ON THE CHOICE OF TEST STATISTIC FOR CONDITIONAL MOMENT INEQUALITES

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On the Choice of Test Statistic for Conditional Moment Inequalities

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Abstract

This paper derives asymptotic power functions for Cramer-von Mises (CvM) style tests for inference on a finite dimensional parameter defined by conditional moment inequalities in the case where the parameter is set identified. Combined with power results for Kolmogorov-Smirnov (KS) tests, these results can be used to choose the optimal test statistic, weighting function and, for tests based on kernel estimates, kernel bandwidth. The results show that KS tests are preferred to CvM tests, and that a truncated variance weighting is preferred to bounded weightings under a minimax criterion, and for a class of alternatives that arises naturally in these models. The results also provide insight into how moment selection and the choice of instruments affect power. Such considerations have a large effect on power for instrument based approaches when a CvM statistic or an unweighted KS statistic is used and relatively little effect on power with optimally weighted KS tests.

1 Introduction

This paper derives power functions for tests for conditional moment inequality models. The results show that, in a broad class of models, Kolmogorov-Smirnov (KS) style statistics, which take the infimum of an objective function, are more powerful than Cramer-von Mises (CvM) style statistics, which integrate or add some function of the negative part of an

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objective function, for detecting local alternatives under conditions that determine the minimax rate and arise naturally in set identified models. Thus, the results also show that KS statistics are preferred to CvM statistics under a minimax criterion in these models.

Combined with results from Armstrong (2015) and Armstrong (2014b), the results in this paper give clear prescriptions for the choice of test statistic in conditional moment inequality models in the set identified case, and provide insights into the choice of critical value as well. To the author's knowledge, this paper is the first to provide a theoretical justification for the choice of test statistic (CvM vs KS) based on power results, and for user defined procedures such as moment selection procedures and bandwidths for CvM statistics in this setting.

The main points can be summarized as follows. In this setting, KS statistics are preferred to CvM statistics in terms of asymptotic power, and a truncated variance weighting for the objective function like the one proposed in Armstrong (2014b) is preferred to bounded weighting functions. The power comparisons are for local alternatives that determine the minimax rate, and can be argued to arise generically in set identified models (see Section A of the appendix). If one prefers CvM statistics for other reasons, but wants them to perform well in the generic set identified case considered here, the results in this paper can be used to choose optimal weightings and, for the case where the CvM statistic is based on kernel estimates, optimal bandwidths (which differ from optimal bandwidths in other settings). If a KS statistic with the truncated variance weighting is used, alleviating nonsimilarity of the test through choice of the critical value has little effect on power. If a bounded weighting is used, alleviating nonsimilarity through the choice of the critical value can have a larger effect on power.

Formally, this paper considers tests of a null hypothesis of the form

$$E(m(W_i, \theta)|X_i) \geq 0 \ \ a.s. \tag{1}$$

where $m : \mathbb{R}^{d_W+d_\theta} \to \mathbb{R}^{d_Y}$ is a known function of data W_i and a parameter $\theta \in \Theta \subseteq \mathbb{R}^{d_\theta}$, and $\geq$ is defined elementwise. Here, W_i is a $\mathbb{R}^{d_W}$ valued random variable and X_i is a $\mathbb{R}^{d_X}$ valued random variable. We are given independent, identically distributed (iid) observations $\{(X_i', W_i')'\}_{i=1}^n$. This defines the identified set

$$\Theta_0 \equiv \{\theta \in \Theta | E(m(W_i, \theta)|X_i) \geq 0 \ \ a.s.\}$$

where $\Theta \subseteq \mathbb{R}^{d_\theta}$ is the parameter space. If Θ_0 contains more than one element, the model is said to be set identified.

The results in this paper are motivated by a setting where (1) is the only condition that is given for the parameter θ, and the goal is to use this condition to obtain a confidence set for θ. By inverting a family of tests of (1), one obtains a confidence region $\mathcal{C}$ that contains each point $\theta_0 \in \Theta_0$ with a prespecified probability, as suggested by Imbens and Manski (2004). This paper derives the asymptotic power of several tests for detecting alternatives of the form $\theta_n = \theta_0 + a_n$, where θ_0 is on the boundary of Θ_0. The local power results in this paper then correspond to statements about the rate at which $\mathcal{C}$ shrinks towards Θ_0.[1]

As an example of the types of problems covered by this setup, consider the interval regression model of Manski and Tamer (2002). We observe (X_i, W_i^L, W_i^H) where $[W_i^L, W_i^H]$ is known to contain the latent variable W_i^*, which follows the linear regression model $E(W_i^*|X_i) = (1, X_i')\theta$. This falls into the setup of this paper with $W_i = (X_i, W_i^L, W_i^H)$ and $m(W_i, \theta) = (W_i^H - (1, X_i')\theta, (1, X_i')\theta - W_i^L)'$. The identified set is then given by

$$\Theta_0 = \{\theta | E(W_i^L|X_i) \leq (1, X_i')\theta \leq E(W_i^H|X_i) \ \ a.s.\}.$$

Thus, a parameter θ_0 in the identified set corresponds to a regression line $(1, x')\theta_0$ that is between the conditional means $E(W_i^L|X_i = x)$ and $E(W_i^H|X_i = x)$ for all x on the support of X_i. If θ_0 is on the boundary of the identified set, it will be equal to one of these regression lines for some value of x. To obtain a tight confidence set $\mathcal{C}$ for Θ_0, we want to invert tests that have good power at alternatives of the form $\theta_0 + a_n$, where θ_0 is on the boundary of the identified set.

