Supplementary Appendix to
“Misinterpreting Others and the Fragility of Social Learning”
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D Proofs for Section 6

D.1 Proof of Proposition 1

We omit the proof of the first part, as it follows the same steps as in Appendix A (for details, see Appendix A of the previous working paper version, Frick, Iijima, and Ishii (2019b)). To prove the second part, define for each \( F, \hat{F} \in F \) and \( \omega \in \Omega \) the set of steady states

\[
\text{SS}(F, \hat{F}, \omega) := \{ \hat{\omega}_\infty \in \Omega : \hat{\omega}_\infty \in \arg\min_{\hat{\omega} \in \Omega} \text{KL}(\alpha F(\theta^*(\hat{\omega}_\infty)) + (1 - \alpha)F(\theta^*(\omega)), \hat{F}(\theta^*(\hat{\omega})) \}.
\] (7)

The following lemma shows that whenever \( \text{SS}(F, \hat{F}, \omega) \) is finite, incorrect agents’ long-run beliefs correspond to steady states.

**Lemma D.1.** Fix any \( F, \hat{F} \) such that \( \text{SS}(F, \hat{F}, \omega) \) is finite for each \( \omega \). Then in all states \( \omega \), there exists some state \( \hat{\omega}_\infty(\omega) \in \text{SS}(F, \hat{F}, \omega) \) such that almost all incorrect agents’ beliefs converge to a point mass on \( \hat{\omega}_\infty(\omega) \).

**Proof.** Since Lemma B.2 continues to characterize incorrect agents’ inferences from observed actions, the proof proceeds in an analogous manner to that of Proposition B.1. Let \( q_0^C(\omega), q_0^I(\omega) \in [0, 1] \) denote the actual fraction of action 0 among correct and incorrect agents in period \( t \) and state \( \omega \), and let \( \bar{q}_t^C(\omega) := \frac{1}{t} \sum_{\tau=1}^{t} q_\tau^C(\omega) \) and \( \bar{q}_t^I(\omega) := \frac{1}{t} \sum_{\tau=1}^{t} q_\tau^I(\omega) \) denote the corresponding time averages.

Note that since by the first part of Proposition 1 almost all correct agents learn the true state as \( t \to \infty \), it follows that \( \lim_{t \to \infty} \bar{q}_t^C(\omega) = \lim_{t \to \infty} q_t^C(\omega) = F(\theta^*(\omega)) \) for all \( \omega \). Moreover, since \( \text{SS}(F, \hat{F}, \omega, \alpha) \) is finite, we can follow the same argument as in the proof of Lemma B.3 to show (using Lemma B.2) that the limit \( R^I(\omega) := \lim_{t \to \infty} \bar{q}_t^I(\omega) \) exists for all \( \omega \). For each \( \omega \), let

\[
\hat{\omega}_\infty(\omega) := \arg\min_{\hat{\omega} \in \Omega} \text{KL}(\alpha R^I(\omega) + (1 - \alpha)F(\theta^*(\omega)), \hat{F}(\theta^*(\hat{\omega}))).
\]

Then by the same argument as in the proof of Proposition B.1, we obtain that conditional on each state \( \omega \), almost all incorrect agents’ beliefs converge to a point mass on \( \hat{\omega}_\infty(\omega) \). But then \( R^I(\omega) = F(\theta^*(\hat{\omega}_\infty(\omega))) \), whence \( \hat{\omega}_\infty(\omega) \in \text{SS}(F, \hat{F}, \omega) \).

Combined with Lemma D.1, the following lemma completes the proof of the proposition.

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Lemma D.2. Fix any analytic $F \in \mathcal{F}$ and $\delta > 0$. There exists $\varepsilon > 0$ such that for any analytic $\hat{F} \neq F$ with $\|F - \hat{F}\| < \varepsilon$ and every $\omega \in \Omega$:

1. $\text{SS}(F, \hat{F}, \omega)$ is finite.
2. $|\omega - \hat{\omega}| < \delta$ for every $\hat{\omega} \in \text{SS}(F, \hat{F}, \omega)$.

Proof. Fix any analytic $F \in \mathcal{F}$ and $\delta > 0$, where we can assume that $\delta < \frac{\varepsilon - \omega}{2}$. Choose $\varepsilon > 0$ sufficiently small such that $\frac{\varepsilon}{1 - \alpha} < |F(\theta^*(\omega)) - F(\theta^*(\omega'))|$ for any pair of states $\omega, \omega'$ with $|\omega - \omega'| \geq \delta$.

