Supplemental Materials to
DYNAMIC RANDOM UTILITY

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Supplementary Appendix to Dynamic Random Utility

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F Proof of Theorem 0

F.1 Preliminaries

In this section we prove Theorem 0 which extends the characterizations of REU representations in Gul and Pesendorfer (2006) and Ahn and Sarver (2013) to allow for an arbitrary separable metric space X of outcomes. Refer to section 2.1 of the main text for all relevant notation and terminology. Throughout, we fix some $y^* \in X$ and let $\tilde{\mathbb{R}}^X = \{0\} \times \mathbb{R}^{X \smallsetminus \{y^*\}}$ denote the set of utility functions u in $\mathbb{R}^X$ that are normalized by $u(y^*) = 0$.

We first define the static analog of S-based representations introduced in Appendix A:

Definition 11. An *S-based REU representation* of ρ is a tuple $(S, \mu, \{U_s, \tau_s\}_{s \in S})$ such that

(i). S is a finite state space and μ is a probability measure on S such that $\mathrm{supp}(\mu) = S$

(ii). for each $s \in S$, the utility $U_s \in \tilde{\mathbb{R}}^X$ is nonconstant and $U_s \not\approx U_{s'}$ for $s \neq s'$

(iii). for each $s \in S$, the tie-breaking rule τ_s is a proper finitely-additive probability measure on $\tilde{\mathbb{R}}^X$ endowed with the Borel σ-algebra

(iv). for all $p \in \Delta(X)$ and $A \in \mathcal{A}$,

$$\rho(p; A) = \sum_{s \in S} \mu(s) \tau_s(p, A),$$

where $\tau_s(p, A) := \tau_s(\{u \in \tilde{\mathbb{R}}^X : p \in M(M(A, U_s), u)\})$.

Analogous arguments as for the DREU part of Proposition 5 yield the equivalence of S-based REU representations and static REU representations.

Proposition 6. Let ρ be a stochastic choice rule on $\mathcal{A}$. Then ρ admits an REU representation if and only if it admits an S-based REU representation.

Proof. Analogous to Proposition 5 (i). ■

Thus, Theorem 0 is equivalent to the following result, which we prove throughout the rest of this section.

Theorem 4. The stochastic choice rule ρ on $\mathcal{A}$ admits an S-based REU representation $(S, \mu, \{U_s, \tau_s\}_{s \in S})$ if and only if ρ satisfies Axiom 0.

Remark 1. Note that because X may be infinite, continuity of each U_s in the representation is not directly implied by linearity. However, in constructing an evolving utility representation in Section C, we can make use of Axiom 6 (iii) (Continuity) to show that each U_{s_t} is continuous (see Lemma 6). In the static setting, an alternative approach is to impose the following axiom.

As in Section 3.3, let $\mathcal{A}^*$ denote the collection of *menus without ties*, i.e., the set of all $A \in \mathcal{A}$ such that for any $p \in A$ and any sequences $p^n \to^m p$ and $B^n \to^m A \setminus \{p\}$, we have $\lim_{n\to\infty} \rho(p^n; B^n \cup \{p^n\}) = \rho(p; A)$.

Axiom 11 (Continuity)**.** $\rho : \mathcal{A}^* \to \Delta(\Delta(X))$ is continuous.

Here $\mathcal{A}$ is endowed with the Hausdorff topology induced by the Prokhorov metric π on $\Delta(X)$, and $\mathcal{A}^*$ with the relative topology. We have the following proposition. Since we do not make use of this result anywhere in the paper, the proof is omitted but available on request.

Proposition 7. Suppose ρ admits an S-based REU representation $(S, \mu, \{U_s, \tau_s\}_{s\in S})$. Then each utility U_s is continuous if and only if ρ additionally satisfies Axiom 11. ▲

Additional notation: For any $Y \subseteq X$, let $\mathcal{A}(Y) := \{A \in \mathcal{A} : \forall p \in A, \text{supp}(p) \subseteq Y\} \subseteq \mathcal{A}$ denote the space of all menus consisting only of lotteries with support in Y. Note that for each $A \in \mathcal{A}$, there is a finite Y such that $A \in \mathcal{A}(Y)$. We denote by ρ^Y the restriction of ρ to $\mathcal{A}(Y)$, which can be seen as a map from $\mathcal{A}(Y)$ to $\Delta(\Delta(Y))$. If $y^* \in Y$, we write $\tilde{\mathbb{R}}^Y := \{0\} \times \mathbb{R}^{Y \setminus \{y^*\}}$.

For any $A \in \mathcal{A}(Y)$ and $p \in \Delta(X)$, let $N_Y(A, p) := \{u \in \tilde{\mathbb{R}}^Y : p \in M(A, u)\}$ and let $N^+_Y(A, p) := \{u \in \tilde{\mathbb{R}}^Y : \{p\} = M(A, u)\}$. Note that $N_Y(\{p\}, p) = N^+_Y(\{p\}, p) = \tilde{\mathbb{R}}^Y$ and that $N_Y(A, p) = N^+_Y(A, p) = \varnothing$ if $p \notin A$. Let $\mathcal{N}(Y) := \{N_Y(A, p) : A \in \mathcal{A}(Y) \text{ and } p \in \Delta(X)\}$, $\mathcal{N}^+(Y) := \{N^+_Y(A, p) : A \in \mathcal{A}(Y) \text{ and } p \in \Delta(X)\}$.

We will consider both the Borel σ-algebra on $\tilde{\mathbb{R}}^Y$ and its subalgebra $\mathcal{F}(Y)$ that is generated by $\mathcal{N}(Y) \cup \mathcal{N}^+(Y)$. A finitely-additive probability measure ν^Y on either of these algebras is called *proper* if $\nu^Y(N_Y(A, p)) = \nu^Y(N^+_Y(A, p))$ for any $A \in \mathcal{A}(Y)$ and $p \in \Delta(X)$. Whenever $Y = X$, we omit Y from the description of $N_Y(A, p)$, $N^+_Y(A, p)$, $\mathcal{N}(Y)$, $\mathcal{N}^+(Y)$, and $\mathcal{F}(Y)$.

F.2 Proof of Theorem 4: Sufficiency

F.2.1 Outline

The proof proceeds as follows:

(i). In section F.2.2, we use conditions (i)–(iv) of Axiom 0 and Theorem 2 in Gul and Pesendorfer (2006) to construct, for each *finite* $Y \subseteq X$, a proper finitely-additive probability measure ν^Y on $\mathcal{F}(Y)$ representing ρ^Y, in the sense that $\rho^Y(p; A) = \nu^Y(N_Y(A, p))$ for all A, p. Given the fact that each ρ^Y is derived from the same ρ, it is easy to check that the family $\{\mathcal{F}(Y), \nu^Y\}$ is Kolmogorov consistent. We can then find a proper finitely-additive probability measure ν on $\mathcal{F}$ extending all the ν^Y (and hence representing ρ).

(ii). The support of ν is defined by

$$\text{supp}(\nu) := \left(\bigcup\{V \in \mathcal{F} : V \text{ is open and } \nu(V) = 0\}\right)^c.$$

In section F.2.3, we use part (v) of Axiom 0 to show that $\operatorname{supp}\nu$ is finite (up to positive affine transformation of utilities) and contains at least one non-constant utility function. While Axiom 0 (v) is similar to the finiteness axiom in Ahn and Sarver (2013), this step requires more work in our setting. A key technical challenge is that unlike in Ahn and Sarver, it is not clear in our infinite outcome space setting how to normalize utilities to ensure that $N(A,p)$-sets are compact. Compact sets C have the useful property (used repeatedly by Ahn and Sarver) that if $C \cap \operatorname{supp}\nu = \varnothing$, then $\nu(C) = 0$. Lemma 22 exploits the geometry of $N(A,p)$-sets to show that this property continues to hold for $N(A,p)$-sets in our setting, even though they are not compact.

(iii). In section F.2.4, we proceed in a similar way to the proof of Theorem S3 in Ahn and Sarver (2013)(again using Lemma 22 to circumvent technical difficulties). Letting $S := \{s_1, \ldots, s_L\}$ denote the equivalence classes of nonconstant utilities in $\operatorname{supp}\nu$, we find separating neighborhoods $B_s \in \mathcal{F}$ of each s such that $\nu(B_s) > 0$. We then define $\mu(s) = \nu(B_s)$ and $\tau_s(V) = \frac{\nu(V \cap B_s)}{\nu(B_s)}$ and show that this yields an S-based REU representation of ρ.

F.2.2 Construction of ν

In this section, we construct a proper finitely-additive probability measure ν on $\mathcal{F}$ that represents ρ, i.e., such that for all $A \in \mathcal{A}$ and $p \in A$, we have

$$\rho(p; A) = \nu(N(A,p)) = \nu(N^+(A,p))).$$

First consider any finite $Y \subseteq X$ with $y^* \in Y$. By Axiom 0 (i)–(iv) (Regularity, Linearity, Extremeness, and Mixture Continuity), Theorem 2 in Gul and Pesendorfer (2006) ensures that there is a proper finitely-additive probability measure ν^Y on $\mathcal{F}^Y$ such that

$$\rho^Y(p; A) = \nu^Y(N_Y(A,p)) = \nu^Y(N_Y^+(A,p))$$

for all $A \in \mathcal{A}(Y)$ and $p \in A$.

Claim 4. For any finite $Y' \supseteq Y \ni y^*$, $(\nu^{Y'}, \mathcal{F}(Y'))$ and $(\nu^Y, \mathcal{F}(Y))$ are Kolmogorov consistent, i.e., for any $E \in \mathcal{F}(Y)$, we have

$$\nu^{Y'}(E \times \mathbb{R}^{Y' \smallsetminus Y}) = \nu^Y(E). \tag{29}$$

Proof. To see this, note first that the LHS of (29) is well-defined, since $E \times \mathbb{R}^{Y' \smallsetminus Y} \in \mathcal{F}^{Y'}$ by Lemma 21 (iv). Note next that by Lemma 21 (iii), E is of the form $\bigcup_{i=1}^n N_Y(A_i, p_i) \cap N_Y^+(B_i, q_i)$ for some finite n and $A_i, B_i \in \mathcal{A}(Y)$. Let E' be obtained from E by replacing each $N_Y(A_i, p_i)$ with $N_Y^+(A_i, p_i)$. By Lemma 21 (ii), $E' = \bigcup_{i=1}^n N_Y^+(C_i, r_i)$ for some family $\{C_i\} \subseteq \mathcal{A}(Y)$. Moreover, since both ν^Y and $\nu^{Y'}$ are proper, we have that $\nu^Y(E) = \nu^Y(E')$ and $\nu^{Y'}(E \times \mathbb{R}^{Y' \smallsetminus Y}) = \nu^{Y'}(E' \times \mathbb{R}^{Y' \smallsetminus Y})$. Hence, it suffices to prove that $\nu^{Y'}(E' \times \mathbb{R}^{Y' \smallsetminus Y}) = \nu^Y(E')$. For this, it is enough to show that for any collection of sets $N_1^+, \ldots, N_n^+ \in \mathcal{N}^+(Y) := \{N^+(A,p) : A \in \mathcal{A}(Y)\}$, we have $\nu^Y(\bigcup_{i=1}^n N_i^+) = \nu^{Y'}(\bigcup_{i=1}^n N_i^+ \times \mathbb{R}^{Y' \smallsetminus Y})$. We prove this by induction. For the base case, note that for any $N^+(A,p) \in \mathcal{N}^+(Y)$, we have

$$\nu^{Y'}(N^+(A,p)) \times \mathbb{R}^{Y' \setminus Y}) = \rho^{Y'}(p,A) := \rho(p;A) =: \rho^Y(p;A) = \nu^Y(N^+(A,p)).$$

Suppose next that the claim is true whenever $m < n$. Then

$$\begin{aligned}
&\nu^Y(\bigcup_{i=1}^{m+1} N_i^+) = \nu^Y(\bigcup_{i=1}^{m} N_i^+) + \nu^Y(N_{m+1}^+) - \nu^Y(\bigcup_{i=1}^{m}(N_i^+ \cap N_{m+1}^+)) = \\
&\nu^{Y'}(\bigcup_{i=1}^{m} N_i^+ \times \mathbb{R}^{Y' \setminus Y}) + \nu^{Y'}(N_{m+1}^+ \times \mathbb{R}^{Y' \setminus Y}) - \nu^{Y'}(\bigcup_{i=1}^{m}(N_i^+ \cap N_{m+1}^+) \times \mathbb{R}^{Y' \setminus Y}) = \\
&\nu^{Y'}(\bigcup_{i=1}^{m+1} N_i^+ \times \mathbb{R}^{Y' \setminus Y}),
\end{aligned}$$

where the second equality follows from the inductive hypothesis and the fact that $N_i^+ \cap N_{m+1}^+ \in \mathcal{N}^+(Y)$ by Lemma 21 (ii). ■

Now define ν on $\mathcal{F}$ by setting $\nu(E) := \nu^Y(\text{proj}_{\tilde{\mathbb{R}}^Y} E)$ for any finite $Y \ni y^*$ such that $E = \text{proj}_{\tilde{\mathbb{R}}^Y} E \times \mathbb{R}^{X \setminus Y}$ and $\text{proj}_{\tilde{\mathbb{R}}^Y} E \in \mathcal{F}^Y$. By Lemma 21 (iv) such a Y exists. Moreover, given Kolmogorov consistency of the family $\{\nu^Y\}_{Y \subseteq X}$, this is well-defined. Finally, it is immediate that ν is a proper finitely-additive probability measure and that ν represents ρ.

F.2.3 Finiteness of $\text{supp}\,\nu$

The support of a finitely-additive probability measure ν is defined by

$$\text{supp}(\nu) := \left(\bigcup\{V \in \mathcal{F} : V \text{ is open and } \nu(V) = 0\}\right)^c.$$

The next lemma invokes Axiom 0 (v) (Finiteness) to show that the support of ν constructed in the previous section contains finitely many equivalence classes of utility functions and contains at least one nonconstant function. We use 0 to denote the unique constant utility function in $\tilde{\mathbb{R}}^X$.

Lemma 18. Let K be as in the statement of the Finiteness Axiom and let $Pref(\Delta(X))$ denote the set of all preferences over $\Delta(X)$. Then

$$\#\{\succsim \in Pref(\Delta(X)) : \succsim \text{ is represented by some } u \in \text{supp}(\nu) \setminus \{0\}\} = L,$$

where $1 \leq L \leq K$.

Proof. We first show that $L \leq K$. If not, then we can find utilities $\{u_1, ..., u_{K+1}\} \subseteq \text{supp}(\nu)$ such that each u_i is non-constant over X and $u_i \not\approx u_j$ for all $i \neq j$. By Lemma 13, we can find a menu $A = \{p^i : i = 1, \dots, K+1\} \in \mathcal{A}$ such that $u_i \in N^+(A, p^i)$ for each i. Take any $B \subseteq A$ with $|B| \leq K$. Then $p^i \notin B$ for some i.

Fix any sequences $p_n^i \to^m p^i$ and $B_n \to^m B$. By definition, this means that there exists $r \in \Delta(X)$ and $\alpha_n \to 0$ such that $p_n^i = \alpha_n r + (1-\alpha)p^i$ for all n, and that for each $q \in B$ there exists $B_q \in \mathcal{A}$ and $\beta_n(q) \to 0$ such that $B_n = \bigcup_{q \in B}(\beta_n(q)B_q + (1-\beta_n(q))\{q\})$ for all n. Now, B and each B_q are finite, and u_i is linear with $u_i \cdot p^i > u_i \cdot q$ for all $q \in B$. Hence, there is N such that for all $n \geq N$, $u_i \cdot p^i_n > u \cdot q_n$ for all $q_n \in B_n$. Thus, $u_i \in N^+(\{p^i_n\} \cup B_n, p^i_n)$ for all $n \geq N$. But since $u_i \in \operatorname{supp}(\nu)$ and $N^+(\{p^i_n\} \cup B_n, p^i_n)$ is an open set in $\mathcal{F}$, the definition of $\operatorname{supp}(\nu)$ then implies that $\nu(N^+(\{p^i_n\} \cup B_n, p^i_n)) > 0$ for all $n \geq N$. But then $\rho(p^i_n; \{p^i_n\} \cup B_n) = \nu(N^+(\{p^i_n\} \cup B_n, p^i_n)) > 0$ for all $n \geq N$, contradicting Finiteness.

Next we show that $L \geq 1$. Indeed, if $L = 0$, then for any $A \in \mathcal{A}$ with $|A| \geq 2$ and for any $p \in A$, we have $(N(p, A) \smallsetminus \{0\}) \cap \operatorname{supp}\nu = \varnothing$. By Lemma 22 below, this implies that $\nu(N^+(p, A)) = 0$ for any $p \in A$. But since ν represents ρ, $\rho(p; A) = \nu(N^+(p, A))$ for any $p \in A$, so we have $\sum_{p \in A} \rho(p; A) = 0$, which is a contradiction. ■

F.2.4 Constructing the REU representation

Let $\succsim_1, \ldots, \succsim_L$ denote all the preferences represented by some non-constant utility in $\operatorname{supp}(\nu)$, where by Lemma 18 we know that L is finite and $L \geq 1$. For each $i = 1, \ldots, L$, pick some $u_i \in \operatorname{supp}\nu$ representing $\succsim_i$. For any $u \in \tilde{\mathbb{R}}^X$, let $[u] := \{u' \in \tilde{\mathbb{R}}^X : u' \approx u\}$. By Lemma 13, we can find $A := \{p_1, \ldots, p_L\} \in \mathcal{A}$ such that $u_i \in N^+(A, p_i)$ for all $i = 1, \ldots, L$. Let $B_{u_i} := N^+(A, p_i)$ for all i. By construction, $[u_i] \subseteq B_{u_i}$ and $B_{u_i} \cap B_{u_j} = \varnothing$ for $j \neq i$. Moreover, by the definition of $\operatorname{supp}(\nu)$, we have $\nu(B_{u_i}) > 0$ for each i, since $B_{u_i} \in \mathcal{F}$ is open and $u_i \in B_{u_i} \cap \operatorname{supp}(\nu) \neq \varnothing$.

Let $S := \{u_1, \ldots, u_L\}$ and define the function $\mu : S \to [0, 1]$ by

$$\mu(s) = \nu(B_s) \text{ for each } s \in S.$$

We claim that μ defines a full-support probability measure on S. For this it remains to show that $\sum_s \mu(s) = 1$. Since $\sum_s \mu(s) = \sum_s \nu(B_s) = \nu(\bigcup_{s \in S} B_s)$, it suffices to prove the following claim:

Lemma 19. $\nu(\bigcup_{s \in S} B_s) = 1$.

Proof. It suffices to prove that $\nu(\tilde{\mathbb{R}}^X \smallsetminus \bigcup_{s \in S} B_s) = 0$. Note that $\tilde{\mathbb{R}}^X = \bigcup_{i=1}^L N(A, p_i)$, since $A = \{p_i, \ldots, p_L\}$. Thus,

$$\tilde{\mathbb{R}}^X \smallsetminus \bigcup_{s \in S} B_s \subseteq \bigcup_{i=1}^{L} (N(A, p_i) \smallsetminus N^+(A, p_i)).$$

By finite additivity of ν, this implies that

$$\nu(\tilde{\mathbb{R}}^X \smallsetminus \bigcup_{s \in S} B_s) \leq \sum_{i=1}^{L} \nu(N(A, p_i) \smallsetminus N^+(A, p_i)) = 0,$$

where the last inequality follows from properness of ν. ■

Next, we define a set function $\tau_s : \mathcal{F} \to \mathbb{R}_+$ for each $s \in S$ by setting

$$\tau_s(V) := \frac{\nu(V \cap B_s)}{\nu(B_s)}$$

for each $V \in \mathcal{F}$. Since $\nu(B_s) > 0$ for all $s \in S$, this is well-defined. Moreover, since ν is a proper finitely-additive probability measure on $\mathcal{F}$, so is τ_s.

Note that for all $A \in \mathcal{A}$ and $p \in \Delta(X)$, $\{u \in \tilde{\mathbb{R}}^X : p \in M(M(A,s),u)\} = N(M(A,s),p) \in \mathcal{F}$, so $\tau_s(\{u \in \tilde{\mathbb{R}}^X : p \in M(M(A,s),u)\})$ is well-defined. The next lemma will allow us to complete the representation:

Lemma 20. For each $s \in S$, $A \in \mathcal{A}$, and $p \in A$,

$$\nu(N(A,p)) = \sum_{s\in S} \mu(s)\tau_s(\{u \in \tilde{\mathbb{R}}^X : p \in M(M(A,s),u)\}).$$

Proof. We first show that for each $s \in S$, $\operatorname{supp}\tau_s \smallsetminus \{0\} = [s]$. To see that $[s] \subseteq \operatorname{supp}\tau_s \smallsetminus \{0\}$, consider any $u \in [s]$ and any open $V \in \mathcal{F}$ such that $u \in V$. By Lemma 21 (iii), V is a finite union of finite intersections of sets in $\mathcal{N} \cup \mathcal{N}^+$. Hence, since each element of $\mathcal{N} \cup \mathcal{N}^+$ is closed under positive affine transformations so is V. Thus, $u \in V$ implies $s \in V$. But then $V \cap B_s \in \mathcal{F}$ is open and contains s, and hence $\nu(V \cap B_s) > 0$ since $s \in \operatorname{supp}\nu$. This proves $u \in \operatorname{supp}\tau_s \smallsetminus \{0\}$.