This paper derives power results for local alternatives of this form under general conditions on $E(m(W_i, \theta)|X_i = x)$ as a function of θ and x. In the case of interval regression, these translate to conditions on $E(W_i^H|X_i = x)$ and $E(W_i^L|X_i = x)$, and this paper gives primitive conditions for this case. These conditions correspond to smoothness conditions used in the nonparametric statistics literature and conditions on the shape of these conditional means near points where one of them is equal to $(1, x')\theta_0$, and can be argued to hold generically.

A common concern, both when dealing with moment inequalities and when dealing with nonparametric smoothness conditions, is that fixing the data generating process (dgp) and then taking asymptotic approximations may not provide good approximations to finite sample quantities. For example, if one derives local power results in the interval regression model

[1] Formally, the sequence of tests and local alternatives in this paper fixes the data generating process P and varies the null parameter value θ_n being tested. The null hypothesis is that the data generating process is such that $\theta_n = \theta_0 + a_n$ is in the identified set, while the alternative is a fixed P such that θ_0 is in the identified set under P, but θ_n is not. This leads directly to statements about the rate at which the confidence set $\mathcal{C}$ shrinks toward the identified set Θ_0 under P.

using smoothness considions on $E(W_i^H|X_i = x)$, these may not be relevant in finite samples when $E(W_i^H|X_i = x)$ is "wiggly" relative to the sample size. In the econometrics literature, this is typically described as an issue of "uniformity in the underlying distribution," where "underlying distribution" refers to the data generating process. A solution to this problem is to specify some class of underlying distributions (for example, fixing a constant C and considering the class of all distributions such that the second derivative is bounded by this constant C and additional regularity conditions hold), and to consider the worst-case performance (in this case, power) over all possible drifting sequences of dgps in this class. This worst-case power is then called the minimax power over the given class of distributions (in this case, the class of distributions with second derivative bounded by C). In the nonparametric statistics literature, relative efficiency results and statements about optimal rates and constants are typically formalized using this approach. This literature is often cited in econometrics when making claims of optimality of nonparametric estimators (for example, Ichimura and Todd 2007 cite minimax bounds in Stone 1982). See Andrews and Guggenberger (2009) and Chapter 1.2.4 in Tsybakov (2009) for discussions of these issues in the context of moment inequalities and nonparametric regression respectively.

While the power results in this paper for CvM statistics are pointwise in the dgp, they give an upper bound for minimax power in certain smoothness classes (since the worst-case power is no better than the power for a given distribution in the smoothness class). Comparing these bounds to minimax rates derived in Armstrong (2014b) for the corresponding KS statistics, it can be seen that the upper bound for power obtained in the present paper for CvM statistics is worse than the minimax power achieved by the corresponding KS statistics. Thus, the results in this paper show that CvM statistics are preferred for inference on θ in a minimax sense. As discussed above, this notion of relative efficiency is the same one that allows one to make formal statements about optimal rates for nonparametric estimators, as in, e.g., Ichimura and Todd (2007) and references therein. See Section A.5 of the appendix for a formal statement.

The result that CvM and KS statistics can be ranked for the local alternatives considered in this paper contrasts with the more abstract setting where one is interested in a null hypothesis of the form $H_0 : E(Y_i|X_i) \geq 0$ almost surely or $H_0 : E(Y_i|X_i) = 0$ almost surely. Indeed, without more structure on the problem and the alternatives are of interest, such a ranking cannot be made (see Chapter 14 of Lehmann and Romano, 2005, and references therein). This paper places additional structure on the problem by considering the case where $Y_i = m(W_i, \theta)$, and we are interested in alternatives of the form $\theta_0 + a_n$ where θ_0 is

on the boundary of the identified set. This reflects the goal of inverting tests on θ to form a confidence set. See Section 1.1 for further references and Armstrong (2014a) for further discussion of this point in the context of a simple example.

The test statistics considered in this paper are as follows. Given a set $\mathcal{G}$ of nonnegative instruments, the null hypothesis (1) implies that $E(m(W_i, \theta)g(X_i)) \geq 0$ for all $g \in \mathcal{G}$. Thus, under (1), the sample analogue

$$E_n(m(W_i, \theta)g(X_i)) \equiv \frac{1}{n}\sum_{i=1}^{n} m(W_i, \theta)g(X_i) \tag{2}$$

should not be too negative for any $g \in \mathcal{G}$. The results in this paper use classes of functions given by kernels with varying bandwidths and location, given by $\mathcal{G} = \{x \mapsto k((x-\tilde{x})/h) | \tilde{x} \in \mathbb{R}^{d_X}, h \in \mathbb{R}_+\}$ for some kernel function k. With this choice of $\mathcal{G}$, (1) holds if and only if $E(m(W_i, \theta)g(X_i)) \geq 0$ for all $g \in \mathcal{G}$, so that (2) can be used to form a consistent test (see Andrews and Shi, 2013, for a discussion of this and other choices of $\mathcal{G}$).