Consider any analytic $\hat{F} \neq F$ with $\|F - \hat{F}\| < \varepsilon$ and any $\omega$. By (7), each $\hat{\omega} \in \text{SS}(F, \hat{F}, \omega)$ satisfies one of the following three cases:

1. $\hat{\omega} \in (\omega, \omega)$ and $\alpha F(\theta^*(\hat{\omega})) + (1 - \alpha)F(\theta^*(\omega)) = \hat{F}(\theta^*(\hat{\omega}))$
2. $\hat{\omega} = \omega$ and $\alpha F(\theta^*(\omega)) + (1 - \alpha)F(\theta^*(\omega)) \leq \hat{F}(\theta^*(\omega))$
3. $\hat{\omega} = \overline{\omega}$ and $\alpha F(\theta^*(\omega)) + (1 - \alpha)F(\theta^*(\omega)) \geq \hat{F}(\theta^*(\omega)).$

We first show that $|\omega - \hat{\omega}| < \delta$ for all $\hat{\omega} \in \text{SS}(F, \hat{F}, \omega)$. We consider only the first case, as the remaining cases are analogous. Note that

$$\alpha F(\theta^*(\hat{\omega})) + (1 - \alpha)F(\theta^*(\omega)) = \hat{F}(\theta^*(\hat{\omega})) \iff F(\theta^*(\omega)) - \hat{F}(\theta^*(\hat{\omega})) = \frac{\alpha}{1 - \alpha}(\hat{F}(\theta^*(\hat{\omega})) - F(\theta^*(\hat{\omega}))),$$

so that $|F(\theta^*(\omega)) - \hat{F}(\theta^*(\hat{\omega}))| \leq \frac{\alpha}{1 - \alpha} \varepsilon$. Thus,

$$|F(\theta^*(\omega)) - \hat{F}(\theta^*(\hat{\omega}))| \leq |F(\theta^*(\omega)) - \hat{F}(\theta^*(\hat{\omega}))| + |\hat{F}(\theta^*(\hat{\omega})) - F(\theta^*(\hat{\omega}))| \leq \frac{\alpha}{1 - \alpha} \varepsilon + \varepsilon = \frac{\varepsilon}{1 - \alpha}.$$

By choice of $\varepsilon$, this implies $|\omega - \hat{\omega}| < \delta$.

To show that $\text{SS}(F, \hat{F}, \omega)$ is finite, it suffices to show that the equality $\alpha F(\theta^*(\hat{\omega})) + (1 - \alpha)F(\theta^*(\omega)) = \hat{F}(\theta^*(\hat{\omega}))$ admits at most finitely many solutions $\hat{\omega} \in [\omega, \overline{\omega}]$. Since $F$ and $\hat{F}$ are analytic and $[\omega, \overline{\omega}]$ is compact, if this equality admits infinitely many solutions, then $\alpha F(\theta^*(\hat{\omega})) + (1 - \alpha)F(\theta^*(\omega)) = \hat{F}(\theta^*(\hat{\omega}))$ holds for all $\hat{\omega} \in [\omega, \overline{\omega}]$. But the latter is impossible since we have shown that $|\omega - \hat{\omega}| < \delta < \frac{\varepsilon - \omega}{2}$ holds for any solution $\hat{\omega}$.

D.2 Proof of Proposition 2

Fix any $F \in \mathcal{F}$, $\hat{\omega} \in \Omega$, $\hat{\alpha}, \alpha > 0$ with $\hat{\alpha} \neq \alpha$ and $\varepsilon > 0$. If $\hat{\alpha} < \alpha$, take $\hat{F} \in \mathcal{F}$ such that $\hat{F} - F$ crosses zero only once at $\theta^*(\hat{\omega})$ from below. If $\hat{\alpha} > \alpha$, take $\hat{F} \in \mathcal{F}$ such that $\hat{F} - F$ crosses zero only once at $\theta^*(\hat{\omega})$ from above. In either case we can additionally require that $\|F - \hat{F}\| < \varepsilon$, as in the proof of Theorem 1. In addition, we can take $\hat{F}$ sufficiently close to $F$ such that the inverse function $F \circ \hat{F}^{-1}$ has a Lipschitz constant less than $\frac{1}{\alpha}$.

