A DESIGN FOR MARKET SOCIALISM

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Abstract. Socialism is conceptualized as a society in which individuals cooperate, distinguished from capitalism, characterized as involving ubiquitous economic competition. Here, I embed a formal model of cooperation in an Arrow-Debreu model, using the Kantian optimization protocol, and define a *Walras-Kant equilibrium*, in which firms maximize profits, consumers choose demands for commodities in the usual utility-maximizing fashion, and the state rents capital to firms. The labor-supply decision of workers, however, is arrived at using the cooperative protocol. Incomes are redistributed through a flat income tax. Walras-Kant equilibria, with *any* desired degree of income equality exist, are decentralizable, and are Pareto efficient.

Key words: market socialism, general equilibrium, first welfare theorem, cooperation, Kantian optimization

JEL codes: D50, D59, D63, P20

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> In my view, the principal problem that faces the socialist ideal is that we do not know how to design the machinery that would make it run. Our problem is not, primarily, human selfishness, but our lack of a suitable organizational technology: our problem is a problem of design. It may be an insoluble design problem, and it is a design problem that is undoubtedly exacerbated by our selfish propensities, but a design problem, so I think, is what we've got. (G.A. Cohen, 2009)

1. The design problem

So wrote the philosopher G.A. Cohen in his last work, published posthumously. To explicate the problem Cohen was describing: we know that, if economic agents are self-regarding, and a suitable set of conditions hold, then decentralized optimization in a capitalist economy by individuals and firms engenders, at equilibrium, a Pareto efficient allocation (the first theorem of welfare economics). Suppose, now, that individuals wish to cooperate with each other, wish to behave in a solidaristic manner. In a word, they possess a 'socialist ethos.' Is there an economic mechanism that can decentralize decision-making in a market economy, harnessing that cooperative impulse, to deliver a Pareto efficient *and equitable* allocation? This is the Cohen design problem.

In this article, I propose such a design. Cooperation is modeled as Kantian optimization in the labor-supply decisions of workers. Kantian optimization as a method of decentralizing cooperative behavior has been studied in Roemer [2010, 2015, in press]. What's new about the present analysis is inserting Kantian optimization into an Arrow-Debreu economy, in which firms are privately owned and maximize profits in the traditional manner.

The models presented can be thought of as describing a blueprint for *market socialism*. The market-socialist tradition proposes to replace central planning with markets – or, in the design of Oscar Lange and Fred Taylor (1938)—with pseudo markets. In all models of market socialism heretofore, firms have been owned by the state, or by collectives of workers, or by other institutions, rather than by private citizens. James Meade's (1962) model of a 'property-owning democracy' achieved a degree of equalization of capital income by providing incentives to distribute estates in a diffuse manner; he advocated both comprehensive education and a eugenics policy as instruments for equalizing labor incomes. In my earlier proposal (Roemer [1994]), all young adults were issued a property right to dividends in the nation's publicly owned firms; firm shares could be traded on a special stock market, but could not be liquidated for cash. The intention was, again, to equalize (roughly) the distribution of capital income. In Giacomo Corneo's (in press) recent contribution, firms are again state-owned by a 'federal shareholder,' profits escheat to the Treasury, and are used to finance demogrants to citizens and investment.

Market-socialist models to date have not modeled cooperation or solidarity among citizens although these features are at the heart of the socialist ideal. Despite the importance of cooperation to the socialist vision, existing models present no explicit conception of how people would *behave* differently (cooperatively) in a socialist society from how they behave (autarchically) in a capitalist economy. In market-socialist models heretofore, agents are presumed to optimize in the same way that Arrow-Debreu agents optimize, maximizing a self-regarding preference order subject to constraints. One might suppose that socialist citizens would possess preferences with an altruistic element in them. However, I have not seen any market-socialist models with this property – and in any case, if an agent is small in the economy, it is unclear whether his having a preference order with an altruistic character would produce equilibria any different from one in which agents are entirely self-regarding. (See Dufwenberg et al [2011].) After all, if an agent is small, what difference would his altruistic contribution make, and would this small contribution outweigh the personal cost he sustains by making it? The preferences of agents are standard and self-regarding in my proposal.

Income taxation is the redistributive mechanism here. The key observation is that Kantian (as opposed to Nash) optimization in the labor-supply decision nullifies the usual deadweight loss incurred with income taxation. Any degree of post-fisc income equality can be achieved without sacrificing Pareto efficiency. The economic mechanism is decentralized, efficient, and as equal as citizens choose it to be, through a presumably democratic choice of the tax rate.

Combining an Arrow-Debreu structure with additive Kantian optimization offers a solution to Cohen's design problem. In section 2, I briefly review the definition of Kantian optimization, and define a *Walras-Kant (*W-K) *equilibrium with taxation.* In section 3, I present the welfare theorem: under general conditions, Walras-Kant equilibria are Pareto efficient, at any tax rate. Section 4 presents an example. Section 5 proves the existence of W-K equilibria at almost any tax rate. Section 6 discusses the psychology of Kantian optimization and concludes.

2. The proposed design[1]

A. Kantian equilibrium in games

I begin by reminding the reader of the definition of Kantian equilibrium in a game (Roemer [2009, 2015]). Consider a game with *n* agents, each of whom plays a strategy E^i chosen from an interval I^i of real numbers. Denote the payoff function of player *i* by $V^i : I^1 \times ... \times I^n \to \Re$.

Definition 2.1 An *additive Kantian equilibrium of the game* $\{\mathbf{V}, n, \mathbf{I}\}$ is a strategy profile $\mathbf{E} = (E^1,...,E^n)$ such that *no* player would like to add *any* scalar to *all* strategies; that is:

$$(\forall i)(0 = \arg\max_{\rho} V^i(E^1 + \rho,...,E^n + \rho) . \qquad (2.1)$$

Mathematically, additive Kantian equilibrium has a similar structure to Nash equilibrium. The difference is in the counterfactuals the agents contemplate when considering a deviation from a given strategy profile. In Nash equilibrium each agent imagines changing her strategy while all other agents stand pat. In Kantian equilibrium, each agent imagines changing all strategies in a symmetric manner. In Nash optimization, agents imagine choosing their best option from *different* sets of counterfactual strategy profiles; in Kantian optimization, all agents imagine choosing their best option from the

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[1] Notational conventions: Scalar variables are italic, vectors and matrices are boldface roman. If $\mathbf{z} = (z^1,...,z^n)$ is a vector, then $z^S \equiv \sum z^i$. Denote the j^{th} partial derivative of any real-valued function *f* by f_j .

same set of counterfactual profiles. Choosing their favorite strategy profile from a *common set* of counterfactuals is the mathematical form that solidarity/cooperation takes.

Definition 2.3 A game $\{\mathbf{V},n,\mathbf{I}\}$ is *(strictly) monotone increasing* if for each *i*, V^i is (strictly) increasing in the strategies of all other players $j \neq i$. A game $\{\mathbf{V},n,\mathbf{I}\}$ is *(strictly) monotone decreasing* if for each *i*, V^i is (strictly) decreasing in the strategies of all other players.

Monotone increasing games are ones with positive externalities – an example being where the strategies are contributions to a public good. Monotone decreasing games are ones with negative externalities – an example being the tragedy of the commons.