Alternatively, one can test (1) by estimating $E(m(W_i, \theta)|X_i = x)$ directly using the kernel estimate

$$\hat{\bar{m}}_j(\theta, x) = \frac{\sum_{i=1}^{n} m(W_i, \theta)k((X_i - x)/h)}{\sum_{i=1}^{n} k((X_i - x)/h)} \tag{3}$$

for some sequence $h = h_n \to 0$ and kernel function k. If the null hypothesis holds for θ, (3) should not be too negative for any x.

Thus, a test statistic of the null that $\theta \in \Theta_0$ can be formed by taking any function that is positive and large in magnitude when (2) is negative and large in magnitude for some $g \in \mathcal{G}$, or when (3) is negative and large in magnitude for some x. One possibility is to use a CvM statistic that integrates the negative part of (2) over some measure μ on $\mathcal{G}$. This CvM statistic is given by

$$T_{n,p,\omega,\mu}(\theta) = \left[\int \sum_{j=1}^{d_Y} |E_n m_j(W_i, \theta)g(X_i)\omega_j(\theta, g)|_-^p \, d\mu(g)\right]^{1/p} \tag{4}$$

for some $p \geq 1$ and weighting ω, where $|t|_- = |\min\{t, 0\}|$. I refer to this as an instrument based CvM (IV-CvM) statistic. The CvM statistic based on the kernel estimate integrates

the negative part of (3) against some weighting ω, and is given by

$$T_{n,p,\text{kern}}(\theta) = \left[\int \sum_{j=1}^{d_Y} \left| \hat{\bar{m}}_j(\theta, x)\omega_j(\theta, x) \right|_-^p \, dx \right]^{1/p} \tag{5}$$

for some $p \geq 1$. I refer to this as a kernel based CvM (kern-CvM) statistic.

For the instrument based CvM statistic, the scaling for the power function will depend on ω. This paper considers both a bounded weighting which, without loss of generality, can be taken to be constant (the measure μ can absorb any weighting that does not change with the sample size)

$$\omega_j(\theta, g) = 1 \text{ all } \theta, g, j \tag{6}$$

as well as the truncated variance weighting used for KS statistics by Armstrong (2014b), Armstrong and Chan (2016) and Chetverikov (2012), which is given by

$$\omega_j(\theta, g) = (\hat{\sigma}_j(\theta, g) \vee \sigma_n)^{-1} \tag{7}$$

where

$$\hat{\sigma}_j(\theta, g) = \{E_n[m_j(W_i, \theta)g(X_i)]^2 - [E_n m_j(W_i, \theta)g(X_i)]^2\}^{1/2}$$

and σ_n is a sequence converging to zero and $a \vee b$ denotes the maximum of a and b for scalars a and b.[2]

The results for CvM statistics derived in this paper can be compared to power results for KS statistics derived in Armstrong (2015) and Armstrong (2014b). A KS statistic based on (2) simply takes the most negative value of that expression over $g \in \mathcal{G}$, and is given by

$$T_{n,\infty,\omega}(\theta) = \max_j \sup_{g \in \mathcal{G}} |E_n m_j(W_i, \theta)g(X_i)\omega_j(\theta, g)|_-. \tag{8}$$

I refer to this as an instrument based KS (IV-KS) statistic. A KS statistic based on (3)

[2]For the critical value of the test, the results covered in this paper cover any critical value that is of the same order of magnitude asymptotically as a critical value based on the distribution where all moments bind. See Section 3 for details.

statistic	weighting function	rate
instrument based CvM	bounded weights	$n^{-\gamma/\{2[d_X+\gamma+(d_X+1)/p]\}}$
instrument based CvM	variance weights	$n^{-\gamma/\{2[d_X/2+\gamma+(d_X+1)/p]\}}$
kernel CvM	-	$\max\{(nh^{d_X})^{-1/[2(1+d_X/(p\gamma))]}, h^\gamma\}$

Table 1: Local Power for CvM Statistics

statistic	weighting function	rate
instrument based KS	bounded weights	$n^{-\gamma/\{2[d_X+\gamma]\}}$
instrument based KS	variance weights	$(n/\log n)^{-\gamma/\{2[d_X/2+\gamma]\}}$
kernel KS	-	$\max\left\{(nh^{d_X}/\log n)^{-1/2}, h^\gamma\right\}$

Table 2: Local Power for KS Statistics (Armstrong, 2015, 2014b)

simply takes the most negative value of that expression over x, and is given by

$$T_{n,\infty,\text{kern}}(\theta) = \max_j \sup_\theta \left|\hat{\bar{m}}_j(\theta,x)\omega_j(\theta,x)\right|_- . \tag{9}$$

I refer to this as a kernel based KS (kern-KS) statistic. As with CvM statistics, the scaling for the local power function for the instrument based KS test depends on whether a bounded weighting or a truncated variance weighting is used.

