# Supplemental Material to
# INFERENCE BASED ON MANY CONDITIONAL MOMENT INEQUALITIES

**By**

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This Appendix is not to be published. It will be made available on the web.

# Appendix

to

# Inference Based on Many Conditional Moment Inequalities


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# A Outline

This Appendix provides proofs of Theorems 5.1 and 6.1 of Andrews and Shi (2010) "Inference Based on Many Conditional Moment Inequalities," referred to hereafter as ASM. In fact, the results given here cover a much broader class of test statistics than is considered in ASM. We let AS1 abbreviate Andrews and Shi (2013a) and AS2 abbreviate Andrews and Shi (2013b).

This Appendix is organized as follows. Section B defines the class of test statistics that are considered. This class includes the statistics that are considered in ASM. Section B also provides the definition of manageability that is used in Assumption PS2. Section C introduces generalized moment selection (GMS) and plug-in asymptotic (PA) critical values, confidence sets (CS's), and tests. Section D establishes the correct asymptotic size of GMS and PA CS's. Theorem 5.1 of ASM is a corollary to Lemmas D.1 and D.2, which are given in Section D. Section E establishes that GMS and PA CS's contain fixed parameter values outside the identified set with probability that goes to zero. Equivalently, the tests upon which the CS's are constructed are shown to be consistent tests. Theorem 6.1 of ASM is a corollary to Theorem E.1, which is given in Section E. Section F provides proofs of Lemma 7.1-8.5 of ASM, which verify Assumptions PS1, PS2, SIG1, and SIG2 in the examples given in ASM.

# B General Form of the Test Statistic

## B.1 Test Statistic

Here we define the general form of the test statistic $T_n(\theta)$ that is used to construct a CS. We transform the conditional moment inequalities/equalities given $X_i$ into equivalent unconditional moment inequalities/equalities by choosing appropriate weighting functions of $X_i$, i.e., $X_i$ instruments. Then, we construct a test statistic based on the instrumented moment conditions.

The instrumented moment conditions are of the form:

$$\begin{aligned} &E_{F_0}[m_j(W_i,\theta_0,\tau)\,g_j(X_i)] \geq 0 \text{ for } j=1,...,p \text{ and} \\ &E_{F_0}[m_j(W_i,\theta_0,\tau)\,g_j(X_i)] = 0 \text{ for } j=p+1,...,k, \text{ for } g=(g_1,...,g_k)' \in \mathcal{G} \text{ and } \tau \in \mathcal{T}, \end{aligned} \tag{B.1}$$

where $\theta_0$ and $F_0$ are the true parameter and distribution, respectively, $g$ is the instrument vector that depends on the conditioning variables $X_i$, and $\mathcal{G}$ is a collection of instruments. Typically $\mathcal{G}$ contains an infinite number of elements.

The identified set $\Theta_{F_0}(\mathcal{G})$ of the model defined by (B.1) is

$$\Theta_{F_0}(\mathcal{G}) := \{\theta \in \Theta : \text{(B.1) holds with } \theta \text{ in place of } \theta_0\}. \tag{B.2}$$

The collection $\mathcal{G}$ is chosen so that $\Theta_{F_0}(\mathcal{G}) = \Theta_{F_0}$, where $\Theta_{F_0}$ is the identified set based on the conditional moment inequalities and equalities defined in (2.2) of ASM. Section B.4 provides conditions for this equality and shows that the instruments defined in (3.6) of ASM satisfy the conditions. Additional sets $\mathcal{G}$ are given in AS1 and AS2.

We construct test statistics based on (B.1). The sample moment functions are defined in (3.2) in ASM. The sample variance-covariance matrix of $n^{1/2}\overline{m}_n(\theta,\tau,g)$ is defined in (3.3) in ASM. The matrix $\widehat{\Sigma}_n(\theta,\tau,g)$ may be singular with non-negligible probability for some $g \in \mathcal{G}$. This is undesirable because the inverse of $\widehat{\Sigma}_n(\theta,\tau,g)$ needs to be consistent for its population counterpart uniformly over $g \in \mathcal{G}$ for the test statistics considered below. Thus, we employ a modification of $\widehat{\Sigma}_n(\theta,\tau,g)$, denoted by $\overline{\Sigma}_n(\theta,\tau,g)$ and defined in (3.4) in ASM, such that the smallest eigenvalue of $\overline{\Sigma}_n(\theta,\tau,g)$ is bounded away from zero.

The test statistic $T_n(\theta)$ is either a Cramér-von-Mises-type (CvM) or a Kolmogorov-Smirnov-type (KS) statistic. The CvM statistic is

$$T_n(\theta) := \sup_{\tau\in\mathcal{T}} \int_{\mathcal{G}} S(n^{1/2}\overline{m}_n(\theta,\tau,g), \overline{\Sigma}_n(\theta,\tau,g))dQ(g), \tag{B.3}$$

where $S$ is a non-negative function and $Q$ is a weight function (i.e., probability measure) on $\mathcal{G}$. The functions $S$ and $Q$ are discussed in Sections B.2 and B.5 below, respectively.

The Kolmogorov-Smirnov-type (KS) statistic is

$$T_n(\theta) := \sup_{\tau\in\mathcal{T}} \sup_{g\in\mathcal{G}} S(n^{1/2}\overline{m}_n(\theta,\tau,g), \overline{\Sigma}_n(\theta,\tau,g)). \tag{B.4}$$

For brevity, the discussion in this Appendix focusses on CvM statistics and all results stated concern CvM statistics. Similar results hold for KS statistics. Such results can be established by extending the results given in Section 13.1 of Appendix B of AS2 and proved in Section 15.1 of Appendix D of AS2.

## B.2 S Function Assumptions

Let $m_I := (m_1, ..., m_p)'$ and $m_{II} := (m_{p+1}, ..., m_k)'$. Let $\Delta$ be the set of $k \times k$ positive-definite diagonal matrices. Let $\mathcal{W}$ be the set of $k \times k$ positive definite matrices.

**Assumption S1.** $\forall (m, \Sigma) \in \{(m, \Sigma) : m \in (-\infty, \infty]^p \times R^v, \Sigma \in \mathcal{W}\}$,

(a) $S(Dm, D\Sigma D) = S(m, \Sigma)\ \forall D \in \Delta$,

(b) $S(m_I, m_{II}, \Sigma)$ is non-increasing in each element of $m_I$,

(c) $S(m, \Sigma) \geq 0$,

(d) $S$ is continuous, and

(e) $S(m, \Sigma + \Sigma_1) \leq S(m, \Sigma)$ for all $k \times k$ positive semi-definite matrices $\Sigma_1$.

It is worth pointing out that Assumption S1(d) requires $S$ to be continuous in $m$ at all points $m$ in the extended vector space $(-\infty, \infty]^p \times R^v$, not only for points in $R^{p+v}$.

Let $\mathcal{M}$ denote a bounded subset of $R^k$. Let $\mathcal{W}_{cpt}$ denote a compact subset of $\mathcal{W}$.

**Assumption S2.** $S(m, \Sigma)$ is uniformly continuous in the sense that

$$\lim_{\delta \downarrow 0} \sup_{\mu \in R_+^p \times \{0\}^v} \sup_{\substack{m, m^* \in \mathcal{M} \\ \|m - m^*\| \leq \delta}} \sup_{\substack{\Sigma, \Sigma^* \in \mathcal{W}_{cpt} \\ \|\Sigma - \Sigma^*\| \leq \delta}} |S(m + \mu, \Sigma) - S(m^* + \mu, \Sigma^*)| = 0.^{10}$$

**Assumption S3.** $S(m, \Sigma) > 0$ if and only if $m_j < 0$ for some $j = 1, ..., k$, where $m = (m_1, ..., m_k)'$ and $\Sigma \in \mathcal{W}$.

**Assumption S4.** For some $\chi > 0$, $S(am, \Sigma) = a^\chi S(m, \Sigma)$ for all scalar $a > 0$, $m \in R^k$, and $\Sigma \in \mathcal{W}$.

It is shown in Lemma 1 of AS1 that the functions $S_1$-$S_3$ in (3.9) satisfy Assumptions S1-S4. The function $S_4$ also does by similar arguments.

## B.3 Definition of Manageability

Here we introduce the concept of manageability from Pollard (1990) that is used in Assumption PS2 in ASM and Assumption M that is introduced in the following section.

[10] It is important that the supremum is only over $\mu$ vectors with non-negative elements $\mu_j$ for $j \leq p$. Without this restriction on the $\mu$ vectors, Assumption S2 would not hold for typical $S$ functions of interest. Also note that Assumption S2 is Assumption S2$'$, rather than Assumption S2, in AS1. Although Assumption S2 in AS1 is seemingly weaker than Assumption S2$'$, the former implies the latter, i.e. the two assumptions are equivalent. The equivalence can be established by adapting the proof of the well-known result that continuous functions defined on compact sets are uniformly continuous.

This condition is used to regulate the complexity of $\mathcal{T}\times\mathcal{G}$. It ensures that $\{n^{1/2}(\overline{m}_n(\theta,\tau,g)-E_{F_n}\overline{m}_n(\theta,\tau,g)):(\tau,g)\in\mathcal{T}\times\mathcal{G}\}$ satisfies a functional central limit theorem (FCLT) under drifting sequences of distributions $\{F_n:n\geq 1\}$. The latter is utilized in the proof of the uniform coverage probability results for the CS's. See Pollard (1990) and Appendix E of AS2 for more about manageability.

**Definition (Pollard, 1990, Definition 3.3).** The *packing number* $D(\xi,\rho,V)$ for a subset $V$ of a metric space $(\mathcal{V},\rho)$ is defined as the largest $b$ for which there exist points $v^{(1)},...,v^{(b)}$ in $V$ such that $\rho(v^{(s)},v^{(s')})>\xi$ for all $s\neq s'$. The *covering number* $N(\xi,\rho,V)$ is defined to be the smallest number of closed balls with $\rho$-radius $\xi$ whose union covers $V$.

It is easy to see that $N(\xi,\rho,V)\leq D(\xi,\rho,V)\leq N(\xi/2,\rho,V)$.

Let $(\Omega,\mathcal{F},\mathbf{P})$ be the underlying probability space equipped with probability distribution $\mathbf{P}$. Let $\{f_{n,i}(\cdot,\tau):\Omega\to R:\tau\in\mathcal{T},i\leq n,n\geq 1\}$ be a triangular array of random processes. Let

$$\mathcal{F}_{n,\omega}:=\{(f_{n,1}(\omega,\tau),...,f_{n,n}(\omega,\tau))':\tau\in\mathcal{T}\}. \tag{B.5}$$

Because $\mathcal{F}_{n,\omega}\subset R^n$, we use the Euclidean metric $\|\cdot\|$ on this space. For simplicity, we omit the metric argument in the packing number function, i.e., we write $D(\xi,V)$ in place of $D(\xi,\|\cdot\|,V)$ when $V\subset\mathcal{F}_{n,\omega}$.

Let $\odot$ denote the element-by-element product. For example for $a,b\in R^n$, $a\odot b=(a_1b_1,...,a_nb_n)'$. Let *envelope functions* of a triangular array of processes $\{f_{n,i}(\omega,\tau):\tau\in\mathcal{T},i\leq n,n\geq 1\}$ be an array of functions $\{F_n(\omega)=(F_{n,1}(\omega),...,F_{n,n}(\omega))':n\geq 1\}$ such that $|f_{n,i}(\omega,\tau)|\leq F_{n,i}(\omega)$ $\forall i\leq n,n\geq 1,\tau\in\mathcal{T},\omega\in\Omega$.

**Definition (Pollard, 1990, Definition 7.9).** A triangular array of processes $\{f_{n,i}(\omega,\tau):\tau\in\mathcal{T},i\leq n,n\geq 1\}$ is said to be *manageable* with respect to the envelopes $\{F_n(\omega):n\geq 1\}$ if there exists a deterministic real function $\lambda$ on $(0,1]$ for which (i) $\int_0^1\sqrt{\log\lambda(\xi)}d\xi<\infty$ and (ii) $D(\xi\|\alpha\odot F_n(\omega)\|,\alpha\odot\mathcal{F}_{n,\omega})\leq\lambda(\xi)$ for $0<\xi\leq 1$, all $\omega\in\Omega$, all $n$-vectors $\alpha$ of nonnegative weights, and all $n\geq 1$.

## B.4 X Instruments

The collection of instruments $\mathcal{G}$ needs to satisfy the following condition in order for the unconditional moments $\{E_F[m(W_i,\theta,\tau,g)]:(\tau,g)\in\mathcal{T}\times\mathcal{G}\}$ to incorporate the same information as the conditional moments $\{E_F[m(W_i,\theta,\tau)|X_i = x] : x \in R^{d_x}\}$.

For any $\theta \in \Theta$ and any distribution $F$ with $E_F[\|m(W_i,\theta,\tau)\|] < \infty$, $\forall \tau \in \mathcal{T}$, let $\mathcal{X}_F(\theta,\tau)$ be defined as in (6.2) in ASM.

**Assumption CI.** For any $\theta \in \Theta$ and distribution $F$ for which $E_F[\|m(W_i,\theta,\tau)\|] < \infty$, $\forall \tau \in \mathcal{T}$, if $P_F(X_i \in \mathcal{X}_F(\theta,\tau_*)) > 0$ for some $\tau_* \in \mathcal{T}$, then there exists some $g \in \mathcal{G}$ such that

$$\begin{aligned} &E_F[m_j(W_i,\theta,\tau_*)g_j(X_i)] < 0 \text{ for some } j \leq p \text{ or} \\ &E_F[m_j(W_i,\theta,\tau_*)g_j(X_i)] \neq 0 \text{ for some } j > p. \end{aligned}$$

Note that CI abbreviates "conditionally identified." The following Lemma indicates the importance of Assumption CI. The proof of the lemma is the same as the proof of Lemma 2 in AS1, which is given in AS2, and in consequence, is omitted.

**Lemma B.1** *Assumption* CI *implies that* $\Theta_F(\mathcal{G}) = \Theta_F$ *for all* $F$ *with* $\sup_{\theta\in\Theta} E_F[\|m(W_i,\theta,\tau)\|] < \infty$.

Collections $\mathcal{G}$ that satisfy Assumption CI contain non-negative functions whose supports are cubes, boxes, or other sets which are arbitrarily small.