Any additive Kantian equilibrium of a strictly monotone game is Pareto efficient in the game. Thus, Kantian optimization decentralizes an efficient solution of games with positive and negative externalities. It solves free-rider problems. This provides a justification for viewing Kantian optimization as a model of cooperative behavior.

B. A labor-supply game

(i) The economic environment

I begin by defining an economic environment. There are two produced private goods and a homogeneous kind of labor, measured in efficiency units. There are two firms, each of which produces one of the goods from inputs of labor and capital, using production functions *G* and *H* respectively, which map $\Re^2_+ \to \Re_+$. Worker *i* is endowed with ω^i units of labor in efficiency units, and receives a profit share θ^{il} from Firm l , for $l = 1,2$. The state owns fractions θ^{0l} of firm $l = 1,2$, and is endowed with K_0 units of the capital good. Good 1 is used both for consumption and capital, and Good 2 is a pure consumption good. The state uses its capital to finance investment in the two firms, and the private agents spend their incomes on consumption of the two goods. Private agent *i* has preferences over the two consumption goods and labor expended (in efficiency units) represented by a utility function $u^i : \Re_+^2 \times [0,\omega^i] \to \Re_+$. All activity takes place in a single period.

For purposes of expositional simplicity, I have specified an environment with two produced commodities and two firms. All the results below generalize to any number of firms producing any number of commodities, in the obvious manner.

Firms are traditional – they are price-takers and demand capital and labor and supply commodities to maximize profits. A flat tax at an exogenous rate $t \in [0,1]$ will be levied on all private incomes, with the tax revenues returned to the population as a demogrant. Given their incomes (which consist of after-tax wages, capital income and the demogrant) and their labor supply, producer-consumers choose the optimal commodity bundle in the classical way. However, the determination of labor supply, and hence of income, is non-traditional – that is to say, the worker does not choose her labor supply in the Nash manner. A vector of labor supplies must be an *additive Kantian equilibrium* of a game to be defined below.

(ii) The game

Let (p_1, p_2, w, r) be a price vector where p_l is the price of commodity *l*, *w* is the wage rate for labor in efficiency units, and r is the interest rate on capital. Let (E^{i1}, E^{i2}) be a labor supply vector by agent *i* to Firms 1 and 2. Thus the vector of labors supplied to Firm *G* is $\mathbf{E}^1 = (E^{11},...,E^{n1})$ and the vector of labors supplied to Firm *H* is $\mathbf{E}^2 = (E^{12},...,E^{n2})$. Fix the capital levels K^l, $l = 1,2,$ of the two firms. Define the *income* of private agent *i* at $(\mathbf{E}^1, \mathbf{E}^2)$ under an affine income tax at rate *t* as:

$$I^i(E^i, E^{S1}, E^{S2}) = (1-t)wE^i + (1-t)(\theta^{i1}\Pi^1(K^1, E^{S1}) + \theta^{i2}\Pi^2(K^2, E^{S2})) + \frac{t}{n}\Big(p_1 G(K^1, E^{S1}) + p_2 H(K^2, E^{S2}) - \theta^{01}\Pi^1(K^1, E^{S1}) - \theta^{02}\Pi^2(K^2, E^{S2}) - rK^1 - rK^2\Big) \tag{2.2}$$

where the profits of the two firms are defined by:

$$\Pi^1(K^1, E^{S1}) \equiv p_1 G(K^1, E^{S1}) - wE^{S1} - rK^1, \;\; \Pi^2(K^2, E^{S2}) \equiv p_2 H(K^2, E^{S2}) - wE^{S2} - rK^2 \tag{2.3}$$

and E^{Sl} is the labor supplied to Firm *l*. The last term on the r.h.s. of (2.2) is the value of the demogrant, equal to the per capita share of total tax revenues (where taxes are levied on all private incomes but not on the state's income).

The income of the state is:

$$I^0 = \theta^{01}\Pi^1(K^1,E^{S1}) + \theta^{02}\Pi^1(K^2,E^{S2}) + r(K^1+K^2)\ . \tag{2.4}$$

That is, the state receives its share of firms' profits plus the return on its investment, but this is not taxed, which explains the specification of the demogrant in equation (2.2).

Now suppose that every (private) agent were to increase her total labor by a constant ρ, positive or negative. Then *i*'s hypothetical income would be:

$$\begin{aligned} &I^i(E^i+\rho, E^{S1}+\lambda n\rho, E^{S2}+(1-\lambda)n\rho) = (1-t)w(E^i+\rho) + \\ &(1-t)(\theta^{i1}\Pi^1(K^1,E^{S1}+\lambda n\rho) + \theta^{i2}\Pi^2(K^2,E^{S2}+(1-\lambda)n\rho)) + \\ &\frac{t}{n}\begin{pmatrix} p_1G(K^1,E^{S1}+\lambda n\rho) + p_2H(K^2,E^{S2}+(1-\lambda)n\rho) - \\ \theta^{01}\Pi^1(K^1,E^{S1}+\lambda n\rho) - \theta^{02}\Pi^2(K^2,E^{S2}+(1-\lambda)n\rho) - r(K^1+K^2) \end{pmatrix} \end{aligned} \tag{2.5}$$

where fraction λ of the total increase in labor $n\rho$ is allocated to Firm 1, and fraction $(1-\lambda)$ to Firm 2. We need not adopt a rule for how each agent would allocate her additional labor ρ between the two firms, as this will turn out not to matter. It is assumed that workers are price takers: in particular, they take the wage *w* as given.

A comment on the logic behind equation (2.5) is in order. A *Nash player,* who chooses his labor supply while assuming all other labor supplies remain fixed, need not consider the effect of his labor-supply decision on either the profits of firms in which he works or owns equity, or upon the demogrant, if the economy is large. Hence, our practice in Nash-type analysis is to ignore these effects. But in Kantian optimization, the counterfactual the worker envisages is that *all* workers change their labor supplies in the same amount as the change he is contemplating, and hence consistency in the thought experiment requires that we alter the labor supplies to firms, and the value of the demogrant, accordingly. Hence, the formulation of equation (2.5).

At this counterfactual labor supply by worker *i*, $E^{i1}+E^{i2}+\rho$, given her income as specified by (2.5) , let the agent compute her commodity demands, which are the solution of the program:

$$\max_{x,y} u^i(x,y,E^i+\rho)$$
$$\text{subj. to} \tag{2.6}$$
$$p_1x+p_2y=I^i(E^i+\rho,E^{S1}+\lambda n\rho,E^{S2}+(1-\lambda)n\rho)$$

Denote the solution to this program by $(x^i(I^i[\rho],E^i+\rho),y^i(I^i[\rho],E^i+\rho))$, where I abbreviate with the notation $I^i[\rho]\equiv I^i(E^i+\rho,E^{S1}+\lambda n\rho,E^{S2}+(1-\lambda)n\rho)$.

We now define the payoff functions of a game. The payoff to agent *i* is his utility at prices (p_1,p_2,w,r) if the capital invested in the firms is (K^1,K^2) , and the vector of labor supplies $(E^1,...,E^n)$ were to determine wage income, profit income, and the value of the demogrant, that is:

$$V_+^i(E^1,...,E^n)=u^i(x^i(I^i[0],E^i),y^i(I^i[0],E^i),E^i) \ . \tag{2.7}$$

Incorporated in the payoff function is the assumption that at her personal part of the community effort vector, agent *i* has chosen her *commodity* demands optimally, given the income generated.