To see that $\operatorname{supp}\tau_s \smallsetminus \{0\} \subseteq [s]$, consider any $u \neq 0$ such that $u \notin [s]$. It suffices to show that there exists an open $V \in \mathcal{F}$ such that $u \in V$ and $\tau_s(V) = 0$. If $u \approx s'$ for some $s' \in S \smallsetminus \{s\}$, then $V = B_{s'}$ is as required since $B_{s'} \cap B_s = \varnothing$ and $u \in B_{s'}$. If there is no $s' \in S \smallsetminus \{s\}$ such that $u \approx s'$, then $u \notin \operatorname{supp}\nu$. But then there exists an open $V \in \mathcal{F}$ such that $u \in V$ and $\nu(V) = 0$, so also $\tau_s(V) = 0$.

By Lemma 23 below, this implies that $\tau_s(N(A,p)) = \tau_s(N(M(A,s),p))$ for any $A \in \mathcal{A}$ and $p \in A$. This implies that for any $A \in \mathcal{A}$ and $p \in A$

$$\begin{aligned}
\sum_{s\in S} \mu(s)\tau_s(\{u \in \tilde{\mathbb{R}}^X : p \in M(M(A,s),u)\}) &= \sum_{s\in S} \mu(s)\tau_s(N(M(A,s),p)) \\
&= \sum_{s\in S} \mu(s)\tau_s(N(A,p)) \\
&= \sum_{s\in S} \nu(N(A,p) \cap B_s) \\
&= \nu(N(A,p) \cap \bigcup_{s\in S} B_s) \\
&= \nu(N(A,p)),
\end{aligned}$$

where the last equality follows from Lemma 19. ■

For any $s \in S = \{u_1, \dots, u_L\}$, we write $U_s := s$. We claim that $(S, \mu, \{U_s, \tau_s\}_{s\in S})$ is an S-based REU representation of ρ. Indeed, by construction, U_s is non-constant for all s, $U_s \not\approx U_{s'}$ for any distinct $s, s' \in S$, and μ is a full-support probability measure on S. Moreover, each τ_s is a proper finitely-additive probability measure on $\tilde{\mathbb{R}}^X$ endowed with the algebra $\mathcal{F}$. By standard arguments (cf. Rao and Rao (2012)), we can extend τ_s to a proper finitely-additive probability measure on the Borel σ-algebra on $\tilde{\mathbb{R}}^X$. Finally, Lemma 20 and the fact that ν represents ρ implies that for all $A \in \mathcal{A}$ and $p \in A$, we have $\rho(p; A) = \sum_{s\in S} \mu(s)\tau_s(p, A)$, as required.

F.3 Proof of Theorem 4: Necessity

Suppose that ρ admits an S-based REU representation $(S, \mu, \{U_s, \tau_s\}_{s\in S})$. We show that ρ satisfies Axiom 0. Observe first that for any finite $Y \subseteq X$ with $y^* \in Y$, $(S, \mu, \{U_s \restriction_Y, \tau_s \restriction_Y\}_{s\in S})$ constitutes an S-based REU representation of ρ^Y, where $U_s \restriction_Y$ denotes the restriction of U_s to Y and $\tau_s \restriction_Y$ is given by $\tau_s \restriction_Y (B) = \tau_s(B \times \mathbb{R}^{X\setminus Y})$ for any Borel set B on $\mathbb{R}^Y$. Thus, by Theorem S3 in Ahn and Sarver (2013), ρ^Y satisfies Regularity, Linearity, Extremeness, and Mixture Continuity.

To show that ρ satisfies Regularity, consider any $p \in A \subsetneq A'$. Pick a finite $Y \subseteq X$ with $y^* \in Y$ such that $A, A' \in \mathcal{A}(Y)$. By definition, $\rho(p; A) = \rho^Y(p; A)$ and $\rho(p; A') = \rho^Y(p; A')$. Hence, by Regularity for ρ^Y, we have $\rho(p; A) \geq \rho(p; A')$, as required. Similarly, we can show that ρ satisfies Linearity, Extremeness, and Mixture Continuity by using the fact that for each finite Y, each ρ^Y satisfies these axioms.

Finally, to show that ρ satisfies Finiteness, let $K := |S|$ and consider any $A \in \mathcal{A}$. For each $s \in S$, pick any $q_s \in M(A, U_s)$, and define $B := \{q_s : s \in S\}$. Note that $|B| \leq K$. If $B = A$, then Finiteness is trivially satisfied. If $B \subsetneq A$, then pick any $p \in A \setminus B$. We can pick a large enough finite $Y \subseteq X$ such that each U_s is non-constant on Y and $U_s \restriction_Y \not\approx U_{s'} \restriction_Y$ for any distinct $s, s' \in S$. Let $r \in \Delta(Y)$ be given by $r(y) := \frac{1}{|Y|}$ for each $y \in Y$. For each $s \in Y$, pick any $y_s \in \operatorname{argmax}_{y\in Y} U_s(y)$. Note that $U_s(y_s) > U_s(r)$. Define $B^n := \frac{n-1}{n}B + \frac{1}{n}\{y_s : s \in S\}$ and $p^n := \frac{n-1}{n}p + \frac{1}{n}r$. Then $B^n \to^m B$ and $p_n \to^m p$. Moreover, for all large enough n, we have $U_s(\frac{n-1}{n}q_s + \frac{1}{n}y_s) > U_s(p^n)$ for each $s \in S$. Thus, $\rho(p^n; \{p^n\} \cup B^n) = 0$, proving Finiteness.

F.4 Additional Lemmas for Section F

F.4.1 Properties of $N(A, p)$ sets

Lemma 21. Fix any $X' \subseteq X$ with $y^* \in X$. For any collection $\mathcal{S}$, we let $\mathcal{U}(\mathcal{S})$ denote the set of all finite unions of elements of $\mathcal{S}$.

(i). If $E \in \mathcal{N}(X')$ (resp. $E \in \mathcal{N}^+(X')$), then $E^c \in \mathcal{U}(\mathcal{N}^+(X'))$ (resp. $E^c \in \mathcal{U}(\mathcal{N}(X'))$.

(ii). If $E_1, E_2 \in \mathcal{N}(X')$ (resp. $E_1, E_2 \in \mathcal{N}^+(X')$), then $E_1 \cap E_2 \in \mathcal{N}(X')$ (resp. $E_1 \cap E_2 \in \mathcal{N}^+(X')$).

(iii). $\mathcal{F}(X')$ is the set of all E such that $E = \bigcup_{\ell\in L} M_\ell \cap N_\ell$ for some finite index set L and $M_\ell \in \mathcal{N}(X')$, $N_\ell \in \mathcal{N}^+(X')$ for each $\ell \in L$.

(iv). $\mathcal{F}(X')$ is the set of all E for which there exists a finite $Y \subseteq X'$ with $y^* \in Y$ and $E^Y \in \mathcal{F}(Y)$ such that $E = E^Y \times \mathbb{R}^{X\setminus Y}$.

Proof.

(i): If $E = N(A, p) \in \mathcal{N}(X')$, then $E^c = \bigcup_{q\in A\setminus\{p\}} N^+(\{p, q\}, q) \in \mathcal{U}(\mathcal{N}^+(X'))$ if $p \in A$ and $E^c = \tilde{\mathbb{R}}^{X'} \in \mathcal{U}(\mathcal{N}^+(X'))$ if $p \notin A$. Similarly, if $E = N^+(A, p) \in \mathcal{N}^+(X')$, then $E^c = \bigcup_{q\in A\setminus\{p\}} N(\{p, q\}, q) \in \mathcal{U}(\mathcal{N}(X'))$ if $p \in A$ and $E^c = \tilde{\mathbb{R}}^{X'} \in \mathcal{U}(\mathcal{N}(X'))$ if $p \notin A$.

(ii): If $N(A_1, p_1), N(A_2, p_2) \in \mathcal{N}(X')$, then $N(A_1, p_1) \cap N(A_2, p_2) = N(\frac{1}{2}A_1 + \frac{1}{2}A_2, \frac{1}{2}p_1 + \frac{1}{2}p_2) \in \mathcal{N}(X')$. The same argument goes through replacing all instances of N with N^+.

(iii): By standard results, $\mathcal{F}(X')$ can be described as follows: Let $\mathcal{F}_0(X')$ denote the set of all elements of $\mathcal{N}(X') \cup \mathcal{N}^+(X')$ and their complements. Let $\mathcal{F}_1(X')$ denote the set of all finite intersections of elements of $\mathcal{F}_0(X')$. Then $\mathcal{F}(X')$ is the set of all finite unions of elements of $\mathcal{F}_1(X')$. By part (i), $\mathcal{F}_0(X) = \mathcal{U}(N(X)) \cup \mathcal{U}(N(X'))$ is the collection of all finite unions of elements of $\mathcal{N}(X')$ and of all finite unions of elements of $\mathcal{N}^+(X')$. By part (ii), $\mathcal{F}_1(X') = \mathcal{F}_0(X) \cup \mathcal{I}(X')$, where $\mathcal{I}(X')$ consists of all finite unions of the form $\bigcup_{\ell\in L} M_\ell \cap N_\ell$, where $M_\ell \in \mathcal{N}(X')$ and $N_\ell \in \mathcal{N}^+(X')$ for each $\ell \in L$. Note that $\tilde{\mathbb{R}}^{X'} \in \mathcal{N}(X') \cap \mathcal{N}^+(X')$, since $\tilde{\mathbb{R}}^{X'} = N_{X'}(\{p\}, p) = N^+_{X'}(\{p\}, p)$ for any $p \in \Delta(X')$. Thus, $\mathcal{F}_0(X) = \mathcal{U}(N(X)) \cup \mathcal{U}(N(X')) \subseteq \mathcal{I}(X)$. Hence, $\mathcal{F}_1(X) = \mathcal{I}(X) = \mathcal{F}(X)$.

(iv): Note first that for any $N_{X'}(A,p) \in \mathcal{N}(X')$ (resp. $N^+_{X'}(A,p) \in \mathcal{N}^+(X')$) and any finite $Y \subseteq X'$ with $y^* \in Y$ and $A \in \mathcal{A}(Y)$, we have $N_{X'}(A,p) = N_Y(A,p) \times \mathbb{R}^{X' \setminus Y}$ (resp. $N^+_{X'}(A,p) = N^+_Y(A,p) \times \mathbb{R}^{X' \setminus Y}$). Now fix any $E \in \mathcal{F}(X')$. By part (iv), we have a finite index set L and $M_\ell \in \mathcal{N}(X')$, $N_\ell \in \mathcal{N}^+(X')$ for each $\ell \in L$ such that $E = \bigcup_{\ell\in L} M_\ell \cap N_\ell$. By the first sentence, we can then pick a finite $Y \subseteq X'$ with $y^* \in Y$ such that for each ℓ, we have $M_\ell = M^Y_\ell \times \mathbb{R}^{X' \setminus Y}$ and $N_\ell = N^Y_\ell \times \mathbb{R}^{X' \setminus Y}$, where $M^Y_\ell \in \mathcal{N}(Y)$ and $N^Y_\ell \in \mathcal{N}^+(Y)$. Then $E = E^Y \times \mathbb{R}^{X \setminus Y}$, where $E^Y := \bigcup_{\ell\in L} M^Y_\ell \cap N^Y_\ell \in \mathcal{F}(Y)$. Conversely, if $E^Y \in \mathcal{F}(Y)$, then by part (iv), E^Y is of the form $\bigcup_{\ell\in L} M^Y_\ell \cap N^Y_\ell \in \mathcal{F}(Y)$ for some finite collection of $M^Y_\ell \in \mathcal{N}(Y)$ and $N^Y_\ell \in \mathcal{N}^+(Y)$. Then by the first sentence, $M_\ell := M^Y_\ell \times \mathbb{R}^{X' \setminus Y} \in \mathcal{N}(X')$ and $N_\ell = N^Y_\ell \times \mathbb{R}^{X' \setminus Y} \in \mathcal{N}^+(X')$, so $E^Y \times \mathbb{R}^{X' \setminus Y} = \bigcup_{\ell=1}^{L} M_\ell \cap N_\ell) \in \mathcal{F}(X')$ by part (iv). ■

F.4.2 Properties of proper finitely-additive probability measures on $\mathcal{F}$

Lemma 22. Let ν be a proper finitely-additive probability measure on $\mathcal{F}$ and suppose that $(N(p, A) \setminus \{0\}) \cap \operatorname{supp} \nu = \varnothing$ for some $A \in \mathcal{A}$ and $p \in A$, where 0 denotes the unique constant utility in $\tilde{\mathbb{R}}^X$. Then $\nu(N^+(A,p)) = \nu(N(A,p)) = 0$.

Proof. Since $(N(A,p) \setminus \{0\}) \cap \operatorname{supp} \nu = \varnothing$, we have

$$N(A,p) \setminus \{0\} \subseteq (\operatorname{supp} \nu)^c := \bigcup \{V \in \mathcal{F} : V \text{ open and } \nu(V) = 0\}.$$

Thus, for some possibly infinite index set I, there exists a family $\{V_i\}_{i\in I}$, with $V_i \in \mathcal{F}$ open and $\nu(V_i) = 0$ for each i such that

$$N(A,p) \setminus \{0\} \subseteq \bigcup_{i\in I} V_i.$$

We now show that there is a finite subset $\{i_1, \ldots, i_n\} \subseteq I$ such that

$$N(A,p) \setminus \{0\} \subseteq \bigcup_{j=1}^{n} V_{i_j}.$$

To see this, define $L(A,p) := (N(A,p) \cap [-1,1]^X) \setminus \{0\}$. Note that since $[-1,1]^X$ is compact in $\mathbb{R}^X$ (by Tychonoff's theorem) and $N(A,p)$ is closed in $\tilde{\mathbb{R}}^X$, $C(A,p)$ is compact in the relative topology on $\tilde{\mathbb{R}}^X \setminus \{0\}$. Hence, since $L(A,p) \subseteq N(A,p) \setminus \{0\}$ is covered by $\bigcup_{i\in I} V_i$ and each V_i is open, it has a finite subcover $\bigcup_{j=1}^{n} V_{i_j}$.

We claim that $N(A,p) \setminus \{0\}$ is also covered by $\bigcup_{j=1}^n V_{i_j}$. To see this, consider any $u^* \in N(A,p) \setminus \{0\}$. We can find a finite $Y \subseteq X$ such that $y^* \in Y$, $u^* \restriction_Y$ is not constant, $N(A,p) = N_Y(A,p) \times \mathbb{R}^{X \setminus Y}$, and for each $j = 1, \ldots, n$, $V_{i_j} = V_{i_j}^Y \times \mathbb{R}^{X \setminus Y}$ for some $V_{i_j}^Y \in \mathcal{F}^Y$ (see Lemma 21 (iv)).

Since Y is finite, there exists $\alpha > 0$ small enough such that $\alpha u^*(y) \in [-1,1]$ for all $y \in Y$. Define $u \in \tilde{\mathbb{R}}^X$ by $u \restriction_Y = \alpha u^* \restriction_Y$ and $u(x) = 0$ for all $x \in X \setminus Y$. Note that $u \in N(A,p)$: Indeed, $u^* \in N(A,p) = N_Y(A,p) \times \mathbb{R}^{X \setminus Y}$, $u \restriction_Y = \alpha u^* \restriction_Y$, and $N_Y(A,p)$ is closed under positive scaling. Moreover, u is not constant, since $u^* \restriction_Y$ is not constant. Finally, $u \in [-1,1]^X$. This shows $u \in L(A,p)$. Since $L(A,p)$ is covered by $\bigcup_{j=1}^n V_{i_j}$, there exists j such that $u \in V_{i_j} = V_{i_j}^Y \times \mathbb{R}^{X \setminus Y}$. But note that $V_{i_j}^Y$ is closed under positive scaling, since by Lemma 21 (iii) it is a finite union of sets which are closed under positive scaling. Since $u \restriction_Y = \alpha u^* \restriction_Y$, this implies $u^* \in V_{i_j}$.

The above shows that $N(A,p) \setminus \{0\}$ is covered by $\bigcup_{j=1}^n V_{i_j}$, and hence so is $N^+(A,p)$. But since $\nu(V_{i_j}) = 0$ for all $j = 1, \ldots, n$ and ν is finitely additive, it follows that $\nu(N^+(A,p)) = 0$. Moreover, by properness of ν, this implies $\nu(N(A,p)) = 0$. ∎

Lemma 23. Suppose ν is a proper finitely-additive probability measure on $\mathcal{F}$ and $\operatorname{supp} \nu \setminus \{0\} = [u]$ for some $u \in \tilde{\mathbb{R}}^X$. Then for any $A \in \mathcal{A}$ and $p \in A$, we have $\nu(N(A,p)) = \nu(N(M(A,u),p))$.

Proof. Fix any $A \in \mathcal{A}$ and $p \in A$. Note first that for any $q \in A$,

$$q \notin M(A,u) \Rightarrow \nu(N(A,q)) = 0. \tag{30}$$

Indeed, if $q \notin M(A,u)$, then $\varnothing = [u] \cap N(A,q) = (N(A,q) \setminus \{0\}) \cap \operatorname{supp} \nu$. But then Lemma 22 implies that $\nu(N(A,q)) = 0$, as claimed.

Suppose now that $p \notin M(A,u)$. Then (30) implies that $\nu(N(A,p)) = 0$. Moreover, $N(B,p) := \varnothing$ if $p \notin B$, so also $\nu(N(M(A,u),p)) = 0$, as required.

Suppose next that $p \in M(A,u)$. Then

$$N(A,p) \subseteq N(M(A,u),p) \subseteq N(A,p) \cup \bigcup_{q \in A \setminus M(A,u)} N(A,q),$$

so that

$$\nu(N(A,p)) \leq \nu(N(M(A,u),p)) \leq \nu(N(A,p)) + \sum_{q \in A \setminus M(A,u)} \nu(N(A,q)) = \nu(N(A,p)),$$

where the last equality follows from (30). This again shows that $\nu(N(A,p)) = \nu(N(M(A,u),p))$, as required. ∎

G Proof of Proposition 5

The following three subsections prove Proposition 5, that is, the equivalence between DREU, evolving utility, gradual learning and their respective S-based analogs.

G.1 DREU

"If" direction: Suppose ρ admits an S-based DREU representation $(S_t, \{\mu_t^{s_{t-1}}\}_{s_{t-1}\in S_{t-1}}, \{U_{s_t}, \tau_{s_t}\}_{s_t\in S_t})_{t=0,\dots,T}$. We will construct a DREU representation $(\hat{\Omega}, \hat{\mathcal{F}}^*, \hat{\mu}, (\hat{\mathcal{F}}_t, \hat{U}_t, \hat{W}_t))$.

Consider the space $G := \prod_{t=0}^T \left(S_t \times \mathbb{R}^{X_t}\right)$ of all sequences of states and tie-breaking utilities. Let $\hat{\Omega} := \{(s_0, W_0, \dots, s_T, W_T) \in G : \prod_{k=0}^t \mu_k^{s_{k-1}}(s_k) > 0\}$. Let $\hat{\mathcal{F}}^*$ be the restriction to $\hat{\Omega}$ of the product sigma-algebra of the discrete sigma-algebra on $\prod_{t=0}^T S_t$ and the product Borel sigma-algebra on $\prod_{t=0}^T \mathbb{R}^{X_t}$. For each $K = (\{s_0\}, K_0, ..., \{s_T\}, K_T) \in \hat{\mathcal{F}}^*$, let $\hat{\mu}(K) = \prod_{t=0}^T \mu_t^{s_{t-1}}(s_t)\tau_{s_t}(K_t)$; by finiteness of $\prod_{t=0}^T S_t$, $\hat{\mu}$ extends to a finitely-additive probability measure on $\hat{\Omega}$ in the natural way.

Let Π_t be the finite partition of $\hat{\Omega}$ whose cells are all the cylinders $C(s_0, \dots, s_t) := \{\hat{\omega} \in \hat{\Omega} : \text{proj}_{S_0\times\ldots\times S_t}(\hat{\omega}) = (s_0, \dots, s_t)\}$. Let $\hat{\mathcal{F}}_t$ be the sigma-algebra generated by Π_t; by definition of $\hat{\Omega}$, $\mu(\hat{\mathcal{F}}_t(\hat{\omega})) > 0$ for all $\hat{\omega} \in \hat{\Omega}$. Also, $\hat{\mathcal{F}}_t(\hat{\omega}) = \bigcup_{\hat{\omega}'\in\hat{\mathcal{F}}_t(\hat{\omega})} \hat{\mathcal{F}}_{t+1}(\hat{\omega}')$, so $(\hat{\mathcal{F}}_t)_{0\le t\le T} \subseteq \hat{\mathcal{F}}^*$ is a filtration. Define $\hat{U}_t : \hat{\Omega} \to \mathbb{R}^{X_t}$ by $\hat{U}_t(\hat{\omega}) = U_{s_t}$ where $\text{proj}_{S_t}(\hat{\omega}) = s_t$. Note that $(\hat{U}_t)$ is adapted to $(\hat{\mathcal{F}}_t)$ and that $\hat{U}_t(\hat{\omega})$ is nonconstant for each $\hat{\omega}$ since each U_{s_t} is nonconstant. Finally, if $\mathcal{F}_{t-1}(\hat{\omega}) = \mathcal{F}_{t-1}(\hat{\omega}')$ and $\mathcal{F}_t(\hat{\omega}) \neq \mathcal{F}_t(\hat{\omega}')$, then $\text{proj}_{S_{t-1}}(\hat{\omega}) = \text{proj}_{S_{t-1}}(\hat{\omega}) = s_{t-1}$ and $\text{proj}_{S_t}(\hat{\omega}) = s_t \neq s_t' = \text{proj}_{S_t}(\hat{\omega}')$ for some $s_{t-1} \in S_{t-1}$ and $s_t, s_t' \in \text{supp}\, \mu_t^{s_{t-1}}$. By DREU1 (a), this implies $\hat{U}_t(\hat{\omega}) := U_{s_t} \not\approx U_{s_t'} =: \hat{U}_t(\hat{\omega}')$. Thus, $(\mathcal{F}_t, U_t)$ are simple.

