, 2015) and Deb and Said (2015) also investigate the sequential screening under limited commitment by the seller.

A more radical departure from the ex-ante or interim participation constraint to ex-post participation constraints is suggested in recent work by Krähmer and Strausz (2015) and Bergemann et al. (2017). These papers re-consider the sequential screening model of Courty and Li (2000). In this two-period setting, where information arrives over time, and the allocation of a single object can be made in the second period, they impose an ex-post participation rather than an ex-ante participation constraint. In consequence the power of sequential screening is diminished and sometimes the optimal mechanism reduces to the solution of the static mechanism. Ashlagi et al. (2016) investigate the performance guarantees that can be given with ex-post participation constraints in a setting where a monopolist sells k items over k periods.

2 Model

2.1 Payoffs and Allocation

We consider a stationary model with a single seller and a single representative buyer (who we think of as representing a unit continuum of buyers). We alternately refer to the seller as the principal (she) and the buyer as the agent (he).

Time is continuous and indexed by $t \in [0, \infty)$. The buyer departs and gets replaced with a newly arriving buyer at rate $\gamma > 0$. We denote by i the buyer who arrived i-th to the market.[2]

The seller and the buyers discount the future at the same rate $r > 0$. At each point in time t, the buyer demands one unit of the good. The flow valuation of buyer i at time t is denoted by $\theta_t^i \in \mathbb{R}_+$, the quantity allocated to buyer i at time t is $x_t^i \in [0, 1]$, and p_t^i is the flow payment from the buyer to the seller. His flow preferences are represented by a (quasi-)linear utility function

$$\theta_t^i x_t^i - p_t^i, \tag{1}$$

We denote the random arrival time of buyer i by $\alpha_i \in \mathbb{R}_+$ and the random departure time by $T_i = \alpha_{i+1} \in \mathbb{R}_+$. The arrival and departure time of each buyer are independent of his valuation process.

The valuation of the buyer i, $\theta_{\alpha_i} \in \mathbb{R}_+$, at the time of his arrival is distributed according to cumulative distribution function

$$F : \left[0, \overline{\theta}\right] \to \mathbb{R},$$

with strictly positive, bounded density $f = F' > 0$ on the support. The prior distribution F is the same for every buyer i and every arrival time α_i.

The valuation of each buyer evolves randomly over time, independent of the valuation of the other buyers. We assume that each buyer's valuation $(\theta_t^i)_{t\in\mathbb{R}_+}$ follows a geometric Brownian motion, i.e. solves the stochastic differential equation:

$$\mathrm{d}\theta_t^i = \sigma\, \theta_t^i \mathrm{d}W_t\,, \tag{2}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a Brownian motion and $\sigma \in \mathbb{R}_+$ is the volatility which measures the speed of information arrival. The solution to the above differential equation is the value θ_t^i at time t:

$$\theta_t^i = \theta_{\alpha_i}^i \exp\left(\frac{\sigma^2}{2}(t - \alpha_i) + \sigma W_t\right). \tag{3}$$

As the geometric Brownian motion is a martingale an agent's valuation today is the agent's

[2] One can equivalently think of a continuum of buyers that is present at every point in time, where buyers arrive and leave with rate γ. The average behavior of such a continuum of buyers will match the expected behavior of a single representative buyer. The main advantage of the representative agent model is that it avoids technical issues due to integration over a continuum of independent random variables, which is formally not well defined in standard probability theory, see e.g. Judd (1985).

best estimate of his valuation at any future point in time, i.e. for all $s \geq t$:

$$\mathbb{E}_t \left[\theta^i_s\right] = \theta^i_t \, .$$

Furthermore, θ_t takes only positive values and so the agent's valuation for the good is always positive.

Each buyer i seeks to maximize his expected net utility given his willingness to pay at arrival time α_i:

$$\mathbb{E}\left[\int_{\alpha_i}^{T_i} e^{-r(t-\alpha_i)} \left(\theta^i_t x^i_t - p^i_t\right) \mathrm{d}t\right].$$

To simplify notation and without loss of generality we assume that the first buyer arrives at time zero $\alpha_0 = 0$. The seller seeks to maximize the expected discounted net revenue collected from her interaction with the sequence of all buyers:

$$\mathbb{E}\left[\sum_{i=0}^{\infty}\int_{\alpha_i}^{T_i} e^{-r\,t} p^i_t \mathrm{d}t\right] .$$

This objective captures equivalently the total discounted revenue from a continuum of agents.[3]

2.2 Stationary Mechanism

A mechanism specifies, after each history, a set of messages for each buyer and the allocation as well as the transfer as a function of the complete history of messages sent by this buyer.[4] The allocation process (x_t) specifies whether or not the buyer consumes the good at any point in time. We assume that the allocation of the object is reversible, i.e. the seller can give the buyer an object for some time and then take it away later.

Definition 1 (Mechanism)**.**
A mechanism (x, p) specifies at every point in time t the allocation $x_t\left((m_s)_{s\leq t}\right)$ as well as the transfer $p_t\left((m_s)_{s\leq t}\right)$ as a function of the messages $(m_s)_{s\leq t}$ sent by the agent prior to time t.

[3] This objective captures the the total discounted revenue from a continuum of agents $\int_0^\infty \left(\int_0^1 e^{-rt} p^i_t \mathrm{d}i\right) \mathrm{d}t$ which is formally not well defined in standard probability theory due to the integration over a continuum of random variables, see e.g. Judd (1985).

[4] The restriction to mechanisms where each agent's allocation and transfer is only a function of his own messages is without loss of generality in the following sense. In the continuum interpretation of the model, the principal cannot detect if an agent misreports his arrival time as the agent's willingness to pay are independent and there is no capacity constraint linking the revenue maximization problem between different agents.

A direct mechanism is a mechanisms where the agent reports his arrival and his valuations to the mechanism.

Definition 2 (Direct Mechanism)**.**
A direct mechanism (x,p) specifies at every point in time t the allocation $x_t\left(\alpha,(\theta_s)_{\alpha\leq s\leq t}\right)$ as well as the transfer $p_t\left(\alpha,(\theta_s)_{\alpha\leq s\leq t}\right)$ as a function of the arrival time α and the valuations $(\theta_s)_{\alpha\leq s\leq t}$ reported by the agent prior to time t.

As our environment is stationary we restrict attention to stationary mechanisms where the allocation and transfers are independent of the arrival time of the buyer. More formally, we require that an agent who arrives at time α and whose valuations follows the path $(\theta_s)_{s\in[\alpha,T]}$, receives the same allocation as an agent who arrives at time α' and his valuations follows the same path of valuations shifted by the difference in arrival times, i.e. $\theta'_s = \theta_{s+(\alpha-\alpha')}$ for all $s\in[\alpha',\alpha'+T-\alpha]$. Thus, the principal cannot discriminate the agent based on his arrival time.

Definition 3 (Stationary Direct Mechanism)**.**
A direct mechanism (x,p) is stationary if for all arrival times α,α' and valuation paths θ:

$$
\begin{aligned}
x_t\left(\alpha,(\theta_s)_{\alpha\leq s\leq t}\right) &= x_{t+(\alpha'-\alpha)}\left(\alpha',(\theta_s)_{\alpha\leq s\leq t}\right),\\
p_t\left(\alpha,(\theta_s)_{\alpha\leq s\leq t}\right) &= p_{t+(\alpha'-\alpha)}\left(\alpha',(\theta_s)_{\alpha\leq s\leq t}\right).
\end{aligned}
$$

2.3 Progressive Mechanism

By the revelation principle we can without loss of generality restrict attention to *direct mechanisms* where it is optimal for the agent to report his arrival time α and his valuation θ_t truthfully at every point in time t. Define the indirect utility $V_\alpha:\mathbb{R}_+\to\mathbb{R}$ of an agent who arrives at time α with a value of θ_α and reports his arrival and his valuations $(\theta_t)_t$ truthfully by:

$$
\begin{aligned}
V_\alpha(\theta_\alpha) &= \mathbb{E}\left[\int_\alpha^T e^{-r\,(t-\alpha)}\left\{\theta_t x_t - p_t\right\}\mathrm{d}t \mid \alpha,\theta_\alpha\right]\\
&= \mathbb{E}\left[\int_\alpha^\infty e^{-(r+\gamma)\,(t-\alpha)}\left\{\theta_t x_t - p_t\right\}\mathrm{d}t \mid \alpha,\theta_\alpha\right].
\end{aligned}
$$

The above equality follows immediately from the law of iterated expectations and the fact that the departure time of the agent T is independent of the arrival time α and the valuation process θ:

$$\begin{aligned}\mathbb{E}\left[\int_\alpha^T e^{-r\,(t-\alpha)}\cdot \mathrm{d}t\right] &= \mathbb{E}\left[\int_\alpha^\infty \mathbf{1}_{\{T\geq t\}} e^{-r\,(t-\alpha)}\cdot \mathrm{d}t\right] = \mathbb{E}\left[\int_\alpha^\infty \mathbb{E}\left[\mathbf{1}_{\{T\geq t\}}\right] e^{-r\,(t-\alpha)}\cdot \mathrm{d}t\right] \\ &= \mathbb{E}\left[\int_\alpha^\infty \mathbb{P}\left[T\geq t\right] e^{-r\,(t-\alpha)}\cdot \mathrm{d}t\right] = \mathbb{E}\left[\int_\alpha^\infty e^{-\gamma\,(t-\alpha)} e^{-r\,(t-\alpha)}\cdot \mathrm{d}t\right].\end{aligned}$$

It is optimal for the agent to report truthfully if

$$V_\alpha(\theta_\alpha) = \sup_{\hat{\alpha}\geq\alpha,(\hat{\theta}_t)} \mathbb{E}\left[\int_{\hat{\alpha}}^\infty e^{-(r+\gamma)\,(t-\alpha)}\left\{\theta_t x_t(\hat{\alpha},(\hat{\theta}_s)_{s<t}) - p_t(\hat{\alpha},(\hat{\theta}_s)_{s<t})\right\}\,\mathrm{d}t \mid \alpha,\theta_\alpha\right], \tag{IC}$$

where the allocation x_t as well as the payment p_t is a function of the reported arrival time $\hat{\alpha}$ as well as all previously reported valuations $(\hat{\theta}_s)_{s\leq t}$. We note here that the supremum in (IC) is taken over stopping times $\hat{\alpha}$ as the agent can condition his reported arrival on his willingness to pay for the good.

We restriction attention to mechanisms where the agent participates voluntarily, i.e. for all arrival times α and all initial values θ_α, the agent's expected utility from participating in the mechanism is non-negative:

$$V_\alpha(\theta_\alpha)\geq 0\,. \tag{PC}$$

While imposing incentive compatibility constraints (IC) as well as participation constraints (PC) is standard in the literature on (dynamic) mechanism design, we note that the incentive compatibility constrained (IC) imposed here is stronger than the one usually imposed in the literature. As the arrival time α is not observable to the principal, she has to provide incentives for the agent to report his arrival truthfully. In fact the incentive constraint (IC) we impose implies the participation constraint (PC) as the agent can always decide to never report his arrival $\hat{\alpha}=\infty$. The seller seeks to maximize her revenue subject to the incentive and participation constraints, and we refer to it as the *progressive mechanism* design problem.

3 Revenue Equivalence

We denote by $\mathcal{M}$ the set of all incentive compatible stationary mechanisms where every agent participates voluntarily. A first observation that follows from the independence of the values across the buyers is that we can rewrite the objective of the principal only in terms of the revenue collected from the interaction with a single buyer.

Lemma 1 (Expected Revenue)**.**
The expected discounted revenue in the optimal mechanism equals

$$\frac{r+\gamma}{r} \max_{(x,p)\in\mathcal{M}} \mathbb{E}\left[\int_{\alpha_i}^{T_i} e^{-r\,(t-\alpha_i)} p_t^i \mathrm{d}t\right],$$

where i is an arbitrary buyer.

The proofs for all the results are relegated to the Appendix. In a stationary direct mechanism the allocation and transfer depend only on the time which elapsed since the agent arrived. We can therefore, without loss of generality, assume that the agent arrived at time zero $\alpha = 0$ (or chose $i = 0$), to determine the revenue the principal derives from her interaction with the agent. To simplify the notation we will drop the sub-index indicating the agent's arrival and denote by $V(\theta_0)$ the indirect utility of the agent when he arrived at time 0 with initial valuation θ_0.

As a first step in the analysis we show that the progressive mechanism design problem we consider can be related to a static auxiliary problem. In this static problem the agent reports only his initial valuation and the principal chooses a discounted expected quantity $q \in \mathbb{R}_+$ to allocate to the agent. It turns out that in any incentive compatible mechanism, both the value of the agent as well as the revenue of the principal are only a function of this quantity.