Thus, given a vector of prices $\mathbf{p}=(p_1,p_2,w,r)$, and the ownership shares of firms, a game whose strategies are effort/labor supplies is defined, denoted $\mathbf{V}_+$. We can define its additive Kantian equilibrium, which is a vector of labor supplies $\mathbf{E}=(E^1,...,E^n)$ satisfying (2.1): that is to say, a vector $\mathbf{E}$ such that:

$$(\forall i)(\arg\max_{\rho} u^i(x^i(I^i[\rho],E^i+\rho),y^i(I^i[\rho],E^i+\rho),E^i+\rho)=0). \tag{2.8}$$

C. <u>Walras-Kant</u> <u>equilibrium</u> <u>with</u> <u>taxation</u>

The data of the economy are $(u^1,...,u^n;G,H;\omega^1,...,\omega^n;\{\theta^{il},i=0,...,n;l=1,2\};K_0)$. K_0 is the state's endowment of the capital good, which will finance the firm's investment. It is useful, for conceptualizing Pareto efficiency, to define the 'utility function of the state,' which is:

$$u^0(x,y)=x \ . \tag{2.9}$$

That is, the state cares only about Good 1, which it uses in its role as the capital good.

We now define a *Walras-Kant (additive) equilibrium at tax rate t,* to consist of:

i. a price vector (p_1, p_2, w, r),

ii. labor and capital demands by the two firms of D^1, D^2 and K^1, K^2, respectively,

iii. labor supplies (E^{i1}, E^{i2}) by all workers *i* to Firms 1 and 2,

iv. for all private agents *i*, commodity demands (x^i, y^i) for the outputs of Firms 1 and 2, resp., *such that*:

v. at given prices, (K^l, D^l) maximizes profits of Firm *l*, for *l* = 1, 2,

vi. the labor supply vector $\mathbf{E} = (E^1, ..., E^n)$, where $E^i = E^{i1} + E^{i2}$, constitutes an additive Kantian equilibrium at the given prices of the game $\mathbf{V}_+$, as defined in (2.7),

vii. (x^i, y^i) maximizes the utility of agent *i*, given prices, her labor supply, and her income, given by (2.2),

viii. a supply of capital K_0 by the state to the two firms that maximizes its utility u^0 subject to its budget constraint $p_1 x^0 = I^0$ at given firm demands of labor, and

ix. all markets clear; that is, $D^l = \sum_i E^{il}$ for $l = 1, 2$, $x^S = G(K^1, D^1)$ $y^S = H(K^2, D^2)$, and $K_0 = K^1 + K^2$.

The depreciation rate of capital is set at zero. Thus, at the beginning of the next period the state's endowment of the capital good will be $K_0 + \frac{I^0}{p_1}$ (see eqn. (2.4)). Indeed condition (viii) of the definition of equilibrium is superfluous, and is included only for completeness. For at equilibrium, the firms have maximized joint profits, at the given price vector, which maximizes the state's income, subject to condition that $K^1 + K^2 = K_0$.

3. The first welfare theorem for market socialism

The appropriate concept of Pareto efficiency will be called *investment constrained Pareto efficiency* (ICPE). An allocation is ICPE if there is no other feasible allocation that makes at least some agent better off without harming any agent, where the state is included as an agent. Since the model is not intertemporal, it is important to qualify the kind of Pareto efficiency that can be realized: citizens cannot trade off future consumption against present consumption in the model, and hence we cannot speak of efficiency in the full sense. To say this more straightforwardly: the state's investment is determined by its endowment of capital, not by any considerations of the population's future welfare. We know that both the Soviet Union and post-1949-China probably invested too much, committing their populations to excessively low consumption. That can happen in this model, too.

It is easy to show that, with differentiability, an interior allocation[2] is ICPE exactly when:

$$\begin{aligned}
&(a)\sum_{i=0}^{n} x^i = G(K^1, E^{1S}) \quad (b)\sum_{i=1}^{n} y^i = H(K^2, E^{2S})\\
&(c)\sum_{i=1}^{n} E^i = E^{1S} + E^{2S} \quad (d) K^1 + K^2 = K_0\\
&(e)\,(\forall i)\; G_1(K^1, E^{1S}) = -\frac{u_3^i(x^i, y^i, E^i)}{u_1^i(x^i, y^i, E^i)} \quad (f)\,(\forall i)\; H_1(K^2, E^{2S}) = -\frac{u_3^i(x^i, y^i, E^i)}{u_2^i(x^i, y^i, E^i)}\\
&(g)\frac{G_2}{G_1} = \frac{H_2}{H_1}
\end{aligned} \tag{3.1}$$

where $G_1 \equiv \frac{\partial G}{\partial K}$, etc.

Conditions (a)-(d) specify feasibility; conditions (e)-(g) specify efficiency.

<u>Theorem</u> <u>1.</u> *Assume differentiability of the production functions and the utility functions. Assume that the production functions are concave and the utility functions are strictly concave. Let* $(p_1, p_2, w, r, \mathbf{E}^1, \mathbf{E}^2, D^1, D^2, K^1, K^2, \mathbf{x}, \mathbf{y})$ *comprise a Walras-Kant (additive) equilibrium at any income tax rate* $t \in [0,1]$. *Then the induced allocation is investment-constrained Pareto efficient.*

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[2] An allocation is called *interior* if all private agents consume positive amounts of both commodities and leisure, and all supply positive amounts of labor (but it is not necessary that any agent supplies labor to both firms).

At a Walras-Kant equilibrium at tax rate t, profit-maximization gives:

$$p_1 G_2(K^1, E^{S1}) = w = p_2 H_2(K^2, E^{S2}) \text{ and } p_1 G_1 = r = p_2 H_1 \,, \tag{3.2}$$

and clearing of the capital market tells us that $K^1 + K^2 = K_0$. Therefore, it follows from (3.1) that an interior equilibrium is ICPE if and only if:

$$(\forall 1 \le i \le n)(-\frac{u_3^i(x^i, y^i, E^i)}{u_1^i(x^i, y^i, E^i)} = \frac{w}{p_1} \;\&\; -\frac{u_3^i(x^i, y^i, E^i)}{u_2^i(x^i, y^i, E^i)} = \frac{w}{p_2}) \quad (3.3)$$

.

<u>Proof of Theorem 1:</u>

1. Although the theorem's statement assumes the equilibrium is interior, this is easy to relax, with a concomitant alteration of the first-order conditions.