The asymptotic power results derived in this paper for the CvM statistics (4) and (5) are summarized in Table 1. For comparison, Table 2 summarizes the corresponding results for KS statistics, which are contained in Armstrong (2015) and Armstrong (2014b). These tables give the fastest rate at which a_n can approach 0 for each test to have power at $\theta_0 + a_n$ for θ_0 on the boundary of the identified set. Here γ is a smoothness parameter that, roughly speaking, corresponds to the number of derivatives, up to 2, of $E(m(W_i,\theta)|X_i = x)$ with respect to x. The power results for the instrument based statistics depend on the set of functions $\mathcal{G}$, and are reported here only for the ones considered in this paper, but broader implications of the results described here (such as KS statistics being more powerful than CvM statistics in this setting) hold more generally.

These power results have several implications for how the choice of test statistic and weighting affect power. First, tests based on KS statistics are more powerful than those based on the corresponding CvM statistic in all of these cases. Second, variance weights lead to more powerful tests than bounded weights both for CvM and KS statistics.

Third, the results can be used to choose the optimal bandwidth for kernel CvM statistics. Some calculation shows that the rate in the third row of Table 1 is optimized when h_n is proportional to $n^{-1/[2(\gamma+d_X/p+d_X/2)]}$, which leads to a rate of $n^{-\gamma/[2(\gamma+d_X/p+d_X/2)]}$. The optimal

bandwidth is larger than the optimal bandwidth for estimating a conditional mean at a point, or for the corresponding KS statistic.

Fourth, it is interesting to note how the choice of the class of instrument functions $\mathcal{G}$ affects power for these statistics. The main point here is that choosing a larger class of instruments by adding instruments that turn out to be irrelevant has less impact on power for KS statistics than it does for CvM statistics. This can be seen by comparing the rates for instrument based statistics to the corresponding rates for kernel based statistics with the bandwidth chosen optimally. The rates reported in these tables for instrument based statistics take $\mathcal{G}$ to be the class of functions given by $x \mapsto k((x - \tilde{x})/h)$ for all $(\tilde{x}, h)$. The kernel version of this statistic essentially uses a subset of this class of functions with $h = h_n$ restricted to a particular value for each n. For KS statistics, as long as variance weights are used, considering this larger class of functions does not lead to a decrease in the rate for local alternatives even if the optimal h_n is known. The rate in the second row of Table 2 for variance weighted instrument based KS statistics is the same as the rate for kernel based KS statistics in the third row if h is chosen optimally. In general, adding more instruments to $\mathcal{G}$ will not lead to a slower rate in the power function for variance weighted KS statistics as long as certain conditions on the complexity of $\mathcal{G}$ hold.

In contrast, considering a larger set of instruments $\mathcal{G}$ will generally decrease the rate for local alternatives if a CvM statistic is used. If a kernel CvM statistic is used instead of an instrument based CvM statistic (which corresponds to restricting $\mathcal{G}$) and prior knowledge of the data generating process is used to choose the bandwidth optimally, the kernel statistic will achieve a $n^{-\gamma/[2(\gamma+d_X/p+d_X/2)]}$ rate, which is faster than the $n^{-\gamma/[2(d_X/2+\gamma+(d_X+1)/p)]}$ rate for the instrument based CvM statistic with variance weights, where $\mathcal{G}$ includes all bandwidths. It can also be shown, using arguments similar to those in this paper, that expanding $\mathcal{G}$ to include d_X-dimensional boxes with sides of different lengths leads to slower rates for power functions with CvM statistics, but not for KS statistics. In general, CvM statistics are more sensitive to adding functions to $\mathcal{G}$ than KS statistics.

These results provide general insight into the type of objective function, weighting, and critical value one should use. However, the class of tests that are optimal for these models (tests based on KS statistics with a truncated variance weighting) still depend on certain user defined parameters. Choosing these user defined parameters for a particular sample size and data set can be done using Monte Carlos and criteria such as maximizing power against a particular sequence of alternatives.

1.1 Related Literature

Tests based on instrument based CvM and KS statistics have been considered by Andrews and Shi (2013), Kim (2008), Khan and Tamer (2009) and Armstrong (2015) for bounded weights, and Armstrong (2014b), Armstrong and Chan (2016) and Chetverikov (2012) for KS statistics with variance weights. The statistics based on instruments with bounded weights use an approach to nonparametric testing problems that goes back at least to Bierens (1982). Aradillas-Lopez et al. (2013) use a slightly different version of an instrument CvM approach. Chernozhukov et al. (2013) consider kernel based KS statistics and Lee et al. (2013) and Lee et al. (2015) consider kernel based CvM statistics. While some of these papers derive local power results for CvM tests under conditions that appear to be common in point identified models, these results do not apply in set identified models except for in very special cases. Indeed, the results in the present paper show that, when one uses a minimax criterion requiring uniformly good power in classes of underlying distributions defined by smoothness properties, the power of CvM tests is much worse (see Section A.5). The results in this paper show that power comparisons in the set identified case considered here are much different than settings that have been studied previously. Armstrong (2015), Armstrong (2011), Armstrong (2014b), Armstrong and Chan (2016), and Chetverikov (2012) derive power results for KS statistics under conditions similar to those used in this paper, but do not consider CvM statistics.

