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Consumers on a Leash: Advertised Sales and Intertemporal Price Discrimination

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Abstract

The Internet allows sellers to track "window shoppers," consumers who look but do not buy, and to lure them back later by targeting them with an advertised sale. This new technology thus facilitates intertemporal price discrimination, but simultaneously makes it too easy for a seller to undercut her regular price. Because buyers know they could be lured back, the seller is forced to set a lower regular price. Advertising costs can, therefore, serve as a form of commitment: a seller can actually benefit from higher costs of advertising. Based on this framework, the impact of commitment on prices, profits, and welfare are analyzed using a dynamic pricing model. Furthermore, it is demonstrated how buyers' time preferences give rise to price fluctuation or an everyday-low-price in equilibrium.

Keywords: Advertising, Coases conjecture, commitment, dynamic pricing, intertemporal price discrimination, online markets, everyday-low-pricing

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1 Introduction

The advent of tracking via the use of "cookies" by online sellers allows those sellers to keep "window shoppers," consumers who look but do not buy, on a leash. Sellers can reel in window shoppers, who are likely to be responsive to a discounted price, through a targeted email announcing a discounted price. The consequences, however, of this kind of tracking and targeting on the dynamics of prices, profits, and welfare are ambiguous. While it allows sellers to intertemporarily price discriminate, it also hinders commitment to future prices because sellers are always tempted to offer a sale when they learn about the interest of a shopper.

This paper proposes a dynamic pricing model in which a monopolist uses targeted sales in order to recall window shoppers and in which the costs of such sales (advertising costs) endogenously provide the monopolist with some ability to commit to high prices for an extended period of time. Specifically, a seller can set prices aimed at high-valuation buyers today, knowing that she can collect low-valuation buyers later via a targeted sale. The profitability of this strategy is partially undermined if the high-valuation buyers, anticipating a future sale, decide to wait. Consequently, the seller faces a trade-off between having frequent sales, allowing her to sell to low-valuation buyers with less delay, and maintaining a high regular price. Finally, if the seller cannot commit to her frequency of sales, she is exposed to the logic of the Coase conjecture (Coase (1972)).[1] In particular, were advertising costless, the seller would like to immediately offer a low price to a buyer after he has revealed his low valuation by not buying. In that case, only a low price can be sustained in equilibrium.

How valuable advertising costs are as a commitment device to the seller and how profitable price discrimination is compared to a constant low price depends critically on buyers' time preferences. While the cost of advertising determines the frequency of sales, buyers' time preferences determine the regular price that induces high-valuation buyers to buy. For example, if buyers are extremely impatient, then the seller can extract the entire surplus from high-valuation buyers by charging a high regular price regardless of the frequency of sales. In that case, commitment is of little value. In contrast, with very patient buyers, the regular price can be set at a profitable level only if the seller has very infrequent sales. Then, infinite advertising costs become profit-maximizing, but moderate levels of advertising costs are of no value to the seller as a commitment device. For intermediately patient buyers, moderate levels of advertising costs provide the seller with a level of commitment that is optimal to the seller.

To formalize these ideas, I consider buyers to arrive randomly over time. Past buyers' arrivals and purchasing decisions are the seller's private information. Although buyers are forward-looking and rational, they are only actively in the market upon arrival and when the seller "reactivates" them through costly advertising. Furthermore, buyers do not observe past price paths and sales, so

[1] See also Stokey (1979), Bulow (1982), Fudenberg et al. (1985), Gul et al. (1986)

they assign the same probability to all possible equilibrium histories given the observed price. Put differently, buyers know the *frequency* of sales but not the exact *timing* of the next sale.

I focus on stationary equilibria in which high-valuation buyers buy upon arrival: hence, the number of low-valuation buyers who have not bought is the relevant state variable. I show that the seller either chooses a constant low price (everyday-low-price or EDLP equilibrium) or a high regular price with occasional sales (advertising equilibrium). In an advertising equilibrium, the seller accumulates a fixed number of low-valuation buyers before having an advertised sale. I call this number the "cutoff-demand". The constant regular price must make high-valuation buyers indifferent between buying and waiting. The regular price and cutoff-demand uniquely describe the equilibrium price path.

The equilibrium frequency of sales in an advertising equilibrium is determined by two conditions: the seller cannot have an incentive to hold a sale before the cutoff-demand is reached, nor any incentive to wait after it is reached. These conditions are independent of the arrival of high-valuation buyers and the regular price. For any given advertising cost, a (generically) unique cutoff-demand satisfies these conditions. In particular, sales are less frequent if advertising costs are high conditional on the existence of an advertising equilibrium. I show that an advertising equilibrium exists if and only if the frequency of sales supported by the cost of advertising results in a high enough regular price to support profits in excess of an EDLP strategy. Otherwise the unique equilibrium is an EDLP equilibrium.

After characterizing the equilibrium outcome, I elaborate on the role of buyers' time preferences and the value of advertising as a commitment device. To that end, I first consider the full commitment benchmark, in which the seller can commit to a frequency of sales at time zero. In that case, the seller only faces a trade-off between frequent sales and a high regular price. Buyers' time preferences specify how sensitive the regular price is to changes in the frequency of sales. The optimal frequency of sales is first decreasing in the discount rate of buyers and then increasing. On the one hand, if buyers are infinitely impatient, the seller can perfectly price discriminate and prefers very frequent sales. On the other hand, if buyers have a finite discount rate, less frequent sales allow the seller to increase the price significantly. If buyers are, however, very patient, the regular price is close to the low-valuation of buyers even with a delay in trade, such that the seller prefers very frequent sales or no sales at all.

Without ex-ante commitment, the seller's benefit from advertising costs as a commitment device is thus critically affected by buyers' time preferences. I distinguish between the following three cases:

1. If buyers are relatively patient, an equilibrium with periodic sales exists only for large enough advertising costs. The profit-maximizing advertising cost is infinite.

2. For intermediate levels of buyer patience, low advertising costs can increase the seller's profit significantly because she can commit to a lower frequency of sales and, hence, raise the regular price sufficiently to make price discrimination profitable. However, for intermediate advertising costs, the cost of advertising outweighs the benefit from increasing the regular price, so the

unique equilibrium is an EDLP equilibrium. For high advertising costs, an equilibrium with sales exists and the seller's profits converge to the static monopoly profit. The seller prefers small or infinite advertising costs.

3. For highly impatient buyers, a positive advertising cost that gives enough commitment power to not have sales after each arrival suffices to sustain periodic sales in equilibrium. As buyers become more impatient, the seller's profit-maximizing advertising cost decreases.

Hence, profits are non-monotonic in advertising costs. In the region in which finite advertising costs are optimal, the optimal advertising cost is first increasing and then decreasing in the impatience of buyers.

Finally, I show how market size (i.e., the rate at which buyers arrive) influences the frequency of sales. This is important from an empirical perspective. It also has normative implications because social welfare is increasing in the frequency of sales. The average time between two sales is decreasing in the market size, even though more buyers are being reactivated at each sale. Hence, the regular price is lower in large markets and consumers capture more surplus. At the same time, in high-volume, low-ticket markets such as headphones, sales are less frequent than in low-volume, high-ticket markets such as furniture, if both markets are about the same size in terms of revenues.

These results not only have implications for prices in online markets, but also for the budget firms want to allocate to email advertising campaigns. While it is unrealistic for sellers to commit to infinite or very large advertising costs, sellers have an incentive to commit to moderate amounts. Depending on the time preferences of her customers, a seller can benefit from developing a reputation for fancy or elaborate - thus expensive - email advertising. Related to this finding, I show that the seller always prefers a fixed advertising cost to a variable advertising cost that depends on the number of people contacted.[2] The significance of email advertising in general is, in part, reflected by the 2 billion dollars that US firms are forecast to spend on it in 2014. [3] Online retail sales in general accounted for 231 billion dollars in 2012 and is predicted to reach 370 billion dollars by 2017. [4]

More broadly, this paper helps to understand the long-run equilibrium implications of the seller's ability to recall buyers. While targeting per se will always benefit consumers, if the static monopoly price is high, it can hurt or benefit the seller. It is a well-known fact that the ability to track and

[2] This suggests that in connection with sales, firms are willing to pay more to an advertising platform that has fixed rates rather than per-click or per-view rates as commonly used. But for the pricing of banner advertising, intermediary agents are involved, such that asymmetric information and moral hazard can play an important role.

[3] Forrester Research, "US Email Marketing Forecast, 2009 To 2014" (accessed on October 18, 2013): `http://www.forrester.com/US+Email+Marketing+Forecast+2009+To+2014/fulltext/-/E-RES53620?docid=53620`

[4] Forrester Research, "US Online Retail Forecast, 2012 To 2017" (accessed on December 30, 2013): `http://www.forrester.com/US+Online+Retail+Forecast+2012+To+2017/fulltext/-/E-RES93281?objectid=RES93281` and `http://www.forrester.com/US+Online+Retail+Sales+To+Reach+370+Billion+By+2017/-/E-PRE4764`

target reduces advertising costs by resolving an old dilemma of advertising, first pinpointed by John Wanamaker with his famous quote:

> "Half the money I spend on advertising is wasted. The trouble is I dont know which half."[5]

However, if buyers are relatively impatient, it is better for a firm not to collect customers' contact information. Nevertheless, if buyers are sufficiently patient, the seller earns more than static monopoly profits with some intermediate advertising costs.

This paper is mostly related to the intertemporal price discrimination papers by Conlisk et al. (1984) and Sobel (1991), in which sellers have limited commitment for exogenous reasons. In their models, the frequency of sales is determined by the length of a period rather than the cost of advertising. Sobel (1991) shows that as period length goes to zero, stationary equilibria satisfy the Coase conjecture. This paper shows that advertising costs may permit sellers to escape the logic of the Coase conjecture by endogenously creating limited commitment. Moreover, my model allows for a detailed analysis of the interplay of the value for commitment and time preferences. Thus, this paper extends our understanding of intertemporal price discrimination in general.

The idea that advertising can serve to activate buyers has been used in static models of advertising, such as Iyer et al. (2005). They consider oligopolistic markets where several sellers compete for buyers through advertising. In my setup, advertising instead creates competition between the seller and her future self by making buyers long-lived, allowing her to intertemporarily price discriminate. A detailed discussion of the related literature can be found in Section 6.

The paper is organized as follows. In Section 2, I introduce the model and equilibrium notion. Section 3 characterizes all stationary equilibria in which high-valuation buyers buy upon arrival. Using these results, I present some comparative statics about the frequency of sales and advertising costs in Section 4. In Section 5, I investigate the role of buyers' time preferences on the equilibrium outcome and the benefit of advertising costs as a commitment device to the seller. Finally, in Section 6, I discuss some assumptions and results and relate the paper to the relevant literature. Section 7 concludes the analysis.

2 Model

2.1 Basics

A monopolist (she) sells a homogeneous good over time. Time is continuous. For simplicity, and in order to abstract away from inventory considerations, I assume that the seller's marginal cost is equal to zero.

[5] See for example Hoffman and Novak (2000).

Buyers arrive according to a Poisson process with arrival rate λ - also referred to as the *market size* from now on. Each buyer (he) wants at most one unit of the good in his lifetime and his valuation is his private information.[6] Valuations of buyers are independently distributed and independent of arrivals. They are high, v_H, with probability π and otherwise low, v_L, where $v_H > v_L > 0$. Hence, low-valuation buyers arrive according to a Poisson process with an arrival rate $\lambda(1-\pi)$ and high-valuation buyers arrive according to an independent Poisson process with arrival rate $\lambda\pi$. The seller privately observes the arrivals of buyers and whether a buyer has bought or not.

Buyers observe the price when they arrive, but they do not observe past or future prices.[7] They believe that all arrival times are equally likely.[8] Hence, all buyers have the same belief about the upcoming price path. Upon arrival, a buyer decides whether to buy or wait. A buyer who waits can only buy if "reactivated" by the seller. The seller can inform window shoppers who have previously visited, but not bought, about a sale via a targeted advertising at a cost $C_A > 0$. For now, I assume that the advertising cost is a fixed cost no matter how many buyers are being reactivated.[9] Call a point in time at which the seller chooses to advertise an *advertised sales period*. All other times are *regular periods*. The seller chooses a price p_t and whether to advertise or not at any point in time t.

The seller's discount rate is r_s and the buyers' r_b. If the seller sells to N_t buyers in period t at a price p_t, then her profit at time t is $\Pi_t = p_t \cdot N_t$ if she does not advertise, and $\Pi_t = p_t \cdot N_t - C_A$ if she advertises. Her expected discounted profits are hence,

$$\Pi = \mathbb{E}\left[\sum_{t\colon N_t \neq 0} e^{-r_s t} \cdot \Pi_t\right].$$

A buyer with valuation v who buys τ periods after arrived, receives an expected payoff

$$U = \mathbb{E}\left[e^{-r_b \tau} \cdot (v - p_\tau)\right]$$

relative to the time of arrival. Note that τ and N_t are random variables.

2.2 Equilibrium

I focus on *Markov equilibria* in which the seller's strategy only depends on her belief about the number of high- and low-valuation buyers in the market, which I will refer to as the state $s \in [0,1]^{\mathbb{N}\times\mathbb{N}}$ of the game. $s(n,m)$ represents the probability that there are n high- and m low-valuation buyers. Moreover precisely, a seller's strategy is given by a pricing strategy $p : [0,1]^{\mathbb{N}\times\mathbb{N}} \to \mathbb{R}$ and and advertising strategy $\sigma : [0,1]^{\mathbb{N}\times\mathbb{N}} \to \{0,1\}$. Buyers' acceptance rules can only depend on the observed prices and the time

[6] I discuss in Section 6 how the assumption that each buyer demands only a single unit can be relaxed.

[7] If buyers can observe past price paths, the qualitative results of the paper will not change. In that case, the regular price is not constant, but continuously decreasing over time until the seller has an advertised sale.

[8] I formalize this when I introduce the equilibrium notion.

[9] I discuss the implications of advertising costs being variable in Section 6.

that has elapsed since arrival. If the seller plays a Markov strategy, a buyer's best-response acceptance strategy must only depend on his belief over s without loss of generality. As arrivals occur according to Poisson processes, the state s evolves according to a Markov process on equilibrium path. Thus, buyers are assumed to believe upon arrival that the state s is distributed according to the stationary distribution of the Markov process governing s whenever it exists.