The collection $\mathcal{G}$ also must satisfy the following "manageability" condition.

**Assumption M.** (a) $0 \leq g_j(x) \leq G \ \forall x \in R^{d_x}, \forall j \leq k, \forall g \in \mathcal{G}$, for some constant $G < \infty$, and

(b) the processes $\{g_j(X_{n,i}) : g \in \mathcal{G},\ i \leq n, n \geq 1\}$ are manageable with respect to the constant function $G$ for $j = 1, ..., k$, where $\{X_{n,i} : i \leq n, n \geq 1\}$ is a row-wise i.i.d. triangular array with $X_{n,i} \sim F_{X,n}$ and $F_{X,n}$ is the distribution of $X_{n,i}$ under $F_n$ for some $(\theta_n, F_n) \in \mathcal{F}_+$ for $n \geq 1$.[11]

Lemma 3 of AS1 establishes Assumptions CI and M for $\mathcal{G}_{\text{c-cube}}$ defined in (3.6) of ASM.[12]

## B.5 Weight Function Q

The weight function $Q$ can be any probability measure on $\mathcal{G}$ whose support is $\mathcal{G}$. This support condition is needed to ensure that no functions $g \in \mathcal{G}$, which might have set-

[11] The asymptotic results given in the paper hold with Assumption M replaced by any alternative assumption that is sufficient to obtain the requisite empirical process results given in Lemma D.2 below.

[12] Lemma 3 of AS1 and Lemma B2 of AS2 also establish Assumptions CI and M of this Appendix for the collections $\mathcal{G}_{\text{box}}$, $\mathcal{G}_{\text{B-spline}}$, $\mathcal{G}_{\text{box,dd}}$, and $\mathcal{G}_{\text{c/d}}$ defined there.

identifying power, are "ignored" by the test statistic $T_n(\theta)$. Without such a condition, a CS based on $T_n(\theta)$ would not necessarily shrink to the identified set as $n \to \infty$. Section E below introduces the support condition formally and shows that the probability measure $Q$ considered here satisfies it.

We now give an example of a weight function $Q$ on $\mathcal{G}_{\text{c-cube}}$.

**Weight Function Q for $\mathcal{G}_{\text{c-cube}}$.** There is a one-to-one mapping $\Pi_{\text{c-cube}} : \mathcal{G}_{\text{c-cube}} \to AR := \{(a, r) : a \in \{1, ..., 2r\}^{d_x} \text{ and } r = r_0, r_0{+}1, ...\}$. Let $Q_{AR}$ be a probability measure on $AR$. One can take $Q = \Pi_{\text{c-cube}}^{-1} Q_{AR}$. A natural choice of measure $Q_{AR}$ is uniform on $a \in \{1, ..., 2r\}^{d_x}$ conditional on $r$ combined with a distribution for $r$ that has some probability mass function $\{w(r) : r = r_0, r_0 + 1, ...\}$. This yields the test statistic

$$T_n(\theta) := \sup_{\tau \in \mathcal{T}} \sum_{r=r_0}^{\infty} w(r) \sum_{a \in \{1,...,2r\}^{d_x}} (2r)^{-d_x} S(n^{1/2}\overline{m}_n(\theta, \tau, g_{a,r}), \overline{\Sigma}_n(\theta, \tau, g_{a,r})), \qquad \text{(B.6)}$$

where $g_{a,r}(x) := 1(x \in C_{a,r}) \cdot 1_k$ for $C_{a,r} \in \mathcal{C}_{\text{c-cube}}$.

The weight function $Q_{AR}$ with $w(r) := (r^2 + 100)^{-1}$ is used in the test statistics in ASM, see (3.7).

## B.6 Computation of Sums, Integrals, and Suprema

The test statistic $T_n(\theta)$ given in (B.6) involves an infinite sum. A collection $\mathcal{G}$ with an uncountable number of functions $g$ yields a test statistic $T_n(\theta)$ that is an integral with respect to $Q$. This infinite sum or integral can be approximated by truncation, simulation, or quasi-Monte Carlo (QMC) methods. If $\mathcal{G}$ is countable, let $\{g_1, ..., g_{s_n}\}$ denote the first $s_n$ functions $g$ that appear in the infinite sum that defines $T_n(\theta)$. Alternatively, let $\{g_1, ..., g_{s_n}\}$ be $s_n$ i.i.d. functions drawn from $\mathcal{G}$ according to the distribution $Q$. Or, let $\{g_1, ..., g_{s_n}\}$ be the first $s_n$ terms in a QMC approximation of the integral with respect to (wrt) $Q$. Then, an approximate test statistic obtained by truncation, simulation, or QMC methods is

$$\overline{T}_{n,s_n}(\theta) := \sup_{\tau \in \mathcal{T}} \sum_{\ell=1}^{s_n} w_{Q,n}(\ell) S(n^{1/2}\overline{m}_n(\theta, \tau, g_\ell), \overline{\Sigma}_n(\theta, \tau, g_\ell)), \qquad \text{(B.7)}$$

where $w_{Q,n}(\ell) := Q(\{g_\ell\})$ when an infinite sum is truncated, $w_{Q,n}(\ell) := s_n^{-1}$ when $\{g_1, ..., g_{s_n}\}$ are i.i.d. draws from $\mathcal{G}$ according to $Q$, and $w_{Q,n}(\ell)$ is a suitable weight when a QMC method is used. For example, in (B.6), the outer sum can be truncated at $r_{1,n}$, in which case, $s_n := \sum_{r=r_0}^{r_{1,n}} (2r)^{d_x}$ and $w_{Q,n}(\ell) := w(r)(2r)^{-d_x}$ for $\ell$ such that $g_\ell$ corresponds to $g_{a,r}$ for some $a$. The test statistics in (3.7) of ASM are of this form when $r_{1,n} < \infty$.[13]

It can be shown that truncation at $s_n$, simulation based on $s_n$ simulation repetitions, or QMC approximation based on $s_n$ terms, where $s_n \to \infty$ as $n \to \infty$, is sufficient to maintain the asymptotic validity of the tests and CS's as well as the asymptotic power results under fixed alternatives.

The KS form of the test statistic requires the computation of a supremum over $g \in \mathcal{G}$. For computational ease, this can be replaced by a supremum over $g \in \mathcal{G}_n$, where $\mathcal{G}_n \uparrow \mathcal{G}$ as $n \to \infty$, in the test statistic and in the definition of the critical value (defined below). The same asymptotic size results and asymptotic power results under fixed alternatives hold for KS tests with $\mathcal{G}_n$ in place of $\mathcal{G}$. For results of this sort for the tests considered in AS1 and AS2, see Section 13.1 of Appendix B in AS2 and Section 15.1 of Appendix D in AS2.

# C GMS and Plug-in Asymptotic Confidence Sets

## C.1 Bootstrap GMS Critical Values

In this section, we define bootstrap GMS critical values and CS's.

It is shown in Theorem D.3 in Section D.3.1 below that when $\theta$ is in the identified set the "uniform asymptotic distribution" of $T_n(\theta)$ is the distribution of $T(h_n)$, where $T(h)$ is defined below, $h_n := (h_{1,n}, h_2)$, $h_{1,n}(\cdot)$ is a function from $\mathcal{T} \times \mathcal{G}$ to $R^p_{[+\infty]} \times \{0\}^v$ that depends on the slackness of the moment inequalities and on $n$, where $R_{[+\infty]} := R \cup \{+\infty\}$, and $h_2(\cdot, \cdot)$ is a $k \times k$ matrix-valued covariance kernel on $(\mathcal{T} \times \mathcal{G})^2$.

For $h := (h_1, h_2)$, define

$$T(h) := \sup_{\tau \in \mathcal{T}} \int S(\nu_{h_2}(\tau, g) + h_1(\tau, g), h_2^\varepsilon(\tau, g, \tau, g)) dQ(g), \tag{C.1}$$

where $h_2^\varepsilon(\tau, g, \tau, g) = h_2(\tau, g, \tau, g) + \varepsilon I_k$, and

$$\{\nu_{h_2}(\tau, g) : (\tau, g) \in \mathcal{T} \times \mathcal{G}\} \tag{C.2}$$

[13] Typically, the supremum over $\tau$ is obtained through smooth optimization techniques and there is no need to approximate $\mathcal{T}$ by a finite set. However, when smooth optimization is not applicable, we can also approximate $\mathcal{T}$ with a finite subset in the same way as approximating $\mathcal{G}$ by a finite subset.

is a mean zero $R^k$-valued Gaussian process with covariance kernel $h_2(\cdot,\cdot)$ on $(\mathcal{T}\times\mathcal{G})^2$, $h_1(\cdot)$ is a function from $\mathcal{T}\times\mathcal{G}$ to $R^p_{[+\infty]}\times\{0\}^v$, and $\varepsilon$ is as in the definition of $\overline{\Sigma}_n(\theta,\tau,g)$ in (3.4).[14] The definition of $T(h)$ in (C.1) applies to CvM test statistics. For the KS test statistic, one replaces $\int\cdots\, dQ(g)$ by $\sup_{g\in\mathcal{G}}\cdots$.

We are interested in tests of nominal level $\alpha$ and CS's of nominal level $1-\alpha$. Let

$$c_0(h,1-\alpha)\ (=c_0(h_1,h_2,1-\alpha)) \tag{C.3}$$

denote the $1-\alpha$ quantile of $T(h)$. If $h_n:=(h_{1,n},h_2)$ was known, we would use $c_0(h_n,1-\alpha)$ as the critical value for the test statistic $T_n(\theta)$. However, $h_n$ is not known and $h_{1,n}$ cannot be consistently estimated. In consequence, we replace $h_2$ in $c_0(h_{1,n},h_2,1-\alpha)$ by a uniformly consistent estimator $\widehat{h}_{2,n}(\theta)$ $(:=\widehat{h}_{2,n}(\theta,\cdot,\cdot))$ of the covariance kernel $h_2$ and we replace $h_{1,n}$ by a data-dependent GMS function $\varphi_n(\theta)$ $(:=\varphi_n(\theta,\cdot))$ on $\mathcal{T}\times\mathcal{G}$ (defined in Section C.2 below) that is constructed to be less than or equal to $h_{1,n}(\tau,g)$ for all $(\tau,g)\in\mathcal{T}\times\mathcal{G}$ with probability that goes to one as $n\to\infty$. Because $S(m,\Sigma)$ is non-increasing in $m_I$ by Assumption S1(b), where $m:=(m_I',m_{II}')'$ and $m_I\in R^p$, the latter property yields a test with asymptotic level less than or equal to the nominal level $\alpha$. The quantities $\widehat{h}_{2,n}(\theta)$ and $\varphi_n(\theta)$ are defined below.

Using $\widehat{h}_{2,n}(\theta)$ and $\varphi_n(\theta)$, in principle, one can obtain an approximation of $c_0(h_1,h_2,1-\alpha)$ using $c_0(\varphi_n(\theta),\widehat{h}_{2,n}(\theta),1-\alpha)$. However, computing $c_0(\varphi_n(\theta),\widehat{h}_{2,n}(\theta),1-\alpha)$ in practice is not easy because it involves the simulation of the Gaussian process $\{\nu_{\widehat{h}_2(\theta)}(\tau,g):\mathcal{T}\times\mathcal{G}\}$. Although we approximate $\mathcal{G}$ by a finite set in ASM, we often do not do so for $\mathcal{T}$. Even when we also use a finite approximation for $\mathcal{T}$, the combined dimension of the approximated set $\mathcal{T}\times\mathcal{G}$ often is large. That creates difficulty for simulating the Gaussian process. Thus, we recommend using a bootstrap version of the critical value instead.

The bootstrap GMS critical value is[15]

$$c^*(\varphi_n(\theta),\widehat{h}^*_{2,n}(\theta),1-\alpha):=c_0^*(\varphi_n(\theta),\widehat{h}^*_{2,n}(\theta),1-\alpha+\eta)+\eta, \tag{C.4}$$

[14] The sample paths of $\nu_{h_2}(\cdot,\cdot)$ are concentrated on the set $U^k_{\rho_{h_2}}(\mathcal{T}\times\mathcal{G})$ of bounded uniformly $\rho_{h_2}$-continuous $R^k$-valued functions on $\mathcal{T}\times\mathcal{G}$, where $\rho_{h_2}$ is the pseudometric on $\mathcal{T}\times\mathcal{G}$ defined by $\rho^2_{h_2}(\eta,\eta^\dagger):=tr(h_2(\eta,\eta)-h_2(\eta,\eta^\dagger)-h_2(\eta^\dagger,\eta)+h_2(\eta^\dagger,\eta^\dagger))$, where $\eta:=(\tau,g)$ and $\eta^\dagger:=(\tau^\dagger,g^\dagger)$.

[15] The constant $\eta$ is an *infinitesimal uniformity factor* (IUF) that is employed to circumvent problems that arise due to the presence of the infinite-dimensional nuisance parameter $h_{1,n}$ that affects the distribution of the test statistic in both small and large samples. The IUF obviates the need for complicated and difficult-to-verify uniform continuity and strict monotonicity conditions on the large sample distribution functions of the test statistic.

where $c_0^*(h,1-\alpha)$ is the $1-\alpha$ conditional quantile of $T^*(h)$ and $T^*(h)$ is defined as in (C.1) but with $\{\nu_{h_2}(\tau,g):(\tau,g)\in\mathcal{T}\times\mathcal{G}\}$ and $\widehat{h}_{2,n}(\theta)$ replaced by the bootstrap empirical process $\{\nu_n^*(\theta,\tau,g):(\tau,g)\in\mathcal{T}\times\mathcal{G}\}$ and the bootstrap covariance kernel $\widehat{h}_{2,n}^*(\theta)$, respectively. The bootstrap empirical process is defined to be

$$\nu_n^*(\theta,\tau,g):=n^{-1/2}\widehat{D}_n(\theta)^{-1/2}\sum_{i=1}^n(m(W_i^*,\theta,\tau,g)-\overline{m}_n(\theta,\tau,g)), \tag{C.5}$$

where $\{W_i^*:i\le n\}$ is an i.i.d. bootstrap sample drawn from the empirical distribution of $\{W_i:i\le n\}$ and $\widehat{D}_n(\theta)$ is defined in (C.8). Also, $\widehat{h}_{2,n}^*(\theta,\tau,g,\tau^\dagger,g^\dagger)$ and $\widehat{\Sigma}_n^*(\theta,\tau,g,\tau^\dagger,g^\dagger)$ are defined as in (C.8) below with $W_i^*$ in place of $W_i$. Note that we do not recompute $\widehat{D}_n(\theta)$ for the bootstrap samples, which simplifies the theoretical derivations below. Also note that $\widehat{h}_{2,n}^*(\theta,\tau,g,\tau^\dagger,g^\dagger)$ only enters $c(\varphi_n(\theta),\widehat{h}_{2,n}^*(\theta),1-\alpha)$ via indices $(\tau,g,\tau^\dagger,g^\dagger)$ such that $(\tau,g)=(\tau^\dagger,g^\dagger)$.