Define $\hat{W}_t : \hat{\Omega} \to \mathbb{R}^{X_t}$ by $\hat{W}_t(\hat{\omega}) = W_t$ where $\text{proj}_{\mathbb{R}^{X_t}}(\hat{\omega}) = W_t$. Note that for all A_t, $\hat{\mu}(\{\hat{\omega} \in \hat{\Omega} : |M(A_t, \hat{W}_t)| = 1\}) = \sum_{(s_0,\dots,s_T)} \left(\prod_{k=0}^T \mu_k^{s_{k-1}}(s_k)\right) \tau_{s_t}(\{W_t \in \mathbb{R}^{X_t} : |M(A_t, W_t)| = 1\}) = 1$, since each τ_{s_t} is proper. Thus, $(\hat{W}_t)$ satisfies part (i) of the properness requirement for DREU. Moreover, for any $\hat{\mathcal{F}}_T(\hat{\omega}) = C(s_0, \dots, s_T)$ and any sequence (B_t) of Borel sets $B_t \subseteq \mathbb{R}^{X_t}$, the definition of $\hat{\mu}$ implies

$$\hat{\mu}\left(\bigcap_{t=0}^T \{\hat{W}_t \in B_t\} | C(s_0, \dots, s_T)\right) = \prod_{t=0}^T \tau_{s_t}(B_t) = \prod_{t=0}^T \hat{\mu}\left(\{\hat{W}_t \in B_t\} \mid C(s_0, \dots, s_t)\right). \tag{31}$$

Since $\hat{\mathcal{F}}_T(\hat{\omega}) = C(s_0, \dots, s_T)$ implies $\hat{\mathcal{F}}_t(\hat{\omega}) = C(s_0, \dots, s_t)$ for all $t \le T$, this shows that $(\hat{W}_t)$ also satisfies parts (ii) and (iii) of the properness requirement.

Finally, to see that $(\hat{\Omega}, \hat{\mathcal{F}}^*, \hat{\mu}, (\hat{\mathcal{F}}_t, \hat{U}_t, \hat{W}_t))$ represents ρ, fix any $h^t = (A_0, p_0, ..., A_t, p_t) \in \mathcal{H}_t$. Then

$$\hat{\mu}(C(h^t)) = \hat{\mu}\left(\textstyle\bigcap_{k=0}^t \{\hat{\omega} \in \hat{\Omega} : p_k \in M(M(A_k, \hat{U}_k(\hat{\omega})), \hat{W}_k(\hat{\omega}))\}\right) =$$
$$\textstyle\sum_{C(s_0,\dots,s_t)\in\Pi_t} \hat{\mu}(C(s_0, ..., s_t))\hat{\mu}\left(\bigcap_{k=0}^t \{\hat{\omega} \in \hat{\Omega} : p_k \in M(M(A_k, \hat{U}_k), \hat{W}_k)\} | C(s_0, ..., s_t)\right) =$$
$$\textstyle\sum_{(s_0,\dots,s_t)\in S_0\times\ldots\times S_t} \prod_{k=0}^t \mu_k^{s_{k-1}}(s_k)\hat{\mu}\left(\bigcap_{k=0}^t \{\hat{\omega} \in \hat{\Omega} : p_k \in M(M(A_k, U_{s_k}), \hat{W}_k)\} | C(s_0, ..., s_t)\right) =$$
$$\textstyle\sum_{(s_0,\dots,s_t)\in S_0\times\ldots\times S_t} \prod_{k=0}^t \mu_k^{s_{k-1}}(s_k)\tau_{s_k}(p_k, A_k)$$

where the third equality follows from the definition of $\hat{\mu}$ and $\hat{U}$, and the final equality follows from (31). Thus, as required, we have

$$\hat{\mu}(C(p_t, A_t)|C(h^{t-1})) = \frac{\hat{\mu}(C(h^t))}{\hat{\mu}(C(h^{t-1})} = \frac{\sum_{(s_0,\ldots,s_t)} \prod_{k=0}^{t} \mu_k^{s_{k-1}}(s_k)\tau_{s_k}(p_k, A_k)}{\sum_{(s_0,\ldots,s_{t-1})} \prod_{k=0}^{t-1} \mu_k^{s_{k-1}}(s_k)\tau_{s_k}(p_k, A_k)} = \rho_t(p_t; A_t|h^{t-1}),$$

where the final equality holds by DREU2.

"Only if" direction: Take any DREU representation $(\Omega, \mathcal{F}^*, \mu, (\mathcal{F}_t, U_t, W_t))$ of ρ. We will construct an S-based DREU representation $(S_t, \{\hat{\mu}_t^{s_{t-1}}\}_{s_{t-1}\in S_{t-1}}, \{\hat{U}_{s_t}, \tau_{s_t}\}_{s_t\in S_t})_{t=0,\ldots,T}$.

For each t, let $S_t := \{\mathcal{F}_t(\omega) : \omega \in \Omega\}$ denote the partition generating $\mathcal{F}_t$, which is finite since $(\mathcal{F}_t)$ is simple. Each $\hat{\mu}_{t+1}^{s_t}$ is defined to be the one-step-ahead conditional of μ, i.e., $\hat{\mu}_0(s_0) := \mu(s_0)$ for all $s_0 \in S_0$ and $\hat{\mu}_{t+1}^{s_t}(s_{t+1}) := \mu(s_{t+1}|s_t)$ for all $s_t \in S_t$, $s_{t+1} \in S_{t+1}$. This is well-defined since $\mu(\mathcal{F}_t(\omega)) > 0$ for all ω. For each $s_t \in S_t$, define $\hat{U}_{s_t} := U_t(\omega)$ if $\omega \in s_t$; this is well-defined as (U_t) is $\mathcal{F}_t$-adapted and each U_{s_t} is nonconstant since each $U_t(\omega)$ is nonconstant. Finally, for any Borel set $B_t \subseteq \mathbb{R}^{X_t}$, define $\tau_{s_t}(B_t) := \mu(\{W_t \in B_t\}|s_t)$. This is well-defined since W_t is $\mathcal{F}^*$-measurable. Moreover, because $\mu(\{\omega \in \Omega : |M(A_t, W_t(\omega)| = 1\} = 1$ for all A_t and $|S_t|$ is finite, it follows that $\tau_{s_t}(N(A_t, p_t)) = \tau_{s_t}(N^+(A_t, p_t))$ for all p_t, i.e., τ_{s_t} is proper. Thus, each $(S_t, \mu_t^{s_{t-1}}, \{U_{s_t}, \tau_{s_t}\}_{s_t\in S_t})$ is an REU form on X_t.

Moreover, (a) for any distinct $s_t, s'_t \in \text{supp}(\mu_t^{s_{t-1}})$, we have ω, ω' such that $\mathcal{F}_{t-1}(\omega) = s_{t-1} = \mathcal{F}_{t-1}(\omega')$ and $\mathcal{F}_t(\omega) = s_t \neq \mathcal{F}_t(\omega') = s'_t$. Thus, $\hat{U}_{s_t} = U_t(\omega) \not\approx U_t(\omega') = \hat{U}_{s'_t}$, since $(U_t, \mathcal{F}_t)$ is simple. Also, since $(\mathcal{F}_t)$ is adapted, the partition S_t refines the partition S_{t-1}, so that (b) for any distinct s_{t-1}, s'_{t-1}, we have $\text{supp}(\hat{\mu}_t^{s_{t-1}}) \cap \text{supp}(\hat{\mu}_t^{s'_{t-1}}) = \varnothing$. Since additionally $\mu(s_t) > 0$ for all $s_t \in S_t$, we have (c) $\bigcup_{s_{t-1}\in S_{t-1}} \text{supp}\, \hat{\mu}_t^{s_{t-1}} = S_t$. Thus, DREU1 is satisfied.

To see that DREU2 holds, observe that for each $h^t = (A_0, p_0, \ldots, A_t, p_t) \in \mathcal{H}_t$, we have

$$\begin{aligned}
\mu(C(h^t) &= \textstyle\sum_{s_T\in S_T} \mu(s_T)\mu\left(C(h^t)|s_T\right) \\
&= \textstyle\sum_{s_T\in S_T} \mu(s_T)\mu\left(\bigcap_{k=0}^{t}\{\omega \in \Omega : p_k \in M(M(A_k, U_k), W_k)\}|s_T\right) \\
&= \sum_{\substack{(s_0,\ldots,s_T)\\ \exists\omega\in\Omega\forall t:\, s_t=\mathcal{F}_t(\omega)}} \mu(s_T)\mu\left(\bigcap_{k=0}^{t}\{p_k \in M(M(A_k, U_{s_k}), W_k)\}|s_t\right) \\
&= \sum_{\substack{(s_0,\ldots,s_T)\\ \exists\omega\in\Omega\forall t:\, s_t=\mathcal{F}_t(\omega)}} \mu(s_T)\prod_{k=0}^{t}\mu\left(\{p_k \in M(M(A_k, U_{s_k}), W_k)\}|s_k\right) \\
&= \sum_{\substack{(s_0,\ldots,s_t)\\ \exists\omega\in\Omega\forall k\leq t:\, s_k=\mathcal{F}_k(\omega)}} \prod_{k=0}^{t}\mu_k^{s_{k-1}}(s_k)\prod_{k=0}^{t}\tau_{s_k}(p_k, A_k) \\
&= \textstyle\sum_{(s_0,\ldots,s_t)\in S_0\times\ldots\times S_t} \prod_{k=0}^{t}\mu_k^{s_{k-1}}(s_k)\prod_{k=0}^{t}\tau_{s_k}(p_k, A_k),
\end{aligned}$$

where the third equality follows from the fact that (U_t) is $\mathcal{F}_t$-adapted, the fourth equality follows from parts (ii) and (iii) of the properness assumption on (W_t), the final equality follows from the fact that $\prod_{k=0}^{t}\mu_k^{s_{k-1}}(s_k) = 0$ whenever $(s_0, \ldots, s_t) \neq (\mathcal{F}_0(\omega), \ldots, \mathcal{F}_t(\omega))$ for all ω, and the remaining equalities hold by definition. Since $\rho_t(p_t; A_t|h^{t-1}) = \frac{\mu(C(h^t))}{\mu(C(h^{t-1})}$ by (3), this shows that DREU2 holds.

G.2 Evolving Utility

"If" direction: Suppose ρ admits an S-based evolving utility representation $(S_t, \{\mu_t^{s_{t-1}}\}_{s_{t-1}\in S_{t-1}}, \{U_{s_t}, u_{s_t}, \tau_{s_t}\}_{s_t\in S_t})_{t=0,\ldots,T}$. Let $(\hat{\Omega}, \hat{\mathcal{F}}^*, \hat{\mu}, (\hat{\mathcal{F}}_t, \hat{U}_t, \hat{W}_t))$ denote the corresponding DREU representation of ρ obtained in the "if" direction for DREU. In addition, define $\hat{u}_t : \hat{\Omega} \to \mathbb{R}^Z$ for each t by $\hat{u}_t(\hat{\omega}) := u_{s_t}$ whenever $\text{proj}_{S_t}(\hat{\omega}) = s_t$. Note that the process $(\hat{u}_t)$ is $\hat{\mathcal{F}}_t$-adapted. Moreover, for each $\hat{\omega} = (s_0, W_0, ..., s_T, W_T)$, we have $\hat{U}_T(\hat{\omega}) = U_{s_T} = u_{s_T} = \hat{u}_T(\hat{\omega})$ and for each $t \leq T-1$ and (z_t, A_{t+1})

$$\begin{aligned}\hat{U}_t(\hat{\omega})(z_t, A_{t+1}) &= U_{s_t}(z_t, A_{t+1}) \\ &= u_{s_t}(z_t) + \sum_{s_{t+1}\in S_{t+1}} \mu_{t+1}^{s_t}(s_{t+1}) \max_{p_{t+1}\in A_{t+1}} U_{s_{t+1}}(p_{t+1}) \\ &= \hat{u}_t(\hat{\omega})(z_t) + \sum_{s_{t+1}\in S_{t+1}} \hat{\mu}(s_{t+1}|s_t) \max_{p_{t+1}\in A_{t+1}} U_{s_{t+1}}(p_{t+1}) \\ &= \hat{u}_t(\hat{\omega})(z_t) + \mathbb{E}[\max_{p_{t+1}\in A_{t+1}} \hat{U}_{t+1}(p_{t+1})|\hat{\mathcal{F}}_t(\hat{\omega})],\end{aligned}$$

where we let $\hat{\mu}(s_{t+1}|s_t) := \hat{\mu}\left(C(s_0,\dots,s_{t+1}) \mid C(s_0,\dots,s_t)\right)$. Thus we constructed an evolving utility representation with $\delta = 1$.

"Only if" direction: Suppose ρ admits an evolving utility representation $(\Omega, \mathcal{F}^*, \mu, (\mathcal{F}_t, U_t, u_t, W_t), \delta)$. We obtain another evolving utility representation $(\Omega, \mathcal{F}^*, \mu, (\mathcal{F}_t, U'_t, u'_t, W_t), 1)$ with discount factor 1 by setting $U'_t := \delta^t U_t$ and $u'_t := \delta^t u_t$ for each t. Clearly this represents the same ρ. Based on this second representation, let $(S_t, \{\hat{\mu}_t^{s_{t-1}}\}_{s_{t-1}\in S_{t-1}}, \{\hat{U}_{s_t}, \tau_{s_t}\}_{s_t\in S_t})_{t=0,\dots,T}$ denote the corresponding S-based DREU representation obtained in the "only if" direction for DREU. In addition, for each s_t, define $\hat{u}_{s_t} \in \mathbb{R}^Z$ by $\hat{u}_{s_t} = u'_t(\omega)$ for any $\omega \in s_t$; this is well-defined as (u'_t) is $\mathcal{F}_t$-adapted. Reversing the argument in the previous part, we can verify that $\hat{u}_{s_T} = \hat{U}_{s_T}$ for each s_T and $\hat{U}_{s_t}(z_t, A_{t+1}) = \hat{u}_{s_t}(z_t) + \sum_{s_{t+1}} \hat{\mu}_{t+1}^{s_t}(s_{t+1}) \max_{p_{t+1}\in A_{t+1}} \hat{U}_{s_{t+1}}(p_{t+1})$ for each s_t with $t \leq T-1$.

G.3 Gradual learning

"If" direction: Suppose ρ admits an S-based gradual learning representation $(S_t, \{\mu_t^{s_{t-1}}\}_{s_{t-1}\in S_{t-1}}, \{U_{s_t}, u_{s_t}, \tau_{s_t}\}_{s_t\in S_t}, \delta)_{t=0,\dots,T}$. Let $(\hat{\Omega}, \hat{\mathcal{F}}^*, \hat{\mu}, (\hat{\mathcal{F}}_t, \hat{U}_t, \hat{u}_t, \hat{W}_t), 1)$ denote the corresponding evolving utility representation obtained in the "if" direction for evolving utility. In addition, define $\hat{\delta} := \delta$. Note that for each $\hat{\omega} = (s_0, W_0, .., s_T, W_T)$ and $t \leq T-1$, we have $\hat{u}_t(\hat{\omega}) = u_{s_t} = \frac{1}{\delta}\sum_{s_{t+1}} \mu_{t+1}^{s_t}(s_{t+1}) u_{s_{t+1}} = \frac{1}{\hat{\delta}}\mathbb{E}[\hat{u}_{t+1}|\hat{\mathcal{F}}_t(\hat{\omega})]$. Iterating expectations, this yields $\hat{u}_t(\hat{\omega}) = \hat{\delta}^{t-T}\mathbb{E}[\hat{u}_T|\hat{\mathcal{F}}_t(\hat{\omega})] = \hat{\delta}^{t-T}\mathbb{E}[\hat{U}_T|\hat{\mathcal{F}}_t(\hat{\omega})]$. Replace $\hat{U}_t$ with $\hat{U}'_t := \hat{\delta}^{T-t}\hat{U}_t$ for each t. By Proposition 1, $(\hat{\Omega}, \hat{\mathcal{F}}^*, \hat{\mu}, (\hat{\mathcal{F}}_t, \hat{U}'_t, \hat{W}_t))$ is still a DREU representation of ρ. Moreover, for each $t \leq T-1$, we have

$$\begin{aligned}\hat{U}'_t(\hat{\omega})(z_t, A_{t+1}) &= \hat{\delta}^{T-t}\hat{u}_t(\hat{\omega})(z_t) + \hat{\delta}^{T-t}\mathbb{E}[\max_{p_{t+1}\in A_{t+1}} \hat{U}_{t+1}(p_{t+1})|\hat{\mathcal{F}}_t(\hat{\omega})] \\ &= \mathbb{E}[\hat{U}'_T(z_t) \mid \hat{\mathcal{F}}_t(\hat{\omega})] + \hat{\delta}\mathbb{E}[\max_{p_{t+1}\in A_{t+1}} \hat{U}'_{t+1}(p_{t+1})|\hat{\mathcal{F}}_t(\hat{\omega})].\end{aligned}$$

Thus, $(\hat{\Omega}, \hat{\mathcal{F}}^*, \hat{\mu}, (\hat{\mathcal{F}}_t, \hat{U}'_t, \hat{W}_t), \hat{\delta})$ is a gradual learning representation of ρ.

"Only if" direction: Suppose that ρ admits a gradual learning representation $(\Omega, \mu, (\mathcal{F}_t, U_t, W_t), \delta)$. Let $U'_t := \delta^{t-T}U_t$ for all t. By Proposition 1, $(\Omega, \mu, (\mathcal{F}_t, U'_t, W_t))$ is still a DREU representation of ρ. Moreover, let $u'_t := \delta^{t-T}u_t$, where $u_t(\omega) := \mathbb{E}[U_T|\mathcal{F}_t(\omega)]$. By (2), for each ω, we have $U'_T(\omega) = U_T(\omega) = u_T(\omega) = u'_t(\omega)$ and for all $t \leq T-1$

$$\begin{aligned} U'_t(\omega)(z_t, A_{t+1}) &= u'_t(\omega)(z_t) + \delta^{t-T}\delta\mathbb{E}[\max_{p_{t+1}\in A_{t+1}} U_{t+1}(p_{t+1}) \mid \mathcal{F}_t(\omega)] \\ &= u'_t(\omega)(z_t) + \mathbb{E}[\max_{p_{t+1}\in A_{t+1}} U'_{t+1}(p_{t+1}) \mid \mathcal{F}_t(\omega)]. \end{aligned}$$

Thus, $(\Omega, \mu, (\mathcal{F}_t, U'_t, u'_t, W_t), 1)$ is an evolving utility representation of ρ. Let $(S_t, \{\hat{\mu}_t^{s_{t-1}}\}_{s_{t-1}\in S_{t-1}}, \{\hat{U}'_{s_t}, \hat{u}'_{s_t}, \tau_{s_t}\}_{s_t\in S_t})_{t=0,\dots,T}$ denote the corresponding S-based evolving utility representation of ρ obtained in the "only if" direction for evolving utility. In addition, define $\hat{\delta} := \delta$. Then for each $t \leq T-1$ and ω with $\mathcal{F}_t(\omega) = s_t$, we have

$$\hat{u}'_{s_t} = u'_t(\omega) = \hat{\delta}^{t-T}\mathbb{E}[U_T|\mathcal{F}_t(\omega)] = \frac{1}{\hat{\delta}}\mathbb{E}[u'_{t+1}|\mathcal{F}_t(\omega)] = \frac{1}{\hat{\delta}}\sum_{s_{t+1}} \hat{\mu}^{s_t}_{t+1}(s_{t+1})\hat{u}'_{s_t}.$$

Thus $(S_t, \{\hat{\mu}_t^{s_{t-1}}\}_{s_{t-1}\in S_{t-1}}, \{\hat{U}'_{s_t}, \hat{u}'_{s_t}, \tau_{s_t}\}_{s_t\in S_t}, \hat{\delta})_{t=0,\dots,T}$ is an S-based gradual learning representation of ρ. ■

H Proof of Proposition 1

H.1 "If" directions:

DREU: Consider any $h^t = (p_0, A_0, \dots, p_t, A_t) \in \mathcal{H}_t$. Then

$$\begin{aligned} \mu(C(h^t)) &= \sum_{\mathcal{F}_T(\omega)\in\Pi_T} \mu(\mathcal{F}_T(\omega))\mu\left(\bigcap_{k=0}^{t}\{p_k \in M(M(A_k, U_k), W_k)\}|\mathcal{F}_T(\omega)\right) \\ &= \sum_{\mathcal{F}_t(\omega)\in\Pi_t}\prod_{k=0}^{t} \mu(\mathcal{F}_k(\omega) \mid \mathcal{F}_{k-1}(\omega))\mu\left(\{W_k \in N(M(A_k, U_k(\omega)), p_k)\}|\mathcal{F}_k(\omega)\right) \\ &= \sum_{\mathcal{F}_t(\omega)\in\Pi_t}\prod_{k=0}^{t} \hat{\mu}(\phi_k(\mathcal{F}_k(\omega)) \mid \phi_{k-1}(\mathcal{F}_{k-1}(\omega)))\hat{\mu}\left(\{\hat{W}_k \in N(M(A_k, U_k(\omega)), p_k)\}|\phi_k(\mathcal{F}_k(\omega))\right) \\ &= \sum_{\hat{\mathcal{F}}_t(\hat{\omega})\in\hat{\Pi}_t}\prod_{k=0}^{t} \hat{\mu}(\hat{\mathcal{F}}_k(\hat{\omega}) \mid \hat{\mathcal{F}}_{k-1}(\omega))\hat{\mu}\left(\{\hat{W}_k \in N(M(A_k, \hat{U}_k(\hat{\omega})), p_k)\}|\hat{\mathcal{F}}_k(\hat{\omega})\right) \\ &= \sum_{\hat{\mathcal{F}}_T(\hat{\omega})\in\hat{\Pi}_T} \hat{\mu}(\hat{\mathcal{F}}_T(\hat{\omega}))\left(\bigcap_{k=0}^{t}\{p_k \in M(M(A_k, \hat{U}_k), \hat{W}_k)\} \mid \hat{\mathcal{F}}_T(\hat{\omega})\right) = \hat{\mu}(\hat{C}(h^t)), \end{aligned}$$

where the second equality follows from properness of (W_t) and $\mathcal{F}_t$-adaptedness of (U_t), the third equality follows from assumptions (i) and (iii), the fourth equality from the fact that ϕ_t is a bijection and assumption (ii), the fifth equality from the properness of $(\hat{W}_t)$ and $\hat{\mathcal{F}}_t$-adaptedness of $(\hat{U}_t)$, and the first and last equalities hold by definiton. Since $\mathcal{D}$ represents ρ and $\hat{\mathcal{D}}$ represents $\hat{\rho}$, this implies $\rho_t(p_t, A_t|h^{t-1}) = \frac{\mu(C(h^t))}{\mu(C(h^{t-1}))} = \frac{\hat{\mu}(\hat{C}(h^t))}{\hat{\mu}(\hat{C}(h^{t-1}))} = \hat{\rho}_t(p_t, A_t|h^{t-1})$. Thus, $\hat{\rho} = \rho$, as required.