Furthermore, I impose the following assumptions:

- If buyers are indifferent between buying and not buying, they buy. (A1)
- All equilibrium prices are greater than or equal to v_L. (A2)

Assumption (A1) essentially restricts equilibria to pure-strategy equilibria. Assumption (A2) is satisfied in all limiting equilibria of the corresponding discrete-time version of the game as the length of the period goes to zero which follows from the argument used in discrete-time bargaining models with incomplete information, such as in Fudenberg et al. (1985). [10]

Lemma 1. *In any Markov equilibrium satisfying Assumptions (A1) and (A2), high-valuation buyers must buy upon arrival*

Proof. Let us assume that high-valuation buyers did not buy immediately in equlibrium, but at a later point in time at a price p during an advertised sale. Then, high-valuation buyers must accept p and offering a regular price of p is a profitable deviation for the seller as the buyer would accept such an offer: If a high-valuation buyer's strategy was to reject a price of p upon arrival, then the state s was the same as on equilibrium path and the continuation equilibrium is unchanged. This would, however, mean that the buyer must be better off accepting p today. □

Consequently, the unique advertised sales price is given by v_L. Thus, on equilibrium path, the state of the game can be reduced to the number, n, of buyers who have arrived but not yet bought and they must be low-valuation buyers in all equilibria. I call such strategies of the seller *reduced Markov-strategies*. Formally, a reduced stationary strategy of the seller consists of a pricing strategy $p : \mathbb{N} \to \mathbb{R}_+$, and an advertising strategy $\sigma : \mathbb{N} \to \{0,1\}$, where $\sigma(n) = 1$ means the seller holds a sale as soon as n buyers have accumulated. Markovian acceptance rule for buyers is given by $d : \{v_L, v_H\} \times \mathbb{R} \times \{0,1\} \to \{0,1\}$ where $d(v,p,\sigma) = 1$ if the buyer of type v accepts a price p given the advertising strategy σ. Given strategies of the seller (p,n), I refer to the number of accumulated buyers necessary to trigger a sale

$$N = \inf\{n : \sigma(n) = 1\}$$

[10] For non-stationary equilibria, this property does not hold and the set of equilibria is very rich: Sobel (1991) shows a folk theorem for such equilibria in a discrete-time model for a durable goods monopolist with arriving buyers.

as the *cutoff-demand*. Then, each strategy of the seller and equilibrium acceptance rule of buyers induce a Markov process on $\mathbb{N}$ that increases by one after every arrival of a low-valuation buyer as long as $n < N$ and drops to zero after an arrival of a low-valuation buyer for $n > N$. Buyers' beliefs about the state n are given by a probability measure μ on $\mathbb{N}$. Upon arrival the belief is the stationary distribution of this Markov process given by

$$\bar{\mu}(n) = \begin{cases} \frac{1}{N} & \text{for } n \in \{0, \dots, N-1\} \\ 0 & \text{otherwise} \end{cases}$$

whenever $N < \infty$. Thus, a Markovian equilibrium can be described by

1. reduced Markov-strategies (p, σ) that maximize the seller's continuation payoff out of all strategies (p', σ') such that $\sigma'(n) = 0$ and $p'(n) = \max\{\tilde{p} : d(v_H, \tilde{p}, 0) = 1\}$ or $\sigma'(n) = 1$ and $p'(n) = v_L$ given the buyers' acceptance rule d,

2. buyers' acceptance rule d that maximizes a buyer's expected payoff given (p, σ) and given each buyer's belief,

3. buyers' beliefs about the reduced state variable n which are updated according to Bayes' rule where the prior upon arrival is given by $\bar{\mu}$.

3 Characterization of the Stationary Equilibrium

In this section, I characterize all stationary equilibria.

3.1 Preliminaries

First, note that given $C_A < \infty$, $N = \infty$ can only be a Markov-equilibrium outcome if the price is v_L as the following lemma shows.

Lemma 2. *In an equilibrium with $N = \infty$, the price must be constant and equal to v_L. Hence, the seller's profit is given by*

$$\Pi_L \equiv \frac{\lambda}{r_s} \cdot v_L.$$

Proof. Let t_n^λ be the random variable describing the n-th arrival time of a Poisson process with arrival rate λ. It can be shown that t_n^λ is distributed according to a gamma distribution $\Gamma\left(n, \frac{1}{\lambda}\right)$. The corresponding moment generating function is given by

$$\mathbb{E}\left[e^{s \cdot t_n^\lambda}\right] = \left(1 - \frac{s}{\lambda}\right)^{-n}. \tag{1}$$

In any equilibrium in which the seller never advertises, all buyers must buy upon arrival. If not, so if some low-valuation buyers do not buy with a positive probability, then the state variable n would reach any arbitrarily high level with positive probability. As a result, the instantaneous profit that the seller can make by advertising $n \cdot v_L - C_A$ can become arbitrarily high with positive probability. Hence, having an advertised sale would be a profitable deviation for the seller for large enough n. Finally, note that all buyers buy upon arrival if and only if the price is constant and equal to $p = v_L$. Hence, the seller's profit is given by

$$\Pi_L = \sum_{i=1}^{\infty} \mathbb{E}\left[e^{-r_s \cdot t_i^{\lambda}}\right] \cdot v_L = \frac{\left(1+\frac{r_s}{\lambda}\right)^{-1}}{1-\left(1+\frac{r_s}{\lambda}\right)^{-1}} \cdot v_L = \frac{\lambda}{r_s} v_L.$$

□

As a result, an equilibrium can be one of only two kinds:

Everyday-low-price equilibrium: $N = \infty$ and a constant price of v_L.

Advertising equilibrium: $N < \infty$ with a regular price $p_r \in (v_L, v_H)$.

Thus, an advertising equilibrium is essentially characterized by the cutoff demand $N < \infty$ and a regular price p_r. The structure of an advertising equilibrium with cutoff-demand $N = 3$ is illustrated in Figure 1. Black circles represent the random arrival times of low-valuation buyers and white circles arrival times of high-valuation buyers. The state variable n increases by one after arrivals of low-valuation buyers. When the cutoff-demand is reached (i.e., $n = N$) the state drops to $n = 0$ and a new cycle starts. The price path is given by a regular price p_r that is accepted by all high-valuation buyers and advertised sales at price v_L at which all N low-valuation buyers buy.

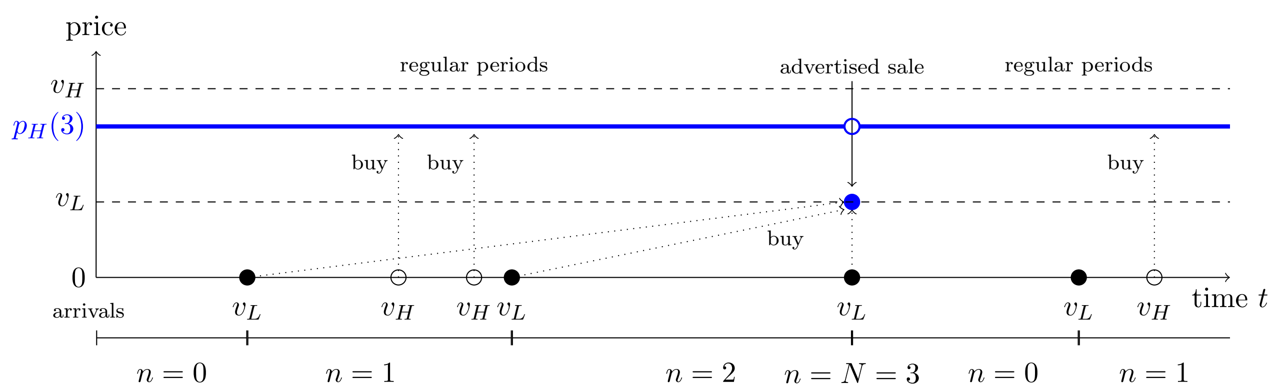

Figure 1: Illustration of a price path with cutoff-demand $N = 3$.

In the following, I focus on properties of the cutoff-demand N and a regular price $p_H(N)$ that makes high valuation buyers just indifferent between buying and waiting in an advertising equilibrium.

Buyers' beliefs. In equilibrium, buyers hold correct beliefs about N, but they do not know when the last sale has happened. Consequently, they do not know the number of buyers, n, who have yet to buy and they assign equal probability to the states $n \in \{0, \dots, N-1\}$. As a result, their belief about the waiting time until the next sale, is the same. The distribution of the equilibrium expected waiting time τ can be characterized by the following lemma.

Lemma 3. *Given the seller's equilibrium cutoff-demand $N < \infty$, a buyer believes in equilibrium that the waiting time until the next sale, τ, conditional on not buying upon arrival is a uniform mixture of gamma distributions $\Gamma\left(N-(n+1), \frac{1}{\lambda(1-\pi)}\right)$ with $n \in \{0, \dots, N-1\}$.*[11]

The intuition for this result is that given the current state is n, if a buyer does not accept the regular price, he has to wait for $N-(n+1)$ low-valuation buyers to arrive because the seller thinks that he has low-valuation. Hence, he has to wait for time $t^{\lambda(1-\pi)}_{N-(n+1)}$ until the next sale. In equlibrium, a buyer assigns equal probability to states $n \in \{0, \dots, N-1\}$

Buyers' indifference rule. Given the equilibrium cutoff-demand N, a high-valuation buyer is willing to pay at most $p_H(N)$ satisfying

$$\underbrace{v_H - p_H(N)}_{\text{surplus from buying today}} = \underbrace{\mathbb{E}[e^{-r_b \tau}] \cdot (v_H - v_L)}_{\text{expected surplus from waiting}} . \tag{2}$$

That is, at a price $p_H(N)$, high-valuation buyers are indifferent between buying today and waiting for the next sale.

Using Lemma 3 and expression(1), for $N < \infty$, I can calculate the *expected discount to a sale*:

$$\mathbb{E}[e^{-r_b \tau}] = \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[e^{-r_b t^{\lambda(1-\pi)}_{N-(n+1)}}\right] = \frac{1}{N} \cdot \frac{1 - \left(1 + \frac{r_b}{\lambda(1-\pi)}\right)^{-N}}{1 - \left(1 + \frac{r_b}{\lambda(1-\pi)}\right)^{-1}}$$

for $N \geq 1$. For $N \in \{0, 1\}$, the expected waiting time is zero, so the buyers expected discount factor is 1.

Seller's profits. First, the seller's expected discount factor between the arrival of two buyers is given by

$$\mathbb{E}\left[e^{-r_s \cdot t_1^{\lambda(1-\pi)}}\right] = (1 + R_s)^{-1}$$

where

$$R_s = \frac{r_s}{\lambda(1-\pi)}$$

is the *seller's arrival-adjusted discount rate.* This is the average discount rate between the arrival of two low-valuation buyers. Furthermore, given n inactive buyers in the market, denote by $\Pi(n; N)$ the

[11] A uniform mixture of distributions $F_1, F_2, \dots, F_n$ is a distribution of a random variable that is distributed according to $F_1, F_2, \dots, F_n$ with equal probability.

expected continuation profit *if* the seller can adhere to a plan to have a sale after unrealized demand N. Then, with n inactive low-valuation buyers, the seller's expected continuation profit is given by

$$\Pi(n;N) = \underbrace{\frac{\lambda\pi}{r_s}\cdot p_H(N)}_{\text{revenue from } v_H\text{-buyers}} + (1+R_s)^{-1}\cdot \underbrace{\left(\Pi(n+1;N) - \frac{\lambda\pi}{r_s}\cdot p_r\right)}_{\substack{\text{expected revenue from advertised sales} \\ \text{after } (n+1)\text{-th arrival of } v_L\text{-buyer}}}$$

for $n < N$ and $\Pi(N;N) = N\cdot v_L + \Pi(0;N) - C_A$.

3.2 Equilibrium Characterization

In an advertising equilibrium with cutoff-demand N, the seller never wishes to have an advertised sale before the N-th low-valuation buyer has arrived and she cannot wish to wait for more low-valuation buyers to arrive. Hence, formally, for any $n < N$ it must hold that

$$\Pi(n;N) \quad > \quad n\cdot v_L + \Pi(0;N) - C_A$$

and, for $n > N$ and all $m \geq 1$,

$$nv_L + \Pi(0;N) - C_A \geq \left(1-(1+R_s)^{-m}\right)\cdot \tfrac{\lambda\pi}{r_s}p_H(N) + (1+R_s)^{-m}\cdot\left((n+m)v_L + \Pi(0;N) - C_A\right).$$

These conditions place upper and lower bounds on the advertising cost C_A consistent with cutoff-demand being N:

Lemma 4. *(i) Given a cutoff-demand N, a seller does not want to drop the price before period N if and only if*

$$C_A \geq \left(N - \left(1-(1+R_s)^{-N}\right)\cdot\left(1+R_s^{-1}\right)\right)\cdot v_L \equiv \underline{C}(N). \tag{3}$$

(ii) Given a cutoff-demand N, a seller does not want to postpone an advertised sale in period N if and only if

$$C_A \leq \left(N - \left(1-(1+R_s)^{-N}\right)\cdot R_s^{-1}\right)\cdot v_L \equiv \overline{C}(N). \tag{4}$$

(iii) Generically, there is a unique integer $N^*(C_A)$ that satisfies (3) *and* (4)*.*

Intuitively, if advertising costs are low, the seller is tempted to have a sale when only a few inactive buyers have accumulated, while if advertising costs are high, she wants to have a sale only when many inactive buyers have accumulated. The bounds on the advertising cost are independent of the value of the regular price and depend only on the seller's arrival-adjusted discount rate R_s, the cutoff-demand N, and v_L. In particular, buyers' time preferences are irrelevant.

The functions $\overline{C}, \underline{C}$ are both increasing in N and $\underline{C}(N) < \overline{C}(N)$ for all N. Let the largest number that satisfies (3) be $\underline{N}$ and the smallest number that satisfies (4) be $\overline{N}$. In other words,

$$\underline{C}(\underline{N}) = C_A = \overline{C}(\overline{N}).$$

The uniqueness of an integer $N^*(C_A) \in [\overline{N}, \underline{N}]$ follows from the observation that $\underline{N} - \overline{N} = 1$. When $\underline{N}, \overline{N}$ are integers, there are two $N^*(C_A)$ that satisfy this condition. Note also that $\underline{N} - \overline{N} = 1$ implies that $\overline{C}(N) = \underline{C}(N+1)$ for all N. Figure 3.2 illustrates $\overline{C}$, $\underline{C}$, $\overline{N}$, $\underline{N}$. In that case, only $N = 28$ satisfies both (3) and (4).

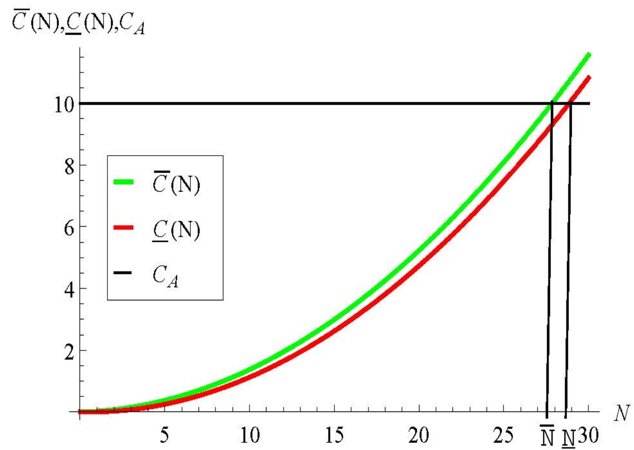

Figure 2: Illustration of Lemma 4 for $C_A = 10$, $v_L = 50$, $\pi = 0.5$, $r_s = 0.5$, $\lambda = 2000$

Using these insights, the (generically) unique Markov-equilibrium can be characterized.