Let $q_t^i(\omega)$ and $q_t^C(\omega)$ denote incorrect and quasi-correct agents’ perceived population fractions of action 0 in period $t$ and state $\omega$. The proof of Lemma 1 applied to incorrect agents’ perceptions
implies that \( \hat{q}_{t}^{I}(\omega) \) is strictly decreasing in \( \omega \) with \( \hat{q}_{\infty}^{I}(\omega) := \lim_{t \to \infty} \hat{q}_{t}^{I}(\omega) = \hat{F}(\theta^{*}(\omega)) \). Likewise, the proof of Proposition 1 applied to quasi-correct agents’ perceptions implies that \( \hat{q}_{\infty}^{C}(\omega) := \lim_{t \to \infty} \hat{q}_{t}^{C}(\omega) \) exists, is strictly decreasing, and satisfies

\[
\hat{q}_{\infty}^{C}(\omega) = \hat{\alpha} F(\theta^{*}(\hat{\omega}_{\omega})) + (1 - \hat{\alpha}) F(\theta^{*}(\omega)) \quad \text{where} \quad \hat{\omega}_{\omega} = \arg\min_{\hat{\omega}} \text{KL} \left( \hat{q}_{\infty}^{C}(\omega), \hat{F}(\theta^{*}(\hat{\omega}')) \right).
\]

Lemma D.3. If \( \hat{\alpha} < \alpha \) (resp. \( \hat{\alpha} > \alpha \)), then \( \hat{F}(\theta^{*}(\omega)) - \hat{q}_{\infty}^{C}(\omega) \) crosses zero only once from below (resp. above) at \( \omega = \hat{\omega} \).

**Proof.** Note that since by construction of \( \hat{F} \) the Lipschitz constant of the the RHS of (8) is less than 1, there is a unique solution \( \hat{q}_{\infty}^{C}(\omega) \) to (8). Given this, we have \( \hat{q}_{\infty}^{C}(\hat{\omega}) = \hat{F}(\theta^{*}(\hat{\omega})) \) as \( F(\theta^{*}(\hat{\omega})) = \hat{F}(\theta^{*}(\hat{\omega})) \). For the remaining claim, we focus on the case \( \hat{\alpha} < \alpha \) as the case \( \hat{\alpha} > \alpha \) follows a symmetric argument.

Take any \( \omega < \hat{\omega} \). Then \( \hat{q}_{\infty}^{C}(\omega) > \hat{q}_{\infty}^{C}(\hat{\omega}) = \hat{F}(\theta^{*}(\hat{\omega})) \), so that \( \hat{\omega}_{\omega} = \arg\min_{\hat{\omega}} \text{KL} \left( \hat{q}_{\infty}^{C}(\omega), \hat{F}(\theta^{*}(\hat{\omega}')) \right) \) must satisfy \( \hat{\omega}_{\omega} < \omega \) and \( \hat{F}(\theta^{*}(\hat{\omega}_{\omega})) \leq \hat{q}_{\infty}^{C}(\omega) \). But since \( F(\theta) < \hat{F}(\theta) \) for all \( \theta > \theta^{*}(\hat{\omega}) \), this implies \( F(\theta^{*}(\hat{\omega}_{\omega})) \in (\hat{F}(\theta^{*}(\hat{\omega}_{\omega})), \hat{q}_{\infty}^{C}(\omega)) \). Since by (8), \( \hat{q}_{\infty}^{C}(\omega) = \hat{\alpha} F(\theta^{*}(\hat{\omega}_{\omega})) + (1 - \hat{\alpha}) F(\theta^{*}(\omega)) \), this implies \( F(\theta^{*}(\hat{\omega}_{\omega})) < \hat{q}_{\infty}^{C}(\omega) < F(\theta^{*}(\omega)) < \hat{F}(\theta^{*}(\omega)) \), as required. Likewise if \( \omega > \hat{\omega} \), then an analogous argument shows \( \hat{q}_{\infty}^{C}(\omega) > \hat{F}(\theta^{*}(\omega)) \).

Let \( q_{t}(\omega) \) denote the actual population fraction of action 0 in period \( t \) at state \( \omega \), and let \( \bar{q}_{t}(\omega) := \frac{1}{t} \sum_{\tau=1}^{t} q_{\tau}(\omega) \) be its time average. The following lemma uses a similar argument as in Lemma B.3 to show that \( \bar{q}_{t} \) converges to \( F(\theta^{*}(\hat{\omega})) \).

**Lemma D.4.** For every \( \omega \), \( \lim_{t \to \infty} \bar{q}_{t}(\omega) = F(\theta^{*}(\hat{\omega})) \).

**Proof.** Fix any \( \omega \). Let \( \overline{R}(\omega) := \limsup_{t \to \infty} \bar{q}_{t}(\omega) \) and \( \underline{R}(\omega) := \liminf_{t \to \infty} \bar{q}_{t}(\omega) \). Suppose for a contradiction that either \( \overline{R}(\omega) > F(\theta^{*}(\hat{\omega})) \) or \( \underline{R}(\omega) < F(\theta^{*}(\hat{\omega})) \). We consider only the first case, as the second case is analogous.