2. Consider the program:

$$\begin{aligned} &\max_{x,y} u^i(x,y,E) \\ &\text{subj. to} \\ &p_1 x + p_2 y = I \end{aligned}$$

where E and I are fixed. Denote the solution $(x^i(I;E), y^i(I;E))$. The f.o.c.s for the solution of the program are:

$$\begin{aligned} &p_2 u_1^i(x^i(I;E), y^i(I;E), E) - p_1 u_2^i(x^i(I;E), y^i(I;E), E) = 0 \\ &p_1 x^i(I;E) + p_2 y^i(I;E) - I = 0 \end{aligned} \tag{3.4}$$

By the implicit function theorem, the functions (x^i, y^i) are differentiable and their derivatives are given by:

$$x_1^i(I,E) = \frac{p_1 u_{22}^i - p_2 u_{12}^i}{(p_2, -p_1)\mathbf{U}^i(p_2, -p_1)^T} \,, \tag{3.5}$$

$$y_1^i(I,E) = \frac{p_2 u_{11}^i - p_1 u_{12}^i}{(p_2, -p_1)\mathbf{U}^i(p_2, -p_1)^T} \,, \tag{3.6}$$

$$x_2^i(I,E) = \frac{p_2(p_1 u_{23}^i - p_2 u_{13}^i)}{(p_2, -p_1)\mathbf{U}^i(p_2, -p_1)^T}, \text{ and} \tag{3.7}$$

$$y_2^i(I,E) = \frac{p_1(p_2 u_{13}^i - p_1 u_{23}^i)}{(p_2, -p_1)\mathbf{U}^i(p_2, -p_1)^T} \,, \tag{3.8}$$

where $\mathbf{U}^i$ is the leading principal sub-matrix of order two of the Hessian of the function u^i, and the superscript ^{T} indicates 'transpose.' Note that the implicit function theorem indeed applies because $\mathbf{U}^i$ is negative definite by the strict concavity of u^i, and so the denominators of equations (3.5)-(3.8) do not vanish.

4. Now the labor-supply vector is an interior additive Kantian equilibrium of the game $\mathbf{V}_+$ if and only if :

   for all *i:*

$$\left.\frac{d}{d\rho}\right|_{\rho=0} u^i(x^i(I^i[\rho],E^i+\rho),y^i(I^i[\rho],E^i+\rho),E^i+\rho)=0 \text{ for } i\geq 1 . \qquad (3.9)$$

This statement reduces to:

$$u_1^i\cdot\left(x_1^i I^{i\prime}[0]+x_2^i\right)+u_2^i\cdot\left(y_1^i I^{i\prime}[0]+y_2^i\right)+u_3^i=0 , \qquad (3.10)$$

where $I^{i\prime}[0]\equiv\left.\frac{dI^i(E^i+\rho,E^{S1}+\lambda n\rho,E^{S2}+(1-\lambda)n\rho)}{d\rho}\right|_{\rho=0}$.

5. From (2.5), calculate that:

$$\begin{aligned}I^{i\prime}[0]&=(1-t)w+(1-t)\left(\theta^{i1}\Pi_2^1(K^1,E^{S1})\lambda n+\theta^{i2}\Pi_2^2(K^2,E^{S2})(1-\lambda)n\right)+\\ &\frac{t}{n}\begin{pmatrix}p_1G_2(K^1,E^{S1})\lambda n+p_2H_2(K^2,E^{S2})(1-\lambda)n-\\ \left(\theta^{01}\Pi_2^1(K^1,E^{S1})\lambda n-\theta^{02}\Pi_2^2(K^2,E^{S2})(1-\lambda)n\right)\end{pmatrix}\end{aligned} \qquad (3.11)$$

Since the two partial derivatives $(\Pi_1^1,\Pi_2^1,\Pi_1^2,\Pi_2^2)$ of the firms' profit functions are zero, by profit maximization, and $p_1G_2=p_2H_2=w$, (3.11) reduces to:

$$I^{i\prime}[0]=(1-t)w+tw=w , \qquad (3.12)$$

for any *t*. It is now evident why we did not have to specify how workers allocate the increment ρ in labor between the two firms: that allocation does not affect the validity of (3.12).

We therefore write the condition for Kantian equilibrium of labor supplies, equation (3.10), as:

$$u_1^i \cdot \left(x_1^i w + x_2^i\right) + u_2^i \cdot \left(y_1^i w + y_2^i)\right) + u_3^i = 0 \ . \tag{3.13}$$

6. We now expand equation (3.13) by making a sequence of substitutions: (i) substitute the expressions for the four derivatives of the x^i and y^i functions from (3.5) through (3.8), and (ii) eliminate p_1 via the substitution $p_1 = \frac{p_2 u_1^i}{u_2^i}$, the f.o.c. from (3.4). So doing reduces (3.13) to:

$$\begin{aligned} &\frac{u_1^i}{d}\left(w\frac{p_2 u_1^i}{u_2^i}u_{22}^i - wp_2 u_{12}^i + p_2(\frac{p_2 u_1^i}{u_2^i}u_{23}^i - p_2 u_{13}^i)\right) + \\ &\frac{u_2^i}{d}\left(wp_2 u_{11}^i - wp_2\frac{u_1^i}{u_2^i}u_{12}^i + \frac{p_2 u_1^i}{u_2^i}(p_2 u_{13}^i - \frac{p_2 u_1^i}{u_2^i}u_{23}^i)\right) + \\ &u_3^i = 0, \end{aligned} \tag{3.14}$$

where $d = (p_2, -p_1)\mathbf{U}^i(p_2, -p_1)^T = p_2^2\left(\frac{u_1^i}{u_2^i}, -1\right)\mathbf{U}^i\left(\frac{u_1^i}{u_2^i}, -1\right)^T$, which is a negative number. Finally, divide both sides of equation (3.14) by the positive number u_2^i, simplify, and calculate that that equation reduces to:

$$\frac{w}{p_2} = -\frac{u_3^i}{u_2^i} \ , \tag{3.15}$$

which is one of the two required efficiency conditions for agent *i*.

7. Now substitute for p_2 in the last equation using $p_2 = \frac{p_1 u_2^i}{u_1^i}$, yielding:

$$\frac{w}{p_1} = -\frac{u_3^i}{u_1^i}. \tag{3.16}$$

By equations (3.15), (3.16) and (3.2), the theorem is proved. ν

The key move in the proof is to show that, regardless of the tax rate, when a worker thinks of all workers as varying their labor supplies in the amount she is contemplating varying her own, she internalizes the externality generated by her labor-supply choice – a choice that affects firm profits and tax revenues. Her own action causes a negligible change in these magnitudes, but of course the aggregate effect of many small changes is significant. The *additive counterfactual* in the universal change in labor supplies and *affine income taxation* combine in such a way as to exactly cancel the deadweight loss of taxation that afflicts Nash optimization in the labor-supply decision. (This is the meaning of equation (3.12), the key to the proof.) This kind of pairing – associating a specific cooperative optimization protocol with a particular allocation rule, where the two together deliver Pareto efficiency – is a feature of Kantian equilibrium in simpler (non-market) environments, as studied in Roemer (in press). What's new here is combining additive Kantian optimization with markets.

A remark on why the incentive problem, causing deadweight losses in the standard model, does not bite here. Consider, for dramatic effect, an income tax rate of one, and suppose every worker is supplying zero labor (as she would in the standard model at this tax rate). But here, by using the Kantian optimization protocol, a worker balances her share of an increase in income that would occur if *all* workers increased their labor supply from zero to some small positive ρ against her (very small) disutility of labor at zero. The trade-off is usually worth it. Consequently, at the Kantian equilibrium, even at a tax rate of unity, (most) workers will supply a positive amount of labor.