The nominal level $1-\alpha$ GMS CS is given by

$$CS_n:=\{\theta\in\Theta:T_n(\theta)\le c_{n,1-\alpha}^*(\theta)\}, \tag{C.6}$$

where the critical value $c_{n,1-\alpha}^*(\theta)$ abbreviates $c^*(\varphi_n(\theta),\widehat{h}_{2,n}^*(\theta),1-\alpha)$.

When the test statistic, $\overline{T}_{n,s_n}(\theta)$, is a truncated sum, simulated integral, or a QMC quantity, a bootstrap approximate-GMS critical value can be employed. It is defined analogously to the bootstrap GMS critical value but with $T^*(h)$ replaced by $T_{s_n}^*(h)$, where $T_{s_n}^*(h)$ has the same definition as $T^*(h)$ except that a truncated sum, simulated integral, or QMC quantity appears in place of the integral with respect to $Q$, as in Section B.6. The same functions $\{g_1,...,g_{s_n}\}$ are used in all bootstrap critical value calculations as in the test statistic $\overline{T}_{n,s_n}(\theta)$.

Next, we define the asymptotic covariance kernel, $\{h_{2,F}(\theta,\tau,g,\tau^\dagger,g^\dagger):(\tau,g),(\tau^\dagger,g^\dagger)\in\mathcal{T}\times\mathcal{G}\}$, of $n^{1/2}(\overline{m}_n(\theta,\tau,g)-E_F\overline{m}_n(\theta,\tau,g))$ after normalization via a diagonal matrix $D_F^{-1/2}(\theta)$. Define

$$\begin{aligned}h_{2,F}(\theta,\tau,g,\tau^\dagger,g^\dagger)&:=D_F^{-1/2}(\theta)\Sigma_F(\theta,\tau,g,\tau^\dagger,g^\dagger)D_F^{-1/2}(\theta),\ \text{where}\\ \Sigma_F(\theta,\tau,g,\tau^\dagger,g^\dagger)&:=Cov_F(m(W_i,\theta,\tau,g),m(W_i,\theta,\tau^\dagger,g^\dagger)'),\\ D_F(\theta)&:=Diag(\sigma_{F,1}^2(\theta),\ldots,\sigma_{F,k}^2(\theta)),\end{aligned} \tag{C.7}$$

and $\sigma^2_{F,j}(\theta)$ is introduced above Assumption PS1.

Correspondingly, the sample covariance kernel $\widehat{h}_{2,n}(\theta)$ $(= \widehat{h}_{2,n}(\theta,\cdot,\cdot))$, which is an estimator of $h_{2,F}(\theta,\tau,g,\tau^\dagger,g^\dagger)$, is defined by

$$\begin{aligned}\widehat{h}_{2,n}(\theta,\tau,g,\tau^\dagger,g^\dagger) &:= \widehat{D}_n^{-1/2}(\theta)\widehat{\Sigma}_n(\theta,\tau,g,\tau^\dagger,g^\dagger)\widehat{D}_n^{-1/2}(\theta), \text{ where}\\ \widehat{\Sigma}_n(\theta,\tau,g,\tau^\dagger,g^\dagger) &:= n^{-1}\sum_{i=1}^n \left(m(W_i,\theta,\tau,g)-\overline{m}_n(\theta,\tau,g)\right)\left(m(W_i,\theta,\tau^\dagger,g^\dagger)-\overline{m}_n(\theta,\tau^\dagger,g^\dagger)\right)',\\ \widehat{D}_n(\theta) &:= Diag(\widehat{\sigma}^2_{n,1}(\theta),\ldots,\widehat{\sigma}^2_{n,k}(\theta)),\end{aligned} \tag{C.8}$$

and $\widehat{\sigma}^2_{n,j}(\theta)$ is a consistent estimator of $\sigma^2_{F,j}(\theta)$ introduced below (3.4).

Note that $\widehat{\Sigma}_n(\theta,\tau,g)$, defined in (3.3), equals $\widehat{\Sigma}_n(\theta,\tau,g,\tau,g)$.

## C.2 Definition of $\varphi_{\mathbf{n}}(\theta)$

Next, we define $\varphi_n(\theta)$. As discussed above, $\varphi_n(\theta)$ is constructed such that $\varphi_n(\theta,\tau,g) \leq h_{1,n}(\tau,g)$ $\forall(\tau,g) \in \mathcal{T}\times\mathcal{G}$ with probability that goes to one as $n\to\infty$ uniformly over $(\theta,F)\in\mathcal{F}$. Let

$$\xi_n(\theta,\tau,g) := \kappa_n^{-1}n^{1/2}\overline{D}_n^{-1/2}(\theta,\tau,g)\overline{m}_n(\theta,\tau,g), \text{ where } \overline{D}_n(\theta,\tau,g) := Diag(\overline{\Sigma}_n(\theta,\tau,g)), \tag{C.9}$$

$\overline{\Sigma}_n(\theta,\tau,g)$ is defined in (3.4), and $\{\kappa_n : n\geq 1\}$ is a sequence of constants that diverges to infinity as $n\to\infty$. The $j$th element of $\xi_n(\theta,\tau,g)$, denoted by $\xi_{n,j}(\theta,\tau,g)$, measures the slackness of the moment inequality $E_F m_j(W_i,\theta,\tau,g)\geq 0$ for $j=1,...,p$.

Define $\varphi_n(\theta,\tau,g) := (\varphi_{n,1}(\theta,\tau,g),...,\varphi_{n,p}(\theta,\tau,g),0,...,0)' \in R^k$ via, for $j\leq p$,

$$\begin{aligned}\varphi_{n,j}(\theta,\tau,g) &:= \overline{h}_{2,n,j}^{1/2}(\theta,\tau,g)B_n 1(\xi_{n,j}(\theta,\tau,g)>1),\\ \overline{h}_{2,n}(\theta,\tau,g) &:= \widehat{D}_n^{-1/2}(\theta)\overline{\Sigma}_n(\theta,\tau,g)\widehat{D}_n^{-1/2}(\theta), \text{ and}\\ \overline{h}_{2,n,j}(\theta,\tau,g) &:= [\overline{h}_{2,n}(\theta,\tau,g)]_{jj}.\end{aligned} \tag{C.10}$$

We assume:

**Assumption GMS1.**(a) $\overline{\varphi}_n(\theta,\tau,g)$ satisfies (C.10) and $\{B_n : n\geq 1\}$ is a nondecreasing sequence of positive constants, and

(b) $\kappa_n\to\infty$ and $B_n/\kappa_n\to 0$ as $n\to\infty$.

In ASM and Andrews and Shi (2014), we use $\kappa_n = (0.3\ln(n))^{1/2}$ and $B_n = (0.4\ln(n)/\ln\ln(n))^{1/2}$, which satisfy Assumption GMS1.

The multiplicand $\overline{h}_{2,n,j}^{-1/2}(\theta,\tau,g)$ in (C.10) is an "$\varepsilon$-adjusted" standard deviation estimator for the $j$th normalized sample moment based on $g$ (see (3.4) for the $\varepsilon$-adjustment in $\overline{\Sigma}_n(\theta,\tau,g)$). It provides a suitable scaling for $\varphi_n(\theta,\tau,g)$.

## C.3 Plug-in Asymptotic Confidence Sets

Next, for comparative purposes, we define plug-in asymptotic (PA) critical values. Subsampling critical values also can be considered, see Appendix B of AS2 for details. We strongly recommend GMS critical values over PA and subsampling critical values for the same reasons as given in AS1 plus the fact that the finite-sample simulations in Sections 7.2 and 8.2 show much better overall performance by GMS critical values than PA and subsampling critical values.

The bootstrap PA critical values are obtained based on the asymptotic null distribution that arises when all conditional moment inequalities hold as equalities a.s. The critical value is

$$c^*(0^k_{\mathcal{T}\times\mathcal{G}}, \widehat{h}^*_{2,n}(\theta), 1-\alpha) := c^*_0(0^k_{\mathcal{T}\times\mathcal{G}}, \widehat{h}^*_{2,n}(\theta), 1-\alpha+\eta)+\eta, \tag{C.11}$$

where $0^k_{\mathcal{T}\times\mathcal{G}}$ denotes the $R^k$-valued function on $\mathcal{T}\times\mathcal{G}$ that is identically $(0,...,0)'\in R^k$, and $\widehat{h}^*_{2,n}(\theta)$ is defined in Section C.1 above (below equation (C.5)). The nominal $1-\alpha$ PA CS is given by (C.6) with the critical value $c^*_{n,1-\alpha}(\theta)$ equal to $c^*(0^k_{\mathcal{T}\times\mathcal{G}}, \widehat{h}^*_{2,n}(\theta), 1-\alpha)$.

# D Asymptotic Size

In this section, we show that the bootstrap GMS and PA CS's have correct uniform asymptotic coverage probabilities, i.e., correct asymptotic size.

## D.1 Notation

First, define

$$
\begin{aligned}
h_{1,n,F}(\theta,\tau,g) &:= n^{1/2}D_F^{-1/2}(\theta)E_F m(W_i,\theta,\tau,g),\\
h_{n,F}(\theta,\tau,g,\tau^\dagger,g^\dagger) &:= (h_{1,n,F}(\theta,\tau,g),h_{2,F}(\theta,\tau,g,\tau^\dagger,g^\dagger)),\\
\widehat{h}_{2,n,F}(\theta,\tau,g,\tau^\dagger,g^\dagger) &:= D_F^{-1/2}(\theta)\widehat{\Sigma}_n(\theta,\tau,g,\tau^\dagger,g^\dagger)D_F^{-1/2}(\theta),\\
\overline{h}_{2,n,F}(\theta,\tau,g) &:= \widehat{h}_{2,n,F}(\theta,\tau,g,\tau,g)+\varepsilon D_F^{-1/2}(\theta)\widehat{D}_n(\theta)D_F^{-1/2}(\theta)\\
&= D_F^{-1/2}(\theta)\overline{\Sigma}_n(\theta,\tau,g)D_F^{-1/2}(\theta), \text{ and} \qquad\qquad (D.1)\\
\nu_{n,F}(\theta,\tau,g) &:= n^{-1/2}\sum_{i=1}^n D_F^{-1/2}(\theta)[m(W_i,\theta,\tau,g)-E_F m(W_i,\theta,\tau,g)],
\end{aligned}
$$

where $m(W_i,\theta,\tau,g)$, $\widehat{\Sigma}_n(\theta,\tau,g,\tau^\dagger,g^\dagger)$, $\overline{\Sigma}_n(\theta,\tau,g)$, $D_F(\theta)$, and $\widehat{D}_n(\theta)$ are defined in (3.2), (3.3), (3.4), and (5.1) of ASM, and (C.8), respectively.

Below we write $T_n(\theta)$ as a function of the quantities in (D.1). As defined, (i) $h_{1,n,F}(\theta,\tau,g)$ is the $k$-vector of normalized means of the moment functions for $(\tau,g)\in\mathcal{T}\times\mathcal{G}$, which measures the slackness of the population moment conditions under $(\theta,F)$, and it has the very useful feature that it is non-negative when $(\theta,F)\in\mathcal{F}$ by (2.1) of ASM, (ii) $h_{n,F}(\theta,\tau,g,\tau^\dagger,g^\dagger)$ contains the approximation to the normalized means of $D_F^{-1/2}(\theta)m(W_i,\theta,\tau,g)$ and the covariances of $D_F^{-1/2}(\theta)m(W_i,\theta,\tau,g)$ and $D_F^{-1/2}(\theta)m(W_i,\theta,\tau^\dagger,g^\dagger)$, (iii) $\widehat{h}_{2,n,F}(\theta,\tau,g,\tau^\dagger,g^\dagger)$ and $\overline{h}_{2,n,F}(\theta,\tau,g)$ are hybrid quantities—part population, part sample—based on the matrices $\widehat{\Sigma}_n(\theta,\tau,g,\tau^\dagger,g^\dagger)$ and $\overline{\Sigma}_n(\theta,\tau,g)$, respectively, and (iv) $\nu_{n,F}(\theta,\tau,g)$ is the sample average of the moment functions $D_F^{-1/2}(\theta)m(W_i,\theta,\tau,g)$ normalized to have mean zero and variance that is $O(1)$, but not $o(1)$. Note that $\nu_{n,F}(\theta,\cdot,\cdot)$ is an empirical process indexed by $(\tau,g)\in\mathcal{T}\times\mathcal{G}$ with covariance kernel given by $h_{2,F}(\theta,\tau,g,\tau^\dagger,g^\dagger)$.

The normalized sample moments $n^{1/2}\overline{m}_n(\theta,\tau,g)$ can be written as

$$
n^{1/2}\overline{m}_n(\theta,\tau,g)=D_F^{1/2}(\theta)(\nu_{n,F}(\theta,\tau,g)+h_{1,n,F}(\theta,\tau,g)). \qquad (D.2)
$$

The test statistic $T_n(\theta)$, defined in (B.3), can be written as

$$
T_n(\theta)=\sup_{\tau\in\mathcal{T}}\int_{\mathcal{G}} S(\nu_{n,F}(\theta,\tau,g)+h_{1,n,F}(\theta,\tau,g),\overline{h}_{2,n,F}(\theta,\tau,g))dQ(g). \qquad (D.3)
$$

Note the close resemblance between $T_n(\theta)$ and $T(h)$ (defined in (C.1)).

Let $\mathcal{H}_1$ denote the set of all functions from $\mathcal{T} \times \mathcal{G}$ to $R^p_{[+\infty]} \times \{0\}^v$.