Evolving utility: By the "if" direction for DREU, $\hat{\mathcal{D}}$ is a DREU representation of ρ. It remains to show that $(\hat{\mathcal{D}}, (\hat{u}_t), \hat{\delta})$ satisfies (1). From assumptions (ii), (iv), and (v) it is immediate that $\hat{U}_T = \hat{u}_T$. Moreover, for all $t \leq T-1$, and $\omega \in \Omega$, $\hat{\omega} \in \phi_t(\mathcal{F}_t(\omega))$, we have

$$
\begin{aligned}
\alpha_t(\omega)\hat{U}_t(\hat{\omega})(z, A_{t+1}) &= U_t(\omega)(z, A_{t+1}) - \beta_t(\omega) \\
&= u_t(\omega)(z) - \beta_t(\omega) + \delta \mathbb{E}_\mu[\max_{p_{t+1} \in A_{t+1}} U_{t+1}(p_{t+1}) \mid \mathcal{F}_t(\omega)] \\
= \alpha_t(\omega)\hat{u}_t(\hat{\omega})(z) - \delta\mathbb{E}_\mu[\beta_{t+1}|\mathcal{F}_t(\omega)] &+ \delta\mathbb{E}_{\hat{\mu}}[\alpha_{t+1} \max_{p_{t+1} \in A_{t+1}} \hat{U}_{t+1}(p_{t+1}) \mid \hat{\mathcal{F}}_t(\hat{\omega})] + \delta\mathbb{E}_\mu[\beta_{t+1}|\mathcal{F}_t(\omega)] \\
&= \alpha_t(\omega)\left(\hat{u}_t(\hat{\omega})(z) + \hat{\delta}\mathbb{E}_{\hat{\mu}}[\max_{p_{t+1} \in A_{t+1}} \hat{U}_{t+1}(p_{t+1}) \mid \hat{\mathcal{F}}_t(\hat{\omega})]\right)
\end{aligned}
$$

where the first equality follows from (ii), the second from (1) for $(\mathcal{D}, (u_t), \delta)$, the third from (i), (ii), and (v) (and the fact ϕ_t is a bijection), and the fourth by (iv). Thus, $(\hat{\mathcal{D}}, (\hat{u}_t), \hat{\delta})$ satisfies (1).

Gradual learning: By the "if" direction for evolving utility, $(\hat{\mathcal{D}}, (\hat{u}_t), \hat{\delta})$ is an evolving utility representation of ρ. Since $\delta = \hat{\delta}$ by (vi), it remains to show that $(\hat{\mathcal{D}}, (\hat{u}_t), \delta)$ satisfies (2). For all $t \leq T-1$ and $\omega \in \Omega$, $\hat{\omega} \in \phi_t(\mathcal{F}_t(\omega))$, we have

$$
\begin{aligned}
\alpha_0(\omega)\hat{u}_t(\hat{\omega}) + \gamma_t(\omega) &= u_t(\omega) \\
&= \mathbb{E}_\mu[U_T|\mathcal{F}_t(\omega)] \\
&= \alpha_0(\omega)\mathbb{E}_{\hat{\mu}}[\hat{U}_T|\hat{\mathcal{F}}_t(\hat{\omega})] + \mathbb{E}_\mu[\beta_T|\mathcal{F}_t(\omega)],
\end{aligned}
$$

where the first equality follows from (iv), (v), and (vi), the second from (2) for $(\mathcal{D}, (u_t), \delta)$, and the third from (i), (ii), (iv), (vi) (and the fact that ϕ_t is a bijection). But since $\gamma_t(\omega) = \beta_t(\omega) - \delta\mathbb{E}_\mu[\beta_{t+1}|\mathcal{F}_t(\omega)]$ by (v) and $\beta_k(\omega) = \frac{1-\delta^{T-k+1}}{1-\delta}\mathbb{E}_\mu[\beta_T \mid \mathcal{F}_k(\omega)]$ for all k by (vi), we have that $\gamma_t(\omega) = \mathbb{E}_\mu[\beta_T|\mathcal{F}_t(\omega)]$. Hence, the above implies that $\hat{u}_t(\hat{\omega}) = \mathbb{E}_{\hat{\mu}}[\hat{U}_T|\hat{\mathcal{F}}_t(\hat{\omega})]$, whence $(\hat{\mathcal{D}}, (\hat{u}_t), \delta)$ satisfies (2) with $\hat{\hat{u}} := \hat{U}_T$.

H.2 "Only if" directions:

DREU: Throughout the proof, for any t and $E_t = \mathcal{F}_t(\omega) \in \Pi_t$, we let $U_t(E_t)$ denote $U_t(\omega)$ and likewise for $\hat{U}$; this is well-defined by adaptedness. We construct the sequence $(\phi_t, \alpha_t, \beta_t)$ inductively, dealing with the base case $t = 0$ and the inductive step simultaneously.

Suppose $t \geq 0$ and that we have constructed $(\phi_{t'}, \alpha_{t'}, \beta_{t'})$ satisfying (i)–(iii) for all $t' < t$ (disregard the latter assumption if $t = 0$). If $t > 0$, fix any $E_{t-1} = \mathcal{F}_{t-1}(\omega^*) \in \Pi_{t-1}$, let $\hat{E}_{t-1} := \phi_{t-1}(E_{t-1})$, and let $\Pi_t(E_{t-1}) := \{E_t = \mathcal{F}_t(\omega) \in \Pi_t : \mathcal{F}_{t-1}(\omega) = E_{t-1}\}$ and $\hat{\Pi}_t(\hat{E}_{t-1}) := \{\hat{E}_t = \hat{\mathcal{F}}_t(\hat{\omega}) \in \hat{\Pi}_t : \hat{\mathcal{F}}_{t-1}(\hat{\omega}) = \hat{E}_{t-1}\}$. As in the proof of Lemma 2, we can repeatedly apply Lemma 13 to find a separating history for $E_{t-1} = \mathcal{F}_{t-1}(\omega^*)$, i.e., a history $h^{t-1} = (B_0, q_0, \ldots, B_{t-1}, q_{t-1}) \in \mathcal{H}^*_{t-1}$ such that $\{\omega \in \Omega : q_k \in M(B_k, U_k(\omega))\} = \mathcal{F}_k(\omega^*)$ for all $k = 0, \ldots, t-1$. By inductive hypothesis h^{t-1} is then also a separating history for $\hat{E}_{t-1}$. Thus, by Lemma 14 (and the translation to S-based DREU in Proposition 5), $C(h^{t-1}) = E_{t-1}$ and $\hat{C}(h^{t-1}) = \hat{E}_{t-1}$. If $t = 0$, then in the following we let $E_{t-1} := \Omega$, $\hat{E}_{t-1} := \hat{\Omega}$, $\Pi_t(E_{t-1}) := \Pi_0$, $\hat{\Pi}_t(E_{t-1}) := \hat{\Pi}_0$, and we disregard all references to the separating history.

Enumerate $\Pi_t(E_{t-1}) = \{E^i_t : i = 1, \ldots, m\}$ with corresponding utilities $U^i_t := U_t(E^i_t)$ and

$\hat{\Pi}_t(\hat{E}_{t-1}) = \{\hat{E}^j_t : j = 1, \ldots, \hat{m}\}$ with corresponding utilities $\hat{U}^j_0 := \hat{U}_t(\hat{E}^j_t)$. Since $(\mathcal{F}_t, U_t)$ and $(\hat{\mathcal{F}}_t, \hat{U}_t)$ are both simple, we have $\mu(E^i_t) > 0$ for all i and $U^i_t \not\approx U^{i'}_t$ for $i \neq i'$, and likewise $\hat{\mu}(\hat{E}^j_t) > 0$ for all j and $\hat{U}^j_t \not\approx \hat{U}^{j'}_t$ for $j \neq j'$. Note that for every j there exists a unique $i(j)$ such $U^{i(j)}_t \approx \hat{U}^j_t$. Indeed, if such an $i(j)$ exists it is unique because all the U^i_t represent different preferences. And the desired $i(j)$ exists, since otherwise by Lemma 13, we can find a menu $B_t = \{q^i_t : i = 1, \ldots, m\} \cup \{\hat{q}^j_t\}$ such that $M(B_t, U^i_t) = \{q^i_t\}$ for each i and $M(B_t, \hat{U}^j_t) = \{\hat{q}^j_t\}$. We can additionally assume (by replacing h^{t-1} with an appropriate mixture if need be) that $h^{t-1} \in \mathcal{H}^*_{t-1}(B_t)$. Since $\mathcal{D}$ and $\hat{\mathcal{D}}$ both represent ρ, we obtain

$$0 = \mu[C(\hat{q}^j_t, B_t)|E_{t-1}] = \rho_t(\hat{q}^j_t; B_t|h^{t-1}) = \hat{\mu}[\hat{C}(\hat{q}^j_t, B_t)|\hat{E}_{t-1}) \geq \hat{\mu}(\hat{E}^j_t|\hat{E}_{t-1}) > 0,$$

a contradiction. Similarly, for every i, there exists a unique $j(i)$ such that $\hat{U}^{j(i)}_t \approx U^i_t$. Thus, defining $\phi_t : \Pi_t(E_{t-1}) \to \hat{\Pi}_t(\hat{E}_{t-1})$ by $\phi_t(E^i_t) = \hat{E}^{j(i)}_t$ yields a bijection. By construction, $U_t(E^i_t) \approx \hat{U}_t(\phi_t(E^i_t))$ for all i, so we can find $\alpha_t(E^i_t) \in \mathbb{R}_{++}$ and $\beta(E^i_t) \in \mathbb{R}$ such that $U_t(E^i_t) = \alpha_t(E^i_t)\hat{U}_t(\phi_t(E^i_t)) + \beta(E^i_t)$. Defining $\alpha(\omega) = \alpha(\mathcal{F}_t(\omega))$ and $\beta(\omega) = \beta(\mathcal{F}_t(\omega))$ this yields $\mathcal{F}_t$-measurable maps $\alpha_t, \beta_t : E_{t-1} \to \mathbb{R}$ such that (ii) holds for all $\omega \in E_{t-1}$. Moreover, applying Lemma 13 again, we can find a menu $D_t = \{r^i_t : i = 1, \ldots, n\}$ such that $M(D_t, U^i_t) = \{r^i_t\}$ for each i. Again, slightly perturbing the separating history h^{t-1} for E_{t-1} if need be, we can assume that $h^{t-1} \in \mathcal{H}^*_{t-1}(D_t)$. Then by the representation, $\mu(E^i_t|E_{t-1}) = \rho_t(r^i_t; D^i_t|h^{t-1}) = \hat{\mu}(\phi_t(E^i_t)|\hat{E}_{t-1})$ for all i, yielding (i).

To show (iii), consider any $p_t \in A_t$, where we can again assume $h^{t-1} \in \mathcal{H}^*_{t-1}(\frac{1}{2}A_t + \frac{1}{2}D_t)$. Let $B^i_t := \{w \in \mathbb{R}^{X_t} : p_t \in M(M(A_t, U_t(E^i_t)), w)\}$. Note that by (ii), $B^i_t = \{w \in \mathbb{R}^{X_t} : p_t \in M(M(A_t, \hat{U}_t(\phi_t(E^i_t))), w)\}$. Thus, $\mu(\{W_t \in B_t\}|E^i_t) = \mu(C(p_t, A_t)|E^i_t)$ and $\hat{\mu}(\{\hat{W}_t \in B_t\}|\phi_t(E^i_t)) = \hat{\mu}(\hat{C}(p_t, A_t)|\phi_t(E^i_t))$. But since $\mathcal{D}$ and $\hat{\mathcal{D}}$ both represent ρ and by choice of D_t,

$$\begin{aligned}\mu(E^i_t|E_{t-1})\mu[C(p_t, A_t)|E^i_t] = \mu[C(\frac{1}{2}p_t + \frac{1}{2}r^i_t, \frac{1}{2}A_t + \frac{1}{2}D_t)|E_{t-1}] =& \\ \rho_t(\frac{1}{2}p_t + \frac{1}{2}r^i_t; \frac{1}{2}A_t + \frac{1}{2}D_t|h^{t-1}) =& \\ \hat{\mu}[\hat{C}(\frac{1}{2}p_t + \frac{1}{2}r^i_t, \frac{1}{2}A_t + \frac{1}{2}D_t)|\hat{E}_{t-1}] = \hat{\mu}(\phi_t(E^i_t)|\hat{E}_{t-1})\hat{\mu}[\hat{C}(p_t, A_t)|\phi_t(E^i_t)]&,\end{aligned}$$

which implies $\mu[C(p_t, A_t)|E^i_t] = \hat{\mu}[\hat{C}(p_t, A_t)|\phi_t(E^i_t)]$, since by (i) we have $\mu(E^i_t|E_{t-1}) = \hat{\mu}(\phi_t(E^i_t)|\hat{E}_{t-1})$. Thus, $\mu(\{W_t \in B_t\}|E^i_t) = \hat{\mu}(\{\hat{W}_t \in B_t\}|\phi_t(E^i_t))$, as required.

Finally, note that the collection $\{\Pi_t(E_{t-1}) : E_{t-1} \in \Pi_{t-1}\}$ partitions Π_t, and similarly $\{\hat{\Pi}_t(\hat{E}_{t-1}) : \hat{E}_{t-1} \in \hat{\Pi}_{t-1}\}$ partitions $\hat{\Pi}_t$. Thus, applying the above construction for every $E_{t-1} \in \Pi_{t-1}$ yields a bijection $\phi_t : \Pi_t \to \hat{\Pi}_t$ and $\mathcal{F}_t$-measurable maps $\alpha_t : \Omega \to \mathbb{R}_{++}$ and $\beta_t : \Omega \to \mathbb{R}$ such that (i)–(iii) are satisfied.

Evolving utility: The "only if" part for DREU yields sequences $(\phi_t, \alpha_t, \beta_t)$ such that (i)–(iii) are satisfied. It remains to show that (iv) and (v) hold. Throughout the proof, for any $E_t = \mathcal{F}_t(\omega) \in \Pi_t$, we sometimes use $U_t(E_t)$, $\alpha_t(E_t)$, $\beta_t(E_t)$ to denote $U_t(\omega)$, $\alpha_t(\omega)$, $\beta_t(\omega)$; this is well-defined since U_t, α_t, β_t are $\mathcal{F}_t$-measurable. We also let $\mathcal{F}_{t-1}(E_t) := \mathcal{F}_{t-1}(\omega)$; this is well-defined since $\mathcal{F}_t(\omega) = \mathcal{F}_t(\omega')$ implies $\mathcal{F}_{t-1}(\omega) = \mathcal{F}_{t-1}(\omega')$, as $(\mathcal{F}_t)$ is a filtration.

For (iv), fix any ω and $t \leq T-1$. Let $E_t := \mathcal{F}_t(\omega)$ and pick any A_{t+1}, B_{t+1} and z_t. Then

$$\begin{aligned} U_t(E_t)(z_t, A_{t+1}) - U_t(E_t)(z_t, B_{t+1}) &= \alpha_t(E_t)(\hat{U}_t(\phi_t(E_t))(z_t, A_{t+1}) - \hat{U}_t(\phi_t(E_t))(z_t, B_{t+1})) \\ &= \alpha_t(E_t)\hat{\delta} \sum_{\hat{E}_{t+1} \in \hat{\Pi}_{t+1}} \hat{\mu}(\hat{E}_{t+1}|\phi_t(E_t))[\max_{A_{t+1}} \hat{U}_{t+1}(\hat{E}_{t+1}) - \max_{B_{t+1}} \hat{U}_{t+1}(\hat{E}_{t+1})] \\ &= \alpha_t(E_t)\hat{\delta} \sum_{E_{t+1} \in \Pi_{t+1}} \hat{\mu}(\phi_{t+1}(E_{t+1})|\phi_t(E_t))[\max_{A_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1})) - \max_{B_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1}))] \\ &= \alpha_t(E_t)\hat{\delta} \sum_{E_{t+1} \in \Pi_{t+1}} \mu(E_{t+1}|E_t)[\max_{A_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1})) - \max_{B_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1}))] \\ &= \alpha_t(E_t)\hat{\delta} \sum_{E_{t+1} \text{ s.t. } \mathcal{F}_t(E_{t+1})=E_t} \mu(E_{t+1}|E_t)[\max_{A_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1})) - \max_{B_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1}))], \end{aligned} \tag{32}$$

where the first equality holds by (ii), the second equality follows from $(\hat{\mathcal{D}}, \hat{\delta})$ being an evolving utility representation, the third equality from the fact that ϕ_t is a bijection, the fourth equality from (i), and the fifth equality from the fact that $\mu(\mathcal{F}_{t+1}(\omega')|E_t) > 0$ iff $\mathcal{F}_t(\omega') = E_t$.

At the same time, we have

$$\begin{aligned} & U_t(E_t)(z_t, A_{t+1}) - U_t(E_t)(z_t, B_{t+1}) \\ &= \delta \sum_{E_{t+1} \in \Pi_{t+1}} \mu(E_{t+1}|E_t)[\max_{A_{t+1}} U_{t+1}(E_{t+1}) - \max_{B_{t+1}} U_{t+1}(E_{t+1}) \\ &= \delta \sum_{E_{t+1} \in \Pi_{t+1}} \mu(E_{t+1}|E_t)\alpha_{t+1}(E_{t+1})[\max_{A_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1})) - \max_{B_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1}))] \\ &= \delta \sum_{E_{t+1} \text{ s.t. } \mathcal{F}_t(E_{t+1})=E_t} \mu(E_{t+1}|E_t)\alpha_{t+1}(E_{t+1})[\max_{A_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1})) - \max_{B_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1}))], \end{aligned} \tag{33}$$

where the first equality follows from $(\mathcal{D}, \delta)$ being an evolving utility representation, the second equality from (ii), and the third equality from the fact that $\mu(\mathcal{F}_{t+1}(\omega')|E_t) > 0$ iff $\mathcal{F}_t(\omega') = E_t$.

Combining (32) and (33), we have that for all A_{t+1} and B_{t+1},

$$\begin{aligned} & \hat{\delta} \sum_{E_{t+1} \text{ s.t. } \mathcal{F}_t(E_{t+1})=E_t} \mu(E_{t+1}|E_t)\alpha_t(E_t)[\max_{A_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1})) - \max_{B_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1}))] \\ &= \delta \sum_{E_{t+1} \text{ s.t. } \mathcal{F}_t(E_{t+1})=E_t} \mu(E_{t+1}|E_t)\alpha_{t+1}(E_{t+1})[\max_{A_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1})) - \max_{B_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1}))]. \end{aligned} \tag{34}$$

Since $(\hat{\mathcal{F}}_t, \hat{U}_t)$ is simple and ϕ_t is a bijection, $\hat{U}_{t+1}(\phi_{t+1}(E_{t+1})) \not\approx \hat{U}_{t+1}(\phi_{t+1}(E'_{t+1}))$ for all distinct E_{t+1}, E'_{t+1} with $\mathcal{F}_t(E_{t+1}) = E_t = \mathcal{F}_t(E'_{t+1})$. So by Lemma 13, we can find a menu $A_{t+1} := \{q_{t+1}^{E_{t+1}} : \mathcal{F}_t(E_{t+1}) = E_t\}$ such that for all E_{t+1} with $\mathcal{F}_t(E_{t+1}) = E_t$ we have $M(A_{t+1}, \hat{U}_{t+1}(\phi_{t+1}(E_{t+1})) = \{q_{t+1}^{E_{t+1}}\}$. Let $E^*_{t+1} := \mathcal{F}_{t+1}(\omega)$ and let $B_{t+1} = A_{t+1} \setminus \{q_{t+1}^{E^*_{t+1}}\}$. Then in (34), $[\max_{A_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1})) - \max_{B_{t+1}} \hat{U}_{t+1}(\phi_{t+1}(E_{t+1}))] \neq 0$ iff $E_{t+1} = E^*_{t+1}$. Hence, (34) implies $\frac{\hat{\delta}}{\delta}\alpha_t(\omega) = \frac{\hat{\delta}}{\delta}\alpha_t(E_t) = \alpha_{t+1}(E^*_{t+1}) = \alpha_{t+1}(\omega)$. Since this is true for all $t \leq T-1$, (iv) follows.