4. An example of Walras-Kant (additive) equilibrium

Because capital allocation is passive in this model, let's simplify by studying an economic environment where the capital inputs are fixed, there is no state, and we model production as a function of labor only:

$$\begin{gathered} G(E) = E - \frac{a}{2}E^2,\ H(E) = E - \frac{b}{2}E^2 \\ u^i(x,y,E) = x^{1/3}y^{1/3}(\omega^i - E)^{1/3},\ \text{for } i = 1,\dots,n \end{gathered} \tag{3.17}$$

Agent *i*'s endowment of efficiency units of labor is ω^i. There are *n* agents, and the total endowment of labor is $\omega^S = \sum_i \omega^i$. We let $\theta^{i1} = \theta^{i2} = \frac{1}{n}$ for all $1 \le i \le n$. We set $\theta^{01} = \theta^{02} = 0$. We normalize the price vector by choosing $w = 1$. There is no market for capital and hence no interest rate.

An interior allocation is a Walras-Kant (additive) equilibrium at income tax rate *t* when the allocation is Pareto efficient, the income of *i* is given by (2.2), and markets clear. (The critical condition that the labor supplies comprise a Kantian equilibrium of the game $\mathbf{V}_+$ is embedded in the efficiency conditions, as the proof of theorem 1 shows.) We write these conditions as:

$$\begin{array}{l} 1-aE^{1S}=\dfrac{1}{p_1}, 1-bE^{2S}=\dfrac{1}{p_2} \quad (\mathrm{MRT}^i = w/p_i) \\ \dfrac{x^i}{\omega^i - E_i}=\dfrac{1}{p_1}, \quad \dfrac{y^i}{\omega^i - E_i}=\dfrac{1}{p_2} \quad (\mathrm{MRT}^i = MRS) \end{array} \tag{3.18}$$

and (2.2) holds for all i. By (3.18), the post-fisc income of agent i is given by $I^i = p_1 x^i + p_2 y^i = 2(\omega^i - E_i)$. Hence, (2.2) can be written:

$$\begin{array}{l} 2(\omega^i - E^i) = (1-t)E^i + \dfrac{1-t}{n}(\Pi^1 + \Pi^2) + \dfrac{t}{n}(G(E^{S1}) + H(E^{S2})) = \\ (1-t)E^i + \dfrac{p_1 G(E^{S1}) + p_2 H(E^{S2})}{n} - \dfrac{(1-t)}{n}(E^{S1} + E^{S2}) \end{array} . \tag{3.19}$$

By adding up the equations over all i in (3.18), we have:

$$\begin{array}{l} p_1 x^S = p_1 G(E^{S1}) = \omega^S - (E^{S1} + E^{S2}), \\ p_2 y^S = p_2 H(E^{S2}) = \omega^S - (E^{S1} + E^{S2}) \end{array} \tag{3.20}.$$

Now using the expressions for commodity prices in (3.18), we write these equations as:

$$\frac{G(E^{S1})}{1-aE^{S1}} = \omega^S - (E^{S1} + E^{S2}), \quad \frac{H(E^{S2})}{1-bE^{S2}} = \omega^S - (E^{S1} + E^{S2}). \tag{3.21}$$

System (3.21) comprises two equations in the two unknowns E^{S1} and E^{S2} ; the solution must be a vector $(E^{S1}, E^{S2}) \in (0, \frac{1}{a}) \times (0, \frac{1}{b})$. Thus total production at Walras-Kant equilibrium for this economy, if such exists, is independent of the tax rate *t*. Profits are also independent of *t*. Taxation simply redistributes a fixed output of commodities.

Parameterize the example with $(a,b) = (0.1, 0.2)$, $\omega^S = 10, n = 100$. We have not yet specified the individual endowments ω^i . We solve (3.21):

$$E^{S1} = 3.28, \quad E^{S2} = 2.63 \ . \tag{3.22}$$

Profits are positive for both firms, and comprise 28% of national income, a reasonable approximation to reality.

To complete the analysis, we must specify the $\{\omega^i\}$ and solve for $\{E^i\}$. Rewrite equation (3.19) as:

$$2\omega^i + E^i(t-3) = \frac{\Pi^1+\Pi^2}{n} + \frac{t}{n}(E^{S1}+E^{S2}) . \tag{3.23}$$

Examination shows that equation (3.23) possesses an interior solution in which $E^i \in (0, \omega^i]$ for all *i* exactly when:

for all *i*,
$$\omega^i > \frac{1}{2n}(\Pi^1+\Pi^2+t(E^{S1}+E^{S2})) . \tag{3.24}$$

If, on the other hand, (3.24) is false for some *i*, then there is no interior equilibrium.

It is of interest to compute the lower bound on the labor endowment that will guarantee an interior Walras-Kant equilibrium at tax rate *t*. From (3.24) , this depends upon the tax rate. We compute this lower bound for various tax rates for our example:

t	min ω
0.	0
0.1	0
0.2	0
0.3	0
0.4	0
0.5	0.00344757
0.6	0.0109318
0.7	0.0184161
0.8	0.0259003
0.9	0.0333846
1.	0.0408689

Table. The minimum value of ω^i supporting an interior Walras-Kant equilibrium as a function of the tax rate

Recall that the average labor endowment with our parameterization is $\frac{\omega^S}{n} = 0.1$. From the table, a Walras-Kant (additive) equilibrium exists where all agents work regardless of the distribution of individual labor endowments, as long as $t \le 0.4$. But as the tax rate rises, the restriction on the distribution of labor endowments bites.

For tax rates larger than 40%, equilibrium still exists, but workers who are insufficiently skilled do not work. We illlustrate with a second paramaterization. The utility functions and production parameters are as before, but we examine an economy with two agents (n =2), where $\omega^1 = 9, \omega^2 = 1$. If both agents work, then E^{S1} **and** E^{S2} are given by (3.22). Let us look for an equilibrium where $t = 1$. Both agents must then have the same after-tax income. Inequality (3.24) is false for agent 1, so there is no equilibrium at $t = 1$ where both agents work. We therefore set agent 2's labor supply to zero: $E^2 = 0$. The other equations characterizing a Walras-Kant equilibrium are:

$$\begin{aligned}
&\frac{x^1}{\omega^1 - (E_1^1 + E_1^2)} = \frac{1}{p_1}, \frac{y^1}{\omega^1 - (E_1^1 + E_1^2)} = \frac{1}{p_2},\\
&G'(E^{11}) = \frac{1}{p_1}, \quad H'(E^{12}) = \frac{1}{p_2}\\
&x^1 = x^2, \quad y^1 = y^2,\\
&G(E^{11}) = x^1 + x^2, H(E^{12}) = y^1 + y^2,\\
&\frac{x^2}{\omega^2} \geq \frac{1}{p_1}, \frac{y^2}{\omega^2} \geq \frac{1}{p_2},
\end{aligned} \tag{3.25}$$

The two equations in the first line say the marginal rates of substition for the agent with positive labor supply equal the correct price ratios; the second line says the marginal rates of transformation equal the correct price ratios; the third line is true because when the tax rate is 1, both agents have the same (post-fisc) income, and so consume the two commodities identically; the fourth line expresses market-clearing for the two commodities; and the fifth line expresses the efficiency condition for the agent who supplies zero labor. The solution is given by:

$$\begin{aligned}
&x^1 = x^2 = 1.513, \quad y^1 = y^2 = 1.024,\\
&p^1 = 1.592, \quad p^2 = 2.352, E_1^1 = 3.717, E_1^2 = 2.874
\end{aligned} \tag{3.26}$$

5. <u>Existence of Walras-Kant equilibrium</u>

We first note:

<u>Proposition 2</u> *Let* $(p_1, p_2, w, r, \mathbf{E}^1, \mathbf{E}^2, D^1, D^2, K^1, K^2, \mathbf{x}, \mathbf{y})$ *be a Walrasian equilibrium at* $t = 0$. *Then it is also an additive Walras-Kant equilibrium at* $t = 0$.