Consider any \( R \in (F(\theta^{*}(\hat{\omega})), \overline{R}(\omega)) \). We first claim that in state \( \omega \) and any period \( t \) if (i) almost all incorrect agents’ beliefs assign probability 1 to \( \hat{\omega}^{I} := \arg\min_{\hat{\omega}} \text{KL} (R, \hat{F}(\theta^{*}(\hat{\omega}'))) \) and (ii) almost all quasi-correct agents’ beliefs assign probability 1 to \( \hat{\omega}^{C} := \arg\min_{\hat{\omega}} \text{KL} (R, \hat{q}_{\infty}^{C}(\hat{\omega}')) \), then \( q_{t}(\omega) < R \).

To show this claim, we consider only the case \( \hat{\alpha} < \alpha \), as the case \( \hat{\alpha} > \alpha \) is analogous. By Lemma D.3, \( \hat{q}_{\infty}^{C}(\omega) > \hat{F}(\theta^{*}(\hat{\omega})) \) if \( \omega < \hat{\omega} \). Hence, we have \( \hat{\omega}^{C} < \omega \) since \( R > F(\theta^{*}(\hat{\omega})) = \hat{F}(\theta^{*}(\hat{\omega})) \). Likewise, \( \hat{\omega}^{I} < \hat{\omega} \). Thus, since \( \hat{F}(\theta^{*}(\omega)) > \hat{q}_{\infty}^{C}(\omega) \) for all \( \omega < \hat{\omega} \), it follows that \( \omega > \hat{\omega}^{I} > \hat{\omega}^{C} \).

By definition of \( \hat{\omega}^{C} \), this leaves two cases to consider:

1. \( R = \hat{q}_{\infty}^{C}(\hat{\omega}^{C}) \)
2. \( R > \hat{q}_{\infty}^{C}(\hat{\omega}^{C}) \) and \( \hat{\omega}^{C} = \omega \).

In either case, \( q_{t}(\omega) = \alpha F(\theta^{*}(\hat{\omega}^{I})) + (1 - \alpha) F(\theta^{*}(\hat{\omega}^{C})) \). Moreover, in case 1, (8) implies \( R = \hat{\alpha} F(\theta^{*}(\hat{\omega}^{I})) + (1 - \hat{\alpha}) F(\theta^{*}(\hat{\omega}^{C})) \), so that \( R > q_{t}(\omega) \) because \( \hat{\alpha} < \alpha \) and \( \hat{\omega}^{I} > \hat{\omega}^{C} \). For case 2, we can extend the domain of function \( \hat{q}_{\infty}^{C} \) from \( \Omega \) to \( \mathbb{R} \) by first extending the domain of function \( \theta^{*} \) from
Ω to \( \mathbb{R} \) (in such a way that \( \theta^* \) is still continuous, strictly decreasing, and has full range) and then defining \( \hat{q}_C^\omega \) by (8) on the whole of \( \mathbb{R} \). It is easy to show (using the same argument as above) that the extended \( \hat{q}_C^\omega \) continues to satisfy Lemma D.3. Choosing \( \hat{\omega}^C < \omega \) such that \( R = \hat{q}_C^\omega(\hat{\omega}^C) \) yields

\[
R = \hat{\alpha}F(\theta^* (\hat{\omega}^I)) + (1 - \hat{\alpha})F(\theta^* (\hat{\omega}^C)) > \hat{\alpha} F(\theta^* (\hat{\omega}^I)) + (1 - \hat{\alpha}) F(\theta^* (\hat{\omega}^C)),
\]

where the equality holds by (8) and the inequality holds since \( \hat{\omega}^C = \omega \). Thus, we again have \( R > q_t(\omega) \) because \( \hat{\alpha} < \alpha \) and \( \hat{\omega}^I > \hat{\omega}^C \).

As a result, by continuity of \( u \) and \( F \), there exist signals \( \gamma < \bar{\gamma} \), intervals of states \( E^I \supseteq \hat{\omega}^I, E^C \supseteq \hat{\omega}^C \) with non-empty interior, and \( \gamma > 0 \) such that in state \( \omega \) and any period \( t \) if (i)’ at least fraction \( 1 - \gamma \) of incorrect agents with private signals \( s \in [\gamma, \bar{\gamma}] \) hold beliefs such that \( H_t(E^I|a^t-1, s) \geq 1 - \gamma \) and (ii)’ at least fraction \( 1 - \gamma \) of quasi-correct agents with private signals \( s \in [\gamma, \bar{\gamma}] \) hold beliefs such that \( H_t(E^C|a^t-1, s) \geq 1 - \gamma \), then \( q_t(\omega) < R - \gamma \).