The results in this paper are also related to the statistics literature on minimax testing of hypotheses of the form $H_{0,=} : f(x) = 0$ all x, $H_{0,\geq} : f(x) \geq 0$ all x, $H_{0,\uparrow} : f(x) \geq f(x')$ all $x < x'$, (and related hypotheses such as convexity of f), where the function f is observed with noise. While much of this literature focuses on the Gaussian white noise model or Gaussian sequence model, the results are closely related to the case where $f(x) = E(Y_i|X_i = x)$, and iid observations of X_i, Y_i are available (which falls into our setup if we take $Y_i = m(W_i, \theta_0)$). To formulate the minimax testing problem considered in this literature, one specifies a smoothness class $\mathcal{F}$ for f and a functional $\psi : \mathcal{F} \to [0, \infty)$ such that $\psi(f)$ is 0 if f satisfies the null and strictly positive otherwise. For example, for $H_{0,=}$, one can take the L_p norm $\psi(f) = [\int f(x)^p \, dx]^{1/p}$ and, for $H_{0,\geq}$, one can take the one-sided L_p norm $\psi(f) = [\int |f(x)|_-^p]$. The minimax testing problem is to obtain tests that have good worst-case power over alternatives f in the smoothness class $\mathcal{F}$ with $\psi(f) \geq a_n$ for $a_n \to 0$ as quickly as possible. Dumbgen and Spokoiny (2001) and Juditsky and Nemirovski (2002) consider $H_{0,\geq}$ with ψ given by the one-sided L_∞ norm $\psi(f) = \sup_x |f(x)|_-$ and the one-sided L_p norm with $p < \infty$ respectively, as well as $H_{0,\uparrow}$ and the hypothesis of convexity with

related distance functions ψ. Lepski and Tsybakov (2000) consider $H_{0,=}$ with $\psi(f)$ given by the L_∞ norm and by $\psi(f) = |f(x_0)|$ for a given point x_0. See Ingster and Suslina (2003) for further results and references to this literature.

In contrast to this literature, the results in this paper have implications for minimax rates of CvM statistics for testing the null that a given value of θ is in the identified set against the alternative that the distance between θ and any point in the identified set is at least a_n (see Appendix A.5 for a formal statement). Since the dimension of θ is finite and fixed, the choice of distance (i.e. whether to use Euclidean distance or sup-norm distance when defining distance of θ from points in the identified set) does not matter for the rate at which a_n can approach zero with the test having good power. This contrasts with the nonparametric testing literature described above, in which the choice of distance function ψ has implications for relative efficiency of different test statistics, and is part of the reason that CvM and KS tests can be ranked in this setting. Interestingly, the problem of minimax inference on θ in the settings considered here appears to be closely related to nonparametric testing with ψ given by the L_∞ norm. See Armstrong (2014a) for further discussion.

Inference on conditional moment inequalities can also be cast as a problem of inference with many unconditional moment inequalities, as considered by Menzel (2010). The results of the present paper can be extended to provide power results for this case by allowing $\mathcal{G}$ to depend on n. This paper also relates to the broader literature on set identified models, including models defined by unconditional moment inequalities. See Armstrong (2015) for additional references to this literature.

This paper is organized as follows. Section 2 gives an intuitive description of the power results in this paper and how they are derived. Section 3 defines the tests considered in this paper. Section 4 states formal the conditions used in this paper, and provides primitive conditions for the interval regression model. Section 5 derives the power results. Section 6 reports the results of a Monte Carlo study. Section 7 concludes. An appendix contains proofs and auxiliary results, including minimax power comparisons as well as primitive conditions for the results in the main text in additional settings.

2 Intuition for the Results

To get some intuition for the results, consider the case of instrument based CvM statistic with bounded weights. This paper considers the case where the class of functions $\mathcal{G}$ is given by the set of kernel functions with varying bandwidths and locations $\{x \mapsto k((x-\tilde{x})/h) | \tilde{x} \in$

$\mathbb{R}^{d_X}, h \in \mathbb{R}_+\}$ for some kernel function k, and the measure μ has a density $f_\mu(\tilde{x}, h)$ with respect to the Lebesgue measure. We also assume that X_i has a density $f_X(x)$. For simplicity, consider the case where $d_Y = 1$.

The test statistic is given by an integral over a sample expectation. We expect that the test will have power when the integral over the corresponding population expectation is large relative to the critical value, which, as discussed below, will be of order $n^{-1/2}$. Thus, to have power at $\theta_n = \theta_0 + a_n$, we expect that

$$\begin{aligned}&\left[\int\int |Em(W_i, \theta_n)k((X_i - \tilde{x})/h)|_-^p f_\mu(\tilde{x}, h)\, d\tilde{x}\, dh\right]^{1/p} \\ &= \left[\int\int \left|\int \bar{m}(\theta_n, x)k((x - \tilde{x})/h)f_X(x)\, dx\right|_-^p f_\mu(\tilde{x}, h)\, d\tilde{x}\, dh\right]^{1/p} \qquad (10)\end{aligned}$$

will have to be large relative to $n^{-1/2}$, where $\bar{m}(\theta_n, x) = E(m(W_i, \theta_n)|X_i = x)$.