We define the "expected aggregate quantity" $q : \Theta \to \mathbb{R}_+$ which is allocated to an agent with initial valuation θ_0 by

$$q(\theta_0) \triangleq \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t} x_t \exp\left(-\frac{\sigma^2}{2}t + \sigma\,W_t\right) \mathrm{d}t \mid \theta_0\right]. \tag{4}$$

The first term inside the integral is simply the discounted quantity in period t:

$$e^{-(r+\gamma)\,t} x_t.$$

The second term is the derivative of the value θ_t in period t with respect to the initial value θ_0:

$$\frac{d\theta_t}{d\theta_0} = \exp\left(-\frac{\sigma^2}{2}t + \sigma\,W_t\right),$$

where we recall–see (3)–that the geometric Brownian motion can be explicitly represented as:

$$\theta_t = \theta_0 \exp\left(\frac{\sigma^2}{2}t + \sigma W_t\right).$$

The above derivative represents the influence that the initial value θ_0 has on the future state θ_t. In Bergemann and Strack (2015), we referred to it as stochastic flow, and it is the analogue of the impulse response function in discrete time dynamic mechanism (see Pavan et al. (2014), Definition 3). Thus, the "expected aggregate quantity" $q(\theta_0)$ weighs the discounted quantity with the corresponding stochastic flow, or information rent that emanates from the initial value.

As the quantity x_t is bounded between 0 and 1 and the exponential term is a martingale, it follows that the aggregate quantity is bounded as well, i.e. for all $\theta \in [0, \overline{\theta}]$

$$0 \leq q(\theta) \leq \frac{1}{r+\gamma}. \tag{5}$$

We can complete the description of the static auxiliary problem with the introduction of the virtual value and denote by:

$$J(\theta) \triangleq \theta - \frac{1-F(\theta)}{f(\theta)} \tag{6}$$

the "virtual flow value" of the buyer upon arrival to the mechanism. We denote by

$$\widehat{\theta} \triangleq \inf\{\theta \colon J(\theta) \geq 0\}, \tag{7}$$

the lowest type with a non-negative virtual value. We assume that the distribution of initial valuations is such that $\theta \mapsto \min\{0, f(\theta)J(\theta)\}$ is non-decreasing.[5]

The expected quantity q and the virtual utility J are useful as they allow us to completely summarize the expected discounted revenue of the principal and the value of the agent:

Proposition 1 (Revenue Equivalence)**.**
The value of the agent in an incentive compatible mechanism when he arrives with initial valuation θ is given by

$$V(\theta) = \int_0^\theta q(z)\mathrm{d}z + V(0) \tag{8}$$

[5] This is a weak technical assumption which is satisfied for most standard distributions like the uniform distribution, the exponential distribution, or the log-normal distribution. For example for the uniform distribution $U([0,\overline{\theta}])$ we have that $f(\theta)J(\theta) = \frac{2\theta-\overline{\theta}}{\overline{\theta}}$ which is increasing in θ. For the exponential distribution with mean $\mu > 0$ we have that $\min\{0, f(\theta)J(\theta)\} = \min\left\{0, \left(\frac{\theta}{\mu}-1\right)\exp\left(-\frac{1}{\mu}\theta\right)\right\}$ which is also increasing in θ.

and the expected discounted revenue of the principal is given by

$$\mathbb{E}\left[\int_0^{\infty} e^{-(r+\gamma)\,t} p_t \mathrm{d}t\right] = \int_0^{\overline{\theta}} J(\theta)q(\theta)\mathrm{d}F(\theta) - V(0)\,. \tag{9}$$

Proposition 1 allows us to express the objective functions of the agent and the principal in terms of the discounted quantities q only. This result is an application of a general revenue equivalence result, stated as Theorem 1 in Bergemann and Strack (2015). The particularly transparent reduction here comes from the presence of the geometric Brownian motion and the unit demand. By contrast, the reduction to an auxiliary static program extends to a wide class of stochastic process and allocation problems. The next result establishes that the function q must be increasing in any incentive compatible mechanism.

Proposition 2 (Monotonicity of Discounted Quantity)**.**
The function q increases in any mechanism that is incentive compatible.

Proposition 1 and 2 follow from the optimality of the agent to report truthfully at time zero. They thus provide only necessary conditions for incentive compatibility, as they ignore the agent's incentive constraints after time zero and the option to misreport his arrival. As we see in the next section this condition is sufficient if arrivals are observable in the sense that for every monotone expected aggregate quantity q there exist a mechanism that implements it (the mechanism where x_t is constant). The monotonicity of q turns out not to be sufficient for incentive compatibility if the agent strategically times when he contracts with the principal as we will see in the next Section.

4 Sales Contract

We will derive the revenue maximizing mechanism for the seller when she does not observe the arrival time of the buyer in Section 5. As a point of reference, it will be instructive for us to first understand what the seller would do if the (individual) arrival time of each buyer would be observable by the seller. With observable arrival, the optimal solution can be implemented by sales contract. We first review these results in Section 4.1 and then investigate in Section 4.2 how a sales contract performs with unobservable arrival .

4.1 Optimal Contract with Observable Arrival

With observable arrival time by the agent, we are in the canonical dynamic mechanism design environment. In earlier work, Bergemann and Strack (2015), we derived the revenue maximizing mechanism for a general environment of allocation problems and stochastic processes in continuous time. The current problem of interest, unit demand with values governed by a geometric Brownian motion was a canonical problem of this general environment. We establish that an implementation of the optimal mechanism is to offer the product for sale at an optimally determined price P, see Proposition 8 of Bergemann and Strack (2015).

We described the revenue of the seller in Proposition 1. By the revenue equivalence formula (9), the revenue of the seller is determined by the flow virtual value when the uncertainty is given by the geometric Brownian motion. Thus the optimal mechanism awards the object to the agent if his virtual value is positive upon arrival, thus if and only if

$$J(\theta_0) \geq 0.$$

Hence, it is optimal to maximize $q(\theta_0)$ if $J(\theta_0) \geq 0$ and minimize it otherwise. The optimal allocation then awards the object to the agent at all times $s \geq 0$ if and only if his initial valuation θ_0 at arrival time $t = 0$ is sufficiently high:

$$x_s = \begin{cases} 1, & \text{if } \theta_0 \geq \widehat{\theta}; \\ 0, & \text{otherwise;} \end{cases}$$

where the critical value threshold $\widehat{\theta}$ is determined by

$$J(\widehat{\theta}) = 0.$$

Thus, the buyer receives the object *forever* whenever his initial valuation θ_0 is above the threshold value $\widehat{\theta}$. With observable arrivals this allocation can be implemented in a sales contract where the principal charges a sales price of

$$\widehat{P} = \frac{\widehat{\theta}}{r + \gamma},$$

which entitles the buyer to ownership and continued consumption at all future times. An revenue-equivalent implementation would be to sell the good at time $t = 0$ and then charge the buyer a constant flow price of

$$\widehat{p} = \widehat{\theta},$$

in all future periods, independent of his future value θ_s, for all $s \geq 0$. Thus, the indirect utility of the agent when his arrival is observable equals

$$V(\theta_0) = \max\left\{0, \frac{\theta_0 - \widehat{\theta}}{r + \gamma}\right\}.$$

4.2 Sales Contract with Unobservable Arrival

We now abandon the restrictive informational assumption of observable arrival and let the arrival time be private information to each buyer. Let us first consider what would happen if the seller were to maintain the above sales policy and offer the object for sale at the flow price p, which could be the optimal observable price $\widehat{p}$, as a stationary contract, at time $t = 0$ and all future times. Any newly arriving buyers would conclude that rather than buy immediately, he should wait until he learns more about his value, and purchase the object if and only if he learned that he has a sufficiently high valuation for the object. Thus the sale would occur (i) later and (ii) to fewer buyers.

When deciding on the optimal purchase time in a sales contract the agent faces an optimal stopping problem. Purchasing the good later forces the agent to delay his consumption, but allows him to learn more about his value for the good and to potentially avoid an ex-post sub-optimal purchasing decision. Recall that we denote by T the random time at which the agent leaves the market. If the agent acquires the good at time t with valuation θ_t his expected continuation utility is given by

$$\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}(\theta_s - p)\,\mathrm{d}s\right] = (\theta_t - p)\;\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\mathrm{d}s\right] = \frac{\theta_t - p}{\gamma + r}.$$

The first equality in the above equation follows from the fact that θ is a martingale (independent of T) and thus the agent's value at time t is his best estimate of his value at later points in time. The second equality follows as the time T at which the agent leaves the market and thus stops consuming the good is (from a time t perspective) exponentially distributed with mean $\alpha + \frac{1}{\gamma}$. The time τ at which the agent optimally purchases the good thus solves the stopping problem:

$$\sup_\tau \frac{1}{\gamma + r}\,\mathbb{E}\left[e^{-r\tau}\,\mathbf{1}_{\{\tau < T\}}\,(\theta_\tau - p)\right].$$

As the agent leaves the market with rate γ this problem is equivalent to the problem where the discount rate is given by $(r+\gamma)$, i.e. the agent solves the stopping problem

$$\sup_{\tau} \frac{1}{\gamma + r} \mathbb{E}\left[e^{-(r+\gamma)\,\tau}\, (\theta_\tau - p)\right] . \tag{10}$$

The stopping problem given in (10) is the classical irreversible investment problem analyzed in Dixit and Pindyck (1994, Chapter 5, p.135 ff.). To simplify notation we define a constant β that summarizes the discount rate r, the renewal rate γ and the variance σ^2 in a manner relevant for the stopping problem :

$$\beta \triangleq \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\,(r+\gamma)}{\sigma^2}} > 1\,. \tag{11}$$

The solution to the irreversible investment problem given in (10) is well known (c.f. p.142 in Dixit and Pindyck, 1994) and leads to a characterization of the buyer's behavior in any sales contract with price p.

Proposition 3 (Sales Contract)**.**
In a sales contract with flow price p, the buyer acquires the object once his valuation exceeds the time independent threshold $\theta^\star$ given by

$$\theta^\star \triangleq \frac{\beta}{\beta - 1} p\,. \tag{12}$$

The buyer's value in this sales contract is given by

$$V(\theta) = \begin{cases} \frac{1}{1+\gamma}\left(\frac{\theta}{\theta^\star}\right)^\beta (\theta^\star - p)\,, & \text{if } \theta \leq \theta^\star; \\ \frac{1}{1+\gamma}(\theta^\star - p)\,, & \text{if } \theta \geq \theta^\star. \end{cases}$$

We assumed that the valuation evolves according to a geometric Brownian motion for two reasons that now become apparent: First, it is invaluable for tractability. It allows us to calculate the option value the agent receives from delaying his decision to enter a contract in closed form. It is exactly for this reason that most of the literature on irreversible investment and the option value has focused on the geometric Brownian motion (c.f. Dixit and Pindyck, 1994). As a second benefit this enables us to relate and interpret some of our results through the lens of this classical literature. In particular, the agent's stopping problem when he is offered a fixed price sales contract becomes a special case of the framework analyzed in Dixit and Pindyck (1994).

We denote by $\tau_{\theta^\star}$ the (random) time at which the agent purchases the good:

$$\tau_{\theta^\star} \triangleq \inf\{t \colon \theta_t \geq \theta^\star\}.$$

As $\theta^\star > p$, the agent only purchases the good once his valuation is sufficiently above the price p charged for the object. Thus, an agent who starts with an initial value of θ_0 below the threshold $\theta^\star$ expects to wait some random time until he hits any given threshold $\theta^\star$. With the geometric Brownian motion we can compute the expected discounted time for an agent with initial value θ_0 to hit any arbitrary threshold x.

Lemma 2 (Expected Discounted Time)**.**
The expected discounted time $\tau_x = \inf\{t \colon \theta_t \geq x\}$ *until an agent's valuation exceed a threshold* x *conditional on the initial valuation* θ_0

$$\mathbb{E}\left[e^{-r\,\tau_x} \mid \theta_0\right] = \min\left\{\left(\frac{\theta_0}{x}\right)^{\beta}, 1\right\}. \tag{13}$$

Thus if the initial value θ_0 exceeds the threshold x, then the expected discounted time is simply 1, in other words there is no waiting at all. By contrast, if the initial value θ_0 is below the threshold x, then the expected discount time is smaller the larger the gap between initial value θ_0 and target threshold x is. The magnitude of the discounting is again determined entirely by the constant β which summarizes the primitives of the dynamic environment, namely r, γ and σ^2, as defined earlier in (11).