To complete the proof, we consider separately the case where \( \overline{R}(\omega) > \overline{R}(\omega) \) and the case where \( \overline{R}(\omega) = \overline{R}(\omega) \). In the former case, we can choose \( R \in (F(\theta^* (\hat{\omega}^I)), \overline{R}(\omega)) \) that additionally satisfies \( R > \overline{R}(\omega) \). Then following a similar argument as in the proof of Lemma B.3 leads to a contradiction. Specifically, for any sufficiently small \( \eta > 0 \), by definition of \( \overline{R}(\omega), \overline{R}(\omega) \) and since \( |\hat{q}_t(\omega) - \hat{q}_{t-1}(\omega)| < \eta \) for all large enough \( t \), we can find an infinite sequence of times \( t_k \) such that \( R - \eta < \hat{q}_{t-k}(\omega) \leq R + \eta \), a contradiction.

Finally, if \( \overline{R}(\omega) = \overline{R}(\omega) \), then we choose \( R = \overline{R}(\omega) = \overline{R}(\omega) > F(\theta^*(\hat{\omega})) \). In this case, by the law of large numbers and Lemma B.2, almost all incorrect agents’ beliefs converge to a point-mass on \( \hat{\omega}^I := \text{argmin}_{\hat{\omega}} KL(R, \hat{F}(\theta^*(\hat{\omega}))) \), and almost all quasi-correct agents’ beliefs converge to a point-mass on \( \hat{\omega}^C := \text{argmin}_{\hat{\omega}} KL(R, \hat{q}_C^\omega(\hat{\omega}^I)) \). Thus, hypotheses (i)’ and (ii)’ are satisfied for all large enough \( t \), whence \( \lim_{t \to \infty} q_t(\omega) \leq \gamma - \gamma \). This contradicts \( \lim_{t \to \infty} \hat{q}_t(\omega) = R \).

To complete the proof of Proposition 2, let \( \hat{\omega}^I := \text{argmin}_{\hat{\omega}} KL(F(\theta^*(\hat{\omega})), \hat{q}_C^\omega(\hat{\omega}^I)) \) and \( \hat{\omega}^C := \text{argmin}_{\hat{\omega}} KL(F(\theta^*(\hat{\omega})), \hat{q}_C^\omega(\hat{\omega}^I)) \). Then Lemmas B.2 and D.4 imply that almost all incorrect agents’ beliefs converge to a point-mass on \( \hat{\omega}^I \) and almost all quasi-correct agents’ beliefs converge to a point-mass on \( \hat{\omega}^C \). Moreover, since \( \hat{q}_C^\omega(\cdot) = \hat{F}(\theta^*(\cdot)) \) and \( \hat{F}(\theta^*(\hat{\omega})) = F(\theta^*(\hat{\omega})) \) by construction, we must have \( \hat{\omega}^I = \hat{\omega} \). Likewise, by Lemma D.3, \( \hat{q}_C^\omega(\theta^*(\hat{\omega})) = \hat{F}(\theta^*(\hat{\omega})) = F(\theta^*(\hat{\omega})) \), so that \( \hat{\omega}^C = \hat{\omega} \).

### E. Omitted Details

#### E.1 Robustness of Single-Agent Active Learning

Consider the active learning model discussed in Section 4.3, whose limit model belief process (see footnote 28) satisfies

\[
\hat{\omega}_t = \text{argmin}_{\hat{\omega} \in \Omega} KL(q(x^*_t, \omega), \hat{q}(x^*_t, \hat{\omega})), \quad x^*_t = x^*(\hat{\omega}_{t-1}).
\]
We measure the amount of misperception by a “bias” parameter $b \in \mathbb{R}$. Specifically, we write $\hat{q}(x, \omega) = r(x, \omega, b)$ for some $C^1$ function $r$ that is strictly decreasing in $(x, \omega)$ and satisfies $q(x, \omega) = r(x, \omega, 0)$. We also assume that $x^*(\omega)$ is $C^1$.

**Proposition E.1.** Fix any $\varepsilon > 0$. There exists $\tilde{b} > 0$ such that if $|b| < \tilde{b}$, then at each $\omega \in \Omega$, process (9) admits a unique steady state $\hat{\omega}_\infty(\omega)$; moreover, $\hat{\omega}_\infty(\omega) \in [\omega - \varepsilon, \omega + \varepsilon]$ and is globally stable.