Proposition 1. *(i) An advertising equilibrium exists if and only if*

$$\Pi_{p_H(N^*(C_A))}(0; N^*(C_A)) \geq \Pi_L. \tag{5}$$

In that case, the unique profit-maximizing advertising equilibrium is given by the cutoff demand $N^*(C_A)$ *and the regular price* $p_H(N^*(C_A))$.
(ii) An EDLP equilibrium exists if and only if $\Pi_{p_H(N^*(C_A))}(0; N^*(C_A)) \geq \Pi_L$.

In order to prove this, I first show that the reduced stationary advertising and EDLP equilibirum exists, respectively, and then prove that the profit maximizing equilibria are also Markov equilibria. The formal proof can be found in the appendix. From now on, I refer to $\Pi(0; N^*(C_A))$ as the *profit from price discrimination*, given cutoff-demand $N^*(C_A)$. [12]

4 Comparative Statics: Advertising Cost and Market Size

Using the equilibrium characterization of Proposition 1, I now relate my model to two well-known benchmark models: the model by Sobel (1991), in which sellers lack any commitment power, that

[12] In a model with a continuous inflow of buyers conditions (3) and (4) can be summarized in one equation, where the upper and lower bound of C_A coincide.

results in the Coasean outcome and a model in which the seller can fully commit to a constant price. Then, I analyze the effect of the arrival rate λ on the frequency of sales. I will refer to λ as the *market size* from now on.

4.1 Advertising Cost

An immediate implication of conditions (3) and (4) is that if advertising is costless ($C_A = 0$), then the unique equilibrium is an EDLP equilibrium. The reason is that the seller wants to drop the price as soon as a buyer does not buy. For $C_A > \overline{C}(1)$, however, advertising costs serve as a commitment device for the seller. In particular, the higher that cost, the higher $N^*(C_A)$, hence the less frequent are sales in an advertising equilibrium. By (3) and (4), every advertising cost in $(\underline{C}(N^*(C_A)), \overline{C}(N^*(C_A)))$ is mapped to a unique cutoff-demand $N^*(C_A)$. As C_A becomes infinitely large, the static monopoly profit can be sustained. More precisely, $\Pi(0; N^*(C_A))$ converges to

$$\Pi_H \equiv \frac{\lambda\pi}{r_s} \cdot v_H,$$

which is the profit the seller makes if she charges v_H in every period without ever advertising. These results are summarized in the following corollary.

Corollary 1. *(i) (Coase conjecture) For $C_A < \overline{C}(1)$, the only equilibrium is an EDLP equilibrium. (ii) The cutoff-demand $N^*(C_A)$ is increasing in advertising costs C_A. (iii) For $C_A = \infty$, the static monopoly profit can be sustained at any point in time.*

Hence, my model encompasses two well-known benchmark cases: (i) specifies the Coasean equilibrium outcome for monopolist without any commitment power; and (iii) captures a monopolist who can fully commit to a constant price. In the following, the goal is to obtain a better understanding of the intermediate case, when $\underline{C}(1) < C_A < \infty$.

As previously noted, the equilibrium cutoff-demand $N^*(C_A)$ is independent of the buyers' discount rates r_b and v_H that only affect the profit from price discrimination through the regular price $p_H(N^*(C_A))$ given by (2). $N^*(C_A)$ is a decreasing function of the arrival-adjusted discount rate $R_s = \frac{r_s}{\lambda(1-\pi)}$ and v_L. Hence, an increase in the market size λ is similar in effect to having a more patient seller.

These observations imply that, everything else the same, in larger markets (i.e., with high λ) the cutoff-demand $N^*(C_A)$ is large, that is the good is sold to more buyers per sale. However, buyers are also accumulated faster. I show that the average *time* between sales is decreasing in the market size λ. Hence, popular products should be on sale more frequently than less popular products. More frequent sales decrease the delay in trade with low-valuation buyers, but they also force the seller to lower the regular price.

In two markets of the same size in terms of revenues (i.e., with same $v_L\lambda$), the market with frequent arrivals and smaller valuations has less frequent sales than the one with less frequent arrivals, but higher valuations . In other words, the model predicts that high-volume, low-ticket goods, such as headphones or groceries, should be on sale less frequently than low-volume, high-ticket goods, such sofas from a specific brand, if the average revenues are approximately the same. The following corollary summarizes these insights.

Corollary 2. *(i) The advertising equilibrium cutoff-demand $N^*(C_A)$ is decreasing in the arrival-adjusted discount rate R_s and in v_L. Furthermore, $\lim_{R_s\to\infty} N^*(C_A) = \frac{C_A}{v_L}$ and $\lim_{R_s\to 0} N^*(C_A) = \infty$.*
(ii) A higher arrival rate decreases the average time between two sales.
(iii) If the arrival rate increases proportionally to a decrease in v_L (i.e., λv_L is constant), then the average time between two sales increases.

The intuition for the second part of (i) is that if the arrival-adjusted discount rate R_s is extremely large, the average cost per customer $\frac{C_A}{N^*(C_A)}$ is approximately equal to the sale price v_L. For small R_s, average costs per customer converge to zero, such that the seller can make high profits during an advertised sale. While an increase in the arrival rate increases the revenues from a sale, delaying this revenue is more costly. Hence, sales are more frequent. (iii) follows from the observation that for high-volume, low-ticket goods, waiting for more low-valuation buyers to arrive is less costly than in low-volume, high-ticket markets.

Finally, for higher advertising costs, the average advertising cost per reactivated buyer $\frac{C_A}{N^*(C_A)}$ is higher. In other words, as the cost of advertising increases, the number of buyers that can be reactivated in equilibrium increases less than proportionally. Hence, a dollar of advertising cost is more valuable as a commitment device if total advertising costs are low than if they are high. The intuition is that, for high $N^*(C_A)$, the revenue during an advertised sales period is higher, so further delay of trade is more costly than if $N^*(C_A)$ is small.

Corollary 3. *The average advertising cost per buyer $\frac{C_A}{N^*(C_A)}$ is increasing in C_A.*

Graphically, this follows from the convexity of $\overline{C}$ and $\underline{C}$ in Figure 3.2.

Proof. This lemma follows by rewriting inequalities (3) and (4) as

$$\underbrace{\sum_{i=0}^{N-1}\left(1-(1+R_s)^{-i}\right)\cdot v_L}_{\text{cost of waiting for } N-1 \text{ arrivals}} \geq \quad C_A \quad \geq \underbrace{\sum_{i=1}^{N}\left(1-(1+R_s)^{-i}\right)\cdot v_L}_{\text{cost of waiting for } N \text{ arrivals}}.$$

Hence, in equilibrium, the advertising cost must equal the cost of waiting for N low-valuation buyers to arrive. The cost of waiting for the N-th arrival is always higher than the cost of waiting for the $(N-1)$-th arrival. □

All in all, my model allows to make testable qualitative predictions about markets with in which intertemporal price discrimination plays a major role. In particular, it shows how in different markets the frequency of sales, which is a crucial for the entire price level, differs.

5 Comparative Statics: Commitment and Buyers' Time Preferences

In this section, I further elaborate the role of advertising costs as a commitment device and how it is related to buyers' time preferences. I first highlight the trade-off coming from intertemporal price discrimination (similar to Sobel (1991)) by assuming that the seller can commit to a fixed cutoff-demand N. Then, I examine the actual model in which advertising costs endogenously generate limited commitment.

5.1 Full Commitment Benchmark

As a benchmark, it is useful to analyze the situation in which the seller can commit to a cutoff-demand N in period zero, that is I assume the seller can attain any profit $\Pi^{FC}(N) \equiv \Pi(0; N)$. Then, the highest profit the seller can attain maximizes

$$\Pi^{FC}(N) \equiv \Pi(0;N) = \underbrace{\frac{\lambda\pi}{r_s} \cdot p_H(N)}_{\text{revenue from } v_H\text{-buyers}} + \underbrace{\frac{(1+R_s)^{-N}}{1-(1+R_s)^{-N}} \cdot (Nv_L - C_A)}_{\text{revenue from } v_L\text{-buyers}} \tag{6}$$

with respect to N.[13] The revenue from high-valuation buyers can be calculated analogously to Π_L, while the revenue from low-valuation buyers $\pi_L(N)$ is recursively given by

$$\pi_L(N) = \underbrace{\mathbb{E}\left[e^{-r_s \cdot t_N^{\lambda(1-\pi)}}\right]}_{(1+R_s)^{-N}} \cdot (Nv_L - C_A + \pi_L(N)),$$

such that $\pi_L(N) = \frac{(1+R_s)^{-N}}{1-(1+R_s)^{-N}} \cdot (Nv_L - C_A)$. The following lemma summarizes the trade-off faced by the seller.

Lemma 5. *(i)* **Cost of frequent sales:** *the regular price $p_H(N)$ and the expected discounted costs from advertising $\frac{(1+R_s)^{-N}}{1-(1+R_s)^{-N}} \cdot C_A$ are increasing and concave in the cutoff-demand N.*
(ii) **Benefit of frequent sales:** *expected discounted revenues from low-valuation buyers $\frac{(1+R_s)^{-N}}{1-(1+R_s)^{-N}} \cdot N \cdot v_L$ are decreasing and convex in the cutoff-demand N.*

[13] Note that the optimal advertising strategy of the seller if she can fully commit to a history-dependent pricing and advertising strategy, is not necessary a function of the state variable n. The seller might want to have deterministic times between two sales. The reason is that if the advertising strategy depends on n, buyers can bring forward sales by rejecting an offers and therefore have a higher incentive to do so.

With frequent sales, i.e. if the cutoff-demand N is low, the seller can pick up low-valuation buyers with less delay which increases her expected profits. This is, however, anticipated by high-valuation buyers. Consequently, the seller has to set a lower regular price $p_H(N)$ in order to satisfy (2). Furthermore, the cost of advertising makes intertemporal price discrimination costly. These effects are smaller for large cutoff-demands N, because the marginal change is further in future and thus discounted more.

Which of the two effects dominates depends on buyers' time preferences. A natural and common benchmark is to assume $r_s = r_b$. In that case, a well-known result is that it is optimal for the seller to set the price equal to the static monopoly price.[14]

Lemma 6. *If the seller and buyer share the same discount rates ($r_s = r_b$), then it is always optimal for the seller to choose the static monopoly price in every period (i.e., v_H if $v_H\pi \geq v_L$ and v_L otherwise).*

Intuitively, if buyers and the seller have the same time preferences given by discount rate r, then they have *approximately* the same expected discount to sale δ.[15] Let us compare the profit that the seller can make if she has sales with a frequency that results in an expected discount to sale δ with the profit that she can make with a constant monopoly price. First, consider the case $v_H\pi \geq v_L$. Then the seller gains less than $\delta v_L \frac{(1-\pi)\lambda}{r}$ because she can sell to low-valuation buyers. However, she has to drop the regular price by $\delta(v_H - v_L)$, that is, she loses $\delta\frac{\lambda(v_H-v_L)\pi}{r}$ in profits. Hence, in total, the seller loses more than $\delta\lambda\frac{v_H\pi - v_L}{r} > 0$. If $v_L > \pi v_H$, then the seller gains by the increased price she can charge to high-valuation buyers $(1-\delta)(v_H - v_L)$, which increases profits by $(1-\delta)\frac{\pi\lambda(v_H-v_L)}{r}$. On the contrary, she loses profits by more than $(1-\delta)v_L\frac{(1-\pi)\lambda}{r}$ because she has to delay trade with low-valuation buyers. Hence, the total loss in profits from having sales is greater than $(1-\delta)\lambda\frac{v_L-\pi v_H}{r}$. The formal proof can be found in the appendix. Figure 3 illustrates profits from price discrimination $\Pi^{FC}(N)$ as a function of cutoff-demand N for $\pi v_H > v_L$. The blue dashed line represents Π_H and the red dotted line represents Π_L. $\Pi^{FC}(N)$ is always lower than Π_H, but it converges to Π_H from below, because with infrequent sales, the profit advertised sales converges to the static monopoly profit.

A more interesting and, arguably, more plausible situation is that the seller is more patient than the buyers, that is, $r_b > r_s$.[16] If buyers are sufficiently impatient relative to the seller, then profits can be increased above static monopoly profits.[17] In that case, the optimal cutoff-demand is finite, which has interesting implications for the profit-maximizing outcome. These insights will be useful for

[14] E.g. Conlisk et al. (1984), Sobel (1991)

[15] In my model, the buyer's expected discount to sale is slightly different because buyers do not know the price history.

[16] Assuming different discount rates of buyers and the seller seems to be a natural assumption. On the one hand, firms usually face lower market interest rates than individuals. On the other hand, it has been shown by many experimental and field studies that individuals' time preferences are represented by relatively high discount rates. See for example, Coller and Williams (1999) or Andreoni and Sprenger (2012).

[17] Landsberger and Meilijson (1985) shows this in a model with finite horizon and without an influx of buyers after period 0 but he does not quantify it. Similarly, Sobel (1984) briefly note that if buyers are more impatient than the seller, the seller can take advantage of the difference in time preferences.

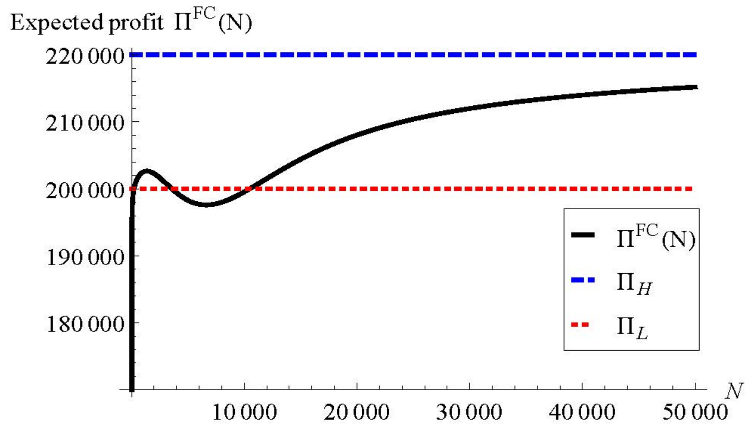

Figure 3: Profit as a function of cutoff-demand $\Pi^{FC}(N)$ with $r_s = r_b = 0.5$, $v_L = 50$, $\pi = 0.5$, $v_H = 110$, $C_A = 100$, $\lambda = 2000$ $r_b = r_s = 0.5$

the analysis of the profit from price discrimination $\Pi(0; N^*(C_A))$ that will be discussed in the next section.

Lemma 7. *(i) As the cutoff-demand approaches infinity, profits $\Pi^{FC}(N)$ approach Π_H from below.*
(ii) The regular price $p_H(N)$ and seller's profit $\Pi^{FC}(N)$ are increasing in buyers' discount rate r_b.
(iii) The seller's profit is decreasing in advertising cost C_A.
(vi) Given parameters $v_H, v_L, \pi, \lambda, C_A, r_s$ there exists a $\overline{r}_b > 0$, such that for all $r_b \geq \overline{r}_b$, there exists a cutoff-demand N for which the seller makes higher profits than if she chooses the static monopoly price forever ($\max_N \Pi^{FC}(N) > \Pi_H$) if and only if

$$\frac{\lambda}{r_s} \cdot (v_L - \pi v_H) < \left[\max_{N \geq 0} \frac{(1+R_s)^{-N}}{1-(1+R_s)^{-N}} \cdot (N v_L - C_A) \right]. \tag{7}$$

For $C_A = 0$, this condition becomes $\pi v_H > 0$ and as $C_A \to \infty$, it converges to $v_L \leq \pi v_H$.