For (v), note that the claim for T is immediate from (ii) and the fact that $U_T = u_T$, $\hat{U}_T = \hat{u}_T$. Next, fix any $\omega \in \Omega$, $\hat{\omega} \in \phi_t(\mathcal{F}_t(\omega))$, $t \leq T-1$, and $(z, \{p_{t+1}\})$. Then

$$\begin{aligned} U_t(\omega)(z, \{p_{t+1}\}) &= u_t(\omega)(z) + \delta \mathbb{E}_\mu[U_{t+1}(p_{t+1})|\mathcal{F}_t(\omega)] \\ &= u_t(\omega)(z) + \alpha_t(\omega)\hat{\delta}\mathbb{E}_{\hat{\mu}}[\hat{U}_{t+1}(p_{t+1})|\hat{\mathcal{F}}_t(\hat{\omega})] + \delta\mathbb{E}_\mu[\beta_{t+1}|\mathcal{F}_t(\omega)], \end{aligned} \tag{35}$$

where the first equality follows from $(\mathcal{D}, (u_t), \delta)$ being an evolving utility representation and the second equality from (i), (ii), (iv) (and the fact that ϕ_t is a bijection). At the same time, we have

$$\begin{aligned} U_t(\omega)(z, \{p_{t+1}\}) &= \alpha_t(\omega)\hat{U}_t(\hat{\omega})(z, \{p_{t+1}\}) + \beta_t(\omega) \\ &= \alpha_t(\omega)\hat{u}_t(\omega)(z) + \alpha_t(\omega)\hat{\delta}\mathbb{E}_{\hat{\mu}}[\hat{U}_{t+1}(p_{t+1})|\hat{\mathcal{F}}_t(\hat{\omega})] + \beta_t(\omega), \end{aligned} \tag{36}$$

where the first equality follows from (ii) and (iv) and the second equality from $(\hat{\mathcal{D}}, (\hat{u}_t), \hat{\delta})$ being an evolving utility representation. Combining (35) and (36) yields the desired claim.

Gradual learning: The "only if" part for evolving utility yields sequences $(\phi_t, \alpha_t, \beta_t)$ such that (i)–(v) are satisfied. It remains to show that (vi) and (vii) hold.

For (vi), note that since $(\mathcal{D}, (u_t), \delta)$ is a gradual learning representation of ρ, combining the proofs of Proposition 5 (equivalence of gradual learning and S-based gradual learning representations) and of the necessity direction of Theorem 3 yields that for all $t \leq T-1$, h^t, and $q_t, r_t \in \Delta(X_t)$, we have $q_t \succsim_{h^t} r_t$ if and only if $U_t(\omega)(q_t) \geq U_t(\omega)(r_t)$ for all $\omega \in C(h^t)$. Fix any $\omega \in \Omega$ and $h^0 = (A_0, p_0) \in \mathcal{H}^*_0$ such that $\{\omega' : p_0 \in M(A_0, U_0(\omega'))\} = \mathcal{F}_0(\omega)$ (which exists by Lemma 13). Then by Lemma 14, $C(h^0) = \mathcal{F}_0(\omega)$. So by Condition 1 applied to h^0, we can find $\ell, m, n \in \Delta Z$ such that $U_0(\omega)(\ell, n, \ldots, n) \neq U_0(\omega)(m, n, \ldots, n)$. Note that by (2) and iterated expectations, we have $U_0(\ell^0, \ldots, \ell^T) = \sum_{k=0}^T \delta^k u_0(\omega)(\ell^k)$ for any stream of consumption lotteries, $(\ell^0, \ldots, \ell^T)$, where $u_0(\omega) := \mathbb{E}[U_T|\mathcal{F}_0(\omega)]$. Hence, we have $u_0(\omega)(\ell) \neq u_0(\omega)(m)$, and $U_0(\omega)(\ell, m, n, \ldots, n) - U_0(\omega)(\eta\ell + (1-\eta)m, \eta\ell + (1-\eta)m, n, \ldots, n) = 0$ if and only if $\eta = \frac{1}{1+\delta}$.

Now pick any $\hat{\omega} \in \phi_0(\mathcal{F}_0(\omega))$. Then since $(\hat{\mathcal{D}}, (\hat{u}_t), \hat{\delta})$ is also a gradual learning representation of ρ, by the same reasoning as above we have that $\hat{U}_0(\hat{\omega})(\ell, m, n, \ldots, n) - \hat{U}_0(\hat{\omega})(\eta\ell + (1-\eta)m, \eta\ell + (1-\eta)m, n, \ldots, n) = 0$ if and only if $\eta = \frac{1}{1+\hat{\delta}}$. By (ii), this implies that $\delta = \hat{\delta}$, proving (vi).

Given (iv) and (vi) and the fact that $u_t(\omega) = \mathbb{E}[u_{t+1}|\mathcal{F}_t(\omega)]$ and $\hat{u}_t(\hat{\omega}) = \mathbb{E}[\hat{u}_{t+1}|\hat{\mathcal{F}}_t(\hat{\omega})]$ by (2), applying (v) inductively then yields that $\beta_t(\omega) = \frac{1-\delta^{T-t+1}}{1-\delta}\mathbb{E}[\beta_T|\mathcal{F}_t(\omega)]$, proving (vii). ■

I Proofs for Section 5.2

I.1 Proof of Proposition 2

(ii)⟹(i): Consider any $(h^{t-1}, A_t, p_t), (h^{t-1}, A_t, q_t) \in \mathcal{H}_t$ and $p_{t+1} \in A_{t+1} \in \mathcal{A}^*_{t+1}(h^{t-1}, A_t, p_t) \cap \mathcal{A}^*_{t+1}(h^{t-1}, A_t, q_t)$ such that A_t and A_{t+1} are atemporal, $A^Z_{t+1} \subseteq A^Z_t$ and $p^Z_{t+1} = p^Z_t =: p$.

Let $\mathcal{U}_t(p) := \{u_t(\omega) : \omega \in C(h^{t-1}, A_t, p_t)\}$ and $\mathcal{U}_t(q) := \{u_t(\omega) : \omega \in C(h^{t-1}, A_t, q_t)\}$. Note that since A_{t+1} features no ties, Lemma 14 implies $C(p_{t+1}, A_{t+1}) = \{\omega : p_{t+1} \in M(A_{t+1}, U_{t+1}(\omega))\}$, which since A_{t+1} is atemporal is in turn equal to $\{\omega : p \in$

$M(A^Z_{t+1}, u_{t+1}(\omega))\}$. Hence, by the representation,

$$\begin{aligned}\rho_{t+1}(p_{t+1}; A_{t+1}|h^{t-1}, A_t, p_t) &= \mu(\{p \in M(A^Z_{t+1}, u_{t+1})\}|C(h^{t-1}, A_t, p_t)) \\ &\geq \min_{u \in \mathcal{U}_t(p)} \mu(\{p \in M(A^Z_{t+1}, u_{t+1})\}|C(h^{t-1}) \cap \{u_t \approx u\}).\end{aligned} \tag{37}$$

Likewise,

$$\begin{aligned}\rho_{t+1}(p_{t+1}; A_{t+1}|h^{t-1}, A_t, q_t) &= \mu(\{p \in M(A^Z_{t+1}, u_{t+1})\}|C(h^{t-1}, A_t, q_t)) \\ &\leq \max_{u' \in \mathcal{U}_t(q)} \mu(\{p \in M(A^Z_{t+1}, u_{t+1})\}|C(h^{t-1}) \cap \{u_t \approx u'\}).\end{aligned} \tag{38}$$

Pick $u \in \mathcal{U}_t(p)$ (respectively, $u' \in \mathcal{U}_t(q)$) which achieve the min (respectively max) in (37) (respectively, in (38)). Let $\{u^1_{t+1}, ..., u^m_{t+1}\} := \{u_t(\omega) : \omega \in C(h^{t-1}, A_t, p_t) \cup C(h^{t-1}, A_t, q_t)$ and $p \in M(A^Z_{t+1}, u_{t+1}(\omega))\}$ and let $D := \text{co}\{u, u^1_{t+1}, ..., u^m_{t+1}\}$. Note that since $A^Z_t \supseteq A^Z_{t+1}$ and since $p^Z_t = p$, we have $p \in M(A^Z_{t+1}, u)$. Hence, $C(h^{t-1}) \cap \{u_t \approx u\} \cap \{p \in M(A^Z_{t+1}, u_{t+1})\} = C(h^{t-1}) \cap \{u_t \approx u\} \cap [D]$, and likewise $C(h^{t-1}) \cap \{u_t \approx u'\} \cap \{p \in M(A^Z_{t+1}, u_{t+1})\} = C(h^{t-1}) \cap \{u_t \approx u'\} \cap [D]$. Thus,

$$\begin{aligned}&\mu(\{p \in M(A^Z_{t+1}, u_{t+1})\}|C(h^{t-1}) \cap \{u_t \approx u\}) = \mu([D]|C(h^{t-1}) \cap \{u_t \approx u\}) \geq \\ &\mu([D]|C(h^{t-1}) \cap \{u_t \approx u'\}) = \mu(\{p \in M(A^Z_{t+1}, u_{t+1})\}|C(h^{t-1}) \cap \{u_t \approx u'\}),\end{aligned} \tag{39}$$

where the inequality holds by (ii). Combining (37), (38), and (39) yields $\rho_{t+1}(p_{t+1}; A_{t+1}|h^{t-1}, A_t, p_t) \geq \rho_{t+1}(p_{t+1}; A_{t+1}|h^{t-1}, A_t, q_t)$, as required.

(i)$\Longrightarrow$(ii): We prove the contrapositive. Suppose that for some $u, u' \in \mathbb{R}^Z$ and h^{t-1} with $C(h^{t-1}) \cap \{u_t \approx u\} \neq \varnothing \neq C(h^{t-1}) \cap \{u_t \approx u'\}$, and convex $D \subseteq \mathbb{R}^Z$ with $u \in D$, we have

$$\mu(\{u_{t+1} \in [D]\}|C(h^{t-1}) \cap \{u_t \approx u\}) < \mu(\{u_{t+1} \in [D]\}|C(h^{t-1}) \cap \{u_t \approx u'\}). \tag{40}$$

Let $\mathcal{U}_{t+1}$ be the set of possible realizations of u_{t+1} conditional on the event $C(h^{t-1}) \cap \{\omega : u_t \approx u \text{ or } u_t \approx u'\}$. Let $\mathcal{U}_t$ be the set of possible realizations of u_t conditional on event $C(h^{t-1})$. Utility functions in these sets are all non-constant by Condition 2 (Uniformly Ranked Pair). Enumerate $\{u^1_{t+1}, ..., u^m_{t+1}\} := \mathcal{U}_{t+1} \cap [D]$ and $\{\hat{u}^1_{t+1}, ..., \hat{u}^n_{t+1}\} := \mathcal{U}_{t+1} \setminus [D]$. Fix any $p^Z \in \text{int}\Delta(Z)$. Note that for any $j = 1, ..., n$, $\hat{u}^j_{t+1}$ does not belong to $[\text{co}\{u, u^1_{t+1}, ..., u^m_{t+1}\}]$. Thus, by Lemma 24, for each $j = 1, ..., n$, we can find a vector $w^j \in \mathbb{R}^Z$ with $\sum_z w^j(z) = 0$ such that $\hat{u}^j_{t+1} \cdot w^j > 0 \geq u^i_{t+1} \cdot w^j, u \cdot w^j$ for any $i = 1, ..., m$. For each j, we construct $q^Z(j) \in \Delta(Z)$ such that the vector $q^Z(j) - p^Z$ (in $\mathbb{R}^Z$) is proportional to w^j.[58] Thus for each $j = 1, ..., n$ and $i = 1, .., m$, we have $\hat{u}^j_{t+1} \cdot (q^Z(j) - p^Z) > 0 \geq u^i_{t+1} \cdot (q^Z(j) - p^Z), u \cdot (q^Z(j) - p^Z)$.

Pick a uniformly ranked pair of consumption lotteries $\overline{\ell}, \underline{\ell} \in \Delta Z$ from Condition 2. Since $u \cdot (\overline{\ell} - \underline{\ell}) > 0$ and $u^i_{t+1} \cdot (\overline{\ell} - \underline{\ell}) > 0$ for each $i = 1, ..., n$, setting $\tilde{p}^Z := (1 - \epsilon)p^Z + \epsilon(\bar{p} - \underline{p})$ for some $\epsilon > 0$, we have $0 > u^i_{t+1} \cdot (q^Z(j) - \tilde{p}^Z), u \cdot (q^Z(j) - \tilde{p}^Z)$ for any $i = 1, ..., m$ and $j = 1, ..., n$. We can pick ϵ sufficiently small so that $\tilde{p}^Z$ is indeed well-defined (i.e., belongs to ΔZ) and that $\hat{u}^j_{t+1} \cdot (q^Z(j) - \tilde{p}^Z) > 0$ holds for each $j = 1, ..., n$. Finally, subject to

[58] Note that such a construction is possible because p^Z is in interior of $\Delta(Z)$.

perturbations of the lotteries $\{\tilde{p}^Z\} \cup \{q^Z(j) : j = 1, ..., n\}$, we can assume without loss[59] that $u_t(q^Z(j)) \neq u_t(q^Z(j'))$ for each $u_t \in \mathcal{U}_t$ and distinct $j, j' = 1, .., n$ while preserving the fact that $\hat{u}^j_{t+1} \cdot (q^Z(j) - \tilde{p}^Z) > 0 > u^i_{t+1} \cdot (q^Z(j) - \tilde{p}^Z), u \cdot (q^Z(j) - \tilde{p}^Z)$ for any $i = 1, ..., m$ and $j = 1, ..., n$.

Construct an atemporal menu A_{t+1} such that $A^Z_{t+1} = \{\tilde{p}^Z\} \cup \{q^Z(j) : j = 1, ..., n\}$ and such that for each $p \in A^Z_{t+1}$ there is a unique $p_{t+1} \in A_{t+1}$ such that $p^Z_{t+1} = p$. Note that since $\hat{u}^j_{t+1} \cdot (q^Z(j) - \tilde{p}^Z) > 0 > u^i_{t+1} \cdot (q^Z(j) - \tilde{p}^Z)$ for any $j = 1, ..., n$ and $i = 1, ..., m$, we have

$$\mu(\tilde{p}^Z \in M(u_{t+1}, A^Z_{t+1}) | C(h^{t-1}), u_t \approx u) = \mu(\{u_{t+1} \in [D]\} | C(h^{t-1}), u_t \approx u)$$

Let $\{[u^1_t], ..., [u^k_t]\}$ denote the collection of equivalence classes of utilities in $\mathcal{U}_t$, and assume without loss that $u \in [u^1_t]$. By Lemma 13, we construct a collection of consumption lotteries $\{r^Z(l) : l = 1, ..., k\}$ such that $u^l_t(r^Z(l)) > u^l_t(r^Z(l'))$ for any distinct $l, l' = 1, ..., k$.

Pick $\epsilon' > 0$ sufficiently small such that $\tilde{p}^Z + \epsilon'(r^Z(l) - r^Z(1)) \in \Delta(Z)$ for all $l = 2, \ldots, k$; such an ϵ' exists as $\tilde{p}^Z$ is in the interior of $\Delta(Z)$. Thus, we can construct an atemporal menu A_t such that

$$A^Z_t := \{\tilde{p}^Z\} \cup \{q^Z(j) : j = 1, ..., n\} \cup \{\tilde{p}^Z + \epsilon'(r^Z(l) - r^Z(1)) : l = 2, ..., k\}$$

and such that for each $p \in A^Z_t$ there is a unique $p_t \in A_t$ with $p^Z_t = p$.

Recall that by construction $u_t(q^Z(j)) \neq u_t(q^Z(j'))$ for each $u_t \in \mathcal{U}_t$ and distinct $j, j' = 1, .., n$. Moreover, for any $u_t \in \mathcal{U}_t$, $u_t(\tilde{p}^Z + \epsilon'(r^Z(l) - r^Z(1)))$ is non-constant in ϵ'. Therefore, for small enough $\epsilon' > 0$, we can guarantee that $|M(u_t, A^Z_t)| = 1$ for all $u_t \in \mathcal{U}_t$. Since A_t is atemporal and each lottery in A_t has a unique corresponding projection in A^Z_t, this ensures that $A_t \in \mathcal{A}^*(h^{t-1})$ (by Lemma 14). Since $u^l_t \cdot (r^Z(l) - r^Z(1)) > 0 > u \cdot (q^Z(j) - \tilde{p}^Z), u \cdot (r^Z(l) - r^Z(1))$ for each $j = 1, .., n$ and $l = 2, ..., k$, we have

$$\mu(\tilde{p}^Z \in M(u_t, A^Z_t) | C(h^{t-1})) = \mu(\{u_t \approx u\} | C(h^{t-1})).$$

Let p_t denote the unique lottery in A_t such that $p^Z_t = \tilde{p}^Z$ and let p_{t+1} denote the unique lottery in A_{t+1} such that $p^Z_{t+1} = \tilde{p}^Z$. Since $u \not\approx u'$, which implies $u'(\tilde{p}^Z + \epsilon'(r^Z(l) - r^Z(1)) > u'(\tilde{p}^Z)$ for the index l such that $u' \in [u^l_t]$, there is a lottery $q_t \in A_t$ different from p_t such that $M(A^Z_t, u') = \{q^Z_t\}$. Also, $|M(u_{t+1}, A^Z_{t+1})| = 1$ for all $u_{t+1} \in \mathcal{U}_{t+1}$, and thus $A_{t+1} \in \mathcal{A}^*_{t+1}(h^{t-1}, A_t, p_t) \cap \mathcal{A}^*_{t+1}(h^{t-1}, A_t, q_t)$.

In case $A_t \notin \mathcal{A}(h^{t-1})$, we can construct another history $\hat{h}^{t-1} = \lambda h^{t-1} + (1 - \lambda) d^{t-1}$ by mixing h^{t-1} with an appropriate degenerate history d^{t-1} (as in the construction of the extended ρ, Definition 3) such that $C(h^{t-1}) = C(\hat{h}^{t-1})$ and $A_t \in \mathcal{A}(\hat{h}^{t-1})$. By the previous paragraphs, we have $A^Z_t \supseteq A^Z_{t+1}$ and $\rho_{t+1}(p_{t+1}; A_{t+1} | \hat{h}^{t-1}, A_t, p_t) = \mu(\{u_{t+1} \in [D]\} | C(h^{t-1}), u_t \approx u)$ and $\rho_{t+1}(p_{t+1}; A_{t+1} | \hat{h}^{t-1}, A_t, q_t) = \mu(\{u_{t+1} \in [D]\} | C(h^{t-1}), u_t \approx u')$. But then (40) implies that $\rho_{t+1}(p_{t+1}; A_{t+1} | \hat{h}^{t-1}, A_t, p_t) < \rho_{t+1}(p_{t+1}; A_{t+1} | \hat{h}^{t-1}, A_t, q_t)$, which is a violation of consumption persistence. ■

Lemma 24. Take any finite Y and a finite set of non-constant utilities $\{u^1, .., u^m\} \subseteq \mathbb{R}^Y$. Then for any non-constant $u \in \mathbb{R}^Y$, the following are equivalent:

[59] Recall that each $u_t \in \mathcal{U}_t$ is non-constant.

(i). for any $w \in \mathbb{R}^Y$ such that $\sum_{y \in Y} w(y) = 0$,

$$[\forall i = 1, ..., m, u^i \cdot w \leq 0] \Rightarrow u \cdot w \leq 0$$

(ii). $u \in [\text{co}\{u^1, ..., u^m\}]$.