**Proof.** We first show that there exists $\tilde{b} > 0$ such that at each $\omega \in \Omega$, process (9) satisfies $\hat{\omega}_t \in [\omega - \varepsilon, \omega + \varepsilon]$ for all $t \geq 2$ whenever $|b| \leq \tilde{b}$. To see this, consider identity

$$r(x, \omega, 0) = r(x, \omega, b)$$

as a function of $\hat{\omega}$. If $b = 0$, then for any $x$ and $\omega$, (10) admits $\hat{\omega} = \omega$ as the unique solution. Thus, by the implicit function theorem, $\frac{db}{d\omega} = -\frac{\partial r(x, \omega, b)}{\partial x} \frac{dx}{d\omega}$ holds at $b = 0$ and $\hat{\omega} = \omega$. But since $r$ is $C^1$ and $X \times \Omega = [0, 1] \times [\omega, \overline{\omega}]$ is compact, $\max_{(x, \omega) \in X \times \Omega} \left| \frac{\partial r(x, \omega, 0)}{\partial x} \right| < \infty$. Hence, there exists $\tilde{b} > 0$ such that for every $b \in [-\tilde{b}, \tilde{b}]$, $x$, and $\omega$, (10) admits a unique solution $\hat{\omega} \in [\omega - \varepsilon, \omega + \varepsilon]$; that is, process (9) satisfies $\hat{\omega}_t \in [\omega - \varepsilon, \omega + \varepsilon]$ for all $t \geq 2$ from any initial point $\hat{\omega}_1$.

Finally, applying the implicit function theorem to $r(x^*(\hat{\omega}_t), \omega, 0) = r(x^*(\hat{\omega}_t), \hat{\omega}_t, b)$, we obtain

$$\frac{d\hat{\omega}_t}{dt} = -\frac{dx^*(\hat{\omega}_t)}{dt} \left( \frac{\partial r(x^*(\hat{\omega}_t), \omega, 0)}{\partial x} \right) \left( \frac{\partial r(x^*(\hat{\omega}_t), \omega, b)}{\partial b} \right) \frac{\partial x}{\partial \omega} \frac{\partial \hat{\omega}_t}{\partial \omega} \frac{\partial \hat{\omega}_t}{\partial b}.$$  

By uniform continuity of the derivatives (which holds by compactness of the domain $X \times \Omega$), we can choose $\tilde{b}$ sufficiently small such that for all $|b| \leq \tilde{b}$ and $\omega$, the right hand side is strictly less than 1 in absolute value at all $t \geq 2$. This guarantees that process (9) is a contraction on $[\omega - \varepsilon, \omega + \varepsilon]$. Hence, it admits a unique steady state $\hat{\omega}_\infty(\omega) \in [\omega - \varepsilon, \omega + \varepsilon]$, to which it converges from any initial point. \qed

### E.2 Misperceptions about Matching Technology

Consider the assortative random matching model from Section 7.1. As in Section 4.2, we set up a limit model where each agent observes the actions of infinitely many matches at the end of each period. To simplify the exposition, we consider the unbounded state space $\Omega = \mathbb{R}$ and assume that $\theta^*(\cdot)$ is unbounded on $\Omega$. Fix any true state $\omega$. If $\hat{P} = P$, then agents learn the true state at the end of the first period; in period 2 and all subsequent periods, agents play a threshold strategy with cutoff type $\theta^*(\omega)$, and each type’s observed fraction of action 0, $P(\theta^*(\omega) \mid \theta)$, matches his expectation.

If $\hat{P} \neq P$, then for simplicity, we continue to assume that in period 2, agents play a threshold strategy according to some cutoff type $\theta^*_t$.\footnote{This simplifying assumption is satisfied whenever $\|\hat{P} - P\|_2$ is sufficiently small. Indeed, while different types $\theta$ might believe in different states $\hat{\omega}_\theta^t$ at the end of period 1, when $\|\hat{P} - P\|_2$ is sufficiently small, all $\hat{\omega}_\theta^t$ are sufficiently close to $\omega$ that $u(1, \theta, \hat{\omega}_\theta^t) - u(0, \theta, \hat{\omega}_\theta^t)$ is increasing in $\theta$. Thus, agents follow a threshold strategy.} Inductively, this induces the following sequence of cutoff types $(\theta^*_t)$ and type-dependent point-mass beliefs $(\hat{\omega}_\theta^t)$. At any $t \geq 2$, if agents play according to cutoff $\theta^*_{t-1}$, then each type $\theta$ observes fraction $P(\theta^*_{t-1} \mid \theta)$ of action 0, and based on this, assigns a point
mass to the state \( \tilde{\omega}_t^\theta = \arg\min_{\tilde{\omega} \in \mathbb{R}} \text{KL}(P(\theta_{t-1}^*|\theta, \hat{P}(\theta^*(\tilde{\omega})|\theta)) \) that best explains this observation. Since \( \theta^*(\cdot) \) is unbounded and \( P(\cdot|\theta) \) is a continuous distribution with full support, \( \tilde{\omega}_t^\theta \) is uniquely given by