Intuitively, the agent has an option value of waiting and learning more about his valuation of the good and only purchases once the forgone utility of not purchasing the good is sufficiently high. This is in sharp contrast to the classical dynamic mechanism design approach where the arrival time of the agent is observable. When the arrival time is observable the seller can commit herself to not sell to the agent in the future if the agent does not purchase the good immediately and thus the agent can not delay his purchasing decision. The agent thus always buys the good immediately if his valuation exceeds the price p. The information rent that the agent gains from his ability to delay his purchasing decision is called the "option value" and equals

$$\mathbb{E}\left[e^{-r\,\tau_{\theta^\star}}(\theta^\star - p)\right] - \max\left\{(\theta^\star - p), 0\right\}. \tag{14}$$

From a dynamic mechanism design perspective the option value given in (14) corresponds to an additional information rent the agent receives due to his ability to delay entering a contractual relation with the seller. As the option value is always positive, the buyer is, for any fixed mechanism, unambiguously better off if he can delay his purchasing decision.

In contrast the effect of the buyer's ability to delay the purchase on the seller's revenue is ambiguous in a sales contract. When the buyer delays his purchase the revenue of the seller decreases. But to the extent, that some types of the buyer who would not have bought the object upon arrival will do now later on, and after a sufficiently large positive shock on their valuation, there are now additional revenues accruing to the seller.

Using the characterization of the purchase behavior of the buyer in Proposition 3 and standard stochastic calculus arguments, we can completely describe the seller's average revenue for a given sales contract.

Proposition 4 (Revenue of Sales Contract)**.**
The flow revenue per time in a sales contract with flow price p is given by

$$R_{sales}(p) = \frac{p}{r}\int_0^\infty \min\left\{\left(\frac{\beta-1}{\beta}\frac{\theta}{p}\right)^\beta, 1\right\} f(\theta)\,\mathrm{d}\theta\,. \tag{15}$$

Equation (15) reduces the problem of finding an optimal sales contract to a simple single dimensional maximization problem over the price. It is worth noting that the revenue up to a linear scaling depends on r, γ, σ only through β which implies that the optimal sales price is only a function of β.

The expression inside the integral of (15) represent the expected quantity to be sold to an agent with initial value θ. In contrast to a standard revenue function under unit demand, the realized quantities are not merely 0 or 1. Rather, the sellers offers positive quantity to all buyers, namely

$$\min\left\{\left(\frac{\beta-1}{\beta}\frac{\theta}{p}\right)^\beta, 1\right\}. \tag{16}$$

This expression reflects the expected discounted time for those buyers who have an initial value below

$$\frac{\beta}{\beta-1}p.$$

By the earlier Proposition 3, the above term identifies the optimal threshold of the buyer when faced with a flow sales price p. The complete expression (16) then follow from Lemma 2 as the expected discounted probability of a sale to a buyer with initial value θ. Thus, an increase in the sales price p uniform lowers the probability of a sale for every value θ. The problem for the seller with unobservable arrivals is therefore how to respond to slower and more selective sales.

4.3 Failure of Incentive Compatibility with Unobservable Arrival

We can now summarize some of our findings regarding the performance of a sales contract in an environment with unobservable arrival. Suppose the seller would like to implement the sales contract $(\widehat{p}, \widehat{\theta})$ that was revenue maximizing under observable arrival in the new environment. This sales contract set the flow price equal to the threshold value, or

$$\widehat{p} = \widehat{\theta}.$$

In this contract any agent with an initial value θ_0 below $\widehat{\theta}$ never gets the object and makes no payment and thus his utility from the contract equals zero. We can now ask whether this contract remains incentive compatible in the environment with unobservable arrival times.

Now, if it were optimal for the buyer to reveal his presence to the mechanism immediately, then the value from revealing his presence at any stopping time $\hat{\alpha}$ must be smaller than revealing his presence at time zero. As the agent can condition the time at which he reports his arrival to the mechanism on his past valuations the following constraint must hold for all stopping times $\hat{\alpha}$ which may depend on the agents valuation path $(\theta_t)_t$:[6]

$$V(\theta_0) \geq \sup_{\hat{\alpha}} \mathbb{E}\left[e^{-(r+\gamma)\,\hat{\alpha}} V(\theta_{\hat{\alpha}}) \mid \theta_0\right]. \tag{IC-A}$$

Now from Proposition 3, we can immediately infer that the constraint (IC-A) is not satisfied as the stopping time τ_w for some $w > \widehat{\theta}$ yields a value higher than the value obtained by reporting the arrival immediately. Thus the contract $(\widehat{p}, \widehat{\theta})$ will not be incentive compatible anymore under unobservable arrival. Namely, every agent with an initial valuation θ_0 lower than

$$\frac{\beta}{\beta-1}\widehat{p} = \frac{\beta}{\beta-1}\widehat{\theta} > \widehat{\theta},$$

would deviate to report his arrival only at the first time τ where his value θ_t matches the above threshold, or

$$w \triangleq \frac{\beta}{\beta-1}\widehat{p}.$$

[6] This is a version of the revelation principle as the principal can replicate every outcome where the agent does not report his arrival immediately in a contract where the agent reveals his arrival immediately, but never gets the object before he would have revealed his arrival in the original contract.

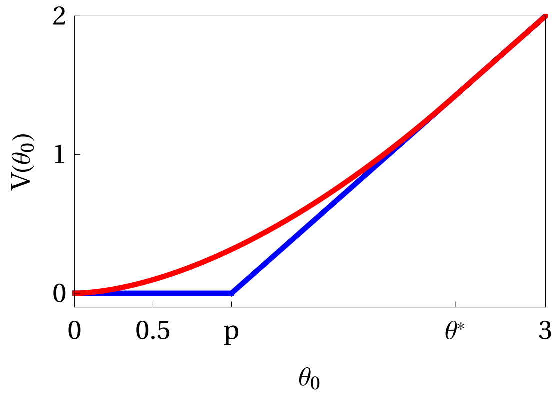


Figure 1: This figure displays the value of the agent as a function of her initial valuation in a sales contract with flow price $p = 1$ when she has to participate immediately (blue) and when she can delay her arrival (red) when $\beta = 1.7$.

The delay resulting from waiting until stopping time τ_w

$$\tau_w \triangleq \inf\{t \geq 0 \colon \theta_t \geq w\}\,,$$

would yield the buyer a strictly positive expected utility of

$$\mathbb{E}\left[e^{-(r+\gamma)\,\tau_w} V(w) \ \mid \theta_0\right] = \mathbb{E}\left[e^{-(r+\gamma)\,\tau_w} \frac{w - \widehat{\theta}}{r+\gamma} \ \mid \theta_0\right] > 0\,,$$

and thus represent a profitable deviation for the buyer.

We can illustrate the payoff consequences by comparing the value functions of the buyer across the two informational environments. The blue line depicts the value function for the buyer in the setting with observable arrival time. The value is zero for all values below the threshold $\widehat{\theta}$ and then a linear function of the initial value. Notably, the value function has a kink at the threshold level $\widehat{\theta}$. The red curve depicts the value function when the sales contract is offered at the above terms as a stationary contract. Now, the value function is smooth everywhere, and coincides with blue curve whenever the initial value weakly exceeds w. Importantly, for all values θ_0 below w, the red curve is above the blue curve, which depicts the option value as expressed by (14). Notably, the value is strictly positive for all initial values which expresses the fact that the option value guarantees every value θ_0 an information, quite distinct from the environment with observable arrival.

Perhaps surprisingly then, using a sequence of relaxation arguments we prove in Section 5 that the optimal mechanism (in the space of all incentive compatible mechanisms when the buyers arrival to the mechanism is unobservable) remains a sales contract. Thus, (15) can be used to identify the optimal mechanism. But importantly, as the current analysis suggests, there is going by a large gap between the optimal flow price $p^\star$ and the optimal threshold $\theta^\star$ with $p^\star < \theta^\star$.

5 The Optimal Mechanism

The discussion in the previous Section illustrates that the first order approach will in general fail once the agent can misreport his arrival time. To solve this problem we will employ the following strategy: First, we will identify particularly tractable necessary conditions for the truthful reporting of arrivals, by considering a specific class of deviations in the arrival time dimension. We then find the optimal mechanisms for the relaxed problem where we impose only these necessary conditions using a novel result on optimization theory we develop. Finally, we will verify that in this mechanism it is indeed optimal to report the arrival time truthfully.

5.1 Truthful Reporting of Arrivals

In the first step we find a necessary condition such that the agent wants to report his arrival immediately. We first show that the agents value function V in any incentive compatible mechanism must be continuously differentiable and convex.

Proposition 5 (Convexity of Value Function)**.**
The value function in any incentive compatible mechanism is continuously differentiable and convex.

The discussion in Subsection 4.3 illustrated that the indirect utility need not to be continuously differentiable in the optimal mechanism if the agents arrival time is observable. Intuitively, the constraint that the agent must find it optimally to report his arrival immediately, (IC-A) implies that there cannot be kinks in the value function as this would imply a first order gain for the agent from the information he would get by waiting to report his arrival.

In the next step we will relax the problem by restricting the agent to a small class of deviations in reporting his arrival. The class of deviations we are going to consider is to have the agent report his arrival the first time his valuation crosses a time independent cut-off

$w > \theta_0$:

$$\tau_w = \inf\{t \geq 0 \colon \theta_t \geq w\}\,.$$

Note, that the optimal deviation of the agent will not (necessarily) be of this form for every direct mechanism. By restricting to deviations of this form we hope that in the optimal mechanism the optimal deviation will be of this form and the restriction is non-binding.

5.2 Information Rents Associated with Unobservable Arrival

We established in Lemma 2 that the payoff from deviating to τ_w when reporting the arrival time, while maintaining to report values truthfully, is given by:

$$\mathbb{E}\left[e^{-(r+\gamma)\,\tau_w} V(v_{\tau_w}) \;\mid \theta_0\right] = V(w)\left(\frac{\theta_0}{w}\right)^{\beta}$$

where $\beta > 1$ was defined in (11). Intuitively,

$$\left(\frac{\theta_0}{w}\right)^{\beta}$$

captures the discount factor caused by the time the agent has to wait to reach a value of w before participating in the mechanism. When the agent then participates in the mechanism he receives the indirect utility $V(w)$ of an agent whose initial value equals w. Now, in any mechanism where (IC-A) is satisfied the agent does not want to deviate to the strategy τ_w we must have

$$V(\theta_0) \geq V(w)\left(\frac{\theta_0}{w}\right)^{\beta} \Leftrightarrow V(w)w^{-\beta} \leq V(\theta_0)\theta_0^{-\beta}. \tag{17}$$

As (17) holds for all θ_0 and $w > \theta_0$, we have that the agent does not want to deviate to any reporting strategy $(\tau_w)_{w>\theta_0}$ if and only if $V(w)w^{-\beta}$ is decreasing. Taking derivatives gives us that this is the case whenever[7]

$$V'(w) \leq \beta \frac{V(w)}{w}\,.$$

As the derivative of the value function is the aggregate quantity q we have the following proposition that derives a necessary condition on the aggregate quantity q for it to be optimal for the agent to report his arrival truthfully.

[7] $0 \geq V'(w)w^{-\beta} - \beta w^{-\beta-1}V(w) \Rightarrow V'(w) \leq \beta \frac{V(w)}{w}\,.$

Proposition 6 (Upper Bound on Discounted Quantities)**.**
The aggregate quantity is bounded from above by

$$q(\theta_0) = V'(\theta_0) \le \beta \frac{V(\theta_0)}{\theta_0} \tag{18}$$

in any mechanism where it is optimal to report arrivals truthfully, i.e. that satisfies (IC-A)*.*

Intuitively, (18) bounds the discounted quantity an agent of initial type θ_0 can receive. Note, that (18) is always satisfied if the value function of all initial values θ_0 of the agent from participating in the mechanism is sufficiently high. Intuitively, due to discounting the agent does not want to delay reporting his arrival when the value from participating is high. As we can always increase the value to all types of the agent, by possibly offering a subsidy to the lowest type, we can reformulate (18) as a lower bound on the value$V(0)$ of the lowest type $\theta_0 = 0$.