It is straightforward that as sales become very infrequent (i.e., $N \to \infty$), $p_H(N)$ approaches v_H and the profits from low-valuation buyers vanish, as already shown in Lemma 5. More interestingly, this limit is always approached from below, that is, for large N, the profit loss due to a lower regular price outweighs the gain from frequent sales.

Intertemporal price discrimination is more profitable in markets with impatient buyers. As the buyers' discount rate increases, the seller can increase the regular price $p_H(N)$ because buyers care less about future sales. Hence, consumer surplus drops and firm's profits increase. In fact, price discrimination can increase profits far above the static monopoly profits as illustrated in Figure 3. (iii) is straightforward because advertising costs do not benefit the seller in any way.

Finally, (iv) specifies a condition on parameters that guarantees that price discrimination is profitable for some discount rates r_b of buyers. With sufficiently large r_b, the seller can always charge a

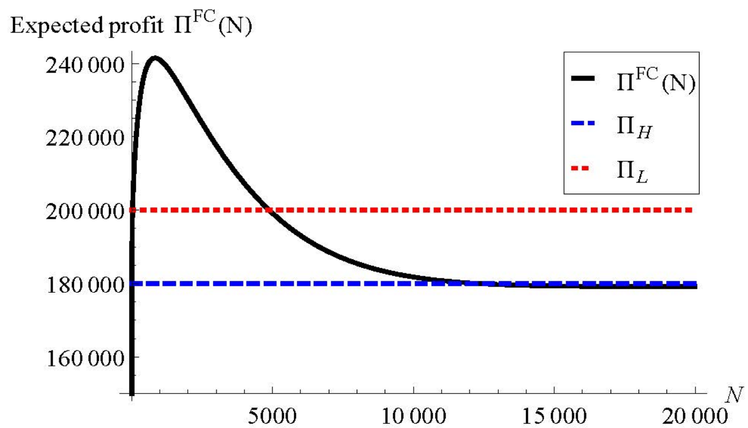

Figure 4: Profit as a function of cutoff-demand N with $r_s = 0.5$, $r_b = 5$, $v_L = 50$, $\pi = 0.5$, $v_H = 90$, $C_A = 100$, $\lambda = 2000$, i.e. $v_L > \pi v_H$.

regular price that is arbitrarily close to v_H. Hence, as long as $\pi v_H \geq v_L$, it is always profitable for her to price discriminate for large enough r_b. Remarkably, price discrimination can also be profitable if $\pi v_H < v_L$. In fact, if advertising is free, price discrimination is always profitable for sufficiently impatient buyers. This is demonstrated in Figure 4. The inequality in (iv) implies that, in this case, the monopoly profit from high-valuation types $\frac{\lambda \pi v_H}{r_s}$ must make up for the cost of delay in trade with low-valuation buyers and advertising

$$\frac{\lambda v_L}{r_s} - \max_{N \geq 0} \frac{1}{(1+R_s)^N - 1} \cdot (N v_L - C_A) .$$

In contrast, as high $C_A \to \infty$, the price discrimination can be profitable only if $v_L < \pi v_H$. Next, I consider some properties of the locally profit-maximizing cutoff-demand $\Pi^{FC}(N)$. Graphically, one can see that $\Pi^{FC}(N)$ is either always increasing everywhere or it has has a single local maximum (and minimum) as illustrated in Figures 3, 4, and 5. I do not prove that there is always a *unique* local maximum is, however, because the fact that the derivatives of Π are intractable analytically.

Lemma 8. *(i) Any locally profit-maximizing cutoff-demand is increasing in r_b for small levels of r_b and decreasing everywhere else.*
(ii) Any locally profit-maximizing cutoff-demand is increasing in advertising costs C_A.

The intuition for Lemma 8 (i) can be understood as follows. For simplicity, let us assume $C_A = 0$. Buyers' time preferences only affect profits through the regular price $p_H(N)$. In particular, buyers' time preferences determine the *sensitivity of $p_H(N)$ to changes in N*, that is, they affect the marginal benefit (MB) of increasing N. The marginal cost (MC) of increasing N is given by the cost of delay and is independent of buyers' time preferences. The MB of increasing N by one is small if buyers are extremely patient or extremely impatient, but larger in between. Moreover, this marginal effect is smaller for an increase in N further in the future (i.e., for large N) than early on (i.e., for small N). The

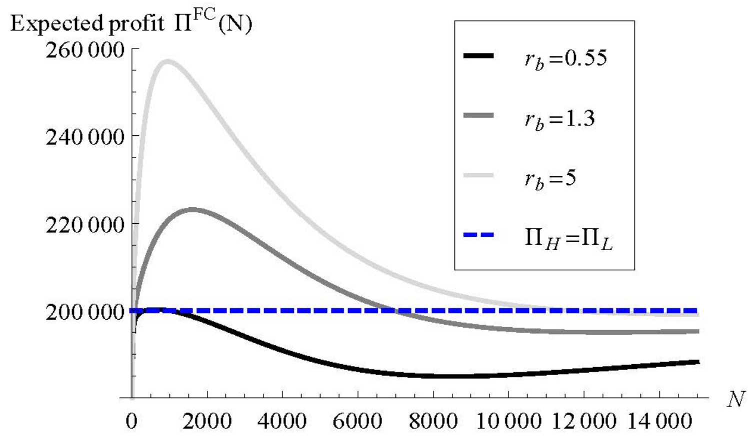

Figure 5: Profit as a function of cutoff-demand N for $v_L = 50$, $\pi = 0.5$, $v_H = 100$, $C_A = 10$, $\lambda = 2000$, $r_s = 0.5$

seller can benefit from waiting for an additional low-valuation buyer as long as the MB is larger than the MC of delay. For very patient and very impatient buyers, the marginal benefit of delay exceeds the marginal cost of delay only for small cutoff-demands, such that the locally profit-maximizing cutoff-demand is small. In contrast, for intermediately patient buyers, higher cutoff-demands become optimal. Hence, the locally profit-maximizing cutoff-demand is first increasing and then decreasing in r_b and is largest in between.

The limiting cases when buyers are extremely impatient ($r_b \to \infty$) and when buyers are extremely patient ($r_b \to 0$) are particularly interesting. If buyers are extremely impatient, then the seller can charge a regular price close to v_H (as long as $N > 1$). Hence, the benefit of frequent sales completely dominates, so an increase in N decreases profits. It is, thus, optimal for the seller to choose the smallest cutoff-demand. In the other extreme, for very patient buyers, the regular price $p_H(N)$ is close to v_L, so the seller can only benefit from extremely large N. Small delays are not beneficial for the seller at all, so the local maximum of $\Pi^{FC}(N)$ is zero.

These results are useful to prove similar properties for $\Pi(0; N^*(C_A))$ as a function of C_A in the next subsection. In order to illustrate these results, Figure 5 shows that the profit-maximizing cutoff-demand is increasing from $r_b = 0.55$ to $r_b = 1$, but decreasing from $r_b = 1.3$ to $r_b = 5$.

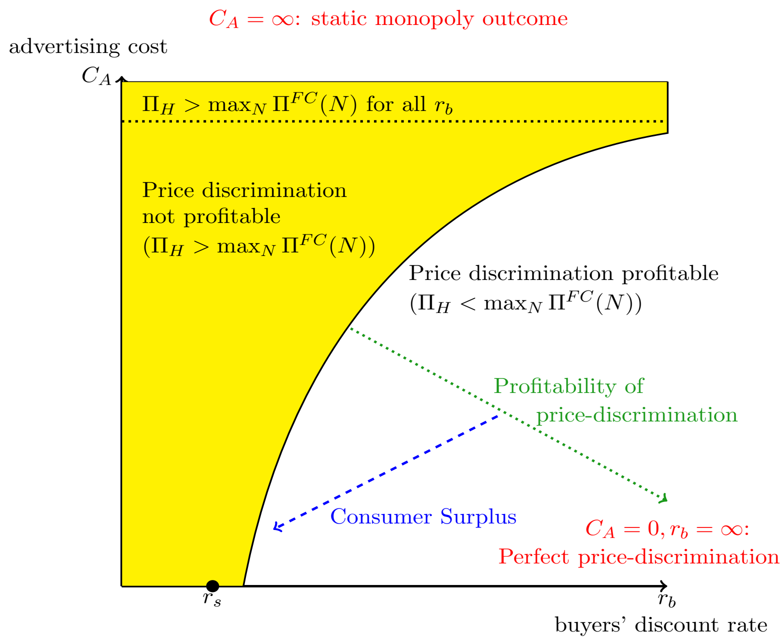

Figure 6: Implication of interaction of C_A and r_b on profit from price discrimination

Finally, the frequency of sales is decreasing in advertising cost C_A, because it is more costly to have advertised sales with higher C_A. Figure 6 summarizes all results. It depicts the region in which the seller prefers a constant monopoly price v_H for different consumer impatience and advertising cost. If it is infinitely costly to recall buyers, then the repeated static monopoly outcome is profit-maximizing. On the other extreme, if it is costless to recall buyers and if they are myopic (i.e., $r_b = \infty$), then the seller can perfectly price discriminate. For intermediate values, it is harder to price-discriminate profitably the higher C_A and the smaller r_b. However, consumers receive more surplus if r_b is low. Hence, as shown in Proposition 8, the effect on welfare is generally not monotone in r_b, because the delay in trade with low-valuation buyers can increase or decrease in r_b. Finally, the greater the gap $v_L - \pi v_H$, the harder it becomes to profitably price discriminate. In particular, if $v_L - \pi v_H > 0$ and advertising costs C_A are above the threshold given by (7), price discrimination is not profitable for any r_b. If $v_H \pi > v_L$, for any finite advertising cost C_A, price discrimination is profitable for sufficiently impatient buyers.

5.2 Endogenous Commitment

In this subsection, I show how the trade-off discussed in the previous subsection affects how much the seller benefits from advertising costs as a commitment device. To this end, I consider the profit from price discrimination, $\Pi(0; N^*(C_A))$, as a function of C_A. By Proposition 1, each C_A pins down a (generically) unique $N^*(C_A)$ and also affects the function $\Pi(0; N)$ directly. Then, I show how buyers' discount rates r_b affect the equilibrium outcome. In particular, I show under which conditions there exists an advertising equilibrium.

Denote the seller's profit if the advertising cost is $C_A = \underline{C}(N)$, i.e., just high enough to sustain cutoff-demand N, by

$$\overline{\Pi}(N) \equiv \Pi(0; N^*(\underline{C}(N))) = \frac{\lambda\pi}{r_s} p_H(N) + \left(1 + R_s^{-1}\right)(1 + R_s)^{-N} v_L \tag{8}$$

and the seller's profit if the advertising cost is $C_A = \overline{C}(N)$ and cutoff-demand N is sustained by

$$\underline{\Pi}(N) \equiv \Pi(0; N^*(\overline{C}(N))) = \frac{\lambda\pi}{r_s} p_H(N) + R_s^{-1}(1 + R_s)^{-N} v_L. \tag{9}$$

A change in advertising cost C_A can have two different effects:

1. If the cost C_A is in $(\underline{C}(N), \overline{C}(N))$ for an integer N, then a small increase in the advertising cost does not help to sustain longer cutoff-demands. Hence, profits decrease linearly.

2. If the cost C_A is $\overline{C}(N) = \underline{C}(N+1)$ for an integer N, then a small increase in the advertising cost results in a longer cutoff-demand $N+1$. Hence, profits jump from $\underline{\Pi}(N)$ to $\overline{\Pi}(N+1)$. This effect can be positive or negative depending on the shape of $\underline{\Pi}$ and $\overline{\Pi}$.

More precisely, the correspondence that maps to each advertising cost the equilibrium profits in an advertising equilibrium $\mathbf{\Pi}(C_A)$ is given by

$$\mathbf{\Pi}(C_A) = \begin{cases} \left\{\overline{\Pi}(N), \underline{\Pi}(N-1)\right\} & \text{if } C_A = \underline{C}(N) = \overline{C}(N-1) \\ \overline{\Pi}(N) - \frac{(1+R_s)^{-N}}{1-(1+R_s)^{-N}}(C_A - \underline{C}(N)) & \text{if } C_A \in (\underline{C}(N), \overline{C}(N)) \end{cases}$$

By Proposition 1, an advertising equilibrium exists if and only if $\mathbf{\Pi}(C_A) \geq \Pi_L$.

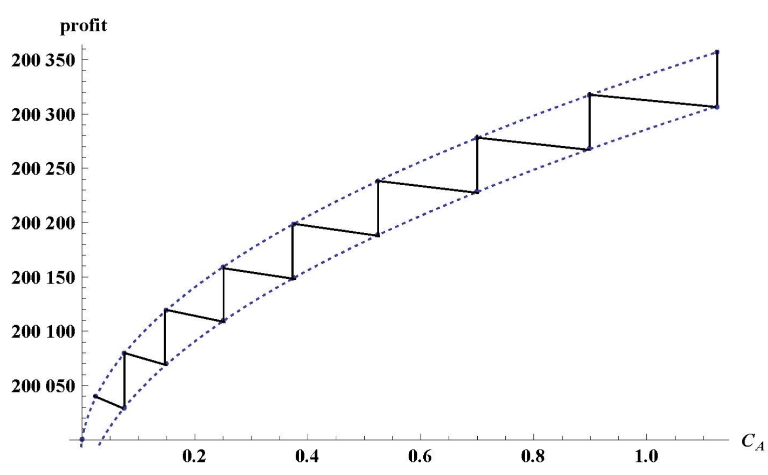

Figure 7: $\mathbf{\Pi}(C_A)$ (solid) and $\overline{\Pi}(\underline{C}^{-1}(C_A)), \underline{\Pi}(\overline{C}^{-1}(C_A))$ (dotted) for parameter values $r_s = 0.5$, $v_H = 110$, $v_L = 50$, $\pi = 0.5$, $\lambda = 2000$, $r_b = 1.5$

Figure 7 illustrates $\mathbf{\Pi}(C_A)$. Moreover, it shows that $\overline{\Pi}$ and $\underline{\Pi}$ define upper and lower bounds of $\mathbf{\Pi}(C_A)$ given by

$$\underline{\Pi}(\overline{C}^{-1}(C_A)) \leq \mathbf{\Pi}(C_A) \leq \overline{\Pi}(\underline{C}^{-1}(C_A)).$$

The inverse of the functions $\overline{C}$ and $\underline{C}$ are well-defined because they are increasing as shown in Section 3.2. Furthermore, on the grid

$$\mathcal{C} = \{C_A | C_A = \underline{C}(N) \text{ with } N \in \{1, 2, 3, \dots\}\},$$

the bounds are binding, that is

$$\underline{\Pi}(\overline{C}^{-1}(C_A)) = \mathbf{\Pi}(C_A) = \overline{\Pi}(\underline{C}^{-1}(C_A)).$$

Hence, I focus the analysis of $\mathbf{\Pi}(C_A)$ on the grid $\mathcal{C}$ where C_A acts as an endogenous commitment device. To this end, it is sufficient to analyze the functions $\overline{\Pi}$ and $\underline{\Pi}$ since their properties carry on to $\mathbf{\Pi}$ on the grid $\{\underline{C}(N) : N \in \{1, 2, 3, \dots\}\}$.