For notational simplicity, for any function of the form $r_F(\theta, \tau, g)$ for $(\tau, g) \in \mathcal{T} \times \mathcal{G}$, let $r_F(\theta)$ denote the function $r_F(\theta, \cdot, \cdot)$ on $\mathcal{T} \times \mathcal{G}$. Correspondingly, for any function of the form $r_F(\theta, \tau, g, \tau^\dagger, g^\dagger)$ for $(\tau, g), (\tau^\dagger, g^\dagger) \in \mathcal{T} \times \mathcal{G}$, let $r_F(\theta)$ denote the function $r_F(\theta, \cdot, \cdot, \cdot, \cdot)$ on $(\mathcal{T} \times \mathcal{G})^2$. Thus, $h_{2,F}(\theta)$ abbreviates the asymptotic covariance kernel $\{h_{2,F}(\theta, \tau, g, \tau^\dagger, g^\dagger) : (\tau, g), (\tau^\dagger, g^\dagger) \in \mathcal{T} \times \mathcal{G}\}$ defined in (C.7). Define

$$\mathcal{H}_2 := \{h_{2,F}(\theta) : (\theta, F) \in \mathcal{F}\}, \tag{D.4}$$

where, as defined at the end of Section 2, $\mathcal{F}$ is the subset of $\mathcal{F}_+$ that satisfies Assumption PS3. On the space of $k \times k$ matrix-valued covariance kernels on $(\mathcal{T} \times \mathcal{G})^2$, which is a superset of $\mathcal{H}_2$, we use the uniform metric $d$ defined by

$$d(h_2^{(1)}, h_2^{(2)}) := \sup_{(\tau,g),(\tau^\dagger,g^\dagger) \in \mathcal{T} \times \mathcal{G}} \|h_2^{(1)}(\tau, g, \tau^\dagger, g^\dagger) - h_2^{(2)}(\tau, g, \tau^\dagger, g^\dagger)\|. \tag{D.5}$$

Let $\Rightarrow$ denote weak convergence. Let $\{a_n\}$ denote a subsequence of $n$. Let $\rho_{h_2}(\theta)$ be the intrinsic pseudometric on $\mathcal{T} \times \mathcal{G}$ for the tight Gaussian process $\nu_{h_2}(\theta)$ with variance-covariance kernel $h_2$:

$$\begin{aligned} &\rho_{h_2}(\tau, g, \tau^\dagger, g^\dagger) \\ &:= tr\left(h_2(\tau, g, \tau, g) - h_2(\tau, g, \tau^\dagger, g^\dagger) - h_2(\tau^\dagger, g^\dagger, \tau, g) + h_2(\tau^\dagger, g^\dagger, \tau^\dagger, g^\dagger)\right). \end{aligned} \tag{D.6}$$

## D.2 Proof of Theorem 5.1

We prove Theorem 5.1 of ASM using two lemmas. The two lemmas together imply the uniform validity of the GMS and PA CS's over $\mathcal{F}$ under Assumptions M, S1, and S2 and, in the case of GMS CS's, Assumption GMS1.

The first lemma below establishes the uniform asymptotic size under two high-level assumptions (given below). The second lemma verifies these two assumptions under Assumptions M, S1, and S2.

**Assumption PS4.** For any subsequence $\{a_n\}$ of $\{n\}$ and any sequence $\{(\theta_{a_n}, F_{a_n}) \in \mathcal{F}_+ :$

$n \geq 1\}$ for which

$$\lim_{n\to\infty} d(h_{2,F_{a_n}}(\theta_{a_n}), h_2) = 0 \tag{D.7}$$

for some $k \times k$ matrix-valued covariance kernel $h_2(\tau, g, \tau^\dagger, g^\dagger)$ on $(\mathcal{T} \times \mathcal{G})^2$, we have

(i) $\nu_{a_n,F_{a_n}}(\theta_{a_n}) \Rightarrow \nu_{h_2}(\cdot)$ and

(ii) $d(\widehat{h}_{2,a_n,F_{a_n}}(\theta_{a_n}), h_2) \to_p 0$ as $n \to \infty$, where $\widehat{h}_{2,a_n,F_{a_n}}(\theta_{a_n})$ is defined in (D.1).

**Assumption PS5.** For any subsequence $\{a_n\}$ of $\{n\}$, conditional on any sample path $\omega$ for which

$$\lim_{n\to\infty} d(\widehat{h}_{2,a_n,F_{a_n}}(\theta_{a_n})(\omega), h_2) = 0, \tag{D.8}$$

for some $k \times k$ matrix-valued covariance kernel $h_2(\tau, g, \tau^\dagger, g^\dagger)$ on $(\mathcal{T} \times \mathcal{G})^2$, we have (i) $\nu^*_{a_n}(\theta) \Rightarrow \nu_{h_2}$ and (ii) $d(\widehat{h}^*_{2,a_n}(\theta_{a_n}), h_2) \to_p 0$.

**Lemma D.1** *Suppose Assumptions* PS4, PS5, S1, S2, *and* SIG1 *hold, and Assumption* GMS1 *holds when considering* GMS *critical values. Then, for any compact subset* $\mathcal{H}_{2,cpt}$ *of* $\mathcal{H}_2$*, the* GMS *and the* PA *CS's satisfy:*

$$\liminf_{n\to\infty} \inf_{\substack{(\theta,F)\in\mathcal{F}: \\ h_{2,F}(\theta)\in\mathcal{H}_{2,cpt}}} P_F(\theta \in CS_n) \geq 1 - \alpha.$$

**Lemma D.2** *Suppose Assumptions* M, S1, *and* S2 *hold. Then,*

(a) *Assumption* PS4 *holds and*

(b) *Assumption* PS5 *holds.*

**Comments. 1.** Lemma D.1(a) shows that GMS and PA CS's have correct uniform asymptotic size. The uniformity results hold whether the moment conditions involve "weak" or "strong" IV's $X_i$.

**2.** Theorem 5.1 of ASM for the case $r_{1,n} = \infty$ is proved by verifying the conditions of Lemma D.2 (that is, by showing that Assumptions M, S1, S2, and GMS1 hold for the $\mathcal{G}_{\text{c-cube}}$ set and the $S$ functions considered in ASM).[16] The functions $S_1$, $S_2$, and $S_3$ in (3.9) of ASM satisfy Assumptions S1 and S2 by Lemma 1 of AS1 and the function $S_4$ of ASM satisfy Assumptions S1 and S2 by similar arguments. Lemma 3 of AS1 establishes Assumption M for $\mathcal{G}_{\text{c-cube}}$ defined in (3.6) of ASM. Assumption GMS1 holds immediately for $\kappa_n$ and $B_n$

[16] The quantity $r_{1,n}$ is the test statistic truncation value that appears in (3.7) of ASM. It satisfies either $r_{1,n} = \infty$ for all $n \geq 1$ or $r_{1,n} < \infty$ and $r_{1,n} \to \infty$ as $n \to \infty$.

used in (4.1) and (4.2) of ASM, respectively. Theorem 5.1 of ASM holds for $r_{1,n}$ such that $r_{1,n} < \infty$ and $r_{1,n} \to \infty$ as $n \to \infty$ by minor alterations to the proofs.

## D.3 Proof of Lemma D.1

### D.3.1 Theorem D.3

The following Theorem provides a uniform asymptotic distributional result for the test statistic $T_n(\theta)$. It is an analogue of Theorem 1 in AS1. It is used in the proof of Lemma D.1.

**Theorem D.3** *Suppose Assumptions* PS4, S1, S2, *and* SIG1 *hold. Then, for all compact subsets* $\mathcal{H}_{2,cpt}$ *of* $\mathcal{H}_2$*, for all constants* $x_{h_{n,F}(\theta)} \in R$ *that may depend on* $(\theta, F)$ *and* $n$ *through* $h_{n,F}(\theta)$*, and all* $\delta > 0$*, we have*

(a) $\limsup_{n\to\infty} \sup_{\substack{(\theta,F)\in\mathcal{F}: \\ h_{2,F}(\theta)\in\mathcal{H}_{2,cpt}}} \left[P_F(T_n(\theta) > x_{h_{n,F}(\theta)}) - P\left(T(h_{n,F}(\theta)) + \delta > x_{h_{n,F}(\theta)}\right)\right] \leq 0$ *and*

(b) $\liminf_{n\to\infty} \inf_{\substack{(\theta,F)\in\mathcal{F}: \\ h_{2,F}(\theta)\in\mathcal{H}_{2,cpt}}} \left[P_F(T_n(\theta) > x_{h_{n,F}(\theta)}) - P\left(T(h_{n,F}(\theta)) - \delta > x_{h_{n,F}(\theta)}\right)\right] \geq 0,$

*where* $T(h)$ *is the function defined in* (C.1)*.*

**Proof of Theorem D.3.** Theorem D.3 is similar to Theorem 1 in AS1. The proof of the latter theorem goes through with the following modifications:

(i) Redefine $SubSeq(h_2)$ to be the set of subsequences $\{(\theta_{a_n}, F_{a_n}) \in \mathcal{F} : n \geq 1\}$ where $\{a_n\}$ is a subsequence of $\{n\}$, such that (D.7) holds.

(ii) Replace $\int \cdots dQ(g)$ by $\sup_{\tau\in\mathcal{T}} \int_{\mathcal{G}} \cdots dQ(g)$. In other instances where $g$ and $\mathcal{G}$ appear, replace $g$ with $(\tau, g)$ and $\mathcal{G}$ with $\mathcal{T} \times \mathcal{G}$.

(iii) Replace "by Lemma A1" with "by Assumption PS4."

(iv) Change the paragraph at the bottom of p. 6 of AS2 to the following:

"Given this and Assumption SIG1, by the almost sure representation theorem, e.g., see Pollard (1990, Thm. 9.4), there exists a probability space and random quantities $\tilde{\nu}_{a_n}(\cdot)$, $\tilde{h}_{2,a_n}(\cdot)$, $\tilde{V}_{a_n}$, and $\tilde{\nu}_0(\cdot)$ defined on it such that (i) $(\tilde{\nu}_{a_n}(\cdot), \tilde{h}_{2,a_n}(\cdot), \tilde{V}_{a_n})$ has the same distribution as $(\nu_{a_n,F_{a_n}}(\theta_{a_n}, \cdot), \widehat{h}_{2,a_n,F_{a_n}}(\theta_{a_n}, \cdot), D_{F_{a_n}}^{-1/2}(\theta_{a_n})\widehat{D}_{a_n}(\theta_{a_n})D_{F_{a_n}}^{-1/2}(\theta_{a_n}))$, (ii) $(\tilde{\nu}_0(\cdot))$ has the same distribution as $\nu_{h_{2,0}}(\cdot)$, and

$$\text{(iii)} \quad \sup_{(\tau,g)\in\mathcal{T}\times\mathcal{G}} \left\| \begin{pmatrix} \tilde{\nu}_{a_n}(\tau, g) \\ \tilde{h}_{2,a_n}(\tau, g) \\ vec(\tilde{V}_{a_n}) \end{pmatrix} - \begin{pmatrix} \tilde{\nu}_0(\tau, g) \\ h_{2,0}(\tau, g) \\ vec(I_k) \end{pmatrix} \right\| \to 0 \text{ as } n \to \infty, \text{ a.s.} \qquad \text{(D.9)}$$

(v) Replace $Diag(\tilde{h}_{2,a_n}(1_k))$ by $\tilde{V}_{a_n}$.

With the above modifications, the proof of Theorem 1 in AS2 up to the proof of (12.7) of AS2 goes through. The proof of (12.7) in AS2, which relies on a dominated convergence argument, does not go through because the test statistic considered in this paper is not of the pure CvM type, and thus, $\tilde{T}_{a_n}$ and $\tilde{T}_{a_n,0}$ are not integrals with respect to $(\tau, g)$.

We change the proof of (12.7) in AS2 to the following.

As in the proof of (12.7) in AS2, we fix a sample path $\omega$ at which $(\tilde{\nu}_{a_n}(\tau, g), \tilde{h}_{2,a_n}(\tau, g))(\omega)$ converges to $(\tilde{\nu}_0(\tau, g), h_{2,0}(\tau, g))(\omega)$ uniformly over $(\tau, g) \in \mathcal{T} \times \mathcal{G}$ as $n \to \infty$ and $\sup_{(\tau,g)\in\mathcal{T}\times\mathcal{G}} \|\tilde{\nu}_0(\tau, g)(\omega)\| < \infty$. Let $\tilde{\Omega}$ be the collection of such sample paths. By (D.9), $P(\tilde{\Omega}) = 1$. For a fixed $\omega \in \tilde{\Omega}$, by Assumption S2, we have

$$\sup_{(\tau,g)\in\mathcal{T}\times\mathcal{G}} \sup_{\mu\in[0,\infty)^p\times\{0\}^v} |S(\tilde{\nu}_{a_n}(\tau, g)(\omega) + \mu, \tilde{h}^{\varepsilon}_{2,a_n}(\tau, g)(\omega)) - S(\tilde{\nu}_0(\tau, g)(\omega) + \mu, h^{\varepsilon}_{2,0}(\tau, g))| \to 0, \tag{D.10}$$

as $n \to \infty$, where $\tilde{h}^{\varepsilon}_{2,n}(\tau, g) := \tilde{h}_{2,n}(\tau, g) + \varepsilon\tilde{V}_{a_n}$, and $h^{\varepsilon}_{2,0}(\tau, g) := h_{2,0}(\tau, g) + \varepsilon I_k$. Thus, for every $\omega \in \tilde{\Omega}$,

$$\begin{aligned} |\tilde{T}_{a_n}(\omega) - \tilde{T}_{a_n,0}(\omega)| \le \sup_{(\tau,g)\in\mathcal{T}\times\mathcal{G}} & |S(\tilde{\nu}_{a_n}(\tau, g)(\omega) + h_{1,a_n,F_{a_n}}(\theta_{a_n}, \tau, g), \tilde{h}^{\varepsilon}_{2,a_n}(\tau, g)(\omega)) - \\ & - S(\tilde{\nu}_0(\tau, g)(\omega) + h_{1,a_n,F_{a_n}}(\theta_{a_n}, \tau, g), h^{\varepsilon}_{2,0}(\tau, g))| \\ & \to 0 \text{ as } n \to \infty. \end{aligned} \tag{D.11}$$

This verifies (12.7) in AS2. $\square$

### D.3.2 Proof of Lemma D.1

Lemma D.1 is similar to Theorem 2(a) of AS1 and we modify the proof of the latter in AS2 to fit the context of Lemma D.1. In addition to notational changes, a substantial modification is needed because Theorem 2 of AS1 does not cover bootstrap critical values.