Proposition 7 (Lower Bound on Information Rent)**.**
In any mechanism which satisfies (IC-A) *we have that*

$$V(0) \ge \sup_{\theta \in \Theta} \left(\frac{\theta q(\theta)}{\beta} - \int_0^\theta q(z) \mathrm{d}z \right).$$

The above result establishes a lower bound on the cost of providing the agent with incentives to report his arrival time truthfully. This lower bound depends only on the allocation q. Intuitively, the principal may need to pay subsidies independent of the agent's type to provide incentives for the agent to report his arrival time truthfully if the quantity q is too convex and the option value of waiting is thus too high. The subsidy would correspond to a payment made to the agent upon arrival and independent of his reported value θ_0. While such a scheme makes delaying the arrival costly to the agent due to discounting it is potentially very costly as it requires the principal to pay the agent just for "showing up". We will show that in the optimal mechanism this issue will not be relevant as the optimal mechanism does not reward the agent merely for arriving.[8]

As a consequence of Proposition 7 we get an upper bound on the revenue in any incentive compatible mechanism.

[8] Such subsidy schemes are also discussed in Gershkov et al. (2015, 2018) in a context where the buyers' values do not evolve over time. In contrast to our result Gershkov et al. (2015) show that to implement the efficient allocation in this context such subsidies are sometimes necessary in order to to incentivize the agents to report their arrival time truthfully.

Corollary 1 (Revenue Bound)**.**
An upper bound on the revenue in any incentive compatible mechanism is given by

$$\int_0^{\overline{\theta}} q(z)J(z)\mathrm{d}F(z) - \max_{\theta\in[0,\overline{\theta}]} \left(\frac{\theta q(\theta)}{\beta} - \int_0^{\theta} q(z)\mathrm{d}z \right) . \tag{19}$$

The upper bound on revenue in (19) is obtained by considering only a small class of deviations. In particular, the agent is only allowed to misreport his arrival via simple threshold strategies, where he enters the mechanism once his valuation is sufficiently high. Economically,

$$V(0) = \max_{\theta\in[0,\overline{\theta}]} \left(\frac{\theta q(\theta)}{\beta} - \int_0^{\theta} q(z)\mathrm{d}z \right)$$

is a lower bound on the information rent the agent must receive to ensure that he reports his arrival truthfully in a mechanism which implements the allocation q. As discussed before, this information rent is payed to the agent in the form of a transfer that is independent of his consumption and thus even those types receive who never consume the object. We note that due to the maximum this information rent can not be rewritten as an expectation and thus is fundamentally different from the classical information rent term. As a consequence pointwise maximization can not be used to find the optimal contract even in the relaxed problem. We next develop the mathematical tools to deal with this type of non-standard maximization problem.

5.3 The Optimal Mechanism

We next characterize the optimal mechanism. To do so we proceed by first finding the allocation q that maximizes the upper bound on revenue (19). Second, we are going to construct an incentive compatible mechanism that implements this allocation. As (19) is an upper bound on the revenue, in any incentive compatible mechanism, we then found a revenue maximizing mechanism.

A mathematical challenge is that, due to the information rent from arrivals, the relaxed problem (19) is non-local and non-linear in the quantity q. A change of the quantity for one type can affect the surplus extracted from all higher and lower types. Consider the relaxed problem of finding the revenue maximizing mechanism such that the agent never wants to misreport his arrival using a cut-off stopping time. By Proposition 6, the indirect utility V

of the agent in this mechanism solves the optimization problem

$$\max_V \int_{\underline{\theta}}^{\overline{\theta}} V'(z)J(z)\, f(z)\, \mathrm{d}z - V(\underline{\theta}), \tag{20}$$

subject to

$$V'(\theta) \in \left[0, \frac{1}{r+\gamma}\right] \text{ for all } \theta, \tag{21}$$

$$V \text{ is convex,} \tag{22}$$

$$V'(\theta) \leq \beta \frac{V(\theta)}{\theta} \text{ for all } \theta\,. \tag{23}$$

We will further relax the problem by ignoring the monotonicity constraint (22). Due to the derivative constraint (23) this problem is to the best of our knowledge not covered by any standard result in optimization theory.[9] While, a non-standard version of the Pontryagin Maximum principle with state dependent control constraints could in principle be used to deal with the derivative constraint (23) there seems to be no obvious way to infer the optimal policy from the resulting ordinary differential equations, and we could make this approach work only in special cases. To deal with this problem we develop in the appendix the following optimization theory result which might be of independent interest.

Proposition 8 (Comparison Principle).
Let $\Phi : \mathbb{R} \times [0, \overline{\theta}] \to \mathbb{R}_+$ be increasing and uniformly Lipschitz continuous in the first variable on every interval $[a, \overline{\theta}]$ for $a > 0$.[10] *Let $\mathcal{J} : [0, \overline{\theta}] \to \mathbb{R}$ be continuous, satisfy $\mathcal{J}(\underline{\theta}) = -1$ and $z \mapsto \min\{\mathcal{J}(z), 0\}$ be non-decreasing. Consider the maximization problem:*

$$\max_w \int_0^{\overline{\theta}} \mathcal{J}(\theta) w'(\theta) \mathrm{d}\theta - w(0)\,. \tag{24}$$

over all absolutely continuous functions $w : [0, \overline{\theta}] \to \mathbb{R}_+$ that satisfy $w'(\theta) \leq \Phi(w(\theta), \theta)$. There exists $\hat{\theta} \in [0, \overline{\theta}]$ such that a solution w to this problem satisfies

$$\begin{cases} w(\theta) = 0, & \text{if } \theta \in [0, \hat{\theta}], \\ w'(\theta) = \Phi(w(\theta), \theta), & \text{if } \theta \in (\hat{\theta}, \overline{\theta}]. \end{cases}$$

[9] This constraint is fundamentally different from the Border constraint appearing in multi-agent mechanism design problems which is a (weak) majorization constraint.

[10] This means that for every $a > 0$ there exists a constant $L_a < \infty$ such that $|\Phi(v, \theta) - \Phi(w, \theta)| \leq L_a \cdot |v - w|$ for all $\theta \in [a, \overline{\theta}]$.

To apply Proposition 8 to the optimization problem given by (20), (21) and (23) we define

$$\mathcal{J}(\theta) \triangleq f(\theta)J(\theta),$$

and

$$\Phi(v,\theta) \triangleq \min\left\{\beta\frac{v}{\theta}, 1\right\}.$$

An immediate observation is that $\mathcal{J}(\underline{\theta}) = -1$ and, that every solution to the ordinary differential equation

$$V'(\theta) = \beta\frac{V(\theta)}{\theta},$$

which satisfies $V(\theta') = 0$ at a point $\theta' > 0$ is constant equal to zero. This yields the following characterization of the relaxed optimal mechanism.

Proposition 9 (Optimal Control)**.**
There exists a θ' such that a solution to the control problem (20)-(23) is given

$$V(\theta) = \begin{cases} \left(\frac{\theta}{\theta'}\right)^{\beta}\frac{\theta'/\beta}{\gamma+r}, & \textit{for } \theta \leq \theta', \\ \frac{\theta'/\beta}{\gamma+r} + \frac{\theta-\theta'}{\gamma+r}, & \textit{for } \theta' \leq \theta, \end{cases}. \tag{25}$$

and satisfies for all $\theta \in [0, \overline{\theta}]$:

$$V'(\theta) = \frac{1}{r+\gamma}\min\left\{\left(\frac{\theta}{\theta'}\right)^{\beta-1}, 1\right\}. \tag{26}$$

The next result proposes a simple indirect implementation of the indirect utility derived in Proposition 9 is implementable in an incentive compatible mechanism.

Proposition 10 (Posted Price Implementation)**.**
The indirect utility given in (25) is implemented by a sales contract with a flow price of

$$p = \frac{\beta-1}{\beta}\theta'.$$

Note, that we arrived at the optimization problem (20)-(23) by relaxing the original mechanism design problem in two ways. First, we allowed the agent to misreport his arrival only using cut-off stopping times. Second, we ignored the monotonicity constraint associated with truthful reporting of the initial value. By Proposition 10, the allocation which maximizes the expected revenue under those relaxed incentive constraints can be implemented using a sales contract. As the revenue with relaxed incentive constraints is an upper bound on the revenue in the original problem and this upper bound is achieved by some sales contract it follows that a sales contract is a revenue maximizing mechanism.

Theorem 1 (Sales Contracts are Revenue Maximizing)**.**
There exists a flow price p such that the associated sales contract is a revenue maximizing mechanism with unobservable arrivals.

We observe that the optimal allocation gives the object to the agent forever. Hence, any irreversibility constraint on the allocation is non-binding and thus the problem of irreversibly selling the agent an object yields the same solution.[11] Thus, our optimal mechanism is also revenue maximizing in a problem where the agent consumes the object once and immediately, the agent is privately informed about his arrival, and the agent's valuation evolves over time.

6 An Example: The Uniform Prior

We illustrate the results now for the case of the uniform prior, thus $\theta_0 \sim \mathcal{U}[0,1]$. With the uniform prior we can then directly compute from the revenue formula (15) the value threshold θ^* and the associated flow price p^*: The resulting values are given by:

$$\theta^* = \frac{1}{2}\frac{1+\beta}{\beta},$$

and

$$p^* = \frac{\beta-1}{\beta}\theta^* = \frac{1}{2}\frac{\beta^2-1}{\beta^2}.$$

In the dynamic mechanism, the value threshold and the associated price are determined exclusively by the virtual value at $t=0$, and thus under the uniform distribution, the corresponding threshold and flow price are given by

$$\widehat{\theta} = \widehat{p} = \frac{1}{2}.$$

Thus the price in the progressive mechanism is below the dynamic mechanism whereas the threshold of the progressive mechanism is above the dynamic mechanism:

$$p^* = \frac{1}{2}\frac{1+\beta}{\beta} < \widehat{p} = \frac{1}{2} = \widehat{\theta} < \frac{1}{2}\frac{1+\beta}{\beta} = \theta^*. \tag{27}$$

[11] For the case of observable arrivals this problem was analyzed in Board (2007) and Kruse and Strack (2015).

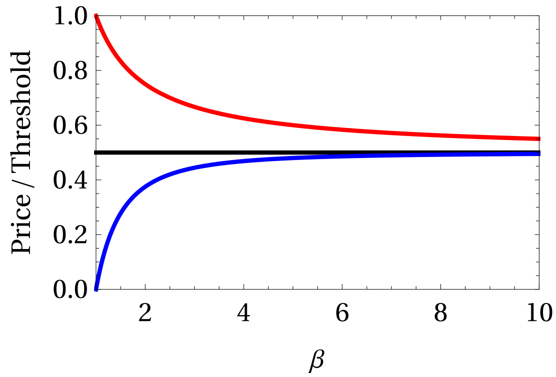
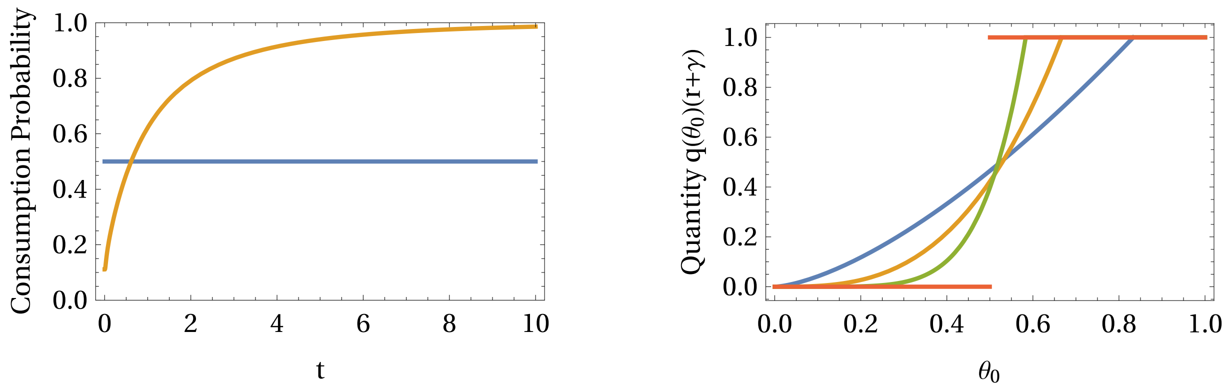


Figure 2: Progressive threshold (red), Dynamic threshold and price (black), and Progressive price (blue)

In Figure 2 we display the behavior of the thresholds and the prices as a function of $\beta \in (1, \infty)$. As β increases, the discounting rate and the renewal rate are increasing, and the buyer becomes less forward-looking. By contrast as β decreases towards one, the gap between the value threshold $\theta^*$and the price p^* increases. As the option value becomes more significant, the buyer chooses to wait until his value has reached a higher threshold, thus he will wait longer to enter into a relationship with the seller. Faced with a more hesitant buyer, the seller decreases the flow price as β decreases. Yet, the decrease in the flow price only partially offsets the option value, and the buyer still waits longer to enter into the relationship with the seller. By contrasts, the threshold value, and the price, in the dynamic mechanism, $\widehat{\theta}$ and $\widehat{p}$, respectively remain invariant with respect to β.