The following lemma summarizes some properties of $\overline{\Pi}$. Since $\underline{\Pi}(N) = \overline{\Pi}(N) - (1 + R_s)^{-N} v_L$, I do not to analyze $\underline{\Pi}$ separately.

Lemma 9. *(i)* $\overline{\Pi}(1) = \Pi_L$ *and* $\lim_{N\to\infty} \overline{\Pi}(N) = \Pi_H$.
(ii) $\overline{\Pi}$ *has at most one local maximum and at most one local minimum.*
(iii) $\overline{\Pi}(N)$ *and* $\overline{\Pi}'(N)|_{N=1}$ *are increasing in* r_b. $\overline{\Pi}'(N)|_{N=1}$ *is negative for small* r_b.

First in the limit, the monopolist makes profits Π_L and Π_H that correspond to constant prices. For intermediate cutoff-demands, there is at most one local maximum and if the seller is sufficiently patient compared to buyers (in particular if $r_s < r_b$), then there is also at most one local minimum. This specifies the shape of $\overline{\Pi}$ and how the seller's trade-offs are resolved. Figure 8 illustrates $\overline{\Pi}$ for different levels of discount rates of buyers.

The discount rate plays an important role for the existence of advertising equilibria. With relatively patient buyers, small C_A means that it is not possible sustain long enough cycle lengths to make price discrimination profitable for the seller because the seller has to leave too much surplus to high-valuation buyers (i.e., $\overline{\Pi}'(N)|_{N=1} < 0$). In that case, $\overline{\Pi}$ is first decreasing and then increasing. Hence, for small advertising costs C_A, the unique equilibrium is an EDLP equilibrium because $\overline{\Pi}(N^*(C_A)) < \Pi_L$. However, as C_A increases, profits converge to Π_H. Thus, an advertising equilibrium exists, as long as $\pi v_H \geq v_L$. In contrast, with impatient buyers, even small C_A can sustain cutoff-demands that make price discrimination profitable ($\overline{\Pi}'(N)|_{N=1} > 0$).

If $\pi v_H > v_L$ and for some intermediate values of r_b, the non-monotonicity of the profit function creates interesting comparative statics: for low C_A, the seller wishes to price discriminate because she can increase the regular price sufficiently to make it profitable. However, in an intermediate region

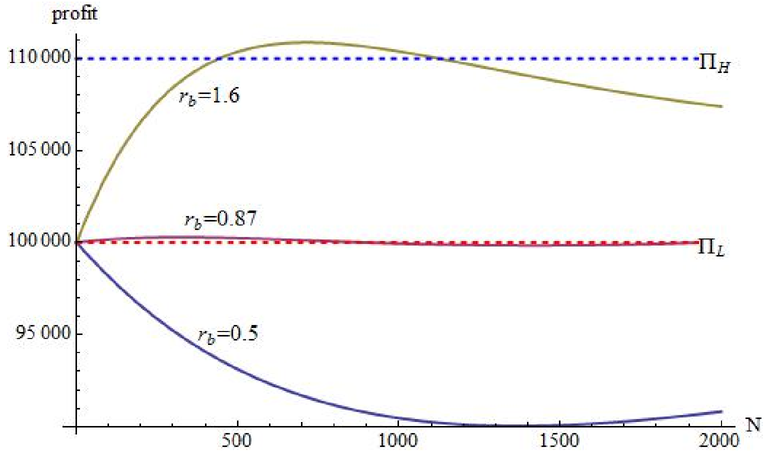

Figure 8: Highest equilibrium profits with cutoff-demand $N^*(C_A)$ for parameter values $r_s = 0.5$, $v_H = 110$, $v_L = 50$, $\pi = 0.5$, $\lambda = 1000$ and different discount rates

of advertising costs, the cost of delaying trade with low-valuation buyers dominates the benefit of sustaining longer cutoff-demands. Hence, the seller prefers to set a constant price of v_L and not to have sales. For sufficiently high C_A, profits converge to the static monopoly profit, so temporary sales are retained from time to time again. Figure 9 illustrates $\mathbf{\Pi}(C_A)$ for the same parameters as used for figure 8. It illustrates how properties of $\overline{\Pi}$ carry over to $\mathbf{\Pi}$.

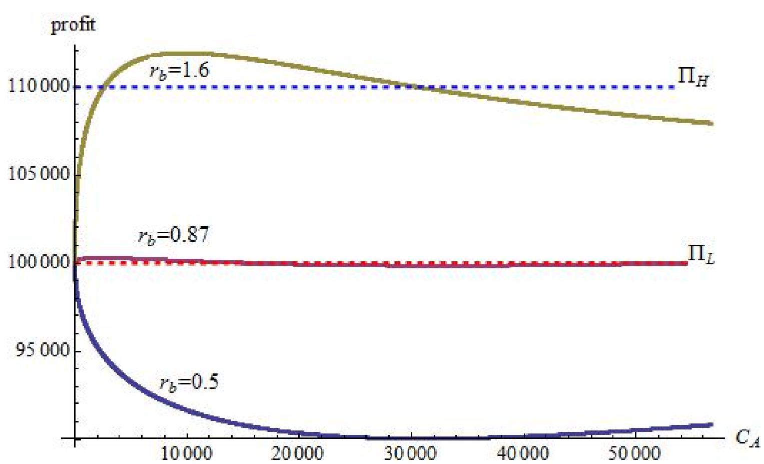

Figure 9: Highest equilibrium profits with cutoff-demand $N^*(C_A)$ for parameter values $r_s = 0.5$, $v_H = 110$, $v_L = 50$, $\pi = 0.5$, $\lambda = 1000$ and different discount rates

Figure 9 also offers a calibration for equilibria given different advertising costs and buyers' time preferences. The seller's discount rate is set to be equal to $r_s = 0.5$ which corresponds to a discount factor of 0.61 per period (e.g. a year). If advertising costs are for example 6000, then if buyers have the same discount rate as the seller, the unique equilibrium is an EDLP equilibrium. In contrast, if the buyers' discount rate is $r_b = 0.87$ (discount factor of 0.42), there is an advertising equilibrium and the seller makes profits close to Π_L. If the buyers' discount rate is $r_b = 1.6$ (discount factor of 0.2), the makes even higher profits than Π_H. This illustrates that with modest advertising costs, the seller's profits are very sensitive to differences in buyers' time preferences.

I define two thresholds

$$T_0(r_b) = \inf\{C_A : \overline{\Pi}(\underline{C}^{-1}(C_A)) < \Pi_L\}$$

and

$$T_1(r_b) = \sup\{C_A : \overline{\Pi}(\underline{C}^{-1}(C_A)) < \Pi_L\}$$

where $\sup \emptyset = \inf \emptyset = 0$. The following proposition follows from Lemma 9.

Proposition 2. *(i) An advertising equilibrium exists given an advertising cost $C_A \in \mathcal{C}$ if and only if*

$$C_A \notin (T_0(r_b), T_1(r_b)).$$

(ii) $T_1(r_b)$ is decreasing in r_b and $\lim_{r_b \to 0} T_1(r_b) = \infty$.
(iii) $T_0(r_b)$ is increasing in r_b as long as $0 < T_0(r_b) < T_1(r_b)$. $T_0(r_b) = 0$ for small r_b.
(iv) The length of the interval $(T_0(r_b), T_1(r_b))$ is decreasing in r_b. If $v_H\pi > v_L$, $T_0(r_b) = T_1(r_b) = 0$ for large enough r_b. If $v_H\pi \leq v_L$, $T_1(r_b) = \infty$ for all r_b.

This proposition characterizes the set parameters r_b and C_A for which advertising equilibria do not exist. If the static monopoly price is high (i.e., $v_H\pi > v_L$), then there are essentially three different cases:

1. For small discount rates r_b, $T_0(r_b) = 0 < T_1(r_b)$, that is advertising equilibria exist only for large enough advertising costs C_A.

2. For intermediate discount rates r_b, $0 < T_0(r_b) < T_1(r_b) < \infty$, that is advertising equilibria do not exist only for some intermediate advertising costs C_A. The intuition is that for intermediate advertising costs, the induced frequency of sales does not increase the regular price enough to outweigh the cost of delay and advertising.

3. For large enough discount rates r_b, $T_0(r_b) = T_1(r_b) = 0$, that is advertising equilibria always exist as long as $C_A \geq \underline{C}(1)$.

In Figure 8 these three different cases for $\pi v_H > v_L$ are illustrated. Indeed, for $r_b = 0.5$, $\overline{\Pi}(N)$ is first smaller than Π_L and then greater. For $r_b = 0.86$, $\overline{\Pi}(N)$ is first greater, then smaller and then again greater than Π_L. For $r_b = 1.7$, $\overline{\Pi}(N) > \Pi_L$ always holds.

If the static monopoly price is low (i.e., $v_H\pi \leq v_L$), then all equilibria are EDLP equilibria for small r_b. For large enough r_b, advertising equilibria exist only for small enough advertising costs C_A. For large advertising cost, the induced profit is always close to, but smaller than $\Pi_H \leq \Pi_L$.

Next, I characterize the optimal advertising cost for the seller if she cannot commit. If the seller can choose C_A, she faces a trade-off between decreasing profits if C_A is high and increasing commitment possibilities. Hence, a seller potentially wants to have an intermediate advertising cost. For example, in Figure 9, the seller would mostly prefer an advertising cost of approximately 8000. As can be seen in Figure 8, this would result in an average frequency of sales of one to two times per period.

Proposition 3. *(i) Let $v_H\pi > v_L$. The profit-maximizing advertising cost is $C_A^* = \infty$ for sufficiently patient buyers. Otherwise, $C_A^* < \infty$ is quasi-concave in r_b.*
(ii) Let $v_H\pi \leq v_L$. Then, if there are advertising equilibria for r_b, $C_A^* < \infty$ is quasi-concave in r_b.
(iii) As $r_b \to \infty$, the profit-maximizing advertising cost C_A^* converges to $\underline{C}(1)$.

The optimal advertising cost C_A^* shows to what advertising costs the sellers would optimally want to commit. For example, a firm can benefit significantly, if it follows a policy that ensures relatively high advertising costs. This can be done by hiring expensive designers and computer specialists who among other tasks design and manage an advertising campaign.

The intuition and proof of Proposition 3 follows from Lemma 8. Recall, that buyers' time preferences determine the *sensitivity of the regular price $p_H(N)$ to changes in cutoff-demand N*. Hence, it determines, how valuable different levels of commitment are to the seller and what the marginal benefit (MB) of additional commitment (i.e., of less frequent sales) to the seller is. It is crucial that for very impatient or very patient buyers, the MB of additional commitment is small, while for intermediately patient buyers it is relatively large. The marginal cost (MC) of additional commitment comes from the cost of advertising and the cost of delay, which are both independent of buyers' time preferences. Hence, for very patient or very impatient buyers, the MB can exceed the MC of commitment only for small levels of commitment. In contrast, if buyers are intermediately patient, the seller can benefit from increasing advertising costs C_A further. For extremely patient buyers, the seller is best-off with infinite advertising costs. The locally-optimal advertising cost is, however, very small.

Hence, moderate advertising costs are only valuable for the seller as a commitment device if buyers are intermediately patient. This is seems to be a plausible assumption in many markets. The literature on intertemporal price discrimination has, however, mostly focused on either myopic buyers ($r_b \to \infty$) or the case in which buyers and seller have the same time preferences. With almost myopic buyers, the seller can almost perfectly price discriminate with very frequent sales, while with very patient buyers only an EDLP can be sustained.

Figure 10 summarizes the properties of equilibria if $v_H\pi > v_L$. The boundaries of the regions are dashed because the boundaries are "thick" and belong in parts to both of the regions.[18]

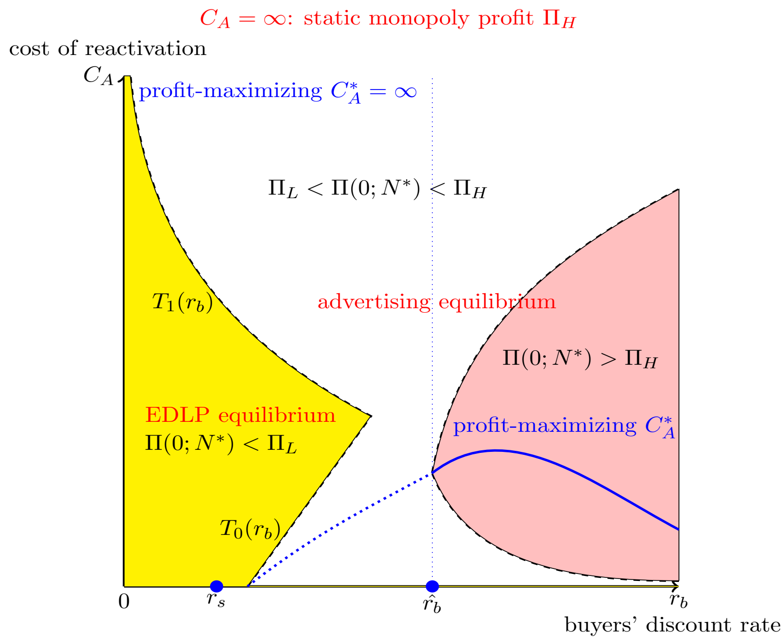

Figure 10: Stylized equilibrium properties for $v_H \pi > v_L$

The yellow region represents the parameters for which the unique equilibrium is an EDLP equilibrium, while for all other parameters, advertising can be sustained in equilibrium. The red region illustrates advertising equilibria for which the seller can make higher profits than with a constant static monopoly price. The solid blue line represents the profit-maximizing advertising cost. The dotted extension of this line illustrates the local maximum of the profit function $\mathbf{\Pi}(C_A)$.

For patient buyers (r_b small), small advertising costs imply EDLP equilibria, while large advertising costs result in an advertising equilibrium. In contrast, if buyers are intermediately patient, EDLP equilibria are sustained for intermediate advertising costs, but for small and large advertising costs, the unique equilibrium is always an advertising equilibrium. The seller prefers infinite advertising costs, but a local maximum of $\Pi(C_A)$ is attained for small advertising costs. Hence, given that infinite advertising costs are hard to sustain, in markets with intermediately patient buyers, sellers have an incentive to invest into a decrease in advertising costs that result in relatively frequent sales.

For patient enough buyers, all equilibria are advertising equilibria except if advertising costs are too small to sustain a cutoff-demand greater than 1 (Coase conjecture). A seller has an incentive to commit to intermediate advertising costs and to have less frequent sales. However, as buyers become very patient, commitment becomes less valuable and small advertising costs become optimal.