Specifically, the proof of Theorem 2(a) in AS2 with the following modifications provides the proof of Lemma D.1.

(i) Replace all references to "Assumption M" of AS1 by references to "Assumption PS4" stated above and Assumptions S1 and S2 of AS1 by Assumptions S1 and S2 stated above. Replace $\int \cdots dQ(g)$ by $\sup_{\tau\in\mathcal{T}} \int_{\mathcal{G}} \cdots dQ(g)$. In other instances where $g$ and $\mathcal{G}$ appear, replace

$g$ with $(\tau, g)$ and $\mathcal{G}$ with $\mathcal{T} \times \mathcal{G}$. Let $\widehat{D}_{a_n}(\theta_{a_n})$ be defined as in (C.8) above, rather than as in AS1 and AS2.

(ii) Replace references to "Theorem 1(a)" of AS1 with references to "Theorem D.3(a)" stated above.

(iii) Redefine $SubSeq(h_2)$ to be the set of subsequences $\{(\theta_{a_n}, F_{a_n}) \in \mathcal{F} : n \geq 1\}$ for which (D.7) holds, where $\{a_n\}$ is a subsequence of $\{n\}$.

(iv) Replace references to "Lemma A1" of AS2 to references to "Assumption PS4" stated above.

(v) In both the statement and the proof of Lemma A3 in AS2, replace $c(\varphi_n(\theta), \widehat{h}_{2,n}(\theta), 1-\alpha)$ with $c_0(\varphi_n(\theta), h_{2,F}(\theta), 1-\alpha)$, and $c(h_{1,n,F}(\theta), \widehat{h}_{2,n}(\theta), 1-\alpha)$ with $c_0(h_{1,n,F}(\theta), h_{2,F}(\theta), 1-\alpha)$. The proof of Lemma A3 given in AS2 goes through with the following changes:

In the 6th and 7th last lines of the proof of Lemma A3 in AS2, delete "$\varepsilon^{-1/2} h_{2,0,j}^{-1/2}(1_k, 1_k)(1 + o_p(1)) =$", and change "by Lemma A1(b) and (5.2)" to "by Assumption SIG1 and (D.1)."

(vi) Replace Lemma A4 in AS2 with Lemma D.4 given immediately below. The proof of the Lemma D.4 given below is self-contained and does not rely on an analogue of Lemma A5 of AS2.

No other changes are needed in the proof of Theorem 2(a) in AS2. $\square$

The following lemma is used in the proof of Lemma D.1 given immediately above.

**Lemma D.4** *Suppose Assumptions* PS4, PS5, S1, S2, *and* GMS1 *hold. Then, for all* $\delta \in (0, \eta)$, *where* $\eta > 0$ *is defined in* (C.4),

$$\lim_{n\to\infty} \sup_{\substack{(\theta,F)\in\mathcal{F}: \\ h_{2,F}(\theta)\in\mathcal{H}_{2,cpt}}} P_F\left(c^*(\varphi_n(\theta), \widehat{h}^*_{2,n}(\theta), 1-\alpha) < c_0(\varphi_n(\theta), h_{2,F}(\theta), 1-\alpha) + \delta\right) = 0.$$

**Prove of Lemma D.4.** The result of the Lemma is equivalent to

$$\lim_{n\to\infty} \sup_{\substack{(\theta,F)\in\mathcal{F}: \\ h_{2,F}(\theta)\in\mathcal{H}_{2,cpt}}} P_F(c_0^*(\varphi_n(\theta), \widehat{h}^*_{2,n}(\theta), 1-\alpha+\eta) + \eta < c_0(\varphi_n(\theta), h_{2,F}(\theta), 1-\alpha) + \delta) = 0. \tag{D.12}$$

By considering a sequence $\{(\theta_n, F_n) \in \mathcal{F} : n \geq 1\}$ that is within $\zeta_n \to 0$ of the supremum in the above display for all $n \geq 1$, it suffices to show that

$$\lim_{n\to\infty} P_{F_n}(c_0^*(\varphi_n(\theta_n), \widehat{h}_{2,n}^*(\theta_n), 1-\alpha+\eta) + \eta < c_0(\varphi_n(\theta_n), h_{2,F_n}(\theta_n), 1-\alpha) + \delta) = 0. \quad \text{(D.13)}$$

Given any subsequence $\{u_n\}$ of $\{n\}$, there exists a further subsequence $\{w_n\}$ such that $d(h_{2,F_{w_n}}(\theta_{w_n}), h_{2,0}) \to 0$ as $n \to \infty$ for some matrix-valued covariance function $h_{2,0}$ by the compactness of $\mathcal{H}_{2,cpt}$. It suffices to show that (D.13) holds with $w_n$ in place of $n$.

By Assumption PS4(ii), $d(h_{2,F_{w_n}}(\theta_{w_n}), h_{2,0}) \to 0$ implies that $d(\widehat{h}_{2,w_n,F_{w_n}}(\theta_{w_n}), h_{2,0}) \to_p 0$, which then implies

$$d(\widehat{h}_{2,w_n}(\theta_{w_n}), h_{2,0}) \to_p 0, \quad \text{(D.14)}$$

where $\widehat{h}_{2,n}(\theta)$ and $\widehat{h}_{2,n,F}(\theta)$ are defined in (C.8) and (D.1), respectively. Then, by a general convergence in probability result, given any subsequence of $\{w_n\}$ there exists a further subsequence $\{a_n\}$ such that

$$d(\widehat{h}_{2,a_n}(\theta_{a_n}), h_{2,0}) \to 0 \text{ a.s.} \quad \text{(D.15)}$$

Hence, it suffices to show (D.13) with $a_n$ in place of $n$. Let $\bar{\Omega}$ be the set of sample paths $\omega$ such that $d(\widehat{h}_{2,a_n}(\theta_{a_n})(\omega), h_2) \to 0$. The above display implies that $P(\bar{\Omega}) = 1$.

Consider an arbitrary sample path $\omega \in \bar{\Omega}$. Below we show that for all constants $x_n \in R$ (possibly dependent on $\omega$) and all $\xi > 0$,

$$\begin{aligned}\limsup_{n\to\infty} \Big[ & P\left(T^*(\varphi_{a_n}(\theta_{a_n}), \widehat{h}_{2,a_n}^*(\theta_{a_n})) \leq x_{a_n}|\omega\right) - \\ & P\left(T(\varphi_{a_n}(\theta_{a_n}), h_{2,a_n,F_{a_n}}(\theta_{a_n})) \leq x_{a_n} + \xi|\omega\right)\Big] \leq 0, \end{aligned} \quad \text{(D.16)}$$

where in the first line $P(\cdot|\omega)$ denotes bootstrap probability conditional on the original sample path $\omega$, in the second line $P(\cdot|\omega)$ denotes $\nu_{h_{2,a_n,F_{a_n}}(\theta_{a_n})}(\cdot)$ probability conditional on the original sample path $\omega$, and $\nu_{h_{2,a_n,F_{a_n}}(\theta_{a_n})}(\cdot)$ is the Gaussian process defined in (C.2) with $h_2 = h_{2,a_n,F_{a_n}}(\theta_{a_n})$, which is taken to be independent of the original sample $\{W_i : i \leq n\}$ and, hence, is independent of $\varphi_{a_n}(\theta_{a_n})$.

The interval $(0, \eta - \delta)$ is non-empty because $\delta \in (0, \eta)$ by assumption. Using (D.16), we obtain, for all $\xi \in (0, \eta - \delta)$,

$$
\begin{aligned}
&\limsup_{n\to\infty} P\left(T^*(\varphi_{a_n}(\theta_{a_n}), \widehat{h}^*_{2,a_n}(\theta_{a_n})) \leq c_0(\varphi_{a_n}(\theta_{a_n}), h_{2,a_n,F_{a_n}}(\theta_{a_n}), 1-\alpha) + \delta - \eta|\omega\right) \\
&\leq \limsup_{n\to\infty} P\left(T(\varphi_{a_n}(\theta_{a_n}), h_{2,a_n,F_{a_n}}(\theta_{a_n})) \leq c_0(\varphi_{a_n}(\theta_{a_n}), h_{2,a_n,F_{a_n}}(\theta_{a_n}), 1-\alpha) + \delta - \eta + \xi|\omega\right) \\
&\leq 1-\alpha, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(D.17)}
\end{aligned}
$$

where the second inequality holds because $\delta - \eta + \xi < 0$ for $\xi \in (0, \eta - \delta)$. For any df $F$ with $1 - \alpha + \eta$ quantile denoted by $q_{1-\alpha+\eta}$, we have $F(q_{1-\alpha+\eta}) \geq 1 - \alpha + \eta$. Hence, if $F(x) < 1 - \alpha + \eta$, then $x < q_{1-\alpha+\eta}$. Combining this with the result in (D.17) implies that given any $\delta \in (0, \eta)$, for $n$ sufficiently large,

$$
c_0(\varphi_{a_n}(\theta_{a_n})(\omega), h_{2,a_n,F_{a_n}}(\theta_{a_n}), 1-\alpha)+\delta-\eta < c_0^*(\varphi_{a_n}(\theta_{a_n}), h_{2,a_n,F_{a_n}}(\theta_{a_n}), 1-\alpha+\eta)(\omega), \quad \text{(D.18)}
$$

where the indexing by $\omega$ denotes that the result holds for fixed $\omega \in \bar{\Omega}$. Because (D.18) holds for all $\omega \in \bar{\Omega}$ and $P(\bar{\Omega}) = 1$, the bounded convergence theorem applies and establishes (D.13).

It remains to prove the result in (D.16). This result follows from an analogous argument to that used to prove Theorem D.3(b). Note the common structure of the original sample and bootstrap sample test statistics:

$$
\begin{aligned}
T_n(\theta_n) &= \sup_{\tau\in\mathcal{T}} \int S(\nu_{n,F_n}(\theta_n,\tau,g) + h_{1,n,F_n}(\theta_n,\tau,g), \overline{h}_{2,n,F_n}(\theta_n,\tau,g))dQ(g), \\
T_n^*(\varphi_n(\theta_n), \widehat{h}^*_{2,n}(\theta_n)) &= \sup_{\tau\in\mathcal{T}} \int S(\nu_n^*(\theta_n,\tau,g) + \varphi_n(\theta_n,\tau,g), \overline{h}^*_{2,n}(\theta_n,\tau,g))dQ(g), \text{ where} \\
\overline{h}^*_{2,n}(\theta,\tau,g) &:= \widehat{h}^*_{2,n}(\theta,\tau,g) + \varepsilon I_k,
\end{aligned}
$$

$\nu_{n,F}$, $h_{1,n,F}$, and $\overline{h}_{2,n,F}$ are defined in (D.1), $\varphi_n(\theta)$ is defined in (C.10), and $\widehat{h}^*_{2,n}(\theta)$ is defined following (C.5) using (C.8) with $W_i^*$ in place of $W_i$.

The result of Theorem D.3(b) with $T_n(\theta_n)$ replaced by $T_n^*(\varphi_n(\theta_n), \widehat{h}^*_{2,n}(\theta_n))$, with $T(h_{n,F}(\theta))$ replaced by $T(\varphi_n(\theta_n), h_{2,n,F_n}(\theta_n))$, and with $\delta$ replaced by $\xi$, when applied to the subsequence $\{(\theta_{a_n}, F_{a_n}) : n \geq 1\}$ is the result of (D.16). The result in (D.16) follows by the same argument as that for Theorem D.3(b) with $\nu_{a_n,F_{a_n}}(\theta_{a_n}, \cdot)$ replaced by $\nu^*_{a_n}(\theta_{a_n}, \cdot)(\omega)$, where $\nu^*_{a_n}(\theta_{a_n}, \cdot)(\omega)$ denotes the bootstrap empirical process given the sample path $\omega$ of the original sample, with

$\widehat{h}_{2,a_n,F_{a_n}}(\theta_{a_n},\cdot,\cdot)$ replaced by $\widehat{h}^*_{2,a_n}(\theta_{a_n},\cdot,\cdot)(\omega)$, and with Assumption PS4 replaced by Assumption PS5, which guarantees that $\nu^*_{a_n}(\theta_{a_n})(\omega) \Rightarrow \nu_{h_{2,0}}$ and $d(\widehat{h}^*_{2,a_n}(\theta_{a_n})(\omega), h_{2,0}) \to_p 0$. $\square$

## D.4 Proof of Lemma D.2

The verification of Assumption PS4 is the same as the proof of Lemma A1 given in Appendix E of AS2 except with some notation changes and with Lemma D.5 below replacing Lemma E1(a) in AS2 in the proof. (Lemma A1 of AS2 is stated in Appendix A of AS2.) The verification of Assumption PS5 is the same as that of Assumption PS4 except that all arguments are conditional on the sample path $\omega$ (specified in Assumption PS5). Details are omitted for brevity.