This paper considers more general classes of data generating processes, but, for simplicity, suppose that $\bar{m}(\theta_0, x) \approx \|x - x_0\|^\gamma$ near some x_0 for some γ, and is bounded from below away from zero elsewhere. This approximation and a first order approximation to $m(\theta_n, x) - m(\theta_0, x)$ suggests that (10) will be approximated well by

$$\left\{\int\int \left|\int \left[\|x - x_0\|^\gamma + \bar{m}_\theta(\theta_0, x)a_n\right] k((x - \tilde{x})/h)f_X(x)\, dx\right|_-^p f_\mu(\tilde{x}, h)\, d\tilde{x}\, dh\right\}^{1/p}$$

where $\bar{m}_\theta(\theta, x)$ denotes the derivative of $\bar{m}(\theta, x)$ with respect to θ. Since the integrand will be nonzero only for x and $\tilde{x}$ close to x_0 and h close to zero, we can further approximate this by

$$\left\{\int\int \left|\int \left[\|x - x_0\|^\gamma + \bar{m}_\theta(\theta_0, x_0)a_n\right] k((x - \tilde{x})/h)f_X(x_0)\, dx\right|_-^p f_\mu(x_0, 0)\, d\tilde{x}\, dh\right\}^{1/p}.$$

Let $a_n = ar_n$ for some sequence r_n to be determined later. By the change of variables $u = (x - x_0)/r_n^{1/\gamma}$, $v = (\tilde{x} - x_0)/r_n^{1/\gamma}$, $\tilde{h} = h/r_n^{1/\gamma}$, the above display can be written as

$$\begin{aligned}&\left\{\int\int \left|\int \left[r_n\|u\|^\gamma + \bar{m}_\theta(\theta_0, x_0)ar_n\right] k((u - v)/\tilde{h})f_X(x_0)r_n^{d_X/\gamma}\, du\right|_-^p f_\mu(x_0, 0)r_n^{d_X/\gamma}\, dv\, r_n^{1/\gamma}\, d\tilde{h}\right\}^{1/p} \\ &= r_n^{[(\gamma+d_X)+(d_X+1)/p]/\gamma}\left\{\int\int \left|\int \left[\|u\|^\gamma + \bar{m}_\theta(\theta_0, x_0)a\right] k((u - v)/\tilde{h})f_X(x_0)\, du\right|_-^p f_\mu(x_0, 0)\, dv\, d\tilde{h}\right\}^{1/p}.\end{aligned}$$

Thus, (10) is of order $r_n^{[(\gamma+d_X)+(d_X+1)/p]/\gamma}$, so we expect to get power when this is large enough relative to $n^{-1/2}$, and equating these gives

$$r_n^{[(\gamma+d_X)+(d_X+1)/p]/\gamma} = n^{-1/2} \iff r_n = n^{-\gamma/\{2[(\gamma+d_X)+(d_X+1)/p]\}}.$$

This is the rate reported in Table 1 and derived formally later in the paper.

A key insight here is that the integral in the CvM statistic has an additional effect on the drift term, and that this effect can be captured through a change-of-variables argument. This contrasts with the corresponding results for KS tests, and leads to a decrease in power relative to these tests (see Section 2.1 of Armstrong, 2014b, for a sketch of the corresponding results for KS tests).

3 Definitions of Tests

To complete the definition of these tests, we need to define a critical value. For tests that use instrument based CvM statistics with bounded weights or inverse variance weights with $p < \infty$, the test $\phi_{n,p,\omega,\mu}$, which rejects when $\phi_{n,p,\omega,\mu} = 1$, is defined as

$$\phi_{n,p,\omega,\mu} = \begin{cases} 1 & \text{if } \sqrt{n}T_{n,p,\omega,\mu} > \hat{c}_{n,p,\omega,\mu} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

for some critical value $\hat{c}_{n,p,\omega,\mu}$. For kernel based CvM statistics, the test $\phi_{n,p,\text{kern}}$, which rejects when $\phi_{n,p,\text{kern}} = 1$, is defined as

$$\phi_{n,p,\text{kern}} = \begin{cases} 1 & \text{if } (nh^{d_X})^{1/2}T_{n,p,\text{kern}} > \hat{c}_{n,p,\text{kern}} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

While all of the new results in this paper are for CvM statistics, I refer to analogous results for KS statistics at some points for comparison. For KS tests with bounded weights, the critical value is defined as in (11). For KS tests based on truncated variance weights, the test $\phi_{n,\infty,(\sigma\vee\sigma_n)^{-1}}$ is defined as

$$\phi_{n,\infty,(\sigma\vee\sigma_n)^{-1}} = \begin{cases} 1 & \text{if } \sqrt{\frac{n}{\log n}}T_{n,\infty,(\sigma\vee\sigma_n)^{-1}} > \hat{c}_{n,\infty,(\sigma\vee\sigma_n)^{-1}} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

for some critical value $\hat{c}_{n,p,\infty,(\sigma\vee\sigma_n)^{-1}}$.

The properties of these tests will depend on the choice of critical value. The only condition needed for upper bounds on power, stated in the following assumption, is that the critical value be of the same order of magnitude as a critical value based on a least favorable asymptotic distribution where all of the moments bind (i.e. $E(m(W_i, \theta)|X_i) = 0$ a.s.).