[18] This is because C_A is not always on the grid $\{\underline{C}(N) : N \in \{1, 2, 3, \dots\}\}$.

For welfare, lower advertising cost are always better in markets with sufficiently impatient and sufficiently patient buyers. For intermediate impatient levels, intermediate levels of advertising can be optimal because they result in an EDLP equilibrium.

Figure 11 summarizes the equilibrium properties for $v_H \pi \leq v_L$. In that case, price discrimination is never profitable for large advertising costs C_A because, as $C_A \to \infty$, profits are close to Π_H, which is less than Π_L by assumption. However, if buyers are sufficiently impatient, small advertising costs result in advertising equilibria.

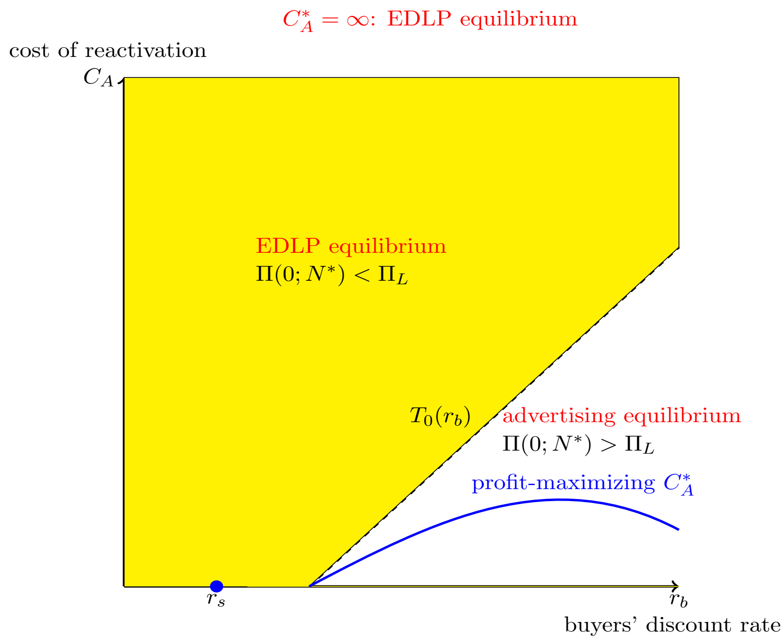

Figure 11: Stylized equilibrium properties for $v_H \pi \leq v_L$

All in all, for sufficiently impatient buyers, equilibria look very similar to the case with $v_H \pi \geq v_l$, while with relatively patient buyers, the equilibria differ dramatically. The reason is that with sufficiently impatient buyers, the seller can benefit from price discrimination even if the static monopoly price is low.

6 Discussion and Related Literature

In this section, I first discuss and relax two assumptions included in the model: fixed advertising cost and unit demand of the seller. In addition, I explain how my model is robust to generalizations. Subsequently, I relate the paper to the existing literature. To this end, I discuss existing dynamic pricing models with full and no commitment and argue why in online markets full commitment is hard to attain. Finally, I relate my paper to other pricing and marketing literature.

6.1 Extensions

6.1.1 Variable Costs

Usually, the fixed cost of advertising consists of the cost of designing the advertising email while the variable cost of sending the email is close to zero. If the firm has to pay an advertising platform per view or per click, then the cost of advertising can have a variable component.

If there is a variable component of advertising costs that depends on the number N of low-valuation buyers reached, the analysis can be generalized easily. In particular, with a linear cost function $\mathcal{C}_{\mathcal{A}}(N) = C_A + c_A \cdot N$, $\underline{N}$ and $\overline{N}$ can be equivalently derived from (3) and (4) with v_L being reduced to $v_L - c_A$. The seller always prefers to have fixed costs to variable costs. In order to see this, note that the same $N^*(\mathcal{C}_{\mathcal{A}})$ results from advertising costs of the form $\mathcal{C}_{\mathcal{A}}(N) = C_A + c_A \cdot N$ as long as $\frac{C_A}{v_L - c_A}$ is constant. Hence, the profits of the seller are only affected in sales periods. The profit during a sale is given by

$$N^*(\mathcal{C}_{\mathcal{A}}) \cdot (v_L - c_A) - C_A \quad = \quad (v_L - c_A) \cdot \underbrace{\left[N^*(C_A) - \frac{C_A}{v_L - c_A} \right]}_{\text{constant}} .$$

Thus, the firm prefers not to face variable costs but rather a fixed advertising cost because it is a cheaper commitment device. In particular, if C_A is zero, the Coase conjecture holds. This can have implications for pricing of advertising platforms. Firms are willing to pay more to an advertising platform that has fixed rates rather than per-click or per-view rates as commonly used.

6.2 Multi-Unit Demand

The assumption that buyers demand only a single unit of the good in their lifetime can be relaxed without changing the results as long as buyers do not have an incentive to stockpile the good for future use during a sale. This is the case if the good is not storable, such as groceries or if buyers do not take calender time into account even if they have learned it during a previous sales period. The latter is a plausible assumption if buyers demand a good very infrequently.

There is a recent theoretical and empirical literature on stockpiling of storable goods. Dudine et al. (2006) show that for storable goods prices are higher and welfare lower with commitment than without commitment. They consider a model with finite time and linear storage cost. Hendel and Nevo (2013) show that buyers anticipate future needs for soft-drinks, and buy big quantities during a sale for future consumption. Other recent empirical studies on storable goods markets have been conducted by Hendel and Nevo (2006b) and Hendel and Nevo (2006a).

If buyers demand several units of the good at a time, but only once in their life time, results remain qualitatively unchanged. However, given an advertising cost, the seller can commit to a higher cutoff-demand as can be seen from the necessary conditions (3) and (4).

6.3 Related Literature

6.3.1 Dynamic Pricing and Commitment

Classic papers on dynamic pricing by Stokey (1979), Conlisk et al. (1984), and Sobel (1991) introduce the idea of intertemporal price discrimination with limited commitment. In these papers, the level of commitment is exogenously given and buyers and sellers are assumed to be equally patient.[19] Hence, the frequency of sales is determined by the period length and if the seller is assumed to be able to change the price at any point in time, the unique equilibrium outcome results in the Coase conjecture. This paper extends this classical theory in two ways: it investigates the trade-off between the value of commitment and the cost of commitment, given by the advertising cost, and it highlights the impact of buyers' time preferences. Furthermore, in my model, buyers do not automatically see all prices, but have to get informed by the seller.

Many recent papers from the dynamic mechanism design literature, such as Board (2008) and Garrett (2012) in contrast assume full commitment power by the seller. Board (2008) focuses on seasonal sales that are driven by demand fluctuations over time, rather than sales solely driven by the incentive to intertemporarily price discriminate. Such sales are predicted by customers and in his model, the seller has to drop the price slowly to abate the effect of buyers delaying purchase in order to wait for a lower price, but increases the price quickly after a sale. This asymmetry between increases and decreases induces a higher total price level and, hence, reduces consumer surplus and social welfare. In contrast, in my model, buyers benefit from sales due to the inability of the seller to commit. Garrett (2012) also assumes full commitment by the seller, but considers a stationary setup as in this paper. In his model, cyclical price paths with slowly decreasing prices and upward jumps after a sale are obtained because buyers' preferences change over time. Besbes and Lobel (2012) take a very different approach by investigating intertemporal price discrimination in a setup without exponential discounting. Instead buyers have heterogeneous finite willingness to wait and the monopolist maximizes the long-run average revenue. The optimal price path is more complicated and incorporates nested sales with largest discount at the end of a cycle.

A common justification for commitment is that in many setups a best-price provision strategy can serve as a commitment device for sellers and yield the same allocation as the best strategy under full commitment as first noted by Butz (1990). This is a reasonable argument in traditional mortar-and-brick stores. In online markets, however, this logic does not apply because the seller can send coupons to buyers (or have instantaneous sales as in my model) which are not seen by buyers who have bought. This possibility makes commitment through best-price provision harder. Nevertheless, some retailers, such as airline companies, have automatized their pricing using complicated algorithms. This can help

[19] This assumption is usually motivated by perfect capital markets, but there are many reasons to assume that buyers and sellers do not share the same time preferences as discussed before.

sellers to commit. For many consumption goods, this automation is, however, not feasible because price offers require an attractive design. This design cannot be easily standardized if one thinks of the product not as a single good but as a product group (such as lamps for example) that is put on sales periodically.

The literature that assumes no commitment power has focused on sellers with a finite inventory and a sales deadline. In that case, the deadline and scarcity of products can provide the seller with sufficient commitment power to guarantee high regular prices and fire sales from time to time. Hörner and Samuelson (2011) study a variant of this setup in which the seller has only a single unit of the good and Dilme and Li (2013) deal with the multi-unit case. If the seller has many units, she is choosing prices and quantities to have on sale simultaneously. Similarly to my model, the equilibrium price path contains sales of large quantities of the good similarly to my model. However, in their model, this is done so as to guarantee higher prices later on by making the good scarcer. Instead, my model considers a monopolist who can produce arbitrary many units, albeit at zero cost.

6.3.2 Other Theories of Pricing

The literature on price flutuation that does not rely on intertemporal price discrimination focuses on the seller's incentive to discriminate between different types of buyers in a static oligopolistic setups (e.g., Shilony (1977),Varian (1980), Salop and Stiglitz (1982), Sobel (1984)). They derive mixed pricing strategies as an equilibrium outcome. In Varian (1980) and Salop and Stiglitz (1982), sellers price discriminate between informed and uninformed customers, facing a decreasing average cost curves. In equilibrium, firms mix among prices according to a smooth distribution. The firm with the lowest price can sell to all types of consumers, while all other firms sell only to a fraction of uninformed consumers. One drawback of these models is that they do not result in a regular price and sales prices. With loss-averse customers, Heidhues and Köszegi (2012) and Rosato (2013) show that even in the absence of competition, mixed-strategy equilibria can lead to a single regular price and sales prices.

I do not consider competitive markets, but I believe that the results remain qualitatively similar if buyers switch between firms with some probability. Prices and frequencies of sales would be pushed down, but the comparative statics should remain similar. If, however, buyers can find out about other firms with a positive search cost, interesting new trade-offs can arise. This is beyond the scope of the present paper and left for future research. Intuitively, with competition, the monopolist is not only competing with its future self, but also with another firm.

This paper also relates to the literature on behavior-based price discrimination. These papers are mainly concerned with products that are repeatedly purchased. In that case, sellers infer from buyers purchasing decision that they have a high valuation. For example, Villas-Boas (2004) considers an overlapping generation model in which buyers live for two periods and the seller can charge two different prices, one for customers who have previously bought and one for new customers. In equilibrium, the price path for new customers is alternating between a high and low price and previous customers always face a high price. A detailed review of the literature on behavior-based price discrimination can be found in Fudenberg and Villas-Boas (2012) and Fudenberg and Villas-Boas (2006). In contrast to these models that focus on sellers learning only from customers who actually buy, in my model, the monopolist learns from the fact that a customer has not bought, assuming she can identify "window shoppers."

6.3.3 Theories of Advertising

Advertising has been interpreted as a technology to signal the quality of the good (Nelson (1970)) or to persuade consumers and change their taste for the good (Dixit and Norman (1978)). A comprehensive review of this classical literature on advertising can be found in Bagwell (2007). In contrast, I focus on informative advertising. Anderson and Renault (2013) and Anderson and Renault (2006) are also concerned with informative advertising, but they consider a static setup in which the content of informative advertising depends on the search cost that buyers need to pay in order to visit the monopolist. In my model, however, the only information that buyers can learn is the discounted price.

Recently, the focus of this field has shifted towards targeted advertising in online markets, but with a focus on static models. Iyer et al. (2005) consider a model with competing firms who can target advertising and pricing to different groups. In equilibrium, firms advertise more to consumers who have a strong preference for their product. In contrast to my model, targeted advertising always increases profits. Bergemann and Bonatti (2011) compare the role of targeted advertising in offline versus that in online markets in a competitive environment with many sellers and many advertising markets (media). They show that an increase in targeting ability leads to an improvement in consumer-product matches, but also to a higher market-power of firms. Finally, in their model, the price of advertising is determined endogenously in equilibrium and is first increasing and then decreasing in targeting capacity. In my environment, however, it seems to be natural to assume that advertising costs are given by the cost of creating the advertising because sellers do not buy advertising space from third parties.

Levin and Milgrom (2010) argue that, even though targeting improves the matching between buyers and advertisers, this benefit has to be traded off with the mutual adverse-selection problems it can create between the advertisers and advertising platforms. In contrast, the present paper examines dynamic effects of advertisings that targets low-valuation buyers in order to inform them about the price and in order to activate them. To the best of my knowledge, this role of advertising is novel to the literature.

Finally, in the marketing literature, EDLP and promotional pricing are the two most important marketing and pricing strategies. Most papers consider oligopolistic setups with differences in customer types. Lal and Rao (1997) explain the coexistence of these two strategies by a game theoretic model in an oligopolistic setup, where buyers with low search cost (cherry pickers) prefer to buy from firms who engage in promotional pricing while buyers with high search cost (time constrained consumers) prefer EDLP. Bell and Lattin (1998) argue that EDLP attracts large basket consumers who prefer a low average price while small basket customers like promotional pricing. [20] I offer an alternative rationale for EDLP.

7 Conclusion

This paper studies a dynamic monopolist who can engage in targeted advertising, but cannot commit to future pricing strategies. The ability to track and target can be a two-edged sword because it allows the seller to intertemporally price discriminate, but it also exposes the seller to the consequences of the Coase conjecture. Costly advertising can benefit the seller because advertising costs act as a deterrent to having overly frequent sales in equilibrium. Depending on the time preferences of buyers and the costs of advertising, there exists an equilibrium with regular high prices and occasional sales. An everyday-low-price equilibrium can always be sustained but yields lower profits than the advertising equilibrium, if the latter exists.

The benefit of the model is that the level of commitment is endogenously determined by the cost of advertising. I derive implications for the frequency of sales, profits, demand for advertising and welfare. The analysis shows that the frequency of sales and the resulting regular price level can be crucially affected by the advertising costs. Sales are more frequent with low advertising costs if price discrimination is profitable compared to EDLP. Because profits are non-monotonic in advertising costs, the existence of an advertising equilibrium is not monotonic in advertising cost. Moreover, the monopolist benefits from different intermediate levels of advertising costs, depending on the impatience of the buyers because buyers' impatience determines the sensitivity of the regular price to changes in the frequency of sales.

This setup is relevant for various online markets and advertising platforms. First, firms and policy makers should be aware of the Coasean force and its implications. My model (as all game-theoretic models) hinges on the assumption that buyers correctly anticipate the frequency of sales. Hence, in the short term, firms might benefit or suffer from wrong beliefs of buyers about the frequency of sales. For example, some retailers offer sales on items in a wish list that have not been bought yet or others have increasing frequencies of sales rather than constant average frequencies of sales. In such markets, as buyers learn about the strategy of the firm, the outcome might converge to equilibria described in this paper in the long-run.

[20] Prominent retailers engaging in EDLP include Walmart, Home Depot, Lowe's, Trader Joe's. JC Penny switched back to a promotional pricing strategy after revenues had dropped significantly. One common explanation for their failure is that people make an inferences about the quality of a good through the regular price and that this plays a greater role for apparel.