An important aspect of the progressive mechanism is that the buyer enters the relationship gradually rather than once and for all, as in the dynamic mechanism. In Figure 3a we plot the probability that an initial type drawn from the uniform distribution purchases the object as a function of time. In the dynamic mechanism, this probability is constant over time. As all values θ_0 above $\widehat{\theta} = 1/2$ buy the object, and all those with initial values $\theta_0 < \widehat{\theta} = 1/2$ never buy the object, the probability of consumption does not change over time, and is always equal to 1/2. By contrast, in the progressive mechanism, the probability of participation is progressing over time, and thus the probability of consumption is increasing over time. The geometric Brownian motion displays sufficient variance, so that eventually every buyer purchases the product.

In Figure 3a we take the expectation over all initial types $\theta_0 \sim \mathcal{U}[0, 1]$. We now zoom in on the purchase behavior of the individual types $\theta_0 \sim \mathcal{U}[0, 1]$ and describe their expected discounted consumption probability, $q(\theta_0)$. In Figure 3b we illustrate the consumption quantity



(a) Consumption Probability over Time, progressive (orange), dynamic (blue).

(b) Quantities assigned in dynamic and progressive mechanism. $\beta = 1.5$ in blue, $\beta = 3$ in yellow, $\beta = 6$ in green, observable arrival in red.

Figure 3

as a function of the initial value $\theta_0 \sim \mathcal{U}[0, 1]$ for various values of β. We find again that in the dynamic mechanism there is a sharp distinction in the consumption quantities between the initial values below and above the threshold of $\widehat{\theta} = 1/2$. By contrast in the progressive mechanism, the consumption quantity is continuous and monotone increasing in the initial value θ_0. As the buyer becomes more patient, and hence as β decreases, the slope of the consumption quantity flattens outs and the threshold θ^* upon which consumption occurs immediately is increasing.

The differing thresholds and allocation probabilities give us some indication regarding the contrasts in welfare properties between progressive and dynamic mechanism. As the price in the progressive mechanism is uniformly lower, this allows us to immediately conclude that the consumer surplus is larger in the progressive mechanism than in the corresponding dynamic mechanism. Conversely, as the seller could have offered the progressive mechanism in the dynamic setting, but did not, it follows that the revenue of the seller is uniformly lower in the progressive mechanism. Thus, the option of the buyer to postpone his allocation is indeed valuable and increases the consumer surplus significantly. This leaves open the question as to how the social surplus is impacted by these different participation constraints. In the current setting with a uniform prior of $\theta_0 \sim \mathcal{U}[0, 1]$, we can give a complete answer, which is summarized in the next result.

Proposition 11 (Welfare Implications)**.**

1. *The sales price and the seller's revenue are uniformly lower in the progressive than in the dynamic mechanism.*

2. *The consumer surplus is uniformly larger in the progressive than in the dynamic mechanism.*

3. *The social welfare is uniformly larger in the progressive than in the dynamic mechanism.*

We should note that the social welfare comparison does extend to all prior distributions. In particular, if there is only a small amount of private information, so that the static virtual utility is non-negative for all initial values, then the dynamic mechanism will not distort the allocation, and thus support the first best social welfare. For example, in the class of uniform distribution on the interval $[a, 1]$, the static virtual utility:

$$\theta - \frac{1 - F(\theta)}{f(\theta)}$$

is positive for all $\theta \in [a, 1]$ if the lower bound a in the support of the distribution is sufficiently large, or $a > 1/2$. In these circumstances, the seller in the dynamic environment will cease to discriminate against any value, and rather the object forever to all initial types $\theta \in [a, 1]$. By contrast, in the progressive mechanism, the option value remains an attractive opportunity for all buyers, and thus the seller will never sell to all buyers irrespective of their initial value $\theta \in [a, 1]$. In consequence, the progressive mechanism is willing to accept some initial inefficiency, and thus will not attain the first best.

7 Discussion

We now contrast the outcomes under the static and the progressive mechanism. The important restriction of the static mechanism is that it makes the flow allocation in a manner that is history-independent. The revenue difference between the progressive and static mechanism is measure of the value of long-term contracting. We then use the difference as measure to compare the dynamic and the progressive mechanism. As the discounting diminishes, the gains from the progressive mechanism eventually vanish, whereas the persist under the dynamic mechanism. This indicates that the gains from the dynamic mechanism are due to the observability of the arrival time rather than any other benefits from long-term contracting.

7.1 Comparison to Static Mechanisms

In this section we will compare the optimal dynamic mechanism derived in Section 5 to the optimal static mechanism. A static mechanism is a mechanism where the mechanism does not condition on the history of a buyer or calendar time. Intuitively, the seller does not discriminate between buyers who purchased the good in the past and those who did not. It follows from standard arguments (Mussa and Rosen 1978; Myerson 1981) that the optimal such mechanism just charges a fixed price at every point in time where an agent consumes the good. Economically, such a mechanism corresponds to a "leasing contract". A leasing contract allows the buyer to decide at every point in time whether or not he wants to consume the good for a fixed price of p. The designers revenue at every point in time is thus given by the product of the price p and the fraction of agent whose valuation at a given point in time exceeds p and thus purchase the good. Using results about the geometric Brownian motion we can compute the expected discounted revenue from any leasing contract:

Proposition 12 (Revenue in a Leasing Contract)**.**
The expected discounted revenue from an agent who arrives at time 0 *in a leasing contract with price* p *equals*

$$(1 - G(p))p$$

where $G : \mathbb{R}_+ \to [0, 1]$ *is given by*

$$G(\theta) = \int_{\mathbb{R}_+} F\left(\frac{\theta}{m}\right) h(m)\,\mathrm{d}m\,,$$

and the function $h(\cdot)$ *is given by:*

$$h(m) \triangleq \frac{2\gamma}{m\sigma\sqrt{m(8\gamma+\sigma^2)}} \exp\left(-\frac{\sqrt{8\gamma+\sigma^2}\,|\log(m)|}{2\sigma}\right).$$

The revenue in a leasing contract is not computed with respect to the distribution of the initial valuation with which buyers arrive to the market F, but with respect to a function G which measures the discounted average valuation a buyer has for the good while he is present in the market. The distribution G is therefore related to–but distinct from– the steady state distribution of the valuation of the buyer.

A leasing contract is arguably the simplest contract to repeatedly sell a good as the contract does not condition on the history a particular buyer has with the seller. By doing so the seller forgoes the opportunity to reward past purchases of the buyer with future

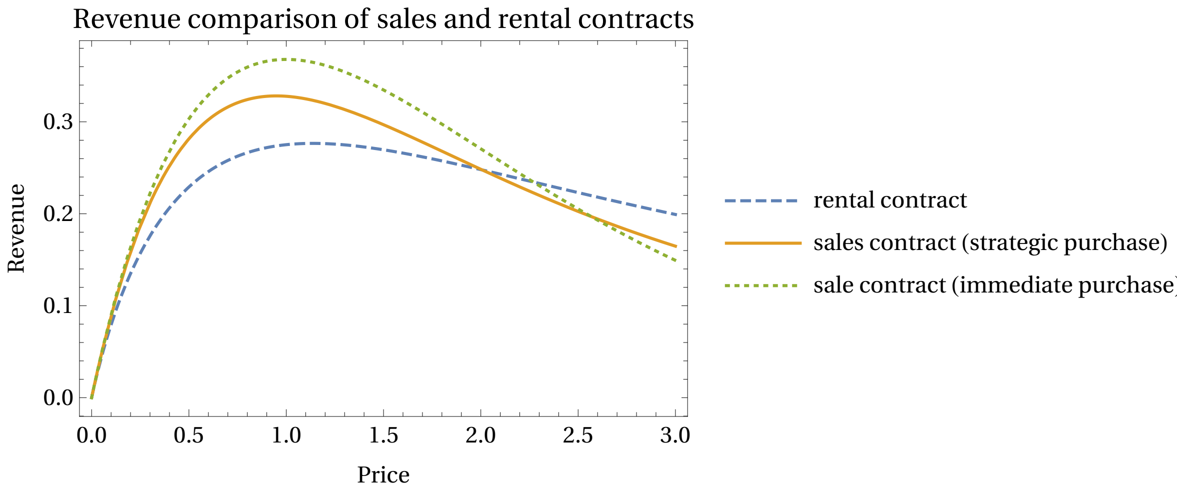


Figure 4: Average revenue under the rental and the sales contract if buyers can and cannot delay their purchase decision. F is the exponential distribution with mean 1, $\sigma = 1, \gamma = 1, r = 1$.

discounts. For different prices, we illustrate the revenue of i) the leasing contract ii) the sales contract when agents strategically decide when to purchase the object as well as iii) the revenue of the sales contract when agents buy upon arrival in Figure 4.

The figure illustrates some economic insights. First, it shows that the optimal revenue in a sales contract if buyers have to contract upon arrival, is greater than the optimal revenue in a sales contract if buyers strategically time their purchase decision which is greater than the revenue of the optimal leasing contract. This mirrors our results that the optimal sales contract with observable arrivals is not incentive compatible with unobservable arrivals and that a sales contract is the optimal dynamic contract.

Second, it shows that the model where agents must contract immediately with the principal for low prices overestimates the revenue if agents can strategically enter the contract and for high prices overestimates it. Intuitively, for low prices, some agents will delay the purchase as they have a high option value and (potentially) leave the market without purchasing the good. For high prices some agents whose initial valuation is below the price will purchase later which increases the sellers revenue.

Finally, there is a substantial difference between the revenue with buyers who purchase immediately and the revenue when buyers strategically time their purchase decision. This illustrates that the classical dynamic mechanism design approach which assumes that arrivals to the market are observable to the designer might substantially overestimate the benefits of dynamic mechanism design.

7.2 Benefits of Dynamic Mechanism Design

We next focus on this aspect and quantify which fraction of revenue gain relative to the optimal static mechanism is due to the assumption that the arrival of agents is observable to the designer. We denote by Δ the difference between the revenue the principal can obtain in the optimal *dynamic* mechanism minus the revenue in the optimal *static* mechanism.

Observable arrivals The dynamic revenue maximizing mechanism is a sales contract if arrivals are observable (see Proposition 8 in Bergemann and Strack, 2015) and the static optimal mechanism is a rental contract. Thus, in the case where arrivals are observable and the principal can commit to never contract with an agent who rejected her initial offer the gain from dynamic mechanism design is given by:

$$\Delta_{\text{observable}} \triangleq \max_p \{p(1 - F(p))\} - \max_p \{p(1 - G(p))\} \ .$$

With the geometric Brownian motion, the optimal sales contract is to offer the object to all those types who have an initial virtual utility that is positive. Thus, the optimal flow price is given by the choice of the optimal monopoly price with respect to the prior distribution $F(\theta_0)$. By contrast, the optimal static policy is to sell at flow price p that maximizes against the discounted average distribution $G(v)$ computed earlier in Proposition 12.

Unobservable arrivals When arrivals are unobservable the gain of the progressive mechanism is given by the difference between the revenue of the optimal sale contract, which we computed in Proposition 4 against the prior distribution $F(\theta_0)$ and the optimal leasing contract against the discounted average distribution $G(v)$:

$$\Delta_{\text{unobservable}} \triangleq \max_p \left\{ \int_0^\infty \min\left\{ \left(\frac{\beta - 1}{\beta} \frac{\theta}{p} \right)^\beta, 1 \right\} p\, f(\theta)\, \mathrm{d}\theta \right\} - \max_p \{p(1 - G(p))\} \ .$$

We can then establish the gains in the dynamic mechanism due to observability of the arrival time by the following ratio:

$$g \triangleq \frac{\Delta_{\text{observable}} - \Delta_{\text{unobservable}}}{\Delta_{\text{observable}}} = 1 - \frac{\Delta_{\text{unobservable}}}{\Delta_{\text{observable}}}$$

The ratio measures the percentage of the increase in revenue that is due to the assumption that arrivals are observable in the dynamic mechanism. If g is high then this means that

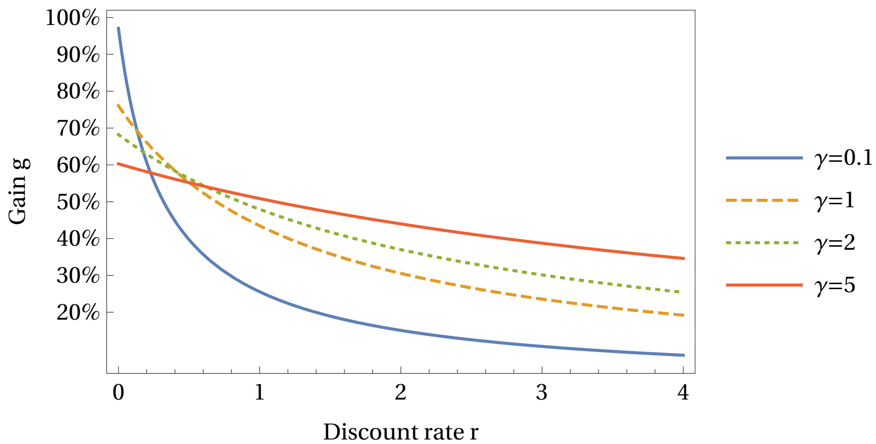


Figure 5: The fraction of the gain of dynamic mechanism design which is due to the fact that arrivals are observable for different discount factors r and replacement rates γ. F is the exponential distribution with mean 1 and $\sigma = 1$.

almost all the benefits of the dynamic mechanism stem from the assumption that arrivals are observable.