Assumption 3.1. *For some $\eta > 0$, the critical value $\hat{c}$ defined in (11) or (12), depending on the weighting and form of the test, satisfies $\hat{c} > \eta$ with probability approaching one.*

Assumption 3.1 holds for the kernel CvM based test of Lee et al. (2013), which uses the least favorable null dgp, as well as the tests using instrument based CvM statistics with bounded weights proposed in Andrews and Shi (2013). Instrument based CvM statistics with variance weights have not been considered in the literature. In Section B.2 of the appendix, I consider critical values for this case and show that critical values based on the least favorable null dgp will satisfy Assumption 3.1.

In the case of the test statistics considered by Andrews and Shi (2013), the normalized test statistic $\sqrt{n}T_{n,p,\omega,\mu}$ has a nondegenerate limiting distribution in the case where $E(m(W_i, \theta)|X_i) = 0$ for X_i on a positive probability set. However, in the case where $E(m(W_i, \theta)|X_i) = 0$ only for X_i on a finite set (which, as discussed further below, will be the case under the conditions considered here), the results of Andrews and Shi (2013) do not give a nondegenerate asymptotic distribution for $T_{n,p,\omega,\mu}$. Rather, they reduce to a statement that $\sqrt{n}T_{n,p,\omega,\mu} \xrightarrow{p} 0$. To deal with these possibilities, Andrews and Shi (2013) propose a critical value of the form $\hat{c} = \tilde{c} + \eta$, where $\tilde{c} \geq 0$ is a conservative estimate of a quantile of the asymptotic distribution, which is always positive but may converge to zero, and η is a fixed value chosen by the researcher. Andrews and Shi (2013) call η the "infinitesimal uniformity factor" and suggest setting $\eta = .001$. Assumption 3.1 holds for this critical value by construction.

The use of the "infinitesimal uniformity factor" is not ideal, since, in certain cases, the justification for controlling size amounts to a statement that $\sqrt{n}T_{n,p,\omega,\mu} \xrightarrow{p} 0$, and that the test rejects only when $\sqrt{n}T_{n,p,\omega,\mu} > \eta$. An alternative approach, which Andrews and Shi (2013) also consider, is to simply base the critical value on the least favorable distribution (where $E(m(W_i, \theta)|X_i) = 0$ almost surely). In the case where $E(m(W_i, \theta)|X_i) = 0$ almost surely, the asymptotic distribution of $\sqrt{n}T_{n,p,\omega,\mu}$ is nondegenerate, so these issues do not arise (and Assumption 3.1 will hold for critical values based on this distribution). Alternatively, it may be possible to find a sequence b_n that increases more quickly than $\sqrt{n}$ such that $b_n T_{n,p,\omega,\mu}$ has a nondegenerate limiting distribution, and base inference on this asymptotic approximation (or some other nondegenerate approximation). This approach would lead to

a critical value that does not satisfy Assumption 3.1, since comparing $b_n T_{n,p,\omega,\mu}$ to a critical value that converges to a constant is equivalent to comparing $\sqrt{n} T_{n,p,\omega,\mu}$ to a critical value that behaves line $\sqrt{n}/b_n \to 0$. While critical values of this form have been considered by Armstrong (2015), Chernozhukov et al. (2013) and Chetverikov (2012) for KS statistics of various forms, such results are not available for instrument based CvM statistics.[3] We leave power calculations when Assumption 3.1 does not hold as a question for future research.

Assumption 3.1 only gives a lower bound for a critical value. This gives bounds on the power function, but to derive the exact local asymptotic power function, we need the following condition, which gives a limiting value for this critical value. Under mild conditions on the data generating process and sequence of local alternatives, this assumption will also hold for the methods of choosing critical values discussed above.

Assumption 3.2. *For the critical value $\hat{c}$ defined in (11) or (13), depending on the weighting and form of the test, and some constant $c > 0$, $\hat{c} \xrightarrow{p} c$.*

The power properties of the test will also depend on the class of functions $\mathcal{G}$ used as instruments. I derive power functions for the case where $\mathcal{G}$ consists of kernel functions with different bandwidths and locations, defined in the following assumption.

Assumption 3.3. *For some bounded, nonnegative function k with finite support and $\int k(u)\,du > 0$, $\mathcal{G} = \{x \mapsto k((x - \tilde{x})/h) | \tilde{x} \in \mathbb{R}^{d_X}, h \in \mathbb{R}_+\}$, and the covering number $N(\varepsilon, \mathcal{G}, L_1(Q))$ defined in Pollard (1984) satisfies $\sup_Q N(\varepsilon, \mathcal{G}, L_1(Q)) \leq A\varepsilon^{-W}$, where the supremum is over all probability measures.*

The covering number assumption in Assumption 3.3 is a technical condition that allows for uniform convergence of kernel estimates over x and h. A sufficient condition is that the kernel k takes the form $k(x) = r(\|x\|)$ where r is a monotone decreasing function on on $[0, \infty)$ (see Pollard, 1984, chapter 2, problem 28).

For CvM statistics, I place the following condition on the measure μ over which the sample means are integrated.