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Appendix

A Proofs

Proof. **(Lemma 4)**

(i) Without commitment power, in an equilibrium with cycle length N, the seller must never have an incentive to have a temporary sale before N-th low valuation buyer has arrived given the buyer's strategies and beliefs. Hence, it must hold for any $n < N$ that

$$p_H(N) \cdot \frac{\lambda\pi}{r_s} + (1+R_s)^{-(N-l)} \cdot \left(v_L N - C_A + \Pi(0;N) - p_H(N)\frac{\lambda\pi}{r_s} \right) \geq v_N n - C_A + \Pi(0;N).$$

This inequality can be simplified to

$$C_A \geq \left(N - (N-n) \cdot \frac{1-(1+R_s)^{-N}}{1-(1+R_s)^{-(N-n)}} \right) v_L.$$

Note that all expressions are independent of $p_H(N)$, because beliefs are fixed and hence, the only trade-off for the seller is between clearing the market more frequently and paying the cost C_A more frequently. The function $x \mapsto \frac{x}{1-(1+R_s)^{-x}}$ is increasing and at 1 equal to $\frac{\lambda(1-\pi)+r_s}{r_s}$. Hence, the first inequality holds for all $n \leq N$ if it holds for $n = N-1$, that is the seller has no incentive to deviate from N by having an earlier sale if and only if

$$C_A \geq \left(N - \frac{\lambda(1-\pi)+r_s}{r_s} \cdot \left(1-(1+R_s)^{-N}\right) \right) \cdot v_L.$$

(ii) A cutoff-demand N can be supported by an equilibrium only if after the arrival of the N-th low valuation buyer, the seller prefers an advertised sale to waiting for more v_L-buyer to arrive. Hence, for any $n \geq N$ that is consistent with an equilibrium of a continuation game, it must hold that

$$v_L N - C_A + \Pi(0;N) \geq p_H(N)\frac{\lambda\pi}{r_s} + (1+R_s)^{-(n-N)} \left(v_L n - C_A + \Pi(0;N) - p_H(N)\frac{\lambda\pi}{r_s} \right).$$

In particular it must hold for $n = N+1$. The inequality can be simplified to

$$\left(N - \frac{1-(1+R_s)^{-N}}{1-(1+R_s)^{-(n-N)}} (1+R_s)^{-(n-N)} \cdot (n-N) \right) v_L \geq C_A$$

The function $x \mapsto \frac{(1+R_s)^{-x}x}{1-(1+R_s)^{-x}}$ is decreasing in x and at 1 equal to $\frac{\lambda(1-\pi)}{r_s}$. Hence, the seller does not have an incentive to accumulate more than N buyers if and only if

$$\left(N - \left(1-(1+R_s)^{-N}\right) \cdot \frac{\lambda(1-\pi)}{r_s} \right) \cdot v_L \geq C_A.$$

(iii) I show that $\overline{N} - \underline{N} = -1$ for all parameter values. From the necessary conditions (3) and (4), the following equality for $\overline{N}$ and $\underline{N}$ follows:

$$\underline{N} - \left(1-(1+R_s)^{-\underline{N}}\right) \cdot \left(1 + \frac{\lambda(1-\pi)}{r_s}\right) = \overline{N} - \left(1-(1+R_s)^{-\overline{N}}\right) \cdot \frac{\lambda(1-\pi)}{r_s}.$$

This equality is equivalent to

$$(1+R_s)^{\overline{N}-\underline{N}+1} = 1 + (\overline{N} - \underline{N} + 1) \cdot (1+R_s)^{\overline{N}} \cdot R_s.$$

One can immediately see that for $\overline{N} - \underline{N} = -1$ the equality is satisfied. Moreover, since $\overline{C}(N) > \underline{C}(N)$ everywhere and $\overline{C}, \underline{C}$ are both increasing functions, for each $\overline{N}$ there should only be one solution of this equality for $\overline{N} - \underline{N} + 1$. Hence, there is only a single integer number $N^*(C_A) \in [\overline{N}, \underline{N}]$. □

Proof. **(Proposition 1)**

Let (5) be satisfied. Then, an advertising equilibrium with regular price $p_H(N^*(C_A))$ and cutoff demand $N^*(C_A)$ exists, where v_H-buyers accept the regular price immediately. v_H-buyers do not wish to deviate as they are indifferent between accepting and rejecting. The seller does not want to deviate by Lemma 4 and because profits are higher than

- offering a price smaller or equal to v_L and
- offering a regular price lower than $p_H(N^*(C_A))$.

An EDLP equilibrium cannot exist because the seller always has an incentive to deviate to $p_H(N^*(C_A))$. This price will be accepted by v_H buyers because they know that the earliest point in time at which they will be called back is in $N^*(C_A)$ periods by Lemma 4.

Next, assume that (5) is not satisfied. Then, an EDLP equilibrium exists because

- for the seller a deviation to a higher price cannot be optimal as the seller knows by Lemma 4 that she will be tempted to have a sale after at most $N^*(C_A)$ arrivals which is not profitable by (5)
- a lower price than v_L will be accepted by all buyers which yields lower profits than EDLP
- buyers do not wish to deviate as they discount and the price is constant over time.

An adveritising equilibrium cannot exist as the seller always want to deviate to v_L. □

Proof. **(Corollary 1)** (i) If $C_A < \overline{C}(1)$, then in an advertising equilibrium, the seller drops the price immediately after she has observed a buyer not buying and $N^* = 1$. Hence, $\Pi(0;1) = \Pi_L - \frac{(1+R_s)^{-1}}{1-(1+R_s)^{-1}} C_A$ and by Proposition 1 there is no advertising equilibrium.

(ii) Because $\overline{C}(N) = \underline{C}(N+1)$ is increasing in N and $C_A \in [\underline{C}(N^*(C_A)), \overline{C}(N^*(C_A))]$, an increase in C_A implies that $N^*(C_A)$ must be greater.

(iii) If $C_A \to \infty$, then $N^* \to \infty$ in an advertising equilibrium, $\Pi(0; N^*) \to \Pi_H$. Hence, if $v_H \pi \geq v_L$, then there exists an advertising equilibrium that sustains the static monopoly profit Π_H. If $v_H \pi < v_L$, then there is no advertising equilibrium, but the static monopoly profit Π_L can be sustained in an EDLP equilibrium. $\square$

Proof. **(Corollary 2)**

By uniqueness of the equilibrium, it suffices to do comparative statics using one of $\overline{C}, \underline{C}$. It is easy to check that $\underline{C}$ is increasing in R_s and increasing in N.

(i) Since $\overline{C}, \underline{C}$ are decreasing in R_s, $N^*(C_A)$ is decreasing in C_A. As $R_s \to \infty$, $\overline{C}(N) = \underline{C}(N+1)$ converges to $N \cdot v_L$. Hence, $N^*(C_A) \to \frac{C_A}{v_L}$. As $R_s \to 0$, $\overline{C}(N) = \underline{C}(N+1)$ converges to 0 because by l'Hôpital

$$\lim_{R_s \to 0} \left(1 - (1+R_s)^{-N}\right) R_s^{-1} = \lim_{R_s \to 0} \frac{(1+R_s)^N - 1}{(1+R_s)^N R_s} = \lim_{R_s \to 0} \frac{N(1+R_s)^{N-1}}{N(1+R_s)^{N-1} R_s + (1+R_s)^N} = N.$$

Hence, $N^*(C_A) \to \infty$ since $\overline{C}(N), \underline{C}(N)$ are increasing in R_s.

(ii) Hence, an increase in λ implies an decrease in R_s, i.e. the equilibrium cutoff-demand $N^*(C_A)$ increases by (i). In order to show that the increase is less than proportional, I analyze the effect of a proportional increase in N and λ. In particular, one can calculate

$$\begin{aligned}
\frac{\partial}{\partial \gamma} \left(N \cdot \gamma - \left(1 - \left(1 + \frac{R_s}{\gamma}\right)^{-N\gamma}\right) \cdot \frac{\gamma}{R_s} \right) \cdot v_L \Big|_{\gamma=1} &= \\
N - \frac{1}{R_s}\left(1 - (1+R_s)^{-N}\right) + N \cdot (1+R_s)^{-N-1} + \frac{1}{R_s} \cdot (1+R_s)^{-N} \cdot \log(1+R_s)^{-N} &> \\
N - \frac{1}{R_s}\left(1 - (1+R_s)^{-N}\right) + N \cdot (1+R_s)^{-N-1} + \frac{1}{R_s} \cdot \left((1+R_s)^{-N} - 1\right) &= \\
N - \frac{2}{R_s}\left(1 - (1+R_s)^{-N}\right) + N \cdot (1+R_s)^{-N-1} > 0 &
\end{aligned}$$

for all $N, R_s > 0$. Since the average length of time between two sales if given by $\frac{N}{\lambda}$, an increase in λ leads to shorter time intervals between two sales in equilibrium.

(iii) One can calculate

$$\begin{aligned}
\frac{\partial}{\partial \gamma} \left(N \cdot \gamma - \left(1 - \left(1 + \frac{R_s}{\gamma}\right)^{-N\gamma}\right) \cdot \frac{\gamma}{R_s} \right) \cdot \frac{v_L}{\gamma} \Big|_{\gamma=1} &= \\
\left(N \cdot (1+R_s)^{-N-1} + \frac{1}{R_s} \cdot (1+R_s)^{-N} \cdot \log(1+R_s)^{-N} \right) v_L &< 0
\end{aligned}$$

for all $N, R_s > 0$. Since the average length of time between two sales if given by $\frac{N}{\lambda}$, an increase in λ with a proportional decrease of v_L leads to longer time intervals between two sales in equilibrium.

$\square$

Proof. **(Lemma 5)**

(i) Note that for all $x > 0$,

$$x \log(x) \geq x - 1 \tag{10}$$

where it holds with equality if only if $x = 1$. It follows that

$$\begin{aligned}\frac{\partial}{\partial N} p_H(N) &= -(v_H - v_L) \cdot \left(1 + \frac{\lambda(1-\pi)}{r_b}\right) \cdot \\ &\quad \frac{\left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N} - 1 - \left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N} \cdot \left(\log\left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N}\right)}{N^2} > 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{(\partial N)^2} p_H(N) &= -(v_H - v_L) \cdot \left(1 + \frac{\lambda(1-\pi)}{r_b}\right) \cdot \frac{1}{N^3} \cdot \\ &\quad \left[\left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N} \cdot \left(\log\left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N}\right)^2 + \right. \\ &\quad \left. 2\left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N} \cdot \log\left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N} + 2\left(1 - \left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N}\right)\right] < 0.\end{aligned}$$

It is immediate that

$$\frac{\partial}{\partial N} \frac{-\left(1+R_s\right)^{-N}}{1-\left(1+R_s\right)^{-N}} = \frac{\left(1+R_s\right)^N \cdot \log\left(1+R_s\right)}{\left(\left(1+R_s\right)^N - 1\right)^2} > 0$$

and

$$\frac{\partial^2}{(\partial N)^2} \frac{-\left(1+R_s\right)^{-N}}{1-\left(1+R_s\right)^{-N}} = -\frac{\left(1+R_s\right)^N \left(\log\left(1+R_s\right)\right)^2}{\left(\left(1+R_s\right)^N - 1\right)^3} < 0.$$

(ii) By inequality (10) it follows that

$$\frac{\partial}{\partial N} \frac{\left(1+R_s\right)^{-N}}{1-\left(1+R_s\right)^{-N}} \cdot N = \frac{\left(1+R_s\right)^N - 1 - \left(1+R_s\right)^N \left(\log\left(1+R_s\right)^N\right)}{\left(\left(1+R_s\right)^N - 1\right)^2} < 0$$

and

$$\begin{aligned}\frac{\partial^2}{(\partial N)^2} \frac{\left(1+R_s\right)^{-N}}{1-\left(1+R_s\right)^{-N}} \cdot N &= \frac{\left(1+R_s\right)^N \cdot \log\left(1+R_s\right)}{\left(\left(1+R_s\right)^N - 1\right)^3} \\ &\cdot \left(-2 \cdot \left(1+R_s\right)^N + \left(1+R_s\right)^N \cdot \log\left(1+R_s\right)^N + \log\left(1+R_s\right)^N + 2\right) > 0\end{aligned}$$

as can be easily checked.[21] □

[21] This can be seen by noting $1 \cdot \log 1 + \log 1 + 2 - 2 \cdot 1 = 0$ and $\frac{\mathrm{d}}{\mathrm{d}y}\left(y \cdot \log y + \log y + 2 - 2 \cdot y = \log y - 1 + \frac{1}{y}\right) \geq 0$ by inequality (10).

Proof. **(Lemma 6)**

Let $r_s = r_b = r$. Then, I can write

$$\Pi(N) = v_H \frac{\lambda\pi}{r} - \frac{\lambda\pi}{r} \cdot \left(1 + \frac{\lambda(1-\pi)}{r}\right) \cdot \frac{1 - \left(1 + \frac{r}{\lambda(1-\pi)}\right)^{-N}}{N} \cdot (v_H - v_L) + \frac{(Nv_L - C_A)\left(1 + \frac{r}{\lambda(1-\pi)}\right)^{-N}}{1 - \left(1 + \frac{r}{\lambda(1-\pi)}\right)^{-N}}$$

which is less than $\frac{\lambda\pi}{r} v_H$ for all N if and only if

$$\left(\frac{\lambda\pi}{r}\right) \frac{r + \lambda(1-\pi)}{r} \cdot (v_H - v_L) \geq \frac{N^2 \left(1 + \frac{r}{\lambda(1-\pi)}\right)^{-N}}{\left(1 - \left(1 + \frac{r}{\lambda(1-\pi)}\right)^{-N}\right)^2} \left(v_L - \frac{C_A}{N}\right) \tag{11}$$

holds for all N. Using that $h : x \mapsto \frac{x^2 \cdot a^{-x}}{(1-a^{-x})}$ is decreasing for all $a > 0$ and for all $x \geq 0$ and $\lim_{N \to 1} \frac{N^2\left(1+\frac{r}{\lambda(1-\pi)}\right)^{-N}}{\left(1-\left(1+\frac{r}{\lambda(1-\pi)}\right)^{-N}\right)^2} = \frac{\frac{\lambda(1-\pi)}{\lambda(1-\pi)+r}}{\left(\frac{r}{\lambda(1-\pi)+r}\right)^2} = \frac{\lambda(1-\pi)}{r} \cdot \frac{\lambda(1-\pi)+r}{r}$, the above inequality is satisfied if

$$\pi \cdot (v_H - v_L) \geq (1 - \pi) \cdot v_L, \tag{12}$$

that is if $v_H\pi \geq v_L$. Next I show that that if $v_H\pi < v_L$, then $\Pi(0; N) < \frac{\lambda}{r} v_L$ for all N.