Proof. The result follows from the utilitarian aggregation theorem, e.g., Theorem 2 in Fishburn (1984). ■

I.2 Proof of Proposition 3

(ii)⟹(i): Consider any $(h^{t-1}, A_t, p_t) \in \mathcal{H}_t$ and $p_{t+1} \in A_{t+1} \in \mathcal{A}^*_{t+1}(h^{t-1}, A_t, p_t)$ such that A_t and A_{t+1} are atemporal with $A^Z_{t+1} \subseteq A^Z_t$ and $p^Z_t = p^Z_{t+1}$. Consider any felicity $u \in \mathbb{R}^Z$ such that $\mu(\{u_t \approx u\} \mid C(h^{t-1}, A_t, p_t)) > 0$. Then $\mu(\{u_t \approx u\} \mid C(h^{t-1})) > 0$, so by (ii) $\mu(\{u_{t+1} \approx u\} \mid C(h^{t-1}) \cap \{u_t \approx u\}) > 0$. Moreover, $u(p^Z_t) \geq u(q^Z_t)$ for all $q^Z_t \in A^Z_t$, so since $A^Z_{t+1} \subseteq A^Z_t$ is atemporal, $p^Z_{t+1} = p^Z_t$, and $A_{t+1} \in \mathcal{A}^*_{t+1}(h^{t-1}, A_t, p_t)$, Lemma 14 implies $u(p^Z_{t+1}) > u(q^Z_{t+1})$ for all $q_{t+1} \in A_{t+1} \setminus \{p_{t+1}\}$. Thus,

$$\rho_{t+1}(p_{t+1}; A_{t+1} | h^{t-1}, A_t, p_t) \geq \mu(\{u_{t+1} \approx u\} | C(h^{t-1}) \cap \{u_t \approx u\}) > 0.$$

(i)⟹(ii): We prove the contrapositive. Suppose that there is some history h^{t-1} and felicity $u \in \mathbb{R}^Z$ such that $\mu(u_t \approx u | C(h^{t-1})) > 0$ but $\mu(\{u_{t+1} \approx u\} | C(h^{t-1}) \cap \{u_t \approx u\}) = 0$. Let $([u^1], ..., [u^m])$ denote the equivalence classes of felicities that can realize in periods t or $t+1$, and suppose without loss that $u \in [u^1]$. Note that they are all non-constant by Condition 1 (Consumption Nondegeneracy). By applying Lemma 13 to consumption lotteries, we construct a collection of consumption lotteries $\{p^Z(i) : i = 1, ..., m\} \subseteq \Delta(Z)$ such that $u^i(p^Z(i)) > u^i(p^Z(j))$ for any distinct pair $i, j = 1, ..., m$.

Construct atemporal menus A_t and A_{t+1} such that $A^Z_t = A^Z_{t+1} = \{p^Z(i) : i = 1, ..., m\}$ and $A_{t+1} \in \text{supp}\, q^A_t$ for all $q_t \in A_t$. Let p_t and p_{t+1} denote the unique lotteries in A_t and A_{t+1} such that $p^Z_t = p^Z_{t+1} = p^Z(1)$. By construction $A_t \in \mathcal{A}^*_t$ and $A_{t+1} \in \mathcal{A}^*_{t+1}$ (Lemma 14). Moreover, conditional on $C(h^{t-1})$, $u_t(\omega) \in M(A_t, p_t)$ if and only if $u_t(\omega) \approx u$, so that $C(h^{t-1}, A_t, p_t) = C(h^{t-1}) \cap \{u_t \approx u\}$ since there is no tie at A_t. In case $A_t \notin \mathcal{A}(h^{t-1})$, by mixing h^{t-1} with an appropriate degenerate history, we choose another history $\hat{h}^{t-1}$ that has the property that $C(h^{t-1}) = C(\hat{h}^{t-1})$ and $A_t \in \mathcal{A}_t(\hat{h}^{t-1})$. But then

$$\rho_{t+1}(p_{t+1}; A_{t+1} | C(\hat{h}^{t-1}, A_t, p_t)) = \mu(\{u_{t+1} \approx u\} | C(h^{t-1}) \cap \{u_t \approx u\}) = 0,$$

so that ρ violates consumption inertia. ■

I.3 Proof of Corollary 1

first part, "only if":

We consider the case $m \geq 2$ as otherwise the desired statement trivially holds with any α. Observe first that for any distinct indices $i, j \in \{1, \ldots, m\}$, consumption persistence and its characterization (Proposition 2) implies

$$M_{ii} = \mu(\{u_1 \in [u^i]\}|u_0 = u^i) \geq \mu(\{u_1 \in [u^i]\}|u_0 = u^j) = M_{ji}. \tag{41}$$

(Note that by assumption both u^i and u^j arise with positive probability in period 0). Moreover, if $D = \text{co}\{u^i, u^j\}$, then by assumption there is no $k \notin \{i,j\}$ such that $u^k \in [D]$. Thus, by consumption persistence and its characterization (Proposition 2),

$$M_{ii} + M_{ij} = \mu(\{u_1 \in [D]\}|u_0 = u^i) = \mu(\{u_1 \in [D]\}|u_0 = u^j) = M_{jj} + M_{ji}. \tag{42}$$

Suppose first that $m = 2$. Since $1 = M_{11} + M_{12} = M_{22} + M_{21}$, we have $M_{11} - M_{21} = M_{22} - M_{12} := \alpha$, which is nonnegative by (41). Since the Markov chain is irreducible, $M_{21}, M_{12} > 0$, which also ensures $\alpha < 1$. Setting $\nu(u^1) := \frac{M_{21}}{1-\alpha}$ and $\nu(u^2) := \frac{M_{12}}{1-\alpha}$, we obtain the desired form.

Suppose next that $m \geq 3$. Take any distinct $i, j, k \in \{1, ..., m\}$ and let $D' = \text{co}\{u^i, u^j, u^k\}$. By assumption there is no $l \notin \{i, j, k\}$ such that $u^l \in [D']$. Thus, by consumption persistence and its characterization (Proposition 2),

$$M_{ii} + M_{ij} + M_{ik} = \mu(\{u_1 \in [D']\}|u_0 = u^i) = \mu(\{u_1 \in [D']\}|u_0 = u^j) = M_{jj} + M_{ji} + M_{jk}.$$

Combined with (42), this implies that $M_{ik} = M_{jk}$ for any distinct i, j, k. Thus, for any k, we can define $\beta_k := M_{ik}$ for some arbitrary $i \neq k$. Here $\beta_k > 0$, because otherwise $\sum_{i \text{ s.t. } i \neq k} M_{ik} = 0$, contradicting irreducibility of the Markov chain. By (42), $M_{ii} - M_{ji} = M_{jj} - M_{ij}$ for any i, j, and thus $M_{ii} - \beta_i = M_{jj} - \beta_j =: \alpha$ for any i, j. By (41) $\alpha \geq 0$, and $\alpha < 1$ as $\beta_k > 0$ for all k. Thus, setting $\nu(u^j) := \frac{\beta_j}{1-\alpha}$ for each j leads to the desired form.

first part, "if":

Take any t and $[u], [u'] \subseteq \mathbb{R}^Z$ that correspond to possible realizations of period t felicities. Then for any h^{t-1} with $\mu(\{u_t \approx u\}|C(h^{t-1})), \mu(\{u_t \approx u'\}|C(h^{t-1})) > 0$ and any convex D with $u \in D$, we have

$$\begin{aligned}
\mu(\{u_{t+1} \in [D]\}|C(h^{t-1}), u_t \approx u) &= \alpha + (1-\alpha) \sum_{u^j \in [D]} \nu(u^j) \\
&\geq \alpha\mu(\{u_t \in [D]\}|C(h^{t-1}), u_t \approx u') + (1-\alpha) \sum_{u^j \in [D]} \nu(u^j) \\
&= \mu(\{u_{t+1} \in [D]\}|C(h^{t-1}), u_t \approx u').
\end{aligned}$$

Thus, ρ features choice persistence by Proposition 2.

Second part:

The proof follows immediately by applying Proposition 3 and thus is omitted. ■

J Proof of Proposition 4

Let $\mathcal{E}$ denote the support of the joint distribution of $(\varepsilon_1^{(x,\{y\})}, \varepsilon_1^{(x,\{z\})})$, which is also the support of the distribution of $(\varepsilon_2^y, \varepsilon_2^z)$. Fix a function $\sigma : \mathcal{E} \to [0,1]$ such that

$$\sigma(\varepsilon^y, \varepsilon^z) \in \operatorname{argmax}_{\alpha\in[0,1]} \alpha(\delta^2 v_2(y) + \delta^2\varepsilon^y) + (1-\alpha)(\delta^2 v_2(z) + \delta^2\varepsilon^z)$$

for each $(\varepsilon^y, \varepsilon^z) \in \mathcal{E}$. Then

$$\begin{aligned} & \mathbb{E}[\max\{\delta^2 v_2(y) + \delta^2\varepsilon_2^y, \delta^2 v_2(z) + \delta^2\varepsilon_2^z\}] \\ = \ & \mathbb{E}[\sigma(\varepsilon_2^y, \varepsilon_2^z)(\delta^2 v_2(y) + \delta^2\varepsilon_2^y) + (1-\sigma(\varepsilon_2^y, \varepsilon_2^z))(\delta^2 v_2(z) + \delta^2\varepsilon_2^z)] \\ = \ & \delta^2(\alpha^* v_2(y) + (1-\alpha^*)v_2(z)) + \delta^2\mathbb{E}[\sigma(\varepsilon_2^y, \varepsilon_2^z)\varepsilon_2^y + (1-\sigma(\varepsilon_2^y, \varepsilon_2^z))\varepsilon_2^z] \end{aligned}$$

where $\alpha^* := \mathbb{E}[\sigma(\varepsilon_2^y, \varepsilon_2^z)]$. Note that since ε_2 has mean zero, $\delta^2(\alpha^* v_2(y) + (1-\alpha^*)v_2(z))$ is the expected value the agent would obtain from A_1^{late} if in period 2 she chooses y with probability α^* regardless of the realization of ϵ_2. This implies that the term $\delta^2\mathbb{E}[\sigma(\varepsilon_2^y, \varepsilon_2^z)\varepsilon_2^y + (1-\sigma(\varepsilon_2^y, \varepsilon_2^z))\varepsilon_2^z]$ is nonnegative. At the same time,

$$\begin{aligned} & \mathbb{E}[\max\{\delta^2 v_2(y) + \delta\varepsilon_1^{(x,\{y\})}, \delta^2 v_2(z) + \delta\varepsilon_1^{(x,\{z\})}\}] \\ \geq \ & \mathbb{E}[\sigma(\varepsilon_1^{(x,\{y\})}, \varepsilon_1^{(x,\{z\})})(\delta^2 v_2(y) + \delta\varepsilon_1^{(x,\{y\})}) + (1-\sigma(\varepsilon_1^{(x,\{y\})}, \varepsilon_1^{(x,\{z\})}))(\delta^2 v_2(z) + \delta\varepsilon_1^{(x,\{z\})})] \\ = \ & \delta^2(\alpha^* v_2(y) + (1-\alpha^*)v_2(z)) + \delta\mathbb{E}[\sigma(\varepsilon_2^y, \varepsilon_2^z)\varepsilon_2^y + (1-\sigma(\varepsilon_2^y, \varepsilon_2^z))\varepsilon_2^z] \end{aligned}$$

where the equality used the i.i.d. assumption on ε_1 and ε_2. Therefore $\mathbb{E}[\max\{\delta^2 v_2(y) + \delta\varepsilon_1^{(x,\{y\})}, \delta^2 v_2(z) + \delta\varepsilon_1^{(x,\{z\})}\}] \geq \mathbb{E}[\max\{\delta^2 v_2(y) + \delta^2\varepsilon_2^y, \delta^2 v_2(z) + \delta^2\varepsilon_2^z\}]$. Thus, the desired claim follows from the i.i.d. assumption on ϵ_0. ■

K Consumption Dependence

K.1 Enriched Primitive

To accommodate consumption dependence, we enrich the stochastic choice data studied in previous sections: We not only keep track of past choices of lotteries from menus, but also of the corresponding realized consumptions.

The dynamic stochastic choice rule ρ is again defined recursively. The observed choice distribution at period 0 is summarized by a map $\rho_0 : \mathcal{A}_0 \to \Delta(\Delta(X_0))$ such that $\sum_{p_0\in A_0} \rho_0(p_0; A_0) = 1$ for all A_0. The set of enriched period 0 histories $\mathbb{H}_0 := \{(A_0, p_0, z_0) : \rho_0(p_0, A_0) > 0 \text{ and } z_0 \in \operatorname{supp} p_0^Z\}$ summarizes all choices p_0 from A_0 and realized consumptions z_0 that jointly occur with positive probability. For any history $\mathbb{h}^0 = (A_0, p_0, z_0) \in \mathbb{H}_0$, let $\mathcal{A}_1(\mathbb{h}^0) := \{A_1 \in \mathcal{A}_1 : (z_0, A_1) \in \operatorname{supp} p_0\}$ denote the set of period 1 menus that follow $\mathbb{h}^0$ with positive probability. For each $t = 1, \dots, T$ and history $\mathbb{h}^{t-1} \in \mathbb{H}_{t-1}$, observed period t choices following $\mathbb{h}^{t-1}$ are summarized by a map $\rho_t(\cdot|\mathbb{h}^{t-1}) : \mathcal{A}_t(\mathbb{h}^{t-1}) \to \Delta(\Delta(X_t))$ such that $\sum_{p_t\in A_t} \rho_t(p_t; A_t \mid \mathbb{h}^{t-1}) = 1$ for all $A_t \in \mathcal{A}_t(\mathbb{h}^{t-1})$. The set of enriched period-t histories is denoted by

$$\mathbb{H}_t := \{(\mathbb{h}^{t-1}, A_t, p_t, z_t) : \mathbb{h}^{t-1} \in \mathbb{H}_{t-1};\ A_t \in \mathcal{A}_t(\mathbb{h}^{t-1});\ \rho_t(p_t; A_t|\mathbb{h}^{t-1}) > 0;\ z_t \in \operatorname{supp} p_t^Z\}.$$

For each $t \leq T-1$, the set of period $t+1$ menus that follow history $\mathbb{h}^t = (\mathbb{h}^{t-1}, A_t, p_t, z_t)$ with positive probability is $\mathcal{A}_{t+1}(\mathbb{h}^t) := \{A_{t+1} \in \mathcal{A}_{t+1} : (z_t, A_{t+1}) \in \operatorname{supp} p_t\}$ and the set of period-t histories that lead to A_{t+1} with positive probability is $\mathbb{H}_t(A_{t+1}) := \{\mathbb{h}^t \in \mathbb{H}_t : A_{t+1} \in \mathcal{A}_{t+1}(\mathbb{h}^t)\}$.

Finally, for each $t = 1, \ldots, T$ and consumption stream $z^{t-1} = (z_0, ..., z_{t-1}) \in Z^t$, let $\mathbb{H}_{t-1}^{z^{t-1}} := \{\mathbb{h}^{t-1} \in \mathbb{H}_{t-1} : \mathbb{h}^{t-1} = (A_0, p_0, z_0, ..., A_{t-1}, p_{t-1}, z_{t-1}) \text{ for some } A_0, p_0, \ldots, A_{t-1}, p_{t-1}\}$ denote the set of histories that give rise to consumption stream z^{t-1}; and let $\mathbb{H}_{t-1}^{z^{t-1}}(A_t) := \mathbb{H}_{t-1}^{z^{t-1}} \cap \mathbb{H}_{t-1}(A_t)$.

K.2 Representations

We define the consumption dependent versions of our representations by extending the S-based representations introduced in Appendix A. Equivalent analogs of the Ω-based representations from the main text can be defined, but are omitted to save space.

Definition 12. A *consumption-dependent DREU (CDREU) representation* of ρ consists of tuples $(S_0, \mu_0, \{U_{s_0}, \tau_{s_0}\}_{s_0 \in S_0})$, $(S_t, \{\mu_t^{s_{t-1}, z_{t-1}}\}_{s_{t-1} \in S_{t-1}, z_{t-1} \in Z}, \{U_{s_t}, \tau_{s_t}\}_{s_t \in S_t})_{1 \leq t \leq T}$ such that for all $t = 0, \ldots, T$, we have:

CDREU1: For all $s_{t-1} \in S_{t-1}$ and $z_{t-1} \in Z$, $(S_t, \mu_t^{s_{t-1}, z_{t-1}}, \{U_{s_t}, \tau_{s_t}\}_{s_t \in S_t})$ is an REU form on X_t such that[60]

(a) $U_{s_t} \not\approx U_{s'_t}$ for any distinct $s_t, s'_t \in \operatorname{supp}(\mu_t^{s_{t-1}, z_{t-1}})$;

(b) $\operatorname{supp}(\mu_t^{s_{t-1}, z_{t-1}}) \cap \operatorname{supp}(\mu_t^{s'_{t-1}, z'_{t-1}}) = \varnothing$ for any distinct pairs $(s_{t-1}, z_{t-1}), (s'_{t-1}, z'_{t-1})$;

(c) $\bigcup_{s_{t-1} \in S_{t-1}, z_{t-1} \in Z} \operatorname{supp} \mu_t^{s_{t-1}, z_{t-1}} = S_t$.

CDREU2: For all p_t, A_t, and $\mathbb{h}^{t-1} = (A_k, p_k, z_k)_{k=0}^{t-1} \in \mathbb{H}_{t-1}(A_t)$,[61]

$$\rho_t(p_t, A_t | \mathbb{h}^{t-1}) = \frac{\sum_{(s_0, \ldots, s_t) \in S_0 \times \ldots \times S_t} \prod_{k=0}^{t} \mu_k^{s_{k-1}, z_{k-1}}(s_k) \tau_{s_k}(p_k, A_k)}{\sum_{(s_0, \ldots, s_{t-1}) \in S_0 \times \ldots \times S_{t-1}} \prod_{k=0}^{t-1} \mu_k^{s_{k-1}, z_{k-1}}(s_k) \tau_{s_k}(p_k, A_k)}.$$

A *consumption-dependent evolving utility representation* of ρ is a CDREU representation such that for all $t = 0, \ldots, T$, we additionally have:

CEVU: For all $s_t \in S_t$, there exists $u_{s_t} \in \mathbb{R}^Z$ such that for all $z_t \in Z, A_{t+1} \in \mathcal{A}_{t+1}$, we have[62]

$$U_{s_t}(z_t, A_{t+1}) = u_{s_t}(z_t) + V_{s_t, z_t}(A_{t+1}),$$

where $V_{s_t, z_t}(A_{t+1}) := \sum_{s_{t+1}} \mu_{t+1}^{s_t, z_t}(s_{t+1}) \max_{p_{t+1} \in A_{t+1}} U_{s_{t+1}}(p_{t+1})$ for $t \leq T-1$ and $V_{s_T, z_T} \equiv 0$.

An *active learning representation* is a consumption-dependent evolving-utility representation such that additionally:

[60] For $t = 0$, we abuse notation by letting $\mu_t^{s_{t-1}, z_{t-1}}$ denote μ_0 for all s_{t-1}, z_{t-1}.

[61] For $t = 0$, we again abuse notation by letting $\rho_t(\cdot | \mathbb{h}^{t-1})$ denote $\rho_0(\cdot)$ for all $\mathbb{h}^{t-1}$.

[62] Note that subject to multiplying U_{s_t} and u_{s_t} by δ^{T-t} for each t and s_t, this yields the representation in equation (9) in the main text.

CGL: There exists $\delta > 0$ such that for all $t = 0, \dots, T-1$, $z_t \in Z$, and $s_t \in S_t$, we have[63]

$$u_{s_t} = \frac{1}{\delta} \sum_{s_{t+1}} \mu_{t+1}^{s_t, z_t}(s_{t+1}) u_{s_{t+1}}.$$

K.3 Characterization

K.3.1 Consumption-Dependent DREU

Consumption-dependent DREU is equivalent to natural analogs of Axioms 1–4 used to characterize DREU. The only difference is that in defining contraction equivalence, linear equivalence, the extended version of ρ, and convergence of histories, we need to restrict attention to histories whose sequences of realized consumptions coincide.

Given $\mathbb{h}^{t-1} = (A_0, p_0, z_0, ..., A_{t-1}, p_{t-1}, z_{t-1}) \in \mathbb{H}_{t-1}$, let $(\mathbb{h}_{-k}^{t-1}, (A'_k, p'_k, z'_k))$ denote the sequence of the form $(A_0, p_0, z_0, ..., A'_k, p'_k, z'_k, ..., A_{t-1}, p_{t-1}, z_{t-1})$.[64] We say that $\mathbb{g}^{t-1} \in \mathbb{H}_{t-1}$ is *contraction equivalent* to $\mathbb{h}^{t-1}$ if for some k, we have $\mathbb{g}^{t-1} = (\mathbb{h}_{-k}^{t-1}, (B_k, p_k, z_k))$, where $A_k \subseteq B_k$ and $\rho_k(p_k, A_k|\mathbb{h}^{k-1}) = \rho_k(p_k, B_k|\mathbb{h}^{k-1})$. We say that a finite set of histories $\mathbb{G}^{t-1} \subseteq \mathbb{H}_{t-1}$ is *linearly equivalent* to $\mathbb{h}^{t-1}$ if

$$\mathbb{G}^{t-1} = \{(\mathbb{h}_{-k}^{t-1}, (\lambda A_k + (1-\lambda) B_k, \lambda p_k + (1-\lambda) q_k, z_k)) : q_k \in B_k\}$$

for some k, B_k, and $\lambda \in (0, 1]$.

Axiom 12 (Contraction History Independence*)**.** If $\mathbb{g}^{t-1} \in \mathbb{H}_{t-1}(A_t)$ is contraction equivalent to $\mathbb{h}^{t-1} \in \mathbb{H}_{t-1}(A_t)$, then $\rho_t(\cdot, A_t|\mathbb{h}^{t-1}) = \rho_t(\cdot, A_t|\mathbb{g}^{t-1})$.

Axiom 13 (Linear History Independence*)**.** If $\mathbb{G}^{t-1} \subseteq \mathbb{H}_{t-1}(A_t)$ is linearly equivalent to $\mathbb{h}^{t-1} \in \mathbb{H}_{t-1}(A_t)$, then $\rho_t(\cdot; A_t|\mathbb{h}^{t-1}) = \rho_t(\cdot; A_t|\mathbb{G}^{t-1})$.