**Lemma D.5** *Let $(\Omega, \mathbb{F}, P)$ be a probability space and let $\omega$ denote a generic element in $\Omega$. Suppose that the row-wise i.i.d. triangular arrays of random processes $\{\phi_{n,i}(\omega, g) : g \in \mathcal{G}, i \leq n, n \geq 1\}$ and $\{c_{n,i}(\omega, \tau) : \tau \in \mathcal{T}, i \leq n, n \geq 1\}$ are manageable with respect to the envelopes $\{F_n(\omega) : \Omega \to R^n : n \geq 1\}$ and $\{C_n(\omega) : \Omega \to R^n : n \geq 1\}$, respectively. Then, $\{\phi_{n,i}(\omega, g)c_{n,i}(\omega, \tau) : (\tau, g) \in \mathcal{T} \times \mathcal{G}, i \leq n, n \geq 1\}$ is manageable with respect to the envelopes $\{F_n(\omega) \odot C_n(\omega) : n \geq 1\}$, where $\odot$ stands for the coordinate-wise product.*

**Proof of Lemma D.5.** For a positive number $\xi$ and a Euclidean space $G$, the packing number $D(\xi, G)$ is defined in Section B.3. For each $\omega \in \Omega$ and each $n \geq 1$, let $\mathcal{F}_{n,\omega} := \{(\phi_{n,1}(\omega, g), ..., \phi_{n,n}(\omega, g))' : g \in \mathcal{G}\}$, and let $\mathcal{C}_{n,\omega} := \{(c_{n,1}(\omega, \tau), ..., c_{n,n}(\omega, \tau))' : \tau \in \mathcal{T}\}$. Let $\lambda_\phi(\varepsilon)$ and $\lambda_c(\varepsilon)$ be the deterministic functions that (i) bound from above $D(\varepsilon||\alpha \odot F_n(\omega)||, \alpha \odot \mathcal{F}_{n,\omega})$ and $D(\varepsilon||\alpha \odot C_n(\omega)||, \alpha \odot \mathcal{C}_{n,\omega})$, respectively, for an arbitrary nonnegative $n$-vector $\alpha$, and (ii) satisfy $\int_0^1 \sqrt{\log \lambda_\phi(x)}dx < \infty$ and $\int_0^1 \sqrt{\log \lambda_c(x)}dx < \infty$. Such functions exist by the assumed manageability of the triangular arrays of random processes in the lemma.

For an arbitrary $\varepsilon > 0$, construct a bound for $D(\varepsilon || \alpha \odot F_n(\omega) \odot C_n(\omega)||, \alpha \odot \mathcal{F}_{n,\omega} \odot \mathcal{C}_{n,\omega})$ as follows:

$$
\begin{aligned}
& D(\varepsilon || \alpha \odot F_n(\omega) \odot C_n(\omega)||, \alpha \odot \mathcal{F}_{n,\omega} \odot \mathcal{C}_{n,\omega}) \\
\leq\ & D((\varepsilon/4) || \alpha \odot F_n(\omega) \odot C_n(\omega)||, \alpha \odot F_n(\omega) \odot \mathcal{C}_{n,\omega}) \\
& \times D((\varepsilon/4) || \alpha \odot F_n(\omega) \odot C_n(\omega)||, \alpha \odot C_n(\omega) \odot \mathcal{F}_{n,\omega}) \\
\leq\ & \sup_{\alpha^* \in R^n_+} D((\varepsilon/4) || \alpha^* \odot C_n(\omega)||, \alpha^* \odot \mathcal{C}_{n,\omega}) \sup_{\alpha^* \in R^n_+} D((\varepsilon/4) || \alpha^* \odot F_n(\omega)||, \alpha^* \odot \mathcal{F}_{n,\omega}) \\
\leq\ & \lambda_\phi(\varepsilon/4) \lambda_c(\varepsilon/4), \qquad \text{(D.19)}
\end{aligned}
$$

where the first inequality holds by the displayed equation following (5.2) in Pollard (1990), the second inequality holds because $\alpha \odot F_n(\omega), \alpha \odot C_n(\omega) \in R^n_+$, and the last inequality holds by the definitions of $\lambda_\phi(\varepsilon)$ and $\lambda_c(\varepsilon)$.

Then, the manageability of $\{\phi_{n,i}(\omega, g) c_{n,i}(\omega, g) : (\tau, g) \in \mathcal{T} \times \mathcal{G}, i \leq n, n \geq 1\}$ with respect to the envelopes $\{F_n(\omega) \odot C_n(\omega) : n \geq 1\}$ is proved by the following calculations:

$$
\begin{aligned}
\int_0^1 \sqrt{\log(\lambda_\phi(x/4)\lambda_c(x/4))} dx &\leq \int_0^1 \sqrt{\log \lambda_\phi(x/4)} dx + \int_0^1 \sqrt{\log \lambda_c(x/4)} dx \\
&= 4 \int_0^{1/4} \sqrt{\log \lambda_\phi(y)} dy + 4 \int_0^{1/4} \sqrt{\log \lambda_c(y)} dy \\
&\leq 4 \int_0^1 \sqrt{\log \lambda_\phi(y)} dy + 4 \int_0^1 \sqrt{\log \lambda_c(y)} dy \\
&< \infty, \qquad \text{(D.20)}
\end{aligned}
$$

where the first inequality holds by the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b > 0$, the equality holds by a change of variables, the second inequality holds because the integrands are nonnegative on $(1/4, 1]$, and the last inequality holds by the definitions of $\lambda_\phi(\varepsilon)$ and $\lambda_c(\varepsilon)$. $\square$

# E Power Against Fixed Alternatives

We now show that the powers of the GMS and PA tests converge to one as $n \to \infty$ for all fixed alternatives. Thus, both tests are consistent tests.

Recall that the null hypothesis is

$$\begin{aligned} H_0 : E_{F_0}[m_j(W_i, \theta_*, \tau)|X_i] &\geq 0 \text{ a.s. } [F_{X,0}] \text{ for } j = 1, ..., p \text{ and} \\ E_{F_0}[m_j(W_i, \theta_*, \tau)|X_i] &= 0 \text{ a.s. } [F_{X,0}] \text{ for } j = p+1, ..., k, \ \forall \tau \in \mathcal{T}, \end{aligned} \tag{E.1}$$

where $\theta_*$ denotes the null parameter value and $F_0$ denotes the fixed true distribution of the data. The alternative is that $H_0$ does not hold. Assumption MFA of ASM specifies the properties of fixed alternatives. For convenience, we restate this assumption here. Recall that $\mathcal{X}_F(\theta, \tau)$, defined in (6.2), is the set of points $x \in R^{d_x}$ such that under $F$ there is a violation of some conditional moment inequality or equality, evaluated at $(\theta, \tau)$, conditional on $X_i = x$.

**Assumption MFA.** The value $\theta_* \in \Theta$ and the true distribution $F_0$ satisfy: (a) for some $\tau_* \in \mathcal{T}$, $P_{F_0}(X_i \in \mathcal{X}_{F_0}(\theta_*, \tau_*)) > 0$ and (b) $(\theta_*, F_0) \in \mathcal{F}_+$.

The following assumption requires the measure $Q$ on $\mathcal{G}$ to have full support. For each $(\theta, F, \tau) \in \mathcal{F}_+ \times \mathcal{T}$, define a pseudometric on $\mathcal{G}$: $d_{(\theta,,F,\tau)}(g, g^\dagger) = \|E_F[m(W_i, \theta, \tau)(g(X_i) - g^\dagger(X_i))]\|$ for $g, g^\dagger \in \mathcal{G}$. Let $\mathcal{B}_{d_{(\theta,F,\tau)}}(g_0, \delta) = \{g \in \mathcal{G} : d_{(\theta,F,\tau)}(g, g_0) \leq \delta\}$.

**Assumption MQ.** The support of $Q$ under $d_{(\theta,F,\tau)}$ is $\mathcal{G}$ for all $(\theta, F, \tau) \in \mathcal{F}_+ \times \mathcal{T}$. That is, $\forall (\theta, F, \tau) \in \mathcal{F}_+ \times \mathcal{T}$, $\forall \delta > 0$, and $\forall g_0 \in \mathcal{G}$, $Q(B_{d(\theta,F,\tau)}(g_0, \delta)) > 0$.

The following theorem shows that the GMS and the PA tests are consistent against all fixed alternatives defined in Assumption MFA.

**Theorem E.1** *Suppose Assumptions* PS4, PS5, MFA, CI, MQ, S1, S3, S4, *and* SIG2 *hold. Then,*

(a) $\lim_{n\to\infty} P_{F_0}(T_n(\theta_*) > c^*(\varphi_n(\theta_*), \widehat{h}^*_{2,n}(\theta_*), 1-\alpha) = 1,$ *and*

(b) $\lim_{n\to\infty} P_{F_0}(T_n(\theta_*) > c^*(0^k_{\mathcal{T}\times\mathcal{G}}, \widehat{h}^*_{2,n}(\theta_*), 1-\alpha) = 1.$

**Comments. 1.** Theorem 6.1 of ASM for the case $r_{1,n} = \infty$ is proved by verifying that the conditions of Theorem E.1 (except Assumption MFA) hold for the $\mathcal{G}_{\text{c-cube}}$ set, the $S$ functions, and the measure $Q_{AR}$ defined as in ASM. (See Section B.5 for the definition of $Q_{AR}$ with weight function $w(r) := (r^2 + 100)^{-1}$.) Assumption CI holds for $\mathcal{G}_{\text{c-cube}}$ defined in (3.6) of ASM by Lemma 3 of AS1. Assumption MQ holds for $\mathcal{G}_{\text{c-cube}}$ and $Q_{AR}$ because $\mathcal{G}_{\text{c-cube}}$ is countable and $Q_{AR}$ has a probability mass function that is positive at each element in $\mathcal{G}_{\text{c-cube}}$. Assumptions S1-S4 hold for the functions $S_1$, $S_2$, and $S_3$ defined in (3.9) of ASM by Lemma 1 of AS1, and for $S_4$ in (3.9) by similar arguments. Assumptions PS4 and PS5 hold by Lemma D.2 provided Assumption M holds. Assumption M holds for $\mathcal{G}_{\text{c-cube}}$ by Lemma 3 of AS1. (Note that Assumption M with $F_0$ in place of $F_n$ in part (b) holds because $\mathcal{C}_{\text{c-cube}}$ is a Vapnik-Cervonenkis class of sets.)

**2.** Theorem 6.1 of ASM holds for $r_{1,n}$ such that $r_{1,n} < \infty$ and $r_{1,n} \to \infty$ as $n \to \infty$ by making some alterations to the proof of Theorem E.1. The alterations required are the same as those described for A-CvM tests in the proof of Theorem B2 in Appendix D of AS2.[17]

**Proof of Theorem E.1.** Part (a) is implied by part (b). Therefore, it suffices to show part (b) only.

Let

$$A(\theta_*) := \sup_{\tau \in \mathcal{T}} \int_{\mathcal{G}} S(E_{F_0}[m(W_i, \theta_*, \tau) g(X_i)], \overline{\Sigma}_{F_0}(\theta_*, \tau, g) dQ(g). \tag{E.2}$$

First, we show that

$$|n^{-\chi/2} T_n(\theta_*) - A(\theta_*)| \to_p 0. \tag{E.3}$$

For any $\delta > 0$,

$$\begin{aligned}
& P_{F_0}(|n^{-\chi/2} T_n(\theta_*) - A(\theta_*)| > \delta) \\
\le\ & P_{F_0}\left( \sup_{\substack{\mu \in [0,\infty) \\ (\tau,g) \in \mathcal{T} \times \mathcal{G}}} \left| S(n^{-1/2} \nu_{n,F_0}(\theta_*, \tau, g) + \mu, \widehat{h}^{\varepsilon}_{2,n,F_0}(\theta_*, \tau, g)) - S(\mu, h^{\varepsilon}_{2,F_0}(\theta_*, \tau, g)) \right| > \delta \right) \\
\le\ & P_{F_0}\left( \sup_{(\tau,g) \in \mathcal{T} \times \mathcal{G}} ||n^{-1/2} \nu_{n,F_0}(\theta_*, \tau, g)|| + \sup_{(\tau,g) \in \mathcal{T} \times \mathcal{G}} ||\widehat{h}^{\varepsilon}_{2,n,F_0}(\theta_*, \tau, g) - h^{\varepsilon}_{2,F_0}(\theta_*, \tau, g)|| > \xi_\delta \right) \\
\to\ & 0 \text{ as } n \to \infty,
\end{aligned} \tag{E.4}$$

where $\widehat{h}^{\varepsilon}_{2,n,F_0}(\theta, \tau, g) := \widehat{h}_{2,n,F_0}(\theta, \tau, g) + \varepsilon D^{-1/2}_{F_0}(\theta_n) \widehat{D}_0(\theta_n) D^{-1/2}_{F_0}(\theta_n)$, $h^{\varepsilon}_{2,F_0}(\theta_*, \tau, g) := h_{2,F_0}(\theta_*, \tau, g) + \varepsilon I_k$, the first inequality uses Assumption S4, and the second inequality holds for some $\xi_\delta > 0$ by Assumptions PS4, S2 and SIG2. This establishes (E.3).

Next we show that $A(\theta_*) > 0$. By Assumption MFA, there exists a $\tau_* \in \mathcal{T}$ and either a $j_0 \le p$ such that $P_{F_0}(E_{F_0}[m_{j_0}(W_i, \theta_*, \tau_*)|X_i] < 0) > 0$ or a $j_0 > p$ such that

[17] The proof of Theorem B2 describes alterations to the proof of Theorem 3 of AS1, which is given in Appendix C of AS2, to accommodate A-CvM tests based on truncation, simulation, or quasi-Monte Carlo computation and KS tests. Theorem 3 of AS1 establishes that the tests in AS1 have asymptotic power equal to one for fixed alternative distributions.

$P_{F_0}(E_{F_0}[m_{j_0}(W_i, \theta_*, \tau_*)|X_i] \neq 0) > 0$. Without loss of generality, assume that $j_0 \leq p$. By Assumption CI, there is a $g_* \in \mathcal{G}$ such that $E_{F_0}[m_{j_0}(W_i, \theta_*, \tau_*) g_{*j_0}(X_i)] < 0$, where $g_{*j_0}(X_i)$ denotes the $j_0$th element of $g_*(X_i)$.

Because $E_{F_0}[m_{j_0}(W_i, \theta_*, \tau_*) g_{*j_0}(X_i)]$ is continuous in $g$ with respect to the pseudometric $d_{(\theta_*, F_0, \tau_*)}$, there exists a $\delta > 0$ such that $\forall g \in \mathcal{B}_{d(\theta_*, F_0, \tau_*)}(g_*, \delta)$, $E_{F_0}[m_{j_0}(W_i, \theta_*, \tau_*) g_{j_0}(X_i)]$ has the same sign as $E_{F_0}[m_{j_0}(W_i, \theta_*, \tau_*) g_{*j_0}(X_i)]$, i.e., $E_{F_0}[m_{j_0}(W_i, \theta_*, \tau_*) g_{j_0}(X_i)] < 0$, $\forall g \in \mathcal{B}_{d(\theta_*, F_0, \tau_*)}(g_*, \delta)$. By Assumption MQ, $Q(\mathcal{B}_{d(\theta_*, F_0, \tau_*)}(g_*, \delta)) > 0$. Therefore,

$$A(\theta_*) \geq \int_{\mathcal{B}_{d(\theta_*, F_0, \tau_*)}(g_*, \delta)} S(E_{F_0}[m(W_i, \theta_*, \tau_*) g(X_i)], \overline{\Sigma}_{F_0}(\theta_*, \tau_*, g) dQ(g) > 0, \tag{E.5}$$

where the second inequality holds by Assumption S3 and $Q(\mathcal{B}_{d(\theta_*, F_0, \tau_*)}(g_*, \delta)) > 0$.