$$\begin{aligned}
\Pi(N) &< \frac{\lambda v_L}{r}\left(1 - \frac{1 - \left(1 + \frac{r_b}{\lambda(1-\pi)}\right)^{-N}}{N} \cdot \left(1 + \frac{\lambda(1-\pi)}{r_b}\right)\right) \\
&\quad + \frac{\lambda\pi v_L}{r} \cdot \frac{1 - \left(1 + \frac{r_b}{\lambda(1-\pi)}\right)^{-N}}{N} \cdot \left(1 + \frac{\lambda(1-\pi)}{r_b}\right) + \frac{N\left(1 + \frac{r}{\lambda(1-\pi)}\right)^{-N}}{1 - \left(1 + \frac{r}{\lambda(1-\pi)}\right)^{-N}} v_L \\
&= \frac{v_L\lambda}{r}\left(1 - (1-\pi)\frac{1 - \left(1 + \frac{r_b}{\lambda(1-\pi)}\right)^{-N}}{N} \cdot \left(1 + \frac{\lambda(1-\pi)}{r_b}\right)\right) + \frac{N\left(1 + \frac{r}{\lambda(1-\pi)}\right)^{-N}}{1 - \left(1 + \frac{r}{\lambda(1-\pi)}\right)^{-N}} v_L
\end{aligned}$$

Since $x \mapsto \frac{x \cdot a^{-x}}{1-a^{-x}}$ is decreasing for all $a > 1, x \geq 0$ and $\lim_{N \to 1} \frac{N\left(1+\frac{r}{\lambda(1-\pi)}\right)^{-N}}{1-\left(1+\frac{r}{\lambda(1-\pi)}\right)^{-N}} = \frac{\lambda(1-\pi)}{r}$ and $x \mapsto \frac{1-a^{-x}}{x}$ is decreasing for all $a > 1, x \geq 0$ and $\lim_{N \to 1} \frac{1-\left(1+\frac{r}{\lambda(1-\pi)}\right)^{-N}}{N} = \frac{r}{r+\lambda(1-\pi)}$, I can further conclude

$$\Pi(N) \quad < \quad v_L \frac{\lambda}{r}\pi + \frac{\lambda(1-\pi)}{r} v_L = v_L \frac{\lambda}{r}.$$

$\square$

Proof. **(Lemma 7)**

(i) follows by noting that $\lim_{N\to\infty}\Pi^{FC}(N)=\Pi_H$ and

$$\Pi_H-\Pi^{FC}(N) \quad > \quad \underbrace{\frac{\lambda\pi(v_H-v_L)}{r_s}\cdot\left(1+\frac{\lambda(1-\pi)}{r_b}\right)\cdot\frac{1-\left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N}}{N}}_{>0,\ \to_{N\to\infty}0}-\underbrace{\frac{N\cdot(1+R_s)^{-N}}{1-(1+R_s)^{-N}}\cdot v_L}_{>0,\ \to_{N\to\infty}0}$$

and $\frac{N\cdot(1+R_s)^{-N}}{1-(1+R_s)^{-N}}=o\left(\frac{1-\left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N}}{N}\right)$, $N\to\infty$.

(ii) First, note that r_b affects the seller's profit only through $p_H(N)$ or the expected discount to a sale $\mathbb{E}\left[e^{-r_b\tau(p_H(N))}\right]=\frac{1}{N}\sum_{n=0}^{N-1}\left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-n}$. This is decreasing in r_b, that is from (2) and (6) it follows that $p_H(N)$ and $\Pi(N)$ are increasing in r_b. Since $\Pi(N)$ is increasing in N for all N, $\max_N\Pi(N)$ is also increasing in r_b.

(iii) is straight-forward, since profit are only affected negatively by C_A.

(iv) First, note that by (2), given all parameters, for any N and all $\epsilon>0$, there exists a $\overline{r}_b(N,\epsilon)$ such that for all $r_b\geq\overline{r_b}(N,\epsilon)$, $p_H(N)>v_H-\epsilon$. Let $\tilde{N}=\arg\max_N\frac{1}{\left(1+\frac{r_s}{\lambda(1-\pi)}\right)^N-1}(Nv_L-C_A)$. Hence, by (6),

$$\Pi^{FC}(\tilde{N})>\frac{\lambda\pi}{r_s}(v_H-\epsilon)+\frac{1}{\left(1+\frac{r_s}{\lambda(1-\pi)}\right)^{\tilde{N}}-1}\cdot(\tilde{N}v_L-C_A).$$

Hence, let us choose

$$\epsilon=\frac{r_s}{\lambda\pi}\cdot\left(\frac{1}{\left(1+\frac{r_s}{\lambda(1-\pi)}\right)^{\tilde{N}}-1}\cdot(\tilde{N}v_L-C_A)-\max\left\{\frac{\lambda(v_L-v_H\pi)}{r_s},0\right\}\right).$$

Then, $\Pi^{FC}(\tilde{N})>\max\left\{\frac{\lambda\pi}{r_s}v_H,\frac{\lambda}{r_s}\right\}$ for all $r_b\geq\overline{r}_b(\tilde{N},\epsilon)$.

If $v_L-\pi v_H\geq\frac{r_s}{\lambda}\max_N\frac{1}{\left(1+\frac{r_s}{\lambda(1-\pi)}\right)^N-1}(Nv_L-C_A)$, then $\Pi(0;N)<\frac{\lambda\pi}{r}v_H+\max_N\frac{1}{\left(1+\frac{r_s}{\lambda(1-\pi)}\right)^N-1}(Nv_L-C_A)\leq\frac{\lambda}{r}v_L$. □

Proof. **(Lemma 8)**

(i) First, note that

$$\begin{aligned}\frac{\partial}{\partial N}\Pi^{FC}(N) \quad = \quad & \frac{\lambda\pi}{r_s}\cdot\left(\frac{\partial}{\partial N}p_H(N)\right)-v_L\cdot\left(\frac{\partial}{\partial N}\frac{\left(1+\frac{r_s}{\lambda(1-\pi)}\right)^{-N}}{1-\left(1+\frac{r_s}{\lambda(1-\pi)}\right)^{-N}}\right)\\ & +C_A\left(\frac{\partial}{\partial N}\frac{-\left(1+\frac{r_s}{\lambda(1-\pi)}\right)^{-N}}{1-\left(1+\frac{r_s}{\lambda(1-\pi)}\right)^{-N}}\right).\end{aligned} \tag{13}$$

Using the expression of $\frac{\partial}{\partial N} p_H(N)$ from the proof of Lemma 5 (i), one can derive

$$\begin{aligned}\frac{\partial^2}{\partial N \partial r_b}\Pi^{FC}(N) &= \frac{\lambda\pi}{r_s}\cdot(v_H - v_L)\cdot\frac{1}{N^2}\cdot\frac{\lambda(1-\pi)}{r_b^2}\cdot \\ &\quad \Bigg[\left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N} \\ &\quad -1-\left(1+\frac{Nr_b}{\lambda(1-\pi)}\right)\cdot\left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N}\cdot\left(\log\left(1+\frac{r_b}{\lambda(1-\pi)}\right)^{-N}\right)\Bigg].\end{aligned}$$

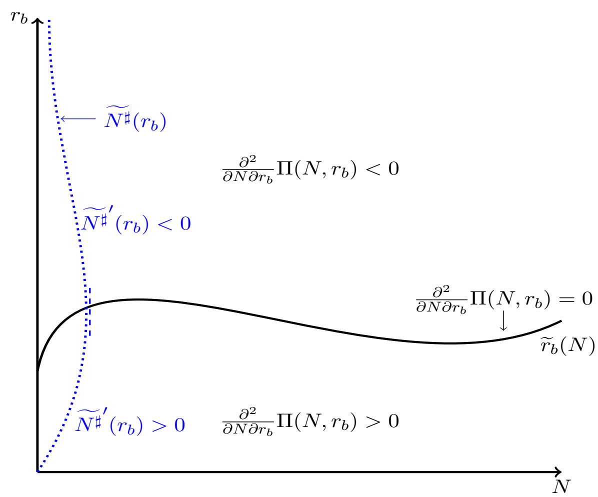

Figure 12: Illustration of the proof of Proposition 8

Step 1: For any fixed N there exists a $\tilde{r}_b(N)$, such that $\frac{\partial^2}{\partial N\partial r_b}\Pi^{FC}(N) > 0$ if and only if $r_b < \tilde{r}_b(N)$ and $\frac{\partial^2}{\partial N\partial r_b}\Pi^{FC}(N) < 0$ if and only if $r_b > \tilde{r}_b(N)$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(y^{-N} - 1 - (1+N(y-1))\cdot y^{-N}\cdot\left(\log y^{-N}\right)\right) = N\cdot y^{-N-1}\cdot(y-1)\cdot\left(N\log y^{-N} - \log y^{-N} + N\right)$$

has only one null for $y > 1$, namely $y = e^{-\frac{1}{1-N}}$ and is positive at $y+\epsilon$ for small $\epsilon > 0$. Moreover, the following holds

- $\lim_{y\to 1}\left(y^{-N} - 1 - (1+N(y-1))\cdot y^{-N}\cdot\left(\log y^{-N}\right)\right) = 0,$
- $\lim_{y\to\infty}\left(y^{-N} - 1 - (1+N(y-1))\cdot y^{-N}\cdot\left(\log y^{-N}\right)\right) = -1 < 0,$
- $\lim_{y\downarrow 1} N\cdot y^{-N-1}\cdot(y-1)\cdot\left(N\log y^{-N} - \log y^{-N} + N\right) = 0^+.$

All together with continuous differentiability of all functions this shows the existence of a unique $\widetilde{r}_b(N) > 0$ for every N and $\widetilde{r}_b(\cdot)$ is a continuous function. The solid line in Figure 12 illustrates $\widetilde{r}_b$

Step 2: Any local maximum cutoff-demand $\widetilde{N^\sharp}(r_b)$ is increasing for small r_b and decreasing for other r_b

By the Implicit Function Theorem, any local maximum $\widetilde{N^\sharp}(r_b)$ of Π is locally differentiable in r_b. $\widetilde{N^\sharp}$ is illustrated by the dotted blue line in Figure 12. Moreover, it follows that $\frac{\partial^2}{\partial N \partial r}(\widetilde{N^\sharp}(r_b), r_b)$ and $\widetilde{N^\sharp}'(r_b)$ must have the same sign. Hence, I can show that $\widetilde{N^\sharp}'(r_b)$ can only change from positive to negative and not the other way around by contradiction. Assume that there exists an $\overline{r_b}$ such that for some $\epsilon_1, \epsilon_2 > 0$, $\widetilde{N^\sharp}'(r) < 0$ for all $r \in (r_b - \epsilon_1, r_b)$ and $\widetilde{N^\sharp}'(r) > 0$ for all $r \in (\overline{r_b}, \widetilde{r_b} + \epsilon_2)$ (and hence $\widetilde{N^\sharp}'(\overline{r_b}) = 0$). Then, there are $r_1 < r_2$ with $\widetilde{N^\sharp}(r_1) = \widetilde{N^\sharp}(r_1) = N$ such that $\widetilde{N^\sharp}'(r_1) < 0 < \widetilde{N^\sharp}'(r_2)$. Hence, such that $\frac{\partial^2}{\partial N \partial r}(N, r_1) < 0 < \frac{\partial^2}{\partial N \partial r}(N, r_2)$. This is, however, a contradiction to Step 1 which concludes the proof.

(ii) follows immediately from Lemma 5.

□

Proof. **(Lemma 9)**

(i) follows immediately from (8).

(ii) Let $\delta_b = \left(1 + \frac{r_b}{\lambda(1-\pi)}\right)^{-1}$, $\delta_s = \left(1 + \frac{r_s}{\lambda(1-\pi)}\right)^{-1}$. Then, one can write

$$\overline{\Pi}(N) - \frac{\lambda}{r_s} v_L = \frac{\lambda(1-\pi)v_L}{r_s} \cdot \left[\underbrace{\frac{\frac{\pi v_H}{v_L} - \pi}{1-\pi}}_{\equiv C} \cdot \left(1 - \frac{1-\delta_b^N}{N \cdot (1-\delta_b)}\right) - \left(1 - \delta_s^{N-1}\right) \right].$$

First, note that $\overline{\Pi}(N)$ is increasing in N for large enough N. Hence, $\overline{\Pi}(N) - \frac{\lambda}{r_s} v_L$ has at most one maximum and at most one minimum, if and only if it is either first increasing, then decreasing and then increasing or first decreasing and then increasing. Therefore, consider the first derivative of $\overline{\Pi}(N) - \frac{\lambda}{r_s} v_L$ which is given by

$$\frac{\partial}{\partial N}\left(\overline{\Pi}(N) - \frac{\lambda}{r_s} v_L\right) = \frac{\lambda(1-\pi)v_L}{r_s \cdot N^2} \cdot \left[\underbrace{C \cdot \frac{1-\delta_b^N}{1-\delta_b} + C \cdot \frac{N\delta_b^N \log(\delta_b)}{1-\delta_b} + N^2 \delta_s^{N-1} \log(\delta_s)}_{\equiv H(N)} \right].$$

I need to show that $H(N)$ is either first positive, then negative and then positive or negative and then positive. Therefore consider

$$\frac{d}{dN} H(N) = N \cdot (\log(\delta_s))^2 \cdot \left[\underbrace{C \cdot \frac{\delta_b}{1-\delta_b} \cdot \left(\frac{\delta_b}{\delta_s}\right)^{N-1} \left(\frac{\log(\delta_b)}{\log(\delta_s)}\right)^2 + \frac{2}{\log(\delta_s)} + N}_{\equiv J(N)} \right].$$

$\frac{d}{dN}H(N)$ is changing signs at most twice because $J(N)$ it is quasi-convex. If it changes signs twice, it is first positive, then negative and then positive again. In that case, since $H(0) = 0$, H is first positive, then negative and then positive again and the claim follows immediately. If J changes signs at most once, then H changes signs at most twice and $\overline{\Pi}(N) - \Pi_L$ changes signs at most twice. Since $\overline{\Pi}(1) - \Pi_L = 0$, the claim follows immediately.

(iii) The derivative of $\overline{\Pi}$ at $N = 1$ is given by

$$\overline{\Pi}'(N)|_{N=1} = \frac{\lambda(1-\pi)}{r_s} v_L \left(C + C \cdot \frac{\delta_b \log \delta_b}{1-\delta_b} + \log \delta_s \right)$$

is decreasing in δ_b, i.e. increasing in r_b. For $r_b \to \infty$, $\delta_b \to 0$ and $\lim \overline{\Pi}'(N)|_{N=1} < 0$. □

Proof. **(Proposition 2)**

For advertising costs $C_A \in \mathcal{C}$, an advertising equilibrium exists if and only if

$$\overline{\Pi}(\underline{C}^{-1}(C_A)) \geq \Pi_L.$$

Hence, the proof follows immediately from Lemma 9, since $\overline{\Pi}$ is either first decreasing and then increasing or, if r_b is large enough, first increasing, then decreasing and then increasing again. Since $\overline{\Pi}$ is increasing in r_b, T_1 is decreasing and T_0 increasing in r_b. □

Proof. **(Proposition 3)**

The proof follows from the fact that $\overline{\Pi}$ has a single maximum and from Proposition 8 (i).

□