Proof:

We know the allocation is Pareto efficient by the (usual) first welfare theorem. The income equation (2.2) holds by definition of Walrasian equilibrium. We need only show that the labor supplies comprise a Kantian equilibrium, which is to say, that equation (3.13) holds. But we have shown that this is equivalent to the efficiency conditions that $MRS^i = MRT$. These conditions hold by hypothesis, and the claim is proved. ν

We assume:

Assumption A

(i) G,H are unbounded, concave, homothetic, and the Inada conditions hold, and

(ii) all consumer preferences are representable by strictly concave, differentiable utility functions, and both commodities are normal goods for all consumers.

Theorem 3 *Let an economic environment* $\{\mathbf{u},\theta,G,H,\omega,K_0\}$ *be given and let Assumption A hold. Suppose that* $\omega^i > 0$ *for all (private) agents and that* $\theta^{01}+\theta^{02} < 2$. *Then a Walras-Kant equilibrium exists for any* $0 \le t < 1$.

Let Δ be the 3-simplex of price vectors $\mathbf{p} = (p_1, p_2, w, r)$. Denote the boundary (interior) of the simplex by $\partial\Delta(\mathrm{int}\,\Delta)$. We define a correspondence on the domain $\mathrm{int}\,\Delta$. Let Q can be any real number, and $Q^{i*}(\cdot)$ a positive continuous function on $\mathrm{int}\,\Delta$. Let:

$$A^i(p_1,p_2,w;Q) = \arg\max_{x,y,E}\{u^i(x,y,E) \mid p_1x + p_2y = wE + Q\}$$
$$B^i(p_1,p_2,w;Q^{i*}) = \{(x,y,E) \in \Re^2_+ \times [0,\omega^i] \mid p_1x + p_2y = (1-t)wE + Q^{i*}(\mathbf{p})\} \qquad (4.1)$$

Now define $\Gamma^i : \mathrm{int}\,\Delta \to\to \Re^3_+$ by:

$$\Gamma^i(p_1,p_2,w;Q^*(\mathbf{p})) = \bigcup_{\infty > Q \ge \tilde{Q}} A^i(p_1,p_2,w;Q) \bigcap B^i(p_1,p_2,w;Q^{i*}(\mathbf{p})) \ , \qquad (4.2)$$

where $\tilde{Q} = Q^{i*}(\mathbf{p}) - tw\omega^i$. $\tilde{Q}$ may be positive, zero, or negative. Finally, define:

$$\Gamma(p_1,p_2,w;Q^*(\mathbf{p})) = \Gamma^1(p_1,p_2,w;Q^{1*}(\mathbf{p})) \times \ldots \times \Gamma^n(p_1,p_2,w;Q^{n*}(\mathbf{p})). \qquad (4.3)$$

Lemma 4 *Let* $t \in (0,1]$ *and* $(p_1, p_2, w, r) \in \text{int}\,\Delta$ *. Let* $Q^{i*} : \text{int}\,\Delta \to \Re_{++}$ *be continuous functions for all i. If Assumption A(ii) holds then* Γ *is a (non-empty) continuous function mapping* $\text{int}\,\Delta \to \prod_{i=1}^{n} (\Re_+^2 \times [0, \omega^i])$.

Proof:

1. It suffices to show that Γ^i is single-valued and continuous for any *i*. By strict concavity of preferences, the correspondence A^i is single-valued and continuous on $\text{int}\,\Delta$. Suppose that Γ^i contains two elements; i.e., there are allocations $(x_\nu, y_\nu, E_\nu) \in A^i(p_1, p_2, w; Q_\nu) \cap B^i(p_1, p_2, w; Q^*(\mathbf{p}))$, for $\nu = 1,2$, with $Q_2 > Q_1$. It follows that:

$$\begin{aligned} p_1(\delta x) + p_2(\delta y) &= w(\delta E) + (\delta Q) \\ p_1(\delta x) + p_2(\delta y) &= (1-t)w(\delta E) \end{aligned} , \tag{4.4}$$

where $\delta x \equiv x_2 - x_1$, etc. Therefore the quantities on the right-hand sides of the two equations in (4.4) are equal, implying that:

$$\delta Q = -tw\delta E \ , \tag{4.5}$$

and so $\delta E < 0$ (note $t > 0$ by assumption). Therefore :

$$p_1 x_2 + p_2 y_2 = (1-t)wE_2 + Q^*(\mathbf{p}) < (1-t)wE_1 + Q^*(\mathbf{p}) = p_1 x_1 + p_2 y_1 \tag{4.6}$$

and so either $x_2 < x_1$ or $y_2 < y_1$. But since $(x_\nu, y_\nu, E_\nu) \in A^i(p_1, p_2, w; Q_\nu)$ for $\nu = 1,2$, it must be that $x_2 > x_1$ and $y_2 > y_1$ because both commodities are normal goods, and the consumer's wealth (check the definition of A^i) is greater at $\nu = 2$ than at $\nu = 1$. This contradiction proves that Γ^i contains at most one element.

2. Next we show Γ^i contains at least one element. $B^i(p_1, p_2, w; Q^{i*})$ is a *planar segment.* We say a point (x, y, E) lies *above* (resp. *below*) the planar segment $B^i(p_1, p_2, w; Q^{i*})$ if it lies in the positive orthant and $p_1 x + p_2 y < (1-t)wE + Q^{i*}$ (resp., $p_1 x + p_2 y > (1-t)wE + Q^{i*}$). Note that the points on planar segment

$$p_1 x + p_2 y = wE + \tilde{Q}, \quad (x, y, E) \in \Re_+^2 \times [0, \omega^i]$$

lie entirely below (or, at one point, on) the planar segment $B^i(p_1, p_2, w; Q^{i*})$ because:

$$wE+\tilde{Q}=wE+Q^{i*}-tw\omega^{i}\le wE+Q^{i*}-twE=(1-t)wE+Q^{i*}\ . \tag{4.7}$$

It therefore follows that $A^{i}(p_1,p_2,w;\tilde{Q})$ lies below (or possibly on) the planar segment $B^{i}(p_1,p_2,w;Q^{i*})$. On the other hand, for large values of Q, the points of

$$p_1x+p_2y=wE+Q,\quad (x,y,E)\in\Re_{+}^{2}\times[0,\omega^{i}]$$

must lie entirely above B^{i}. Since $A^{i}(p_1,p_2,w;Q)$ is a continuous function of Q, by the Berge maximum theorem, it follows that there exists at least one value of Q such that $A^{i}(p_1,p_2,w;Q)\cap B^{i}(p_1,p_2,w;Q^{i*})\neq\varnothing$. Thus, Γ^{i} is a well-defined function.

3. Continuity of Γ^{i} follows from Berge's maximum theorem. ν

<u>Proof of Theorem 3:</u>

0. The theorem is true for $t=0$ by Proposition 2, since a Walrasian equilibrium exists at $t=0$ under the stated premises. Henceforth, we assume $0<t<1$.