Analogous arguments to those used to establish (14.34) of AS2 show

$$c^*(0^k_{\mathcal{T} \times \mathcal{G}}, \widehat{h}^*_{2,n}(\theta_*), 1 - \alpha) = O_p(1). \tag{E.6}$$

Equations (E.3), (E.5), and (E.6) give

$$\begin{aligned}
& P_{F_0}(T_n(\theta_*) > c^*(0^k_{\mathcal{T} \times \mathcal{G}}, \widehat{h}^*_{2,n}(\theta_*), 1 - \alpha)) \\
= \; & P_{F_0}(n^{-\chi/2} T_n(\theta_*) > n^{-\chi/2} c^*(0^k_{\mathcal{T} \times \mathcal{G}}, \widehat{h}^*_{2,n}(\theta_*), 1 - \alpha)) \\
= \; & P_{F_0}(A(\theta_*) + o_p(1) > o_p(1)) \\
\to \; & 1 \text{ as } n \to \infty,
\end{aligned} \tag{E.7}$$

which establishes part (b). $\square$

# F Proofs of Results Concerning the Examples

**Proof of Lemma 7.1.** Assumption PS1(a) holds because $\Theta = \{0\}$. Assumptions PS1(b) holds by the condition given in the lemma. Assumption PS1(c) holds because $\sigma^2_{F,1}(0) = 1$. Assumption PS1(d) holds because $|1\{Y_2 \leq \tau\} - 1\{Y_1 \leq \tau\}| \leq 1$. Assumption PS1(e) holds because

$$E_F M^{2+\delta}(W) = 1/\sigma_F^{2+\delta}(0) = 1. \tag{F.1}$$

Next, we verify Assumption PS2. For $j = 1, 2$, consider the set $\mathcal{M}_{n,j,y_j} := \{(-1\{y_{j,i} \leq$

$\tau\})_{i=1}^n \in R^n : \tau \in \mathcal{T}\}$ for an arbitrary realization $\{y_{j,i} : i \leq n\}$ of the random vector $\{Y_{j,i} : i \leq n\}$. The set has pseudodimension (defined on p. 15 of Pollard (1990)) at most one by Lemma 4.4 of Pollard (1990). Then, by Corollary 4.10 of Pollard (1990), there exist constants $c_1 \geq 1$ and $c_2 > 0$ (not depending on $j$, $n$, $\varepsilon$, or $\{y_{j,i} : i \leq n\}$) such that

$$D(\varepsilon||\alpha||, \alpha \odot \mathcal{M}_{n,j,y_j}) \leq c_1 \varepsilon^{-c_2} \tag{F.2}$$

for $\varepsilon \in (0, 1]$, every rescaling vector $\alpha \in R^n_+$, and $j = 1, 2$. In consequence, by the stability of the $L_2$ packing numbers (see Pollard (1990, p. 22)), we have

$$\begin{aligned} &D(2\varepsilon||\alpha||, (\alpha \odot \mathcal{M}_{n,1,y_1}) \oplus (\alpha \odot \mathcal{M}_{n,2,y_2})) \\ &\leq D(\varepsilon||\alpha||, \alpha \odot \mathcal{M}_{n,1,y_1}) D(\varepsilon||\alpha||, \alpha \odot \mathcal{M}_{n,2,y_2}) \\ &\leq c_1^2 \varepsilon^{-2c_2}, \end{aligned} \tag{F.3}$$

where $A \oplus B = \{a + b : a \in A, b \in B\}$ for any two sets $A, B \subset R^n$.

Now consider the set $\mathcal{M}_{n,y_1,y_2} := \{(1\{y_{2,i} \leq \tau\} - 1\{y_{1,i} \leq \tau\})_{i=1}^n \in R^n : \tau \in \mathcal{T}\}$. By definition, $\alpha \odot \mathcal{M}_{n,y_1,y_2} \subset (\alpha \odot \mathcal{M}_{n,1,y_1}) \oplus (\alpha \odot \mathcal{M}_{n,2,y_2})$. Thus,

$$D(2\varepsilon||\alpha||, \alpha \odot \mathcal{M}_{n,y_1,y_2}) \leq c_1^2 \varepsilon^{-2c_2}. \tag{F.4}$$

Lastly, because $c_1$ and $c_2$ do not depend on $n$ or $\{(Y_{1i}, Y_{2i}) : i \leq n\}$, the manageability of $\{1\{Y_{2,i} \leq \tau\} - 1\{Y_{1,i} \leq \tau\} : \tau \in \mathcal{T}, i \leq n, n \geq 1\}$ holds by the following calculations:

$$\int_0^1 \sqrt{\log(c_1^2 \varepsilon^{-2c_2})} d\varepsilon = \int_{\sqrt{\log(A)}}^{\infty} (2A^{1/W}/W) x^2 e^{-x^2/W} dx < \infty, \tag{F.5}$$

where $A := c_1^2$, $W := 2c_2$, $\log(A) \geq 0$ because $c_1 \geq 1$, and the equality holds by change of variables with $x = \sqrt{\log(A\varepsilon^{-W})}$ or, equivalently, $\varepsilon = A^{1/W} e^{-x^2/W}$, which yields $d\varepsilon = (2A^{1/W}/W) x e^{-x^2/W} dx$. This completes the proof. $\square$

**Proof of Lemma 7.2.** We prove part (a) first. Assumption PS1(a) holds because $\Theta = \{0\}$. Assumptions PS1(b) and PS1(c) hold by conditions (i) and (ii) of the lemma, respectively.

Assumption PS1(d) holds because

$$\begin{aligned}&|(\tau - Y_2)^{s-1}1\{Y_2 \le \tau\} - (\tau - Y_1)^{s-1})1\{Y_1 \le \tau\}|\\ &\le (\tau - Y_2)^{s-1}1\{Y_2 \le \tau\} + (\tau - Y_1)^{s-1})1\{Y_1 \le \tau\}\\ &\le (B - Y_2)^{s-1} + (B - Y_1)^{s-1}. \end{aligned} \tag{F.6}$$

Assumption PS1(e) holds because

$$M(W) \le 2(B - (-B))^{s-1}/\sigma_{F,1}(0) \le 2^s B^{s-1}/\underline{\sigma}. \tag{F.7}$$

Next, we verify Assumption PS2. Consider the set $\mathcal{M}_{n,1,y_1} := \{(-(\tau - y_{1,i})^{s-1}1\{y_{1,i} \le \tau\})_{i=1}^n \in R^n : \tau \in \mathcal{T}\}$ for an arbitrary realization $\{y_{1,i} : i \le n\}$ of the random vector $\{Y_{1,i} : i \le n\}$. First, we show that this set has pseudodimension (as defined in Pollard (1990, p. 15)) at most one. Suppose not. Then, there exists a vector $x = (x_1, x_2)' \in R^2$ and a pair $(i, i')$ such that $\{(-(\tau - y_{1,i})^{s-1}1\{y_{1,i} \le \tau\}, (-(\tau - y_{1,i'})^{s-1}1\{y_{1,i'} \le \tau\}) : \tau \in \mathcal{T}\}$ surrounds $x$.[18] Thus, there exists $\tau_1, \tau_2 \in \mathcal{T}$ such that

$$\begin{aligned}(\tau_1 - y_{1,i})^{s-1}1\{y_{1,i} \le \tau_1\} &> x_1,\\ (\tau_1 - y_{1,i'})^{s-1}1\{y_{1,i'} \le \tau_1\} &< x_2,\\ (\tau_2 - y_{1,i})^{s-1}1\{y_{1,i} \le \tau_2\} &< x_1, \text{ and}\\ (\tau_2 - y_{1,i'})^{s-1}1\{y_{1,i'} \le \tau_2\} &> x_2.\end{aligned} \tag{F.8}$$

This yields

$$\begin{aligned}(\tau_1 - y_{1,i})^{s-1}1\{y_{1,i} \le \tau_1\} &> (\tau_2 - y_{1,i})^{s-1}1\{y_{1,i} \le \tau_2\} \text{ and}\\ (\tau_1 - y_{1,i'})^{s-1}1\{y_{1,i'} \le \tau_1\} &< (\tau_2 - y_{1,i'})^{s-1}1\{y_{1,i'} \le \tau_2\}.\end{aligned} \tag{F.9}$$

Due to the monotonicity of the function $G_s(y, \tau) := (\tau - y)^{s-1}1\{y \le \tau\}$ in $\tau$ for any $y$, the first inequality in the equation above implies that $\tau_1 > \tau_2$, and the second inequality implies that $\tau_1 < \tau_2$, which is a contradiction. Therefore, $\mathcal{M}_{n,1,y_1}$ has pseudodimension at most one.

[18] As defined in Pollard (1990, p. 15), a set $A \subset R^2$ *surrounds* $x$ if there exists points $a, b, c, d \in A$, where $a = (a_1, a_2)'$ etc., such that $a_1 > x_1$, $a_2 > x_2$, $b_1 > x_1$, $b_2 < x_2$, $c_1 < x_1$, $c_2 > x_2$, $d_1 < x_1$, and $d_2 < x_2$.

The remainder of the proof of part (a) is the same as the corresponding part of the proof of Lemma 7.1 and, hence, for brevity, is omitted.

To prove part (b), consider an arbitrary sequence $\{F_n : n \geq 1\}$ such that $(0, F_n) \in \mathcal{F}_+$ for all $n$. Under this sequence, we have for any $\zeta > 0$ and $j = 1, 2$,

$$\begin{aligned}
\Pr_{F_n}(|\overline{Y}_{j,n} - E_{F_n}(Y_j)| > \zeta) &\leq \frac{E_{F_n}(\overline{Y}_{j,n} - E_{F_n}(Y_j))^2}{\zeta^2} \\
&= \frac{E_{F_n}(Y_j - E_{F_n}(Y_j))^2}{n\zeta^2} \\
&\leq \frac{E_{F_n}(Y_j^2)}{n\zeta} \\
&\leq B^2/(n\zeta) \to 0 \text{ as } n \to \infty, \qquad \text{(F.10)}
\end{aligned}$$

where the last inequality holds because the support of $Y_j$ is contained in $\mathcal{T}$ and $\mathcal{T}$ is contained in $[-B, B]$ by condition (iii) of the lemma. Similarly, we have under the sequence $\{F_n : n \geq 1\}$,

$$n^{-1} \sum_{i=1}^{n} (Y_{j,i} - E_{F_n}(Y_j))^{2(s-1)} - E_{F_n}(Y_j - E_{F_n}(Y_j))^{2(s-1)} \to_p 0, \qquad \text{(F.11)}$$

for $j = 1, 2$. Therefore, we have

$$\begin{aligned}
& n^{-1} \sum_{i=1}^{n} (Y_{j,i} - \overline{Y}_{j,n})^{2(s-1)} - E_{F_n}(Y_j - E_{F_n}(Y_j))^{2(s-1)} \\
&= n^{-1} \sum_{i=1}^{n} (Y_{j,i} - E_{F_n}(Y_j))^{2(s-1)} - E_{F_n}(Y_j - E_{F_n}(Y_j))^{2(s-1)} \\
&\quad + \sum_{b=0}^{2(s-1)-1} \binom{2(s-1)}{b} \left[ n^{-1} \sum_{i=1}^{n} (Y_{j,i} - E_{F_n}(Y_j))^b \right] (E_{F_n}(Y_j) - \overline{Y}_{j,n})^{2s-2-b}, \\
&= o_p(1) \\
&\quad + \sum_{b=0}^{2(s-1)-1} \binom{2(s-1)}{b} \left[ n^{-1} \sum_{i=1}^{n} (Y_{j,i} - E_{F_n}(Y_j))^b \right] (E_{F_n}(Y_j) - \overline{Y}_{j,n})^{2s-2-b}, \\
&= o_p(1), \qquad \text{(F.12)}
\end{aligned}$$

where the second equality holds by (F.11), and the last equality holds by (F.10) and the boundedness of $Y_j$.

Therefore,

$$
\begin{aligned}
|\widehat{\sigma}_{n,1}^2(0) - \sigma_{F_n,1}^2(0)|/\sigma_{F_n,1}^2(0) &\le \underline{\sigma}^{-2}|\widehat{\sigma}_{n,1}^2(0) - \sigma_{F_n,1}^2(0)| \\
&\le \underline{\sigma}^{-2}\sum_{j=1}^{2}\left| n^{-1}\sum_{i=1}^{n}(Y_{j,i} - \overline{Y}_{j,n})^{2(s-1)} - E_{F_n}(Y_j - E_{F_n}(Y_j))^{2(s-1)}\right| \\
&\to_p 0, \text{ as } n \to \infty. \qquad \text{(F.13)}
\end{aligned}
$$

Because this holds for an arbitrary sequence $\{F_n : n \ge 1\}$ such that $(0, F_n) \in \mathcal{F}_+$, it establishes both Assumptions SIG1 and SIG2. Thus, part (b) holds. $\square$

**Proof of Lemma 8.1.** The lemma follows from the polar coordinate representation of half-spaces (with boundary hyperplanes going through the origin). $\square$

**Proof of Lemma 8.3.** Assumption PS1(a) holds because $\theta \in \Theta$. Assumptions PS1(b) holds by condition (ii) of the lemma. Assumption PS1(c) holds by $\sigma_{F,1}^2(\theta) = 1$. Assumption PS1(d) holds because $|F_\beta(\mathcal{S}(\tau), \theta) - 1\{A \in [\tau_1, \tau_2]\}| \le 1$. Assumption PS1(e) holds because

$$E_F M^{2+\delta}(W) = 1/\sigma_{F,1}^{2+\delta}(0) = 1. \qquad \text{(F.14)}$$

Next, we verify Assumption PS2. Consider the set $\mathcal{M}_{n,\alpha_1} := \{(F_\beta(\mathcal{S}(\tau), \theta_n) - 1\{a_i \in [\tau_1, \tau_2]\})_{i=1}^n \in R^n : \tau_1 \le \tau_2,\ \tau_1, \tau_2 \in [0, 2\pi)\}$ for an arbitrary realization $\{a_i : i \le n\}$ of $\{A_i : i \le n\}$. The set has pseudodimension at most two by Lemma 4.4 of Pollard (1990). Then, by Corollary 4.10 of Pollard (1990), there exist constants $c_1 \ge 1$ and $c_2 > 0$ (not depending on $n$, $\{a_i : i \le n\}$, or $\varepsilon$) such that

$$D(\varepsilon\|\alpha\|, \alpha \odot \mathcal{M}_{n,\alpha_1}) \le c_1\varepsilon^{-c_2} \qquad \text{(F.15)}$$

for all $0 < \varepsilon \le 1$ and every rescaling vector $\alpha \in R_+^n$. In consequence, the manageability of $\{F_\beta(\mathcal{S}(\tau), \theta_n) - 1\{A_i \in [\tau_1, \tau_2]\} : \tau_1, \tau_2 \in [0, 2\pi), \tau_1 \le \tau_2, i \le n, n \ge 1\}$ follows from the calculations in (F.5) with $A := c_1$ and $W := c_2$, which completes the proof. $\square$

**Proof of Lemma 8.4.** We show that parts (a) and (b) are equivalent by solving a linear programming problem. We show that parts (b) and (c) are equivalent by employing the convex polyhedral cone representation of linear inequalities developed in Gale (1951).