Figure 5 depicts the gain g for different discount factors. It shows that for relatively patient agents $60\% - 70\%$ of the benefit of dynamic mechanism design are due to the assumption that arrivals are observable. We observe that as the renewal rate γ becomes smaller, the ratio g increases eventually as the discount factor r becomes smaller. We suspect that quite generally, in the limit as r and γ converge towards zero, then g converges towards one. In other words, the progressive mechanism converges against the optimal static contract.

8 Conclusion

We considered a dynamic mechanism problem where each agent is described by two dimensions of private information, his willingness-to-pay (which may change over time) *and* his arrival time. We considered a stationary environment–in which the buyers arrive and depart at random–and a stationary contract. In this arguably more realistic setting for revenue management, the seller has to guarantee both interim incentive as well as interim participation constraints. As the buyer has the valuable option of delaying his participation, the mechanism has to offer incentives to enter into the relationship.

One challenge in our environment is that the first-order approach and other standard methods fail as global incentive constraints bind in the optimal contract. We were able to solve this multi-dimensional incentive problem by rephrasing the participation decision of the buyer as a stopping problem, and then solve a new optimal control problem. We developed the methodology to the problem of when to allocate a single object to a single buyer, thus a classic revenue maximization problem.

We decomposed the progressive mechanism problem into an inter temporal participation (entry) problem and an intertemporal incentive problem. With the separability between these two problems, our approach can possible extended to allocation problems beyond the unit demand problem considered here. There are (at least) three natural directions to extend the analysis. First, the stochastic evolution of the value was governed by the geometric Brownian motion, and clearly other stochastic process could be considered. Second, the allocation problem could be extended to nonlinear allocation problems rather than the unit demand problem considered here. Third, a natural next step is to extend the techniques developed here to multi-agent environments, say competing bidders for a scarce resource. The final generalization will pose new challenges as we will have to investigate whether the solution of the individual stopping problem can be decentralized or distributed in a consistent manner across the agents. This is a problem similar to the reduced form auction as posed by Border (1991) but now in dynamic rather than static allocation problem.

References

Ashlagi, I., C. Daskalakis, and N. Haghpanah (2016): "Sequential Mechanisms with Ex-Post Participation Guarantees," in *ACM 2016 Conference on Economics and Computation.*

Baron, D. and D. Besanko (1984): "Regulation and Information in a Continuing Relationship," *Information Economics and Policy*, 1, 267–302.

Battaglini, M. (2005): "Long-Term Contracting with Markovian Consumers," *American Economic Review*, 95, 637–658.

Bergemann, D., F. Castro, and G. Weintraub (2017): "The Scope of Sequential Screening with Ex-Post Participation Constraints," in *18th ACM Conference on Economics and Computation.*

Bergemann, D. and P. Strack (2015): "Dynamic Revenue Maximization: A Continuous Time Approach," *Journal of Economic Theory*, 159, 819–853.

Bergemann, D. and J. Valimaki (2019): "Dynamic Mechanism Design: An Introduction," *Journal of Economic Literature*, 57.

Besanko, D. (1985): "Multi-Period Contracts Between Principal and Agent with Adverse Selection," *Economics Letters*, 17, 33–37.

Board, S. (2007): "Selling Options," *Journal of Economic Theory*, 136, 324–340.

Border, K. (1991): "Implementation of Reduced Form Auctions: A Geometric Approach," *Econometrica*, 59, 1175–1187.

Courty, P. and H. Li (2000): "Sequential Screening," *Review of Economic Studies*, 67, 697–717.

Deb, R. (2014): "Intertemporal Price Discrimination with Stochastic Values," Tech. rep., University of Toronto.

Deb, R. and M. Said (2015): "Dynamic Screening with Limited Commitment," *Journal of Economic Theory*, 159, 891–928.

Dixit, A. and R. Pindyck (1994): *Investment under Uncertainty*, Princeton: Princeton University Press.

Eső, P. and B. Szentes (2007): "Optimal Information Disclosure in Auctions," *Review of Economic Studies*, 74, 705–731.

Garrett, D. (2016): "Intertemporal Price Discrimination: Dynamic Arrivals and Changing Values," *American Economic Review*, 106, 3275–3299.

——— (2017): "Dynamic Mechanism Design: Dynamic Arrivals and Changing Values," *Games and Economic Behavior*, 104, 595–612.

Gershkov, A., B. Moldovanu, and P. Strack (2015): "Efficient Dynamic Allocation with Strategic Arrivals," Tech. rep., Hebrew University.

——— (2018): "Revenue Maximizing Mechanisms with Strategic Customers and Unknown Markovian Demand," *Management Science*, 64, 2031–2046.

Judd, K. (1985): "The Law of Large Numbers with a Continuum of i.i.d. Random Variables," *Journal of Economic Theory*, 35, 19–25.

Krähmer, D. and R. Strausz (2015): "Optimal Sales Contracts with Withdrawal Rights," *Review of Economic Studies*, 82, 762–790.

Kruse, T. and P. Strack (2015): "Optimal Stopping with Private Information," *Journal of Economic Theory*, 159, 702–727.

Lobel, I. and R. Paes Leme (2019): "Dynamic Mechanism Desing under Positive Commitment," in *Proceedings of Conference on Artificial Intelligence.*

Mussa, M. and S. Rosen (1978): "Monopoly and Product Quality," *Journal of Economic Theory*, 18, 301–317.

Myerson, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

Pavan, A., I. Segal, and J. Toikka (2014): "Dynamic Mechanism Design: A Myersonian Approach," *Econometrica*, 82, 601–653.

Skreta, V. (2006): "Sequentially Optimal Auctions," *Review of Economic Studies*, 73, 1085–1111.

——— (2015): "Optimal Auction Design under Non-Commitment," *Journal of Economic Theory*, 159, 854–890.

A Appendix

Proof of Lemma 1. As each agent's allocation is only a function of his own reports and the willingness to pay is independent between different agents the law of iterated expectations implies that the revenue can be rewritten as

$$\mathbb{E}\left[\sum_{i=0}^{\infty}\int_{\alpha_i}^{\alpha_{i+1}} e^{-r\,t} p_t^i \mathrm{d}t\right] = \mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\alpha_i}\mathbb{E}\left[\int_{\alpha_i}^{\alpha_{i+1}} e^{-r\,(t-\alpha_i)} p_t^i \mathrm{d}t\right]\right] .$$

As agent are ex-ante identical they are necessarily treated the same in the optimal mechanism which yields that the revenue equals

$$\max_{(x,p)\in\mathcal{M}} \mathbb{E}\left[\sum_{i=0}^{\infty}\int_{\alpha_i}^{\alpha_{i+1}} e^{-r\,t} p_t^i \mathrm{d}t\right] = \max_{(x,p)\in\mathcal{M}} \mathbb{E}\left[\int_{\alpha_i}^{\alpha_{i+1}} e^{-r\,(t-\alpha_i)} p_t^i \mathrm{d}t\right] \mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\alpha_i}\right] .$$

Note, that $\alpha_{i+1} - \alpha_i = \tau_i - \alpha_i$ are independently and identically exponentially distributed with rate γ it follows from this that

$$\mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\alpha_i}\right] = \mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\alpha_0} \prod_{j=0}^{i-1} e^{-r(\alpha_{j+1}-\alpha_j)}\right] = \mathbb{E}\left[e^{-r\alpha_0}\right] \sum_{i=0}^{\infty} \prod_{j=0}^{i-1} \mathbb{E}\left[e^{-r(\alpha_{j+1}-\alpha_j)}\right]$$
$$= \sum_{i=0}^{\infty} \mathbb{E}\left[e^{-r(\alpha_{j+1}-\alpha_j)}\right]^i = \sum_{i=0}^{\infty} \left(\frac{\gamma}{r+\gamma}\right)^i = \frac{r+\gamma}{r}.$$

This yields the result.

For $\theta_0 \geq x$ the agent stops immediately and thus the statement is true. For $\theta_0 < x$ we have that

$$\mathbb{E}\left[e^{-(r+\gamma)\,\tau_x} \mid \theta_0\right] = \mathbb{E}\left[e^{-(r+\gamma)\,\tau_x} \left(\frac{\theta_{\tau_x}}{\theta_{\tau_x}}\right)^\beta \mid \theta_0\right] = \mathbb{E}\left[e^{-(r+\gamma)\,\tau_x} \left(\frac{\theta_0 e^{-\frac{\sigma^2}{2}\tau_x + \sigma W_{\tau_x}}}{x}\right)^\beta \mid \theta_0\right]$$
$$= \mathbb{E}\left[e^{-\left[(r+\gamma)\,-\frac{\sigma^2}{2}\beta\right]\tau_x + \beta\sigma W_{\tau_x}} \left(\frac{\theta_0}{x}\right)^\beta \mid \theta_0\right]$$
$$= \mathbb{E}\left[e^{-\left[(r+\gamma)\,+\frac{\sigma^2}{2}\beta - \frac{\sigma^2\beta^2}{2}\right]\tau_x} e^{-\frac{\sigma^2\beta^2}{2}\tau_x + \beta\,\sigma\,W_{\tau_x}} \left(\frac{\theta_0}{x}\right)^\beta \mid \theta_0\right].$$

As $(r+\gamma) + \frac{\sigma^2}{2}\beta - \frac{\sigma^2\beta^2}{2} = 0$ and $t \mapsto e^{-\frac{\sigma^2\beta^2}{2}t + \beta\,\sigma\,W_t}$ is a uniformly integrable martingale it follows from Doob's optional sampling theorem that

$$\mathbb{E}\left[e^{-r\,\tau_x} \mid \theta_0\right] = \mathbb{E}\left[\left(\frac{\theta_0}{x}\right)^\beta \mid \theta_0\right]. \qquad \square$$

Proof of Proposition 1. As $(\theta_t)_{t\geq 0}$ is a geometric Brownian motion it can be explicitly represented as:

$$\theta_t = \theta_0 \, \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right). \tag{28}$$

The first part of the Proposition is a special case of Proposition 1 in Bergemann and Strack (2015) and follows by applying the envelope theorem. As it is optimal for the agent to report his initial value θ_0 truthfully we have that the derivative of the agent's indirect utility can be calculated by treating (x, p) as independent of the agents report

$$
\begin{aligned}
V'(\theta) &= \frac{\partial}{\partial \theta_0} \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t}\,\{x_t\,\theta_t - p_t\}\,\mathrm{d}t \mid \theta_0\right] = \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t} x_t \left(\frac{\partial}{\partial \theta_0}\,\theta_t\right)\mathrm{d}t \mid \theta_0\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t} x_t \left(\frac{\partial}{\partial \theta_0}\left\{\theta_0 \cdot \exp\left(-\frac{\sigma^2}{2} t + \sigma W_t\right)\right\}\right)\mathrm{d}t \mid \theta_0\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t} x_t \,\exp\left(-\frac{\sigma^2}{2} t + \sigma W_t\right)\mathrm{d}t \mid \theta_0\right] = q(\theta_0)\,.
\end{aligned}
$$