As before, $\rho_t(\cdot; A_t|\mathbb{G}^{t-1})$ is shorthand for

$$\rho_t(\cdot; A_t|\mathbb{G}^{t-1}) := \sum_{\mathbb{g}^{t-1} \in \mathbb{G}^{t-1}} \rho_t(\cdot, A_t|\mathbb{g}^{t-1}) \frac{\rho(\mathbb{g}^{t-1})}{\sum_{\mathbb{f}^{t-1} \in \mathbb{G}^{t-1}} \rho(\mathbb{f}^{t-1})},$$

where for any $\mathbb{g}^{t-1} = (\hat{A}_0, \hat{p}_0, \hat{z}_0, \dots, \hat{A}_{t-1}, \hat{p}_{t-1}, \hat{z}_{t-1})$, $\rho(\mathbb{g}^{t-1}) := \prod_{k=0}^{t-1} \rho_k(\hat{p}_k, \hat{A}_k|\mathbb{g}^{k-1})$.[65]

Define the set of degenerate period-t histories by $\mathbb{D}_{t-1} := \{\mathbb{d}^{t-1} \in \mathbb{H}_{t-1} : \mathbb{d}^{t-1} = (\{q_k\}, q_k, z_k)_{k=0}^{t-1}$ where $q_k \in \Delta(X_k) \, \forall k \leq t-1\}$.

Definition 13. For any $t \geq 1$, $A_t \in \mathcal{A}_t$, z^{t-1}, and $\mathbb{h}^{t-1} \in \mathbb{H}_{t-1}^{z^{t-1}}$, define

$$\rho_t^{\mathbb{h}^{t-1}}(\cdot; A_t) := \rho_t(\cdot; A_t|\lambda \mathbb{h}^{t-1} + (1-\lambda)\mathbb{d}^{t-1}) \tag{43}$$

[63] As in footnote 62, subject to multiplying U_{s_t} and u_{s_t} by δ^{T-t} for each t and s_t, this yields the representation in equation (10) in the main text.

[64] In general this is not a history in $\mathbb{H}_{t-1}$, but it is if $(z_{k-1}, A'_k) \in \operatorname{supp} p_{k-1}$ and $(z_k, A_{k+1}) \in \operatorname{supp} p'_k$ and $\rho_k(p'_k; A'_k|\mathbb{h}^{k-1}) > 0$.

[65] As for the weights $\rho(g^{t-1})$ in Section 3.1, here we again do not keep track of the probabilities $\hat{p}_k(\hat{z}_k, \hat{A}_{k+1})$, as these do not reveal any private information to the analyst.

for some $\lambda \in (0,1]$ and $\mathbb{d}^{t-1} \in \mathbb{D}_{t-1}^{z^{t-1}}$ such that $\lambda \mathbb{h}^{t-1} + (1-\lambda)\mathbb{d}^{t-1} \in \mathbb{H}_{t-1}^{z^{t-1}}(A_t)$.[66]

As before, it follows from Axiom 13 that the RHS of (43) does not depend on the specific choice of λ and $\mathbb{d}^{t-1}$. Moreover, $\rho_t^{\mathbb{h}^{t-1}}(\cdot; A_t)$ coincides with $\rho_t(\cdot; A_t|\mathbb{h}^{t-1})$ whenever $\mathbb{h}^{t-1} \in \mathbb{H}_{t-1}(A_t)$. Thus, in the following we again do not distinguish between the extended and nonextended version of ρ_t and use $\rho_t(\cdot; A_t|\mathbb{h}^{t-1})$ to denote both.

Axiom 14 (History-dependent REU*)**.** For all $\mathbb{h}^{t-1} \in \mathbb{H}_{t-1}$, conditions (i)–(v) of Axiom 3 hold after replacing all instances of h^{t-1} with $\mathbb{h}^{t-1}$.

For any $0 \leq t \leq T$, we define the set $\mathcal{A}_t^*(\mathbb{h}^{t-1})$ of period-t menus without ties conditional on history $\mathbb{h}^{t-1} \in \mathbb{H}_{t-1}$ and the set $\mathbb{H}_t^*$ of period-t histories without ties as in the main text, replacing each instance of h and $\mathcal{H}$ in Definition 4 with $\mathbb{h}$ and $\mathbb{H}$. We extend convergence in mixture to histories in $\mathbb{H}_t$ by requiring additionally that each history in the sequence gives rise to the same sequence of consumptions: We write $\mathbb{h}_t^n \to^m \mathbb{h}_t$ for any $\mathbb{h}^t = (A_0, p_0, z_0, ..., A_t, p_t, z_t), \mathbb{h}_t^n = (A_0^n, p_0^n, z_0, ..., A_t^n, p_t^n, z_t) \in \mathbb{H}_t$ such that $A_k^n \to^m A_k$ and $p_k^n \to^m p_k$ for each k.

Axiom 15 (History Continuity*)**.** For all $0 \leq t \leq T-1$, A_{t+1}, p_{t+1}, and $\mathbb{h}^t \in \mathbb{H}_t$,

$$\rho_{t+1}(p_{t+1}; A_{t+1}|\mathbb{h}^t) \in \text{co}\{\lim_n \rho_{t+1}(p_{t+1}; A_{t+1}|\mathbb{h}^{t,n}) : \mathbb{h}^{t,n} \to^m \mathbb{h}^t, \mathbb{h}^{t,n} \in \mathbb{H}_t^*\}.$$

Theorem 5. The following are equivalent:

(i). ρ satisfies Axioms 12–15.

(ii). ρ admits a CDREU representation.

K.3.2 Consumption-Dependent Evolving Utility

Consumption-dependent evolving utility does not in general satisfy Separability, as current consumption affects the expected utility over continuation menus through its effect on the transition distributions $\mu_{t+1}^{s_t, z_t}$. Instead, it is fully characterized by analogs of Axioms 6 (DLR Menu Preference) and 7 (Sophisticaton). To state these, we make use of the same history dependent dominance relation as in Definition 5, except that histories now live in the enriched space $\mathbb{H}_t$:

Definition 14. For each $t \leq T-1$ and $\mathbb{h}^t = (\mathbb{h}^{t-1}, A_t, p_t, z_t) \in \mathbb{H}_t$, define relation $\succsim_{\mathbb{h}^t}$, $\sim_{\mathbb{h}^t}$, and $\succ_{\mathbb{h}^t}$ on $\Delta(X_t)$ as follows. For any $q_t, r_t \in \Delta(X_t)$, $q_t \succsim_{\mathbb{h}^t} r_t$ if there exist $q_t^n \to^m q_t$ and $r_t^n \to^m r_t$ such that

$$\rho_t(\frac{1}{2}p_t + \frac{1}{2}r_t^n; \frac{1}{2}A_t + \frac{1}{2}\{q_t^n, r_t^n\}|\mathbb{h}^{t-1}) = 0$$

for all n. Let $\sim_{\mathbb{h}^t}$ and $\succ_{\mathbb{h}^t}$ respectively denote the symmetric and asymmetric component of $\succsim_{\mathbb{h}^t}$.

[66] As before, we define $\lambda \mathbb{h}^{t-1} + (1-\lambda)\mathbb{d}^{t-1} := (\lambda A_k + (1-\lambda)\{q_k\}, \lambda p_k + (1-\lambda)q_k, z_k)_{k=0}^{t-1}$, where $\mathbb{h}^{t-1} = (A_k, p_k, z_k)_{k=0}^{t-1}$ and $\mathbb{d}^{t-1} = (\{q_k\}, q_k, z_k)_{k=0}^{t-1}$.

As before, $q_t \succsim_{\mathbb{h}^t} r_t$ reveals that at all states s_t consistent with $\mathbb{h}^t$ the agent prefers q_t to r_t. Note that since the final realized consumption z_t does not reveal any additional information about s_t, $\succsim_{\mathbb{h}^t}$ depends only on $(\mathbb{h}^{t-1}, A_t, p_t)$ and not on z_t.

We have the following analogs of Axioms 6 and 7:

Axiom 16 (DLR Menu Preference*)**.** For all $t \leq T-1$ and $\mathbb{h}^t$, conditions (i)–(iv) of Axiom 6 hold after replacing all instances of h^t with $\mathbb{h}^t$.

Axiom 17 (Sophistication*)**.** For all $t \leq T-1$, $\mathbb{h}^t = (\mathbb{h}^{t-1}, A_t, p_t, z_t) \in \mathbb{H}_t$, and $A_{t+1} \subseteq B_{t+1} \in \mathcal{A}^*_{t+1}(\mathbb{h}^t)$, the following are equivalent:

(i). $\rho_{t+1}(p_{t+1}; B_{t+1}|\mathbb{h}^t) > 0$ for some $p_{t+1} \in B_{t+1} \setminus A_{t+1}$

(ii). $(z_t, B_{t+1}) \succ_{\mathbb{h}^t} (z_t, A_{t+1})$.

A slight departure from Axiom 7 is that because both the agent's preference over period $t+1$ menus and her period $t+1$ choices may be influenced by her period t consumption, point (ii) of Axiom 17 only imposes a preference for B_{t+1} over A_{t+1} when the corresponding period t consumption is the consumption z_t that arises under $\mathbb{h}^t$.

Theorem 6. Suppose that ρ admits a CDREU representation. The following are equivalent:

(i). ρ satisfies Axioms 16–17.

(ii). ρ admits a Consumption-Dependent Evolving Utility representation.

K.3.3 Active Learning

As with gradual learning, active learning is characterized by additional restrictions on the agent's preference over streams of consumption lotteries. In contrast with consumption-dependent evolving utility, active learning restores a weak form of Separability, requiring that the agent's payoff to today's consumption be independent of tomorrow's consumption lottery stream. This captures the idea that since consumption lottery streams only entail degenerate future choices, today's consumption does not have any informational value in this case and hence yields the same payoff regardless of the future stream:

Axiom 18 (Stream Separability)**.** For all $\mathbb{h}^t$, $z_t, x_t \in Z$, $(\ell_k)_{k=t+1}^T, (m_k)_{k=t+1}^T \in \Delta(Z)$,

$$\frac{1}{2}(z_t, \ell_{t+1}, \ldots, \ell_T) + \frac{1}{2}(x_t, m_{t+1}, \ldots, m_T) \sim_{\mathbb{h}^t} \frac{1}{2}(x_t, \ell_{t+1}, \ldots, \ell_T) + \frac{1}{2}(z_t, m_{t+1}, \ldots, m_T).$$

Additionally, active learning entails analogs of Axioms 8–9 that we used to characterize gradual learning:

Axiom 19 (Stationary Consumption Preference*)**.** For all $t \leq T-1$, $\ell, m, n \in \Delta(Z)$, and $\mathbb{h}^t$, $(\ell, n, \ldots, n) \succ_{\mathbb{h}^t} (m, n, \ldots, n)$ if and only if $(n, \ell, n, \ldots, n) \succ_{\mathbb{h}^t} (n, m, n, \ldots, n)$.

We say that $\ell, m \in \Delta(Z)$ are $\mathbb{h}^t$-*nonindifferent* if $(\ell, n, \ldots, n) \not\sim_{\mathbb{h}^t} (m, n, \ldots, n)$ for some $n \in \Delta(Z)$.

Axiom 20 (Constant Intertemporal Tradeoff*)**.** If ℓ, m are $\mathbb{h}^t$-nonindifferent and $\hat{\ell}, \hat{m}$ are $\mathbb{g}^\tau$-nonindifferent, then for all $\alpha \in [0,1]$ and $n \in \Delta Z$:

$$(\ell, m, n, \ldots, n) \sim_{\mathbb{h}^t} (\alpha\ell + (1-\alpha)m, \alpha\ell + (1-\alpha)m, n, \ldots, n)$$
$$\iff$$
$$(\hat{\ell}, \hat{m}, n, \ldots, n) \sim_{\mathbb{g}^\tau} (\alpha\hat{\ell} + (1-\alpha)\hat{m}, \alpha\hat{\ell} + (1-\alpha)\hat{m}, n, \ldots, n).$$

As before, the axiom is vacuous unless we impose Consumption Nondegeneracy:

Condition 3 (Consumption Nondegeneracy*)**.** For all $t \leq T-1$ and $\mathbb{h}^t$, there exist $\mathbb{h}^t$-nonindifferent $\ell, m \in \Delta(Z)$.

When Condition 3 is satisfied, Axioms 18–20 fully characterize the additional behavioral content of active learning relative to consumption-dependent evolving utility:

Theorem 7. Suppose ρ admits a consumption-dependent evolving utility representation and satisfies Condition 3. The following are equivalent:

(i). ρ satisfies Axioms 18–20.

(ii). ρ admits an active learning representation.

K.4 Proof Sketches for Theorems 5–7

K.4.1 Theorem 5

We first extend the terminology from Section A.3 to the enriched setting. Suppose that $(S_{t'}, \{\mu_{t'}^{s_{t'-1}, z_{t'-1}}\}_{(s_{t'-1}, z_{t'-1}) \in S_{t'-1} \times Z}, \{U_{s_{t'}}, \tau_{s_{t'}}\}_{s_{t'} \in S_{t'}})$ satisfy CDREU1 and CDREU2 from Definition 12 for each $t' \leq t$.

Fix any state $s_t^* \in S_t$ and consumption $z_t^* \in Z$. We let $\text{pred}(s_t^*)$ denote the unique *predecessor sequence* $(s_0^*, z_0^*, \ldots, s_{t-1}^*, z_{t-1}^*) \in S_0 \times Z \times \ldots \times S_{t-1} \times Z$, given by assumptions CDREU1 (b) and (c), such that $s_{k+1}^* \in \text{supp}(\mu_{k+1}^{s_k^*, z_k^*})$ for each $k = 0, ..., t-1$. Given any history $\mathbb{h}^t = (A_0, p_0, z_0, \ldots, A_t, p_t, z_t)$, we say that (s_t^*, z_t^*) is *consistent* with $\mathbb{h}^t$ if $\prod_{k=0}^t \tau_{s_k^*}(p_k, A_k) > 0$ and $z_k = z_k^*$ for all $k = 0, \ldots, t$. A *separating history* for (s_t^*, z_t^*) is a history $\mathbb{h}^t = (B_0, q_0, z_0, \ldots, B_t, q_t, z_t) \in \mathbb{H}_t^*$ such that $\mathcal{U}_{s_{k-1}^*, z_{k-1}^*}(B_k, q_k) = \{U_{s_k^*}\}$ and $z_k = z_k^*$ for all $k = 0, \ldots, t$, where $\mathcal{U}_{s_{k-1}^*, z_{k-1}^*}(B_k, q_k) := \{U_{s_k} : s_k \in \text{supp}\, \mu_k^{s_{k-1}^*, z_{k-1}^*} \text{ and } q_k \in M(B_k, U_{s_k})\}$.[67]

Analogs of Lemmas 1 and 2 are readily established; in particular, any $(s_t^*, z_t^*) \in S_t \times Z$ admits a separating history.

Sufficiency:

The sufficiency direction of the proof proceeds as in Section B.1. Specifically, assuming that for some $t \leq T-1$ we have constructed $(S_{t'}, \{\mu_{t'}^{s_{t'-1}, z_{t'-1}}\}_{(s_{t'-1}, z_{t'-1}) \in S_{t'-1} \times Z}, \{U_{s_{t'}}, \tau_{s_{t'}}\}_{s_{t'} \in S_{t'}})$ satisfying CDREU1 and CDREU2 for each $t' \leq t$, we construct $(S_{t+1}, \{\mu_{t+1}^{s_t, z_t}\}_{(s_t, z_t) \in S_t \times Z}, \{U_{s_{t+1}}, \tau_{s_{t+1}}\}_{s_{t+1} \in S_{t+1}})$ satisfying CDREU1 and CDREU2 as follows:

[67] As usual, $\mathcal{U}_{s_{-1}^*, z_{-1}^*}(B_0, q_0)$ denotes $\mathcal{U}_0(B_0, q_0) := \{U_{s_0} : s_0 \in S_0 \text{ and } q_0 \in M(B_0, U_{s_0})\}$.

For any $(s_t, z_t) \in S_t \times Z$, we define $\rho_{t+1}^{s_t,z_t}(\cdot, A_{t+1}) := \rho_{t+1}(\cdot, A_{t+1}|\mathbb{h}^t(s_t, z_t))$, where $\mathbb{h}^t(s_t, z_t)$ is an arbitrary separating history for (s_t, z_t). Using Axiom 14 and Theorem 4, we then find $(S_{t+1}, \{\mu_{t+1}^{s_t,z_t}\}_{(s_t,z_t)\in S_t\times Z}, \{U_{s_{t+1}}, \tau_{s_{t+1}}\}_{s_{t+1}\in S_{t+1}})$ such that CDREU1 holds and

$$\rho_{t+1}^{s_t,z_t}(p_{t+1}, A_{t+1}) = \sum_{s_{t+1}\in S_{t+1}} \mu_{t+1}^{s_t,z_t}(s_{t+1})\tau_{s_{t+1}}(p_{t+1}, A_{t+1}). \tag{44}$$

Analogous arguments as in Section B.1 yield analogs of Lemma 3, showing that for any history $\mathbb{h}^t$ that can only arise at (s_t, z_t), we have $\rho_{t+1}^{s_t,z_t} = \rho_{t+1}(\cdot|\mathbb{h}^t)$; and of Lemma 4, showing that for any $\mathbb{h}^t = (A_0, p_0, z_0, \ldots, A_t, p_t, z_t) \in \mathbb{H}_t(A_{t+1})$, we have

$$\rho_{t+1}(p_{t+1}, A_{t+1}|\mathbb{h}^t) = \frac{\sum_{(s_0,\ldots,s_t)\in S_0\times\cdots\times S_t} \prod_{k=0}^t \mu_k^{s_{k-1},z_{k-1}}(s_k)\tau_{s_k}(A_k, p_k)\rho_{t+1}^{s_t,z_t}(p_{t+1}, A_{t+1})}{\sum_{(s_0,\ldots,s_t)\in S_0\times\cdots\times S_t} \prod_{k=0}^t \mu_k^{s_{k-1},z_{k-1}}(s_k)\tau_{s_k}(A_k, p_k)}. \tag{45}$$

Combining (45) and (44) shows that $(S_{t+1}, \{\mu_{t+1}^{s_t,z_t}\}_{(s_t,z_t)\in S_t\times Z}, \{U_{s_{t+1}}, \tau_{s_{t+1}}\}_{s_{t+1}\in S_{t+1}})$ also satisfies CDREU2.

Necessity:
Necessity of Axioms 12–15 is established using analogous arguments to Section B.2.

K.4.2 Theorem 6

We first record the following analog of Lemma 5, which admits an analogous proof. Given $\mathbb{h}^t = (\mathbb{h}^{t-1}, A_t, p_t, z_t)$, we say that s_t is consistent with $\mathbb{h}^t$ if (s_t, z_t) is consistent with $\mathbb{h}^t$, and we call $\mathbb{h}^t$ a separating history for s_t if it is a separating history for (s_t, z_t).

Lemma 25. Suppose that ρ admits CDREU representation. Consider any $t \leq T-1$, $\mathbb{h}^t = (A_0, p_0, z_0, \ldots, A_t, p_t, z_t) \in \mathbb{H}_t$, and $q_t, r_t \in \Delta(X_t)$.

(i). If $q_t \succsim_{\mathbb{h}^t} r_t$, then $U_{s_t}(q_t) \geq U_{s_t}(r_t)$ for all s_t consistent with $\mathbb{h}^t$.

(ii). Suppose there exist $g, b \in \Delta(X_t)$ such that $U_{s_t}(g) > U_{s_t}(b)$ for all s_t consistent with $\mathbb{h}^t$. If $U_{s_t}(q_t) \geq U_{s_t}(r_t)$ for all s_t consistent with $\mathbb{h}^t$, then $q_t \succsim_{\mathbb{h}^t} r_t$.

(iii). If $\mathbb{h}^t$ is a separating history for s_t, then $q_t \succsim_{\mathbb{h}^t} r_t$ if and only if $U_{s_t}(q_t) \geq U_{s_t}(r_t)$.

Sufficiency:
We proceed by an analogous inductive argument as in Section C.2. Assume that for some $t \leq T-1$, we have obtained $(S_{t'}, \{\mu_{t'}^{s_{t'-1},z_{t'-1}}\}_{(s_{t'-1},z_{t'-1})\in S_{t'-1}\times Z}, \{U_{s_{t'}}, \tau_{s_{t'}}\}_{s_{t'}\in S_{t'}})$ such that CDREU1 and CDREU2 hold for each $t' \leq t$ and CEVU holds for each $t' \leq t-1$.

For any $(s_t, z_t) \in S_t \times Z$, define $u_{s_t} \in \mathbb{R}^Z$ and $V_{s_t,z_t} : \mathcal{A}_{t+1} \to \mathbb{R}$ by $u_{s_t} \equiv 0$ and $V_{s_t,z_t}(A_{t+1}) := U_{s_t}(z_t, A_{t+1})$. Applying Axiom 16 yields the following analog of Lemma 6, with an analogous proof. Note that because we do not impose Separability, V_{s_t,z_t} depends on z_t.

Lemma 26. For all (s_t, z_t), V_{s_t,z_t} is continuous, monotone, and linear. Moreover, there exist $C'_{t+1}, C_{t+1} \in \mathcal{A}_{t+1}$ such that $V_{s_t,z_t}(C_{t+1}) > V_{s_t,z_t}(C'_{t+1})$ for all (s_t, z_t).

Applying the "moreover" part of Lemma 26 also yields the obvious analog of Corollary C.1.