1. The proof is an adaptation of Mas-Colell, Winston and Green's (1995) proof of existence of competitive equilibrium (Proposition 17.C.1). We will define a correspondence that maps the 3-simplex Δ into itself. We first define the correspondence on $\text{int}\,\Delta$. Given a price vector $(p_1,p_2,w,r)\in\text{int}\,\Delta$ define (D^1,D^2,K^1,K^2) to be the solution of:

$$\begin{aligned}(K^1,D^1)&=\underset{(K,E)}{\arg\max}(p_1G(K,E)-wE-rK)\\(K^2,D^2)&=\underset{(K,E)}{\arg\max}(p_2H(K,E)-wE-rK)\end{aligned} \tag{4.8}$$

Note that, by Assumption A(i) the solution exists and satisfies:

$$G_2(K^1,D^1)=\frac{w}{p_1},\quad H_2(K^2,D^2)=\frac{w}{p_2},G_1(K^1,D^1)=\frac{r}{p_1},H_1(K^2,D^2)=\frac{r}{p_2}\ .$$

2. The profits of the two firms and the value of the demogrant are defined at (D^1,D^2,K^1,K^2). Profits are positive for any price vector $(p_1,p_2,w,r)\in\text{int}\,\Delta$.

We now consider the budget constraints of individuals:

$$\begin{aligned}&p_1x+p_2y=(1-t)wE+(1-t)\left(\theta^{i1}\Pi^1(K^1,D^1)+\theta^{i2}\Pi^2(K^2,D^2)\right)+\\&\frac{t}{n}(p_1G(K^1,D^1)+p_2H(K^2,D^2)-\theta^{01}\Pi^1(K^1,D^1)-\theta^{02}\Pi^2(K^2,D^2)-r(K^1+K^2)\end{aligned} \tag{4.9}$$

and the budget constraint of the state at the firms' demands:

$$p_1 x^0 = \theta^{01}\Pi^1(K^1,D^1) + \theta^{02}\Pi^2(K^2,D^2) + r(K^1+K^2)\ . \tag{4.10}$$

Let $Q^{i*}(\mathbf{p})$ equal the sum of the last two terms on the r.h.s. of (4.9). By the theorem's premise, all private agents have positive income at any $(p_1,p_2,w,r)\in \mathbf{int}\,\Delta$, because the state does not receive all the firms' profits by assumption, and the tax rate is positive.

$Q^{i*}(\cdot)$ are positive continuous functions, and so the premises of Lemma 4 hold; therefore the functions $\Gamma^i(p_1,p_2,w;Q^{i*}(\mathbf{p}))$ are defined and continuous. Henceforth, we write $\Gamma^i(p_1,p_2,w;Q^{i*}) \equiv \Gamma^i(\mathbf{p})$.

4. Define the excess demand functions at a vector $\mathbf{p} = (p_1,p_2,w,r) \in \mathbf{int}\,\Delta$:

$$\begin{aligned} &\Delta E = D^1 + D^2 - \sum E^i,\ \Delta x = \sum_{i=0}^{n} x^i - G(K^1,D^1),\\ &\Delta y = \sum_{i=1}^{n} y^i - H(K^2,D^2), \quad \Delta K = K^1 + K^2 - K_0 \end{aligned}\ , \tag{4.11}$$

where the points $(x^i,y^i,E^i) = \Gamma^i(\mathbf{p})$ for $i \geq 1$. Define the excess demand function for the economy by:

$$z(\mathbf{p}) = (\Delta x, \Delta y, (1-t)\Delta E, (1-t)\Delta K)\ . \tag{4.12}$$

Next, define the correspondence $\Phi : \mathbf{int}\,\Delta \to \Delta$ by:

$$\Phi(\mathbf{p}) = \{\mathbf{q} \in \Delta \,|\, z(\mathbf{p})\cdot\mathbf{q} \geq z(\mathbf{p})\cdot\mathbf{q}',\ \text{for all } \mathbf{q}' \in \Delta\}\ .$$

$\Phi(\mathbf{p})$ is non-empty and convex because it consists of the maximanda of a linear function on a compact set.

5. Finally, for $\mathbf{p} \in \partial\Delta$, define:

$$\Phi(\mathbf{p}) = \{\mathbf{q} \in \Delta \,|\, \mathbf{p}\cdot\mathbf{q} = 0\}\ . \tag{4.13}$$

6. By summing the budget constraints in (4.9) and (4.10), we calculate Walras' Law for this economy, defined on $\mathbf{int}\,\Delta$:

$$p_1\Delta x + p_2\Delta y + (1-t)w\Delta E + (1-t)r\Delta K = z(\mathbf{p})\cdot\mathbf{p} = 0. \tag{4.14}$$

7. Suppose $\mathbf{p}=(p_1,p_2,w,r)$ is a fixed point of Φ. $\mathbf{p}$ cannot be on the boundary of the simplex, since we would have $\mathbf{p}\cdot\mathbf{p}=0$, a contradiction. Therefore $\mathbf{p}\in\operatorname{int}\Delta$. It follows by the definition of Φ and Walras' Law that $0\geq z(\mathbf{p})\cdot\mathbf{q}$ for all $\mathbf{q}\in\Delta$. This implies that $z(\mathbf{p})\leq 0$. Suppose one of the components of $z(\mathbf{p})$ were negative. This would imply the failure of Walras' Law, a contradiction. Hence $z(\mathbf{p})=0$. Consequently $\Delta x=\Delta y=\Delta E=\Delta K=0$, and all markets clear. We deduce $\Delta E=\Delta K=0$ from the premise that $1-t>0$.

8. Associated with these prices is an allocation $(\mathbf{x},\mathbf{y},\mathbf{E})$, with $(x^i,y^i,E^i)\in\Gamma^i(\mathbf{p})$ for all $1\leq i\leq n$. We must show that $(E^1,...,E^n)$ is an additive Kantian equilibrium at prices **p**. This follows immediately from the definition of the functions Γ^i , because the first-order conditions for Kantian equilibrium, which were derived in steps 5,6, and 7 of the proof of Theorem 1, follow from the definition of Γ^i, given that

$$G_2(K^1,D^1)=\frac{w}{p_1} \text{ and } H_2(K^2,D^2)=\frac{w}{p_2}.$$

9. Thus, a fixed point of Φ is a Walras-Kant equilibrium at tax rate *t*. To show the existence of a fixed point, we need to check that the premises of Kakutani's fixed point theorem hold. Φ is obviously convex-valued. Upper-hemi-continuity of Φ at any point in $\operatorname{int}\Delta$ follows quickly.

Finally, we examine u.h.c. of Φ at points on the boundary of the simplex. Suppose $\mathbf{p}^j=(p_1^j,p_2^j,w^j,r^j)\rightarrow\mathbf{p}\in\partial\Delta$. Suppose the sign pattern of $\mathbf{p}$ is (+,+,0,+). We have $\Phi(p_1,p_2,w,r)=\{(0,0,1,0)\}$. . Eventually p_1^j,p_2^j,r^j are positive and bounded away from zero, and $w^j\rightarrow 0$.