First, we show the equivalence between parts (a) and (b).

For a set $A \subset R^{d_\beta}$, let $A^c$ denote the complement of $A$ in $R^{d_\beta}$. By basic set operations, the statement in part (a) is equivalent to

$$\cap_{j=1}^{m} H(c_j)^c \subset H(\overline{c})^c. \tag{F.16}$$

Because $m$ is finite and $H(c)^c$ is an open set for any $c \in R^{d_\beta} \backslash \{0\}$, (F.16) is equivalent to

$$\cap_{j=1}^{m} cl(H(c_j)^c) \subset cl(H(\overline{c})^c). \tag{F.17}$$

Note that $cl(H(c)^c) = \{b \in R^{d_\beta} : b'c \leq 0\}$ and (F.17) is equivalent to the redundancy of the inequality restriction $b'\overline{c} \leq 0$ on $b$ relative to the system of linear inequalities $b'c_j \leq 0$ for $j = 1, ..., m$. In turn, the latter is equivalent to the statement that $V = 0$, where

$$V := \max_{b \in R^{d_\beta}} b'\overline{c} \text{ subject to } b'c_j \leq 0 \text{ for } j = 1, ..., m \text{ and } b'\overline{c} \leq 1. \tag{F.18}$$

Now we solve the linear programming problem in (F.18) using the Lagrange multiplier method. It is well known that

$$V = \min_{\lambda_j \geq 0 : j = 1, ..., m+1} \max_{b \in R^{d_\beta}} \left( b'\overline{c} - \sum_{j=1}^{m} \lambda_j b'c_j - \lambda_{m+1}(b'\overline{c} - 1) \right). \tag{F.19}$$

Because the maximization over $b$ is unconstrained, for any $\lambda_1, ..., \lambda_{m+1} \geq 0$ such that $(1 - \lambda_{m+1})\overline{c} - \sum_{j=1}^{m} \lambda_j c_j \neq 0$, the maximum is infinite. But, $V \leq 1$ by the inequality $b'\overline{c} \leq 1$ in (F.18). Thus, the optimal $\lambda_1, ..., \lambda_{m+1}$ must satisfy

$$(1 - \lambda_{m+1})\overline{c} - \sum_{j=1}^{m} \lambda_j c_j = 0. \tag{F.20}$$

This implies that

$$V = \min_{\lambda_j \geq 0 : j = 1, ..., m+1} \lambda_{m+1} \text{ subject to } (1 - \lambda_{m+1})\overline{c} - \sum_{j=1}^{m} \lambda_j c_j = 0. \tag{F.21}$$

Now we show that $V = 0$ iff there exist $\lambda_1, ..., \lambda_m \geq 0$ such that

$$\overline{c} = \sum_{j=1}^{m} \lambda_j c_j, \tag{F.22}$$

which establishes the equivalence between parts (a) and (b). Suppose that there exists $\lambda_1, ..., \lambda_m \geq 0$ such that (F.22) holds, then $V \leq \min\{\lambda_{m+1} \geq 0 : \lambda_{m+1}\overline{c} = 0^{d_\beta}\} = 0$ by (F.21). However, by (F.18) (with $b = 0^{d_\beta}$), $V \geq 0$. Thus, $V = 0$. Conversely, suppose that $V = 0$, then there exists $\lambda_1, ..., \lambda_m \geq 0$ such that $(1-0)\overline{c} - \sum_{j=1}^{m} \lambda_j c_j = 0^{d_\beta}$ by (F.21), which implies (F.22).

Next, we establish the equivalence between parts (b) and (c). Using the terminology in Gale (1951), let $P(c_1, ..., c_m)$ denote the convex polyhedral cone generated by the vectors $(c_1, ..., c_m)$. That is,

$$P(c_1, ..., c_m) := \left\{ c \in R^{d_\beta} : c = \sum_{j=1}^{m} \lambda_j c_j \text{ for some } \lambda_1, ..., \lambda_m \geq 0 \right\}.$$

Then, part (b) is equivalent to $\overline{c} \in P(c_1, ..., c_m)$.

If $rk([c_1, ..., c_m]) = d_\beta$, then by Weyl's Theorem (see Theorem 1 of Gale (1951)), $P(c_1, ..., c_m)$ is an intersection of at most $\binom{m}{d_\beta - 1}$ half-spaces or, in other words, there exist $b^1, ..., b^N \in R^{d_\beta}$, where $N := \binom{m}{d_\beta - 1}$, such that $P(c_1, ..., c_m) = \{c : [b^1, \cdots, b^N]'c \geq 0\}$. Then, the equivalence between parts (b) and (c) is established with $B(c_1, ..., c_m) := [b^1, \cdots, b^N, 0^{d_\beta \times (M-N)}]$ for an arbitrary $[b^1, \cdots, b^N]$ that satisfies $P(c_1, ..., c_m) = \{c : [b^1, \cdots, b^N]'c \geq 0\}$.

If $rk([c_1, ..., c_m]) < d_\beta$, let $L(c_1, ..., c_m)$ be the linear subspace spanned by $c_1, ..., c_m$. Let the dimension of this linear subspace be $d_L$. Applying Weyl's Theorem on $L(c_1, ..., c_m)$, we have that there exist $b^1, ..., b^{N_1} \in R^{d_\beta}$, where $N_1 := \binom{m}{d_L - 1}$ , such that $P(c_1, ..., c_m) = \{c : [b^1, \cdots, b^{N_1}]'c \geq 0\} \cap L(c_1, ..., c_m)$. Moreover, by the property of linear subspaces, there exist $b^{N_1+1}, ..., b^{N_2} \in R^{d_\beta}$, where $N_2 := N_1 + d_\beta - d_L$, such that $L(c_1, ..., c_m) = \{c : [b^{N_1+1}, \cdots, b^{N_2}]'c = 0\}$. Therefore,

$$P(c_1, ..., c_m) = \{c : [b^1, \cdots, b^{N_2}, -b^{N_1+1}, \cdots, -b^{N_2}]'c \geq 0\}. \tag{F.23}$$

Then, the equivalence between parts (b) and (c) holds with $B(c_1, ..., c_m) := [b^1, \cdots, b^{N_2},$

$-b^{N_1+1}, \cdots, -b^{N_2}, 0^{d_\beta \times (M-2N_2+N_1)}]$ for arbitrary $b^1, ..., b^{N_2}$ that satisfy (F.23). $\square$

The proof of Lemma 8.5 is omitted because it is very similar to the proof of Lemma 8.3.

**Proof of Lemma 9.1** Assumptions PS1(a)-(c) and (e) hold by assumption. Next, we show Assumption PS1(d) holds. We have

$$|h(Q_\theta(w,v),u)| = |\sup_{q \in Q_\theta(w,v)} q'u| \leq \sup_{q \in Q_\theta(w,v)} \|q\|\|u\| \leq \sup_{q \in Q_\theta(w,v)} \|q\| \leq M(w)/2, \tag{F.24}$$

where the first inequality holds by the Cauchy-Schwarz inequality, the second inequality holds because $u$ satisfies $\|u\| \leq 1$, and the last inequality holds by condition (iii) of the lemma. Assumption PS1(d) follows from the following calculations:

$$\begin{aligned}
\left| \int h(Q_\theta(W,v),u) dF_{V|X}(v,X;\theta) - u'q(X) \right| &\leq \int |h(Q_\theta(W,v),u)| dF_{V|X}(v,X;\theta) + |u'q(X)| \\
&\leq M(W)/2 + |u'q(X)| \\
&\leq M(W)/2 + \|u\|\|q(X)\| \\
&\leq M(W), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (F.25)
\end{aligned}$$

where the second inequality holds by (F.24), the third inequality holds by the Cauchy-Schwarz inequality and the last inequality holds by $\|u\| \leq 1$ and condition (iv) of the lemma.

Now, we show that Assumption PS2 holds. Let $m(W,\theta,u) := \int h(Q_\theta(W,v),u) dF_{V|X}(v,X;\theta) - u'q(X)$. Consider an arbitrary sequence $(\theta_n, F_n)$ that satisfy the conditions in the lemma. Arguments similar to those for Assumption PS1(d) above show that $m(W,\theta,u)$ is Lipschitz continuous in $u$ with Lipschitz constant $M(W)$ for all $\theta$. Given the Lipschitz continuity, for any nonnegative $n$-vector $\alpha := (\alpha_1, \ldots, \alpha_n)$, any $u_1 \in R^d$, $u_2 \in R^d$ such that $\|u_1\| \leq 1$ and $\|u_2\| \leq 1$, and any $n$ realizations $(w_1, \ldots, w_n)$ of $W$ (under $F_n$), we have

$$\sum_{i=1}^n (\alpha_i m(w_i,\theta_n,u_1) - \alpha_i m(w_i,\theta_n,u_2))^2 \leq \left( \sum_{i=1}^n (\alpha_i M(w_i))^2 \right) \|u_1 - u_2\|^2. \tag{F.26}$$

Let $\mathcal{F}_{n(w_1,\ldots,w_n)} = \{(m(w_i,\theta_n,u)_{i=1}^n : u \in R^d, \|u\| \leq 1\}$ and let $\vec{M}_n(w_1,\ldots,w_n) = (M(w_1), \ldots,$

$M(w_n))'$. Then, (F.26) implies that, for all $\xi \in (0,1]$,

$$D(\xi\|\alpha \odot \vec{M}_n(w_1,\ldots,w_n)\|, \alpha \odot \mathcal{F}_{n(w_1,\ldots,w_n)}) \le D(\xi, \{u \in R^d : \|u\| \le 1\}) \le C/\xi^d, \quad \text{(F.27)}$$

for some constant $C < \infty$. Assumption PS2 holds because $\int_0^1 \sqrt{\log C - d\log\xi}\,d\xi = 2C^{1/d}d^{-1}$ $\int_0^\infty x^2 e^{-x^2/d}dx < \infty$. $\square$

**Proof of Lemma 9.2.** We prove part (a) first. Assumptions PS1(a)-(e) hold by assumption. Now we verify Assumption PS2.

Let $(\theta_n, F_n)$ be an arbitrary sequence that satisfies all the conditions of the lemma. Consider $n$ arbitrary realizations $(w_1,\ldots,w_n)$ of $W$ under $F_n$ for arbitrary $n \ge 1$. By (9.2) and conditions (iii) and (iv) of the lemma, $\{M(w_i) : i \le n, n \ge 1\}$ are envelopes for the triangular array of processes $\{m(w_i,\theta_n,\tau) : i \le n, n \ge 1, \tau = 1,2,\ldots\}$. Let

$$\mathcal{F}_{n(w_1,\ldots,w_n)} = \{(m(w_1,\theta_n,\tau),\ldots,m(w_n,\theta_n,\tau))' : \tau = 1,2,\ldots\}. \quad \text{(F.28)}$$

Let $\vec{M}_n(w_1,\ldots,w_n) = (M(w_1),\ldots,M(w_n))'$. Then, for any $\xi \in (0,1]$ and any nonnegative $n$-vector $\alpha$, $D(\xi\|\alpha\odot\vec{M}_n(w_1,\ldots,w_n)\|, \alpha\odot\mathcal{F}_{n(w_1,\ldots,w_n)}) \le \lambda_{\mathcal{T}}(\xi)$ because $\alpha\odot(m(w_1,\theta_n,\tau),\ldots,$ $m(w_n,\theta_n,\tau))'$ belongs to the $\xi\|\alpha\odot\vec{M}_n(w_1,\ldots,w_n)\|$-neighborhood of $0^n$ for all $\tau \ge \lambda_{\mathcal{T}}(\xi)$. The latter holds because, for all $\tau \ge \lambda_{\mathcal{T}}(\xi)$,

$$\begin{aligned}\sum_{i=1}^n (\alpha_i m(w_i,\theta,\tau))^2 &= w_{\mathcal{T}}^2(\tau)\sum_{i=1}^n(\alpha_i\widetilde{m}(w_i,\theta,\tau))^2 \\ &\le w_{\mathcal{T}}^2(\tau)\sum_{i=1}^n(\alpha_i M(w_i))^2 \\ &\le \xi^2\|\alpha\odot\vec{M}_n(w_1,\ldots,w_n)\|^2, \end{aligned} \quad \text{(F.29)}$$

where the last inequality holds because $\tau \ge \lambda_{\mathcal{T}}(\xi)$ iff $w_{\mathcal{T}}(\tau) \le \xi$ (because $\lambda_{\mathcal{T}}(\xi)$ is the inverse function of the decreasing function $w_{\mathcal{T}}(\xi)$). By assumption, $\int_0^1 \sqrt{\log(\lambda_{\mathcal{T}}(\xi))}d\xi < \infty$. Hence, Assumption PS2 holds.

The proof of part (b) is similar to that of Lemma 7.2(b) and is omitted for brevity. $\square$

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