Since ρ admits a CDREU representation, we can obtain $(S_{t+1}, \{\mu_{t+1}^{s_t,z_t}\}_{(s_t,z_t)\in S_t\times Z}, \{\tilde{U}_{s_{t+1}}, \tau_{s_{t+1}}\}_{s_{t+1}\in S_{t+1}})$ satisfying CDREU1 and CDREU2 at $t+1$. For all $(s_t, z_t) \in S_t \times Z$, define $\rho_{t+1}^{s_t,z_t}$ by $\rho_{t+1}^{s_t,z_t}(p_{t+1}, A_{t+1}) := \sum_{s_{t+1}} \mu_{t+1}^{s_t,z_t}\tau_{s_{t+1}}(p_{t+1}, A_{t+1})$.

Let $\succsim_{s_t,z_t}$ denote the preference over $\mathcal{A}_{t+1}$ induced by V_{s_t,z_t}. Using Axiom 17, an analogous argument as for Lemma 7 shows that for all s_t, z_t, the pair $(\succsim_{s_t,z_t}, \rho_{t+1}^{s_t,z_t})$ satisfies AS's Axioms 1 and 2, and an analogous argument as for Lemma 8 establishes that $\succsim_{s_t,z_t}$ satisfies AS's Axiom DLR 6. Given this, we proceed as in Section C.2.5 (replacing each instance of s_t with an instance of (s_t, z_t)) to obtain $\alpha_{s_{t+1}} > 0$ and $\beta_{s_{t+1}} \in \mathbb{R}$ such that $U_{s_{t+1}} := \alpha_{s_{t+1}}\tilde{U}_{s_{t+1}} + \beta_{s_{t+1}}$ satisfies

$$V_{s_t,z_t}(A_{t+1}) = \sum_{s_{t+1}} \mu_{t+1}^{s_t,z_t}(s_{t+1}) \max_{p_{t+1}\in A_{t+1}} U_{s_{t+1}}(p_{t+1}).$$

Thus, replacing $\tilde{U}_{s_{t+1}}$ with $U_{s_{t+1}}$, we have that $(S_{t+1}, \{\mu_{t+1}^{s_t,z_t}\}_{(s_t,z_t)\in S_t\times Z}, \{U_{s_{t+1}}, \tau_{s_{t+1}}\}_{s_{t+1}\in S_{t+1}})$ satisfies not only CDREU1 and CDREU2, but also CEVU, as required.

Necessity:

As in Section C.3, we first show that for all $t \leq T-1$, there exist $g_t, b_t \in \Delta(X_t)$ such that $U_{s_t}(g_t) > U_{s_t}(b_t)$ for all $s_t \in S_t$, for which it is sufficient to find $C'_{t+1}, C_{t+1} \in \mathcal{A}_{t+1}$ such that $V_{s_t,z_t}(C'_{t+1}) > V_{s_t,z_t}(C_{t+1})$ for all $(s_t, z_t) \in S_t \times Z$. As before, we let $C'_{t+1} := \{g_{t+1}(s_{t+1}), b_{t+1}(s_{t+1}) : s_{t+1} \in S_{t+1}\}$ where $g_{t+1}(s_{t+1}), b_{t+1}(s_{t+1}) \in \Delta(X_{t+1})$ are such that $U_{s_{t+1}}(g_{t+1}(s_{t+1})) > U_{s_{t+1}}(b_{t+1}(s_{t+1}))$. For each s_t, z_t, we set $A_{t+1}(s_t, z_t) := \{b_{t+1}(s_{t+1})\}$ for some $s_{t+1} \in \mathrm{supp}\mu_{t+1}^{s_t,z_t}$. Then $V_{s_t,z_t}(C'_{t+1}) \geq V_{s_t,z_t}(A_{t+1}(s'_t, z'_t))$ for all s_t, z_t, s'_t, z'_t, with strict inequality for $(s_t, z_t) = (s'_t, z'_t)$. Hence, letting $C_{t+1} := \sum_{(s_t,z_t)\in S_t\times Z} \frac{1}{|S_t\times Z|} A_{t+1}(s_t, z_t)$, we have $V_{s_t,z_t}(C'_{t+1}) > V_{s_t,z_t}(C_{t+1})$ for all s_t, z_t.

Given this, we can proceed as in Section C.3, using part (ii) of Lemma 25 to derive Axiom 16 (i), (ii), (iv) and Axiom 17. Part (iii) of Axiom 16 is also established in an analogous manner to Section C.3.

K.4.3 Theorem 7

We first show that when ρ admits a consumption dependent evolving utility (CEVU) representation, Axiom 18 (Stream Separability) is equivalent to the following form of separability over consumption lottery streams:

Lemma 27. Suppose ρ admits a CEVU representation with utilities U_{s_t}. Then ρ satisfies Axiom 18 if and only if for all $t \leq T-1$, s_t, z, z' and $(\ell_k)_{k=t+1}^T$, $(m_k)_{k=t+1}^T \in \Delta(Z)$, we have

$$U_{s_t}(z, \ell_{t+1}, \ldots, \ell_T) - U_{s_t}(z', \ell_{t+1}, \ldots, \ell_T) = U_{s_t}(z, m_{t+1}, \ldots, m_T) - U_{s_t}(z', m_{t+1}, \ldots, m_T). \quad (46)$$

Proof. Fix any $t \leq T-1$. By the necessity direction of the proof of Theorem 6, there exist $g_t, b_t \in \Delta(X_t)$ such that $U_{s_t}(g_t) > U_{s_t}(b_t)$ for all $s_t \in S_t$. Then Lemma 25 (i)–(ii) implies that for all $\mathbb{h}^t$ and q_t, r_t, we have $q_t \sim_{\mathbb{h}^t} r_t$ if and only if $U_{s_t}(q_t) = U_{s_t}(r_t)$ for all s_t consistent with $\mathbb{h}^t$.

Then Axiom 18 holds if and only if for all $t \leq T-1$, s_t, z, z' and $(\ell_k)_{k=t+1}^T$, $(m_k)_{k=t+1}^T \in \Delta(Z)$, we have $\frac{1}{2}U_{s_t}(z, \ell_{t+1}, \ldots, \ell_T) + \frac{1}{2}U_{s_t}(z, m_{t+1}, \ldots, m_T) = \frac{1}{2}U_{s_t}(z', \ell_{t+1}, \ldots, \ell_T) + \frac{1}{2}U_{s_t}(z', m_{t+1}, \ldots, m_T)$, which in turn is equivalent to (46). ■

Sufficiency:
Suppose that ρ admits a CEVU representation $(S_t, \{\mu_t^{s_{t-1},z_{t-1}}\}_{s_{t-1}\in S_{t-1},z_{t-1}\in Z}, \{U_{s_t}, u_{s_t}, \tau_{s_t}\}_{s_t\in S_t})_{0\le t\le T}$ and satisfies Condition 3 and Axioms 18–20. Steps 1 and 2 below first perform two normalizations. Step 3 then proceeds in an analogous manner to the sufficiency direction of Theorem 3.

Step 1: We first show that replacing each U_{s_t} and u_{s_t} with a suitable $\hat{U}_{s_t}$ and $\hat{u}_{s_t}$ continues to yield a CEVU representation of ρ which additionally satisfies

$$\hat{U}_{s_t}(\ell_t,\dots,\ell_T) = \hat{u}_{s_t}(\ell_t) + \hat{W}_{s_t}(\ell_{t+1},\dots,\ell_T) \tag{47}$$

for all s_t and $(\ell_k)_{k=t}^T \in \Delta(Z)$, where $\hat{W}_{s_t}(\ell_{t+1},\dots,\ell_T) := \sum_{s_{t+1}} \mu_{t+1}^{s_t,z_t}(s_{t+1})\hat{U}_{s_{t+1}}(\ell_{t+1},\dots,\ell_T)$ does not depend on z_t.

To see this, fix any $z_0^*,\dots,z_T^* \in Z$. For all s_0, define $\hat{U}_{s_0} := U_{s_0}$ and $\hat{u}_{s_0}(z_0) := \hat{U}_{s_0}(z_0, z_1^*,\dots,z_T^*) - \hat{U}_{s_0}(z_0^*, z_1^*,\dots,z_T^*)$ for all z_0. For all s_1, define $\hat{U}_{s_1} := U_{s_1} + (u_{s_0}(z_0) - \hat{u}_{s_0}(z_0))$ and $\bar{u}_{s_1} := u_{s_1} + (u_{s_0}(z_0) - \hat{u}_{s_0}(z_0))$, where (s_0, z_0) are the unique state-outcome pair such that $s_1 \in \operatorname{supp}\mu_1^{s_0,z_0}$. Note that replacing each U_{s_0} with $\hat{U}_{s_0}$, u_{s_0} with $\hat{u}_{s_0}$, U_{s_1} with $\hat{U}_{s_1}$, and u_{s_1} with $\bar{u}_{s_1}$, and keeping all utilities in periods $t \ge 2$ the same continues to yield a CEVU representation of ρ: Indeed, U_{s_i} and $\hat{U}_{s_i}$ represent the same preference over $\Delta(X_i)$ so CDREU1 and CDREU2 remain satisfied. Moreover, CEVU holds at s_1 because $\hat{U}_{s_1}$ and $\bar{u}_{s_1}$ are obtained from U_{s_1} and u_{s_1} by adding the same constant, and CEVU holds at s_0 because $\hat{U}_{s_0} = U_{s_0}$ and $\hat{U}_{s_1} = U_{s_1} + (u_{s_0}(z_0) - \hat{u}_{s_0}(z_0))$ for all $s_1 \in \operatorname{supp}\mu_1^{s_0,z_0}$ implies

$$\hat{U}_{s_0}(z_0, A_1) = u_{s_0}(z_0) + \sum_{s_1} \mu_1^{s_0,z_0}(s_1) \max_{p_1\in A_1} U_{s_1}(p_1) = \hat{u}_{s_0}(z_0) + \sum_{s_1} \mu_1^{s_0,z_0}(s_1) \max_{p_1\in A_1} \hat{U}_{s_1}(p_1).$$

Then, for all z_0 and $(\ell_k)_{k=1}^T \in \Delta(Z)$, we have $\sum_{s_1} \mu_1^{s_0,z_0}\hat{U}_{s_1}(\ell_1,\dots,\ell_T) = \hat{U}_{s_0}(z_0,\ell_1,\dots,\ell_T) - \hat{u}_{s_0}(z_0) = \hat{U}_{s_0}(z_0^*,\ell_1,\dots,\ell_T)$, where the last equality follows from the fact that $\hat{u}_{s_0}(z_0) := \hat{U}_{s_0}(z_0, z_1^*,\dots,z_T^*) - \hat{U}_{s_0}(z_0^*, z_1^*,\dots,z_T^*) = \hat{U}_{s_0}(z_0,\ell_1,\dots,\ell_T) - \hat{U}_{s_0}(z_0^*,\ell_1,\dots,\ell_T)$ by Lemma 27. Thus, for all $(\ell_k)_{k=0}^T \in \Delta(Z)$, we have $\hat{U}_{s_0}(\ell_0,\ell_1,\dots,\ell_T) = \hat{u}_{s_0}(\ell_0) + \hat{W}_{s_0}(\ell_{t+1},\dots,\ell_T)$, where $\hat{W}_{s_0}(\ell_1,\dots,\ell_T) := \sum_{s_1} \mu_1^{s_0,z_0}(s_1)\hat{U}_{s_1}(\ell_1,\dots,\ell_T)$ does not depend on z_0, whence (47) holds at s_0.

Next, suppose that for some $t \ge 1$, we have obtained a CEVU representation of ρ by replacing each $U_{s_0}, u_{s_0},\dots,U_{s_t}, u_{s_t}$ with $\hat{U}_{s_0}, \hat{u}_{s_0},\dots,\hat{U}_{s_t}, \bar{u}_{s_t}$ and keeping utilities in periods $t+1,\dots,T$ the same, and suppose that (47) holds for all $t' < t$. For all s_t and z_t, define $\hat{u}_{s_t}(z_t) := \hat{U}_{s_t}(z_t, z_{t+1}^*,\dots,z_T^*) - \hat{U}_{s_t}(z_t^*, z_{t+1}^*,\dots,z_T^*)$. For all s_{t+1}, define $\hat{U}_{s_{t+1}} := U_{s_{t+1}} + (\bar{u}_{s_t}(z_t) - \hat{u}_{s_t}(z_t))$ and $\bar{u}_{s_{t+1}} := u_{s_{t+1}} + (\bar{u}_{s_t}(z_t) - \hat{u}_{s_t}(z_t))$, where (s_t, z_t) are the unique state-outcome pair such that $s_{t+1} \in \operatorname{supp}\mu_{t+1}^{s_t,z_t}$. Then the same argument as in the previous paragraph shows that after replacing $\bar{u}_{s_t}$ with $\hat{u}_{s_t}$ and $U_{s_{t+1}}$ and $u_{s_{t+1}}$ with $\hat{U}_{s_{t+1}}$ and $\hat{u}_{s_{t+1}}$, we continue to have a CEVU representation of ρ which now additionally satisfies (47) in period t. Proceeding inductively, we obtain the desired representation.

Step 2: Next, we show that replacing each $\hat{U}_{s_t}$ and $\hat{u}_{s_t}$ with a suitable $\tilde{U}_{s_t}$ and $\tilde{u}_{s_t}$ continues to yield a CEVU representation of ρ which again satisfies (47) and such that additionally

$$\sum_{z\in Z} \tilde{u}_{s_t}(z) = 0 \tag{48}$$

holds for all s_t. Indeed, let $\tilde{u}_{s_t}(z_t) := \hat{u}_{s_t}(z_t) - \gamma_{s_t}$ for all s_t and z_t, where $\gamma_{s_t} := \frac{1}{|Z|}\sum_{z\in Z}\hat{u}_{s_t}(z)$. Then (48) is immediate. Inductively define $\tilde{U}_{s_t}$ by $\tilde{U}_{s_T} := \tilde{u}_{s_T}$ and $\tilde{U}_{s_t}(z_t, A_{t+1}) := \tilde{u}_{s_t}(z_t) + \sum_{s_{t+1}} \mu_{t+1}^{s_t,z_t} \max_{p_{t+1}\in A_{t+1}} \tilde{U}_{s_{t+1}}(p_{t+1})$ for all s_t. To show that $\tilde{U}_{s_t}$ and $\tilde{u}_{s_t}$ are as required, it suffices to prove that for all $t \leq T-1$ and s_t,

$$\tilde{U}_{s_t} = \hat{U}_{s_t} - (\gamma_{s_t} + \beta_{s_t}), \tag{49}$$

where $\beta_{s_t} := |Z|^{t-T}\sum_{(z_{t+1},\dots,z_T)\in Z^{T-t}} \hat{W}_{s_t}(z_{t+1},\dots,z_T)$. Indeed, (49) implies that $\tilde{U}_{s_t}$ and $\hat{U}_{s_t}$ represent the same preference over $\Delta(X_t)$, so that replacing each $\hat{U}_{s_t}$ and $\hat{u}_{s_t}$ with $\tilde{U}_{s_t}$ and $\tilde{u}_{s_t}$ continues to yield a CEVU representation of ρ. Moreover, (49) implies that $\sum_{s_{t+1}} \mu_{t+1}^{s_t,z_t}(s_{t+1})\tilde{U}_{s_{t+1}}(\ell_{t+1},\dots,\ell_T) = \hat{W}_{s_t}(\ell_{t+1},\dots,\ell_T) - \beta_{s_t}$ is independent of z_t for all s_t, so that (47) continues to hold as well.

To show (49), note first that for each s_{T-1} we have

$$\tilde{U}_{s_{T-1}}(z_{T-1}, A_T) = \hat{u}_{s_{T-1}}(z_{T-1}) - \gamma_{s_{T-1}} + \sum_{s_T} \mu_T^{s_{T-1},z_{T-1}}(s_T)\big(\max_{p_T\in A_T} \hat{u}_{s_T}(p_T) - \gamma_{s_T}\big) =$$

$$\hat{U}_{s_{T-1}}(z_{T-1}, A_T) - \gamma_{s_{T-1}} - |Z|^{-1}\sum_{z\in Z}\sum_{s_T} \mu_T^{s_{T-1},z_{T-1}}(s_T)\hat{u}_{s_T}(z) = \hat{U}_{s_{T-1}}(z_{T-1}, A_T) - \left(\gamma_{s_{T-1}} + \beta_{s_{T-1}}\right),$$

where the final equality holds because $\mu_T^{s_{T-1},z_{T-1}}(s_T)\hat{u}_{s_T}(z) = \hat{W}_{s_{T-1}}(z)$ by (47). Next, assume that (49) holds for all $t' > t$. By (47), we have for all s_{t+1} that

$$\begin{aligned} &\sum_{(z_{t+1},\dots,z_T)\in Z^{T-t}} \hat{U}_{s_{t+1}}(z_{t+1},\dots,z_T) \\ &= |Z|^{T-(t+1)}\sum_{z_{t+1}} \hat{u}_{s_{t+1}}(z_{t+1}) + |Z|\sum_{(z_{t+2},\dots,z_T)} \hat{W}_{s_{t+1}}(z_{t+2},\dots,z_T) = |Z|^{T-t}\left(\gamma_{s_{t+1}} + \beta_{s_{t+1}}\right). \end{aligned} \tag{50}$$

Thus,

$$\begin{aligned} \tilde{U}_{s_t}(z_t, A_{t+1}) &= \hat{U}_{s_t}(z_t, A_{t+1}) - \gamma_{s_t} - \sum_{s_{t+1}} \mu_{t+1}^{s_t,z_t}(s_{t+1})\left(\gamma_{s_{t+1}} + \beta_{s_{t+1}}\right) \\ &= \hat{U}_{s_t}(z_t, A_{t+1}) - \gamma_{s_t} - |Z|^{t-T}\sum_{s_{t+1}} \mu_{t+1}^{s_t,z_t}(s_{t+1})\sum_{(z_{t+1},\dots,z_T)\in Z^{T-t}} \hat{U}_{s_{t+1}}(z_{t+1},\dots,z_T) = \\ &\hat{U}_{s_t}(z_t, A_{t+1}) - (\gamma_{s_t} + \beta_{s_t}), \end{aligned}$$

where the first equality holds by (49) applied to $\tilde{U}_{s_{t+1}}$, the second equality follows from (50), and the final equality holds because $\sum_{s_{t+1}} \mu_{t+1}^{s_t,z_t}(s_{t+1})\hat{U}_{s_{t+1}}(z_{t+1},\dots,z_T) = \hat{W}_{s_t}(z_{t+1},\dots,z_T)$ by (47). Thus, (49) holds at t as well.

Step 3: Finally, we argue as in Section D.1 that the CEVU representation with utilities $\tilde{U}_{s_t}$ and $\tilde{u}_{s_t}$ is an active learning representation, i.e., that there exists $\delta > 0$ such that for all s_t and z_t we have

$$\tilde{u}_{s_t} = \frac{1}{\delta}\sum_{s_{t+1}} \mu_{t+1}^{s_t,z_t}(s_{t+1})\tilde{u}_{s_{t+1}}. \tag{51}$$

To see this, note first that by (47) and Condition 3, $\tilde{u}_{s_t}$ is nonconstant for each s_t. Moreover, for any $s_{T-1} \in S_{T-1}$ and $\ell_{T-1}, \ell_T \in \Delta(Z)$, we have

$$\tilde{U}_{s_{T-1}}(\ell_{T-1}, \ell_T) = \tilde{u}_{s_{T-1}}(\ell_{T-1}) + \mathbb{E}[\tilde{u}_T(\ell_T)|s_{T-1}],$$

where $\mathbb{E}[\tilde{u}_T(\ell_T)|s_{T-1}] := \sum_{s_T} \mu_T^{s_{T-1}(s_T), z_{T-1}} \tilde{u}_{s_T}(\ell_T)$ does not depend on z_{T-1} by (47). Arguing exactly as in Lemma 10, we can invoke Axiom 19 to show that $\mathbb{E}[\tilde{u}_T|s_{T-1}]$ and $\tilde{u}_{s_{T-1}}$ represent the same preference over $\Delta(Z)$. By (48) and because $\tilde{u}_{s_{T-1}}$ is nonconstant, this implies that there exists $\delta_{s_{T-1}} > 0$ for each s_{T-1} such that $\mathbb{E}[\tilde{u}_T|s_{T-1}] = \delta_{s_{T-1}} \tilde{u}_{s_{T-1}}$. Invoking Axiom 20 and arguing as in Lemma 11, we can show that $\delta_{s_{T-1}} = \delta_{s'_{T-1}} =: \delta$ for all s_{T-1}, s'_{T-1}.

Next, assuming that we have established (51) for all $t' \geq t$, (47) implies that

$$\tilde{U}_{s_{t-1}}(\ell_{t-1}, \dots, \ell_T) = \tilde{u}_{s_{t-1}}(\ell_{t-1}) + \sum_{k=0}^{T-t} \delta^k \mathbb{E}[\tilde{u}_t(\ell_{t+k})|s_{t-1}],$$

where $\mathbb{E}[\tilde{u}_t|s_{t-1}] := \sum_{s_t} \mu_t^{s_{t-1}(s_t), z_{t-1}} \tilde{u}_{s_t}$ does not depend on z_{t-1} by (47). Again invoking (48), Axioms 19–20, and similar arguments as in Lemmas 10–11, we can then show that (51) also holds at $t-1$.

Necessity:
Suppose ρ admits an active learning representation. Then for each $t \leq T-1$, s_t, and $(\ell_k)_{k=t}^T \in \Delta(Z)$, we have $U_{s_t}(\ell_t, \dots, \ell_T) = \sum_{k=0}^{T-t} \delta^k u_{s_t}(\ell_{t+k})$. Then (46) holds, which by Lemma 27 implies Axiom 18. Moreover, Axioms 19–20 are verified in an analogous manner to the necessity direction of Theorem 3.