\[
P(\theta_{t-1}^*|\theta) = \hat{P}(\theta^*(\tilde{\omega}_t^\theta)|\theta).
\]

(11)

Given this, we claim that in period \( t + 1 \), agents follow a threshold strategy with cutoff type \( \theta_t^* \) given by

\[
P(\theta_{t-1}^*|\theta_t^*) = \hat{P}(\theta_t^*|\theta_t^*).
\]

(12)

Note that (12) uniquely pins down \( \theta_t^* \), because by assumptions (i) and (ii) in Section 7.1, the left-hand side is weakly decreasing in \( \theta_t^* \) but the right-hand side is strictly increasing in \( \theta_t^* \). To see that agents behave according to cutoff \( \theta_t^* \) in period \( t + 1 \), consider any \( \theta > \theta_t^* \). Then \( P(\theta_{t-1}^*|\theta) \leq P(\theta_{t-1}^*|\theta_t^*) = \hat{P}(\theta_t^*|\theta_t^*) < \hat{P}(\theta|\theta). \) Thus, (11) implies that \( \theta^*(\tilde{\omega}_t^\theta) < \theta \), whence type \( \theta \) plays action 1 in period \( t + 1 \). Analogously, we can verify that any type \( \theta < \theta_t^* \) chooses action 0 in period \( t + 1 \).

Note that by (12), \( \theta_t^* \) is strictly increasing in \( \theta_{t-1}^* \). Indeed, for any \( \eta > 0 \), we have \( P(\theta_{t-1}^* + \eta|\theta_t^*) > \hat{P}(\theta_t^*|\theta_t^*) \), and the left-hand side is decreasing in \( \theta_t^* \) and the right-hand side is strictly increasing in \( \theta_t^* \). Given this, recursion (12) either converges to a steady state \( \theta_t^* \) with

\[
P(\theta_{t-1}^*|\theta_t^*) = \hat{P}(\theta_t^*|\theta_t^*)
\]

(13)

or diverges, and in the former case, each type \( \theta \)'s steady-state belief \( \tilde{\omega}_t^\theta \) satisfies

\[
P(\theta^*|\theta) = \hat{P}(\theta^*(\tilde{\omega}_t^\theta)|\theta).
\]

(14)

The following example illustrates a natural misperception, assortativity neglect, under which the steady-state beliefs \( \tilde{\omega}_t^\theta \) are state-independent and increasing in types.

**Example 4** (Assortativity neglect in a Gaussian setting). Suppose that \( P \) and \( \hat{P} \) are symmetric bivariate Gaussian distributions whose mean, variance, and correlation coefficient are given by \((\mu, \sigma^2, \rho)\) and \((\hat{\mu}, \hat{\sigma}^2, \hat{\rho})\) respectively, with \( \rho, \hat{\rho} \geq 0 \) (reflecting assortativity). To model assortativity neglect, we suppose that \( \hat{\rho} < \rho, \hat{\mu} = \mu, \) and \( \hat{\sigma} = \sigma; \) that is, agents underestimate the correlation in the matching technology, but are correct about the marginal type distribution. Letting \( G \) denote the cdf of the standard Gaussian distribution, equation (13) yields \( G \left[ \sqrt{\frac{1-\rho \theta_t^* \hat{\sigma}}{1+\rho}} \right] = G \left[ \sqrt{\frac{1-\hat{\rho} \theta_t^* \sigma}{1+\hat{\rho}}} \right] \), which admits the unique solution \( \theta_t^* = \mu \). Thus, by (14), each type \( \theta \)'s steady state belief is a state-independent point mass \( \tilde{\omega}_t^\theta \) such that \( \theta^*(\tilde{\omega}_t^\theta) = \frac{\sqrt{1-\rho}}{\sqrt{1+\rho}} (\mu - \rho \theta - (1-\rho)\mu) + \hat{\rho} \theta + (1-\hat{\rho}) \mu \). Since the right-hand side of the latter equation is decreasing in \( \theta \), beliefs \( \tilde{\omega}_t^\theta \) are increasing in types.