We must show that $\lim_{j\to\infty}\Phi(\mathbf{p}^j)=(0,0,1,0)$. Without loss of generality, we may assume that $\mathbf{p}^j\in\operatorname{int}\Delta$ for all *j*. Denote the excess demands at $\mathbf{p}^j$ by $\Delta x(j),\Delta y(j),\Delta E(j)$ and $\Delta K(j)$. We will show that, for *j* sufficiently large

$$\Delta E(j)>\max[\Delta x(j),\Delta y(j),\Delta K(j)], \tag{4.15}$$

and this will imply that, for sufficiently large *j*, $\Phi(\mathbf{p}^j) = (0,0,1,0)$. To show (4.15), we will show that $\frac{\Delta z(j)}{\Delta E(j)} \to 0$, for $z \in \{x, y, K\}$.

We show that $\frac{\Delta K(j)}{\Delta E(j)} \to 0$. We know $\Delta E(j) \to \infty$, because $w^j \to 0$, and so the firms will demand unbounded amounts of labor, while the supply of labor is bounded. If $\Delta K(j)$ were bounded above, we would be done. So we suppose that $\Delta K(j)$ is unbounded. It follows that for at least one firm – say the *G* firm -- $K^{1j} \to \infty$ and $D^{1j} \to \infty$. But by profit maximization, $\frac{G_2(K^{1j}, D^{1j})}{G_1(K^{1j}, D^{1j})} = \frac{w^j}{r^j} \to 0$. By homotheticity of *G* (Assumption *A*(i)), the points (K^{1j}, D^{1j}) must eventually lie below any ray in the positive quadrant of (K, D) space. This implies that $\frac{\Delta K(j)}{\Delta E(j)} \to 0$, as required.

To show $\frac{\Delta z(j)}{\Delta E(j)} \to 0$ for $z \in \{x, y\}$, it suffices to show $\frac{\Pi(j)}{D(j)} \to 0$ because the demand for the two commodities cannot grow faster than total profits (wage income goes to zero). We show $\frac{G(K^1(j), D^1(j))}{D^1(j)} \to 0$. Let *j* be large and $J > j$. Then:

$$G(K^1(J), D^1(J)) - G(K^1(j), D^1(j)) \le G_1(K^1(j), D^1(j))(\delta K) + G_2(K^1(j), D^1(j))(\delta D), \quad (4.16)$$

by concavity of *G*, where $\delta K \equiv K(J) - K(j)$, etc., and so:

$$\frac{G(K^1(J), D^1(J)) - G(K^1(j), D^1(j))}{\delta D} \le G_1(K^1(j), D^1(j))\frac{\delta K}{\delta D} + G_2(K^1(j), D^1(j)). \quad (4.17)$$

Now let $j, J \to \infty$, but *j* more slowly than *J*. We know from above that $\frac{\delta K}{\delta D} \to 0$, and $G_1(K^1(j), D^1(j)) = \frac{r^j}{p_1^j} \to \frac{r^*}{p_1^*}$, and $G_2(K^1(j), D^1(j)) = \frac{w^j}{p_1^j} \to 0$. Therefore the right-hand side of (4.17) approaches zero, and so $\frac{G(K^1(J), D^1(J))}{D^1(J)} \to 0$, as was to be proved.

We examine one more case on the boundary of the simplex. Suppose the sign pattern of $\mathbf{p}^*$ is (0,+,0,+). Then $\Phi(\mathbf{p}^*) = \{(a, 0, 1-a, 0) \mid 0 \le a \le 1\}$. We know that

$p_1^j, w^j \to 0$ and eventually p_2^j and r^j are bounded away from zero. If eventually $\Delta E(j)$ is greater than $\Delta y(j), \Delta x(j)$ and $\Delta K(j)$, then eventually $\Phi(\mathbf{p}^j) = (0,0,1,0) \in \Phi(\mathbf{p}^*)$. Firm 2 eventually demands huge amounts of labor, because the wage goes to zero but the price of output is significantly positive. The profits of Firm 1 go to zero since $p_1^j \to 0$. These facts imply that $\frac{\Pi(j)}{\Delta E(j)} \to 0$ and so, as in the first case examined above, $\Delta E(j)$ dominates the other excess demands, as required.

The other cases of points on $\partial\Delta$ yield to similar analysis. Hence, the premises of Kakutani's theorem hold, and a fixed point in $\text{int}\,\Delta$, which is a Walras-Kant equilibrium, exists. ν

A comment on investment in the model is called for. In the approach I've taken, only the state invests. Could private agents invest in the firms as well, and preserve the efficiency result? The answer is yes, if the profile of investments is also an additive Kantian equilibrium. I elected not to follow this route here, both for reasons of simplicity, and because it strikes me as more credible that workers can learn to adopt Kantian optimization in their labor-supply decisions than in their investment decisions. Perhaps I am here influenced by the observation that workers have a history of cooperation, and investors do not, at least to the same extent.

6. The psychology of Kantian optimization

The *differentia specifica* of the model here proposed, within the class of market-socialist models, is Kantian optimization in the labor-supply decision. I have argued that, because socialism has always been conceived of as a society with enhanced cooperation among its members, this formalization of what cooperation entails should be welcome. That a more robust form of cooperation has not been included in designs for market socialism heretofore is due, I suggest, to the lack of a formal representation of cooperative behavior.

It will surely be the case that skepticism regarding my proposal will focus upon the realism of supposing that a large population of producers can learn to optimize their labor-supply decisions in the Kantian manner. There are, I think, three necessary conditions for the feasibility of such behavior: desire, understanding, and trust. Citizens must *desire* to cooperate with each other, they must view themselves as part of a solidaristic society, whose members believe that cooperation in economic decisions is the *modus operandi*. But why should the Kantian optimization protocol appeal to people as the preferred mode of cooperation? I think the motivation must be in the *conception of fairness or solidarity* embodied in the statement, "I should only reduce (increase) my labor supply if I would like all others to reduce (increase) their labor supplies in like manner[3]." Our brains love symmetry, and fairness always, I believe, involves a conception of symmetrical treatment. Secondly, people must *understand* that cooperation in the labor-supply decision is achieved by each knowing that if all increase their labor supply by a small increment, each person's *income* increases by that increment, because what a worker loses in the tax on her wage, she gets back in the increased demogrant. They should also understand that, if workers optimize according to Kantian protocol, efficiency is separated from distributional considerations. Thirdly, they must *trust* that others will behave cooperatively as well, and will not take advantage of their own cooperative behavior, by optimizing in the Nash manner. If these three conditions are met, then the *method* of implementing cooperative behavior is not difficult: each worker should choose his labor supply to equalize his marginal rates of substitution between commodities and labor to his gross wage, rather than his after-tax wage. Rather than thinking "Is the disutility of an extra day's work worth to me the *after-tax* wage increment?" the worker should ask whether it is worth the *gross* wage increment. If we believe people are capable of optimizing in the Nash manner, optimizing in the Kantian manner is no more cognitively demanding, if the necessary conditions are met.

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[3] But why should workers conceive of symmetric treatment as a *translation* of the labor vector rather than a *rescaling* of it (multiplication by positive constant)? I have no good answer to this question, except to say that there are examples of it in history. 'Doing one's bit' in the Second World War in Britain arguably involved a translation of the labor vector, not a rescaling. Unfortunately, there is no simple income tax function that will combine with 'rescaling' as the Kantian protocol to produce Pareto efficient allocations with any degree of income equality.

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