As shown in Bergemann and Strack (2015, Theorem 1, Proposition 8, and Equation 27) the virtual value is given by $\theta_t\left(1 - \frac{1-F(\theta_0)}{f(\theta_0)}\right)$ and the expected revenue of the principal equals

$$
\begin{aligned}
\mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t} p_t \mathrm{d}t\right] &= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t}\,x_t\,\theta_t\left(1 - \frac{1-F(\theta_0)}{f(\theta_0)}\right)\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t}\,x_t\,\frac{\theta_t}{\theta_0}\left(\theta_0 - \frac{1-F(\theta_0)}{f(\theta_0)}\right)\mathrm{d}t\right] - V(0) \\
&= \int \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t}\,x_t\,\frac{\theta_t}{\theta_0}\left(\theta_0 - \frac{1-F(\theta_0)}{f(\theta_0)}\right)\mathrm{d}t \mid \theta_0\right] f(\theta_0)\mathrm{d}\theta_0 - V(0) \\
&= \left(\theta_0 - \frac{1-F(\theta_0)}{f(\theta_0)}\right)\int \mathbb{E}\left[\int_0^\infty e^{-r\,t}\,x_t\,\frac{\theta_t}{\theta_0}\mathrm{d}t \mid \theta_0\right] f(\theta_0)\mathrm{d}\theta_0 - V(0)\,.
\end{aligned}
$$

Plugging in the explicit representation of θ_t given by (28) yields that the expected revenue satisfies

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t} p_t \mathrm{d}t\right] \\
&= \int_0^{\overline{\theta}} J(\theta_0)\underbrace{\mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t}\,x_t\,\exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right)\,\mathrm{d}t \mid \theta_0\right]}_{q(\theta_0)} f(\theta_0)\mathrm{d}\theta_0 - V(0)\,.
\end{aligned}
$$

□

Proof of Lemma 2. The result follows as q plays the same role as the quantity in a static allocation problem. □

Proof of Proposition 4. Recall that by Proposition 3 the buyer acquires the object once his valuation exceeds $\theta^\star = \frac{\beta}{\beta-1}p$. By Lemma 1 the expected revenue the seller generates from a single buyer with initial valuation θ_0 is given by

$$\begin{aligned}\frac{r+\gamma}{r}\mathbb{E}\left[\int_{\tau_{\theta^\star}}^{\infty} e^{-(r+\gamma)\,t} p \,\mathrm{d}t \mid \theta_0\right] &= \frac{1}{r}\mathbb{E}\left[e^{-(r+\gamma)\,\tau_{\theta^\star}}\, p \mid \theta_0\right] = \frac{p}{r}\,\mathbb{E}\left[e^{-(r+\gamma)\;\tau_{\theta^\star}} \mid \theta_0\right] \\ &= \frac{p}{r}\,\min\left\{\left(\frac{\theta_0}{\theta^\star}\right)^{\beta}, 1\right\} = \frac{p}{r}\,\min\left\{\left(\frac{\beta-1}{\beta}\frac{\theta_0}{p}\right)^{\beta}, 1\right\}.\end{aligned}$$

Consequently, the expected discounted revenue from buyer with random initial valuation distributed according to F is given by

$$\frac{p}{r}\int_0^\infty \min\left\{\left(\frac{\beta-1}{\beta}\frac{\theta}{p}\right)^{\beta}, 1\right\} f(\theta)\,\mathrm{d}\theta\,. \qquad \square$$

Proof of Proposition 5. It follows from the envelope theorem that the value function is continuous and convex in any mechanism where truthfully reporting the initial valuation is incentive compatible. Furthermore, the envelope theorem implies that V is absolutely continuous, thus any non-differentiability must take the form of a convex kink. As it is never optimal to stop in a convex kink it follows that V is differentiable. $\square$

Proof of Proposition 7. By Proposition 1 and 6 we have that IC-A implies that for all θ

$$\begin{aligned}\frac{\theta\, q(\theta)}{\beta} &\le V(\theta) = V(0) + \int_0^\theta q(z)\mathrm{d}z \\ \Leftrightarrow \frac{\theta\, q(\theta)}{\beta} - \int_0^\theta q(z)\mathrm{d}z &\le V(0)\,.\end{aligned}$$

Taking the supremum over θ yields the results. $\square$

Lemma 3 (Comparison Principle)**.**
Let $g, h : [0,\overline{\theta}] \to \mathbb{R}$ be absolutely continuous and satisfy $g'(\theta) \le \Phi(g(\theta),\theta)$ and $h'(\theta) \ge \Phi(h(\theta),\theta)$ where $\Phi : \mathbb{R}\times[0,\overline{\theta}] \to \mathbb{R}$ is uniformly Lipschitz continuous in the first variable. If $g(0) \le h(0)$ we have that $g(\theta) \le h(\theta)$ for all $\theta \in [0,\overline{\theta}]$.

Proof. Define $\Delta \equiv g - h$. Suppose, that there exists a point θ' such that $\Delta(\theta') > 0$. As $\Delta(0) \le 0$ and by the absolute continuity of Δ there exists a point θ'' such that $\Delta(\theta'') = 0$ and as $\Delta' \ge 0$ we have that $\Delta(\theta) \ge 0$ for all $\theta \in [\theta'', \theta']$. This implies that there exists a constant $L > 0$ such that for all $\theta \in [\theta'', \theta']$

$$\begin{aligned}\Delta'(\theta) = g'(\theta) - h'(\theta) &\leq \Phi(g(\theta), \theta) - \Phi(h(\theta), \theta) \leq |\Phi(g(\theta), \theta) - \Phi(h(\theta), \theta)| \\ &\leq L\,|g(\theta) - h(\theta)| = L\,|\Delta(\theta)| = L\,\Delta(\theta)\,.\end{aligned}$$

By Gronwall's inequality we thus have that $\Delta(\theta') \leq \Delta(\theta'')e^{L(\theta' - \theta'')} = 0$ which contradicts the assumption that $\Delta(\theta') > 0$. □

Lemma 4 (Generalized Comparison Principle)**.**
Let $g, h : [0, \overline{\theta}] \to \mathbb{R}$ be absolutely continuous and satisfy $g'(\theta) \leq \Phi(g(\theta), \theta)$ and $h'(\theta) \geq \Phi(h(\theta), \theta)$ where $\Phi : \mathbb{R} \times [0, \overline{\theta}] \to \mathbb{R}$ is uniformly Lipschitz continuous in the first variable. If $g(\hat{\theta}) = h(\hat{\theta})$ we have that $g(\theta) \leq h(\theta)$ for all $\theta \in [\hat{\theta}, \overline{\theta}]$ and $g(\theta) \geq h(\theta)$ for all $\theta \in [0, \hat{\theta}]$.

Proof. The first part of the result follows by considering the functions $\tilde{g}(s) = g\left(\hat{\theta} + s\right), h(s) = \bar{y}\left(\hat{\theta} + s\right)$ and applying Lemma 3. The second part follows by considering the functions $\tilde{g}(s) = -g\left(\hat{\theta} - s\right), \tilde{h}(s) = -h\left(\hat{\theta} - s\right)$ for $s \in [0, \hat{\theta}]$ and applying Lemma 3 which implies that for all $s \in [0, \hat{\theta}]$

$$\tilde{g}(s) \leq \tilde{h}(s) \Leftrightarrow -g\left(\hat{\theta} - s\right) \leq -h\left(\hat{\theta} - s\right) \Leftrightarrow g\left(\hat{\theta} - s\right) \geq h\left(\hat{\theta} - s\right)\,.$$ □

Lemma 5.
Suppose that $\mathcal{J} : [0, \overline{\theta}]$ is a non-decreasing function with $\mathcal{J}(\overline{\theta}) \geq 0$, and $g, h : [0, \overline{\theta}] \to \mathbb{R}$ are absolutely continuous with $h \leqq g$ then

$$\int_0^{\overline{\theta}} \mathcal{J}(\theta) g'(\theta) dt + \mathcal{J}(0) g(0) \leq \int_0^{\overline{\theta}} \mathcal{J}(\theta) h'(\theta) d\theta + \mathcal{J}(0) h(0)\,.$$

Proof. The result follows from partial integration, the assumption that $\mathcal{J}(\overline{\theta}) \leq 0$

$$\begin{aligned}\int_0^{\overline{\theta}} \mathcal{J}(\theta)g'(\theta)dt + \mathcal{J}(0)g(0) &= [\mathcal{J}(\theta)g(\theta)]_{\theta=0}^{\theta=\overline{\theta}} - \int_0^{\overline{\theta}} g(\theta)d\mathcal{J}(\theta) + \mathcal{J}(0)g(0) \\ &= \mathcal{J}(\overline{\theta})g(\overline{\theta}) - \mathcal{J}(0)g(0) - \int_0^{\overline{\theta}} g(\theta)d\mathcal{J}(\theta) + \mathcal{J}(0)g(0) \\ &\leq \mathcal{J}(\overline{\theta})h(\overline{\theta}) - \mathcal{J}(0)h(0) - \int_0^{\overline{\theta}} h(\theta)d\mathcal{J}(\theta) + \mathcal{J}(0)h(0) \\ &= [\mathcal{J}(\theta)h(\theta)]_{\theta=0}^{\theta=\overline{\theta}} - \int_0^{\overline{\theta}} h(\theta)d\mathcal{J}(\theta) + \mathcal{J}(0)h(0) \\ &= \int_0^{\overline{\theta}} \mathcal{J}(\theta)h'(\theta)d\theta + \mathcal{J}(0)h(0).\end{aligned}$$

□

Proof of Proposition 8. Let g be a solution to the optimization problem (24). Define $\theta^\star = \inf\{\theta\colon \mathcal{J}(\theta) \geq 0\}$. As $\mathcal{J}$ is continuous $\mathcal{J}(\theta^\star) = 0$. Let $h : [0, \overline{\theta}] \to \mathbb{R}$ be the solution to

$$\begin{aligned}h'(\theta) &= \Phi(h(\theta), \theta), \\ h(\theta^\star) &= g(\theta^\star)\,.\end{aligned}$$

Define $\tilde{h}(\theta) = \max\{h(\theta), 0\}$ for $\theta \in [0, \overline{\theta}]$. Note, that by construction $\tilde{h} \geqq 0$ and $\tilde{h}'(\theta) \leqq \Phi(h(\theta), \theta)$ and thus $\tilde{h}$ is a feasible control. The proof proceeds in two step: first we establish that $\tilde{h}$ leads to a higher value of the integral (24) above $\theta^\star$ and in the second step we establish the analogous result below $\theta^\star$.

Step 1: As $g'(\theta) \leq \Phi(g(\theta), \theta)$ it follows from Lemma 4 that $g(\theta) \leq h(\theta)$ for $\theta \in [\theta^\star, \overline{\theta}]$ and $g(\theta) \geq h(\theta)$ for $\theta \in [a, \theta^\star]$ for every $a > 0$. As g and h are continuous it follows that $g(0) \geq h(0)$. The monotonicity of Φ in the first variable implies that for $\theta \geq \theta^\star$

$$g'(\theta) \leq \Phi(g(\theta), \theta) \leq \Phi(h(\theta), \theta) = h'(\theta)\,.$$

As $\mathcal{J}(\theta^\star) = 0$ and $\theta \mapsto \min\{\mathcal{J}(\theta), 0\}$ is non-decreasing we have that $\mathcal{J}(\theta) \geq 0$ for $\theta \geq \theta^\star$. This implies that

$$\int_{\theta^\star}^{\overline{\theta}} \mathcal{J}(\theta)g'(\theta)d\theta \leq \int_{\theta^\star}^{\overline{\theta}} \mathcal{J}(\theta)h'(\theta)d\theta\,.$$

Note, that as $g(\theta^\star) \geq 0$ and $\Phi \geqq 0$ we have that for all $\theta \in [\theta^\star, \overline{\theta}]$

$$h(\theta) \geq h(\theta^\star) = g(\theta^\star) \geq 0,$$

and thus $h(\theta) = \tilde{h}(\theta)$ for $\theta \in [\theta^\star, \overline{\theta}]$ which establishes that

$$\int_{\theta^\star}^{\overline{\theta}} \mathcal{J}(\theta) g'(\theta) d\theta \leq \int_{\theta^\star}^{\overline{\theta}} \mathcal{J}(\theta) \tilde{h}'(\theta) d\theta \,. \tag{29}$$

Step 2: Note, that by Lemma 4 $g(\theta) \geq h(\theta)$ for $\theta \leq \theta^\star$. As $g(\theta) \geq 0$ this implies that $g(\theta) \geq \max\{h(\theta), 0\} = \tilde{h}(\theta)$ for all $\theta \leq \theta^\star$. Furthermore, by definition of $\theta^\star$ we have that $\mathcal{J}(\theta) = \min\{\mathcal{J}(\theta), 0\}$ for $\theta \leq \theta^\star$. As $\theta \mapsto \min\{\mathcal{J}(\theta), 0\}$ is non-decreasing $\mathcal{J}(\theta)$ is non-decreasing for $\theta \leq \theta^\star$. Lemma 5 implies that

$$\int_0^{\theta^\star} \mathcal{J}(\theta) g'(\theta) d\theta + \mathcal{J}(0) g(0) \leq \int_0^{\theta^\star} \mathcal{J}(\theta) \tilde{h}'(\theta) d\theta + \mathcal{J}(0) \tilde{h}(0) \,. \tag{30}$$