**E.3 Continuous Actions**

This section considers a continuous action space version of our model. We perform steady state analysis (under the limit model) to illustrate why our main insights do not rely on a finite action space. Throughout, we assume that the action space is an interval \( A = [a, \bar{a}] \subseteq \mathbb{R} \), with \(-\infty \leq \bar{a} \leq \infty \).
\( \mathfrak{a} < \overline{\mathfrak{a}} \leq \infty \). Let \( u(a, \theta, \omega) \) denote type \( \theta \)'s utility to choosing action \( a \) in state \( \omega \). We assume that for every type \( \theta \in \mathbb{R} \) and state \( \omega \in \Omega := [\underline{\omega}, \overline{\omega}] \), there exists a unique optimal action \( a^*(\theta, \omega) := \arg\max_{a \in A} u(a, \theta, \omega) \) which is continuous and strictly increasing in \( (\theta, \omega) \) and such that \( a^*(\cdot, \omega) \) has full range for all \( \omega \).

Given any true and perceived type distributions \( F, \hat{F} \in \mathcal{F} \), we briefly analyze the set of steady states \( \text{SS}(F, \hat{F}) \) of this model. For each state \( \omega \), let \( G(\cdot, \omega) \in \Delta(A) \) denote the true cdf over actions in the population when (almost all) agents assign probability 1 to state \( \omega \) and let \( g(\cdot, \omega) \) denote the corresponding density. Likewise, let \( \hat{G}(\cdot, \omega) \) and \( \hat{g}(\cdot, \omega) \) denote the corresponding perceived action distribution and density when agents assign probability 1 to \( \omega \). Note that \( G(a, \omega) = F(\theta^*(a, \omega)) \) and \( \hat{G}(a, \omega) = \hat{F}(\theta^*(a, \omega)) \), where \( \theta^*(a, \omega) \) satisfies \( a = a^*(\theta^*(a, \omega), \omega) \). Let \( \text{KL}(H, \hat{H}) := \int \log \left[ \frac{h(a)}{\hat{h}(a)} \right] h(a) \, da \) denote the KL divergence between continuous distributions \( H \) and \( \hat{H} \) with densities \( h \) and \( \hat{h} \). As in the binary action space setting, we define a steady state \( \hat{\omega}^* \) to be a solution to

\[
\hat{\omega}^* \in \arg\min_{\hat{\omega}} \text{KL}(G(\cdot, \hat{\omega}^*), \hat{G}(\cdot, \hat{\omega})).
\]

Thus, as before, in a steady state agents assign probability 1 to a state that minimizes the KL divergence between the corresponding observed action distribution and agents’ perceived action distribution. At interior steady states \( \hat{\omega}^* \), the first-order condition yields

\[
\int \frac{g(a, \hat{\omega}^*) \partial g(a, \hat{\omega}^*)}{\hat{g}(a, \hat{\omega}^*)} \, da = 0. \tag{15}
\]

Thus, the set of steady states \( \text{SS}(F, \hat{F}) \) is finite whenever there are at most finitely many \( \hat{\omega}^* \) that satisfy (15). A sufficient condition for this is that the left-hand side of (15) is analytic in \( \hat{\omega}^* \) and not constantly equal to 0; similar to the logic behind Theorem 2, this is ensured if \( F \neq \hat{F} \) are analytic and \( \theta^*(a, \cdot) \) is analytic. Moreover, similar to the logic behind Theorem 1, it is easy to construct examples where \( \hat{F} \) is arbitrarily close to \( F \) but there is only a single (state-independent) steady state, as the following illustrates:

**Example 5.** Consider the quadratic-loss utility \( u(a, \theta, \omega) = -(a - \theta - \omega)^2 \), which implies that the optimal action takes the form \( a^*(\theta, \omega) = \theta + \omega \). Suppose that \( F \) and \( \hat{F} \) are cdfs of the Gaussian distributions \( N(\mu, \sigma^2) \) and \( N(\hat{\mu}, \hat{\sigma}^2) \). Then the left-hand side of (15) is given by \( \int \frac{\hat{\mu} - \theta}{\sigma} \exp\left[-\frac{(\theta - \mu)^2}{2\sigma^2}\right] d\theta = \frac{\hat{\mu} - \mu}{\sigma^2} \). Thus, there is no interior steady state, and whenever \( \mu > \hat{\mu} \) (respectively, \( \mu < \hat{\mu} \)), the unique steady state is given by \( \overline{\omega} \) (respectively, \( \underline{\omega} \)), paralleling Example 1 in the binary action setting. \( \square \)