Combining the inequalities (29) and (30) with the assumption that $\mathcal{J}(0) = -1$ yields that

$$\int_0^{\overline{\theta}} \mathcal{J}(\theta) g'(\theta) dt - g(0) \leq \int_0^{\overline{\theta}} \mathcal{J}(\theta) \tilde{h}'(\theta) d\theta - \tilde{h}(0) \,.$$

As $\tilde{h}(\theta) = \max\{h(\theta), 0\}$ and h is non-decreasing it thus follows that for any policy there exist $\hat{\theta} = \min\{\theta \colon h(\theta) = 0\}$ such that the policy $\tilde{h}$ which solves

$$\begin{cases} \tilde{h}(\theta) = 0 & \text{for } \theta < \hat{\theta}, \\ \tilde{h}'(\theta) = \Phi(h(\theta), \theta) & \text{for } \theta \geq \hat{\theta} \end{cases}$$

does weakly better. □

Proof of Proposition 9. Define $\mathcal{J}(\theta) = J(\theta) f(\theta)$ and $\theta^\star = \min\{\theta \colon \mathcal{J}(\theta) = 0\}$. We first note, that $\mathcal{J}(\tilde{\theta})$ is negative and $\mathcal{J}(0) = -1$. Consider the problem of solving

$$\max_V \int_0^{\overline{\theta}} V'(z) \mathcal{J}(z) \, \mathrm{d}z - V(0) \,.$$
$$\text{subject to } V'(\theta) \leq \Phi(V(\theta), \theta) \text{ for all } \theta \in \left[\underline{\theta}_k, \overline{\theta}\right],$$

where $\Phi(v,\theta) = \min\left\{\beta\frac{v}{\theta}, \frac{1}{\gamma+r}\right\}$. By Proposition 3 we have that there exists a $\hat{\theta}$ such that the solution to this problem is given by

$$\begin{cases} V(\theta) = 0 & \text{for } \theta \le \hat{\theta}, \\ V'(\theta) = \min\left\{\beta\frac{v}{\theta}, \frac{1}{\gamma+r}\right\} & \text{for } \theta \ge \hat{\theta}. \end{cases} \tag{31}$$

We distinguish two cases. First, $\hat{\theta} > 0$ in this case the unique solution to (31) is given by $V \equiv 0$. Second, if $\hat{\theta} = 0$ we have that all solutions to the ODE (31) are of the form

$$V(\theta) = \begin{cases} \left(\frac{\theta}{\theta'}\right)^{\beta} V(\theta') & \text{for } \theta \le \theta' \\ V(\theta') + \frac{\theta-\theta'}{\gamma+r} & \text{for } \theta \le \theta' \end{cases}$$

where $\frac{1}{\gamma+r} = V'(\theta') = \frac{\beta}{\theta'}V(\theta')$. Thus, plugging in $V(\theta')$ yields that

$$V(\theta) = \begin{cases} \left(\frac{\theta}{\theta'}\right)^{\beta} \frac{\theta'/\beta}{\gamma+r} & \text{for } \theta \le \theta' \\ \frac{\theta'/\beta}{\gamma+r} + \frac{\theta-\theta'}{\gamma+r} & \text{for } \theta \le \theta' \end{cases}.$$

Taking derivatives yields that

$$V'(\theta) = \begin{cases} \left(\frac{\theta}{\theta'}\right)^{\beta-1} \frac{1}{\gamma+r} & \text{for } \theta \le \theta' \\ \frac{1}{\gamma+r} & \text{for } \theta \le \theta' \end{cases}.$$

□

Proof of Proposition 10. Consider the sales contract where the object is sold at a flow price of $p = \frac{\beta-1}{\beta}\theta'$. Proposition 3 yields that the agent's value is given by

$$V(\theta) = \begin{cases} \frac{1}{1+\gamma}\left(\frac{\theta}{\theta'}\right)^{\beta} \frac{1}{\beta}\theta' & \text{for } \theta \le \theta' \\ \frac{1}{1+\gamma}\left(\theta - \frac{\beta-1}{\beta}\theta'\right) & \text{for } \theta \ge \theta' \end{cases}.$$

This establishes the result. □

Proof of Proposition 12. Recall, that agent's get replaced at a rate γ by a new agent whose valuation is again drawn from F. Let $\alpha_1, \alpha_2 \ldots$ be the times at which the agents valuation is redrawn and recall that $\alpha_0 = 0$. The expected discounted revenue from an agent arriving at time 0 equals a leasing contract is given by

$$\mathbb{E}\left[\int_0^T e^{-rt}\mathbf{1}_{\{\theta_t \geq p\}}\mathrm{d}t\right] = \int_0^\infty e^{-(r+\gamma)t}\mathbb{P}\left[\theta_t \geq p\right]\mathrm{d}t\,.$$

Define $m_t = {}^{\theta_t}/_{\theta_0}$ and note that, as θ_t is log-normal distributed with mean θ_0, m_t is independent of θ_0. Furthermore, m_t is log-normal distributed with mean 1 and variance $\sigma^2 t$, i.e.. density

$$\phi_t(m) = \frac{1}{\sqrt{2\pi}}\frac{1}{m\,\sigma\sqrt{t}}\exp\left(-\frac{\left(\log m + \frac{1}{2}\sigma^2 t\right)^2}{2\sigma^2 t}\right).$$

We thus have that

$$\begin{aligned}\mathbb{P}\left[\theta_t \geq x\right] = \mathbb{P}\left[\theta_0\, m_t \geq x\right] &= \int_{\mathbb{R}_+}\mathbb{P}\left[\theta_0 \geq \frac{x}{m}\right]\,\phi_t(m)\,\mathrm{d}m\,. \\ &= \int_{\mathbb{R}_+}\left[1 - F\left(\frac{x}{m}\right)\right]\frac{1}{\sqrt{2\pi}}\frac{1}{m\,\sigma\sqrt{t}}\exp\left(-\frac{\left(\log m + \frac{1}{2}\sigma^2 t\right)^2}{2\sigma^2 t}\right)\mathrm{d}m\,.\end{aligned}$$

This implies that:

$$\mathbb{E}\left[\int_0^T e^{-rt}\mathbf{1}_{\{\theta_t \geq p\}}\mathrm{d}t\right]$$

$$\begin{aligned}&= \int_0^\infty e^{-(r+\gamma)t}\int_{\mathbb{R}_+}\left[1 - F\left(\frac{x}{m}\right)\right]\frac{1}{\sqrt{2\pi}}\frac{1}{m\,\sigma\sqrt{t}}\exp\left(-\frac{\left(\log m + \frac{1}{2}\sigma^2 t\right)^2}{2\sigma^2 t}\right)\mathrm{d}m\,\mathrm{d}t \\ &= \int_{\mathbb{R}_+}\left[1 - F\left(\frac{x}{m}\right)\right]\int_0^\infty e^{-(r+\gamma)t}\frac{1}{\sqrt{2\pi}}\frac{1}{m\,\sigma\sqrt{t}}\exp\left(-\frac{\left(\log m + \frac{1}{2}\sigma^2 t\right)^2}{2\sigma^2 t}\right)\mathrm{d}t\,\mathrm{d}m \\ &= \int_{\mathbb{R}_+}\left[1 - F\left(\frac{x}{m}\right)\right]\frac{2(\gamma + r)}{m\sigma\sqrt{m(8(\gamma + r) + \sigma^2)}}\exp\left(-\frac{\sqrt{8(\gamma + r) + \sigma^2}\,|\log(m)|}{2\sigma}\right)\mathrm{d}m\,. \quad \square\end{aligned}$$

Proof of Proposition 11. The revenue function (15) gives us the flow probability of sale to the consumer:

$$R_{sales}(p) = \int_0^\infty \min\left\{\left(\frac{\beta - 1}{\beta}\frac{\theta}{p}\right)^\beta, 1\right\}\,p\,f(\theta)\,\mathrm{d}\theta.$$

The flow price which implements a threshold policy is

$$\frac{\beta - 1}{\beta}\frac{\theta}{p} = 1 \Leftrightarrow p = \frac{\beta - 1}{\beta}\theta.$$

For the uniform distribution, we can compute the optimal price with a threshold r

$$\int_0^r \left\{ \left(\frac{\beta-1}{\beta} \frac{\theta}{p} \right)^\beta \right\} \frac{\beta-1}{\beta} r \mathrm{d}\theta + \frac{\beta-1}{\beta} r(1-r) = \left(\int_0^r \left(\frac{\theta}{r} \right)^\beta \mathrm{d}\theta + (1-r) \right) \frac{\beta-1}{\beta} r$$
$$\left\{ \left[\frac{\theta^{\beta+1}}{(\beta+1) r^\beta} \right]_0^r + (1-r) \right\} \frac{\beta-1}{\beta} r = \left\{ 1 - \frac{\beta r}{\beta+1} \right\} \frac{\beta-1}{\beta} r \,.$$

From the first order conditions we get

$$1 - \frac{2\beta r}{\beta+1} = 0,$$

or

$$r = \frac{1}{2} \frac{1+\beta}{\beta} \qquad p = \frac{\beta-1}{\beta} r. \tag{32}$$

Thus we have a gap between threshold and price of:

$$r - p = \frac{1}{2} \frac{1+\beta}{\beta} - \frac{\beta-1}{\beta} \frac{1}{2} \frac{1+\beta}{\beta} = \frac{1}{2\beta} \frac{1+\beta}{\beta} \,.$$

Then we can compute the welfare of the consumer as follows:

$$\begin{aligned}
&(r-p) \int_0^r \left(\frac{\theta}{r} \right)^\beta d\theta + (1-r) \left(\frac{1+r}{2} - \frac{\beta-1}{\beta} r \right) \\
&= (r-p) \frac{r}{\beta+1} + (1-r) \left(\frac{1+r}{2} - \frac{\beta-1}{\beta} r \right) \\
&= \frac{r}{\beta+1} \left(\frac{1}{2\beta} \frac{1+\beta}{\beta} \right) + (1-r) \left(\frac{1+r}{2} - \frac{\beta-1}{\beta} r \right) \\
&= \frac{\frac{1}{2} \frac{1+\beta}{\beta}}{\beta+1} \left(\frac{1}{2\beta} \frac{1+\beta}{\beta} \right) + (1 - \frac{1}{2} \frac{1+\beta}{\beta}) \left(\frac{1 + \frac{1}{2} \frac{1+\beta}{\beta}}{2} - \frac{\beta-1}{\beta} \frac{1}{2} \frac{1+\beta}{\beta} \right) \\
&= \frac{1}{2\beta} \left(\frac{1}{2\beta} \frac{1+\beta}{\beta} \right) + (1 - \frac{1}{2} \frac{1+\beta}{\beta}) \left(\frac{1 + \frac{1}{2} \frac{1+\beta}{\beta}}{2} - \frac{\beta-1}{\beta} \frac{1}{2} \frac{1+\beta}{\beta} \right) \\
&= \frac{\beta^2+3}{8\beta^2} \,,
\end{aligned}$$

which converges to $1/4$ as $\beta \to 1$. We can compare this with the surplus from the dynamic pricing which is in flow terms:

$$\frac{1}{2} \left(\frac{3}{4} - \frac{1}{2} \right) = \frac{1}{8}.$$

Thus, the consumer surplus in the uniform case is uniformly larger in the progressive contract then in the dynamic contract.

We can then compute the social welfare, which is just:

$$r\frac{r}{\beta+1}+(1-r)\left(\frac{1+r}{2}\right)=\left(\frac{1}{2}\frac{1+\beta}{\beta}\right)\frac{\frac{1}{2}\frac{1+\beta}{\beta}}{\beta+1}+\left(1-\frac{1}{2}\frac{1+\beta}{\beta}\right)\left(\frac{1+\frac{1}{2}\frac{1+\beta}{\beta}}{2}\right)$$
$$=\frac{1}{8\beta^2}\left(3\beta^2+1\right).$$

By contrast, the social surplus in the dynamic mechanism is given by:

$$\frac{1}{2}\frac{3}{4}=\frac{3}{8},$$

and again the social surplus is uniformly higher in the progressive mechanisms than in the dynamic mechanism. □