Assumption 3.4. *The measure μ has bounded support, and has a density $f_\mu(\tilde{x}, h)$ with respect to the Lebesgue measure on $\mathbb{R}^{d_X} \times [0, \infty)$ that is bounded and continuous.*

Relaxing this assumption would lead to different power properties, although the general point that L_p statistics perform worse in these models than supremum statistics would go through.

[3] In recent work, Lee et al. (2015), propose critical values for kernel based CvM statistics that do not satisfy Assumption 3.1, although they are still based on degenerate asymptotic distributions in some cases.

4 Conditions for Local Alternatives

This section presents the main assumptions on the model and dgp used in this paper. The conditions are similar to those used in Armstrong (2015), Armstrong (2014b) and Armstrong and Chan (2016). I first provide high level conditions, and then verify them for the interval regression model in Section 4.1. Section 4.2 provides a discussion of the difference between these conditions and other settings, such as point identified models. Section A in the appendix verifies the conditions in this section for additional settings. I assume throughout that the data are iid.

I place the following conditions on the data generating process when $m(W_i, \theta)$ is evaluated at θ_0 and $\theta_n = \theta_0 + a_n$. In these conditions, γ is a smoothness parameter that is generally given by the minimum of the number of derivatives of the conditional mean and 2. The truncation of the smoothness parameter at 2 comes from the fact that the test statistics here use positive kernels or instruments.

Assumption 4.1. *For each j, the conditional mean $E(m_j(W_i, \theta_0)|X = x) = \bar{m}_j(\theta, x)$ takes its minimum only on a finite set $\{x|E(m_j(W_i, \theta_0)|X = x) = 0 \text{ some } j\} = \mathcal{X}_0 = \{x_1, \ldots, x_\ell\}$. For each k from 1 to ℓ, let $J(k)$ be the set of indices j for which $E(m_j(W_i, \theta_0)|X = x_k) = 0$. Assume that there exist neighborhoods $B(x_k)$ of each $x_k \in \mathcal{X}_0$ such that the following assumptions hold.*

i.) There exists $\eta > 0$ such that, for θ in a neighborhood of θ_0, we have (a) $\bar{m}_j(\theta, x) > \eta$ for $j \notin J(k)$ for $x \in B(x_k)$ and (b) $\bar{m}_j(\theta, x) > \eta$ for all j for $x \notin \cup_{k=1}^{\ell} B(x_k)$.

ii.) For $j \in J(k)$, $\bar{m}_j(\theta_0, x)$ is continuous on the closure of $B(x_k)$ and satisfies

$$\sup_{\|x - x_k\| \leq \delta} \left\| \frac{\bar{m}_j(\theta_0, x) - \bar{m}_j(\theta_0, x_k)}{\|x - x_k\|^{\gamma(j,k)}} - \psi_{j,k}\left(\frac{x - x_k}{\|x - x_k\|}\right) \right\| \overset{\delta \to 0}{\to} 0$$

for some $\gamma(j, k) > 0$ and some function $\psi_{j,k} : \{t \in \mathbb{R}^{d_X} | \|t\| = 1\} \to \mathbb{R}$ with $\overline{\psi} \geq \psi_{j,k}(t) \geq \underline{\psi}$ for some $\overline{\psi} < \infty$ and $\underline{\psi} > 0$. For future reference, define $\gamma = \max_{j,k} \gamma(j, k)$ and $\tilde{J}(k) = \{j \in J(k) | \gamma(j, k) = \gamma\}$.

iii.) X has a continuous density f_X on $B(x_k)$.

iv.) For $j \in J(k)$, $s_j^2(x, \theta) \equiv var(m_j(W_i, \theta)|X_i = x)$ is strictly positive and continuous at (x_k, θ_0).

v.) For x in the closure of $B(x_k)$ and θ in a neighborhood of θ_0, $\bar{m}(\theta, x)$ has a derivative as a function of θ that is continuous as a function of (θ, x). Let $\bar{m}_{\theta,j}(\theta, x)$ denote the jth row of this derivative matrix (i.e. the derivative of $\bar{m}_j(\theta, x)$ with respect to θ).

Assumption 4.2. *The data are iid and for some fixed $\overline{Y} < \infty$ and θ in a some neighborhood of θ_0, $|m(W_i, \theta)| \leq \overline{Y}$ with probability one.*

The deterministic bound in Assumption 4.2 allows for the use of certain technical results that are useful in the proofs. It may be possible to relax this assumption, although additional technical arguments would be needed in some places.

The following assumption, which is used for kernel based statistics, ensures that the kernel estimators do not encounter boundary problems (cf. Assumption 1(iii) in Lee et al., 2013).

Assumption 4.3. *X_i has a density f_X that is bounded away from infinity, and the weighting function $\omega_j(\theta, x)$ is continuous for all j and, for some $\varepsilon > 0$, is equal to zero whenever $f_X(\tilde{x}) < \varepsilon$ for some $\tilde{x}$ with $\|\tilde{x} - x\| < \varepsilon$.*

4.1 Discussion and Primitive Conditions for Interval Regression

In discussing these assumptions, it is useful to keep in mind the interval regression model introduced in the introduction, in which $W_i = (X_i, W_i^L, W_i^H)$ and $m(W_i, \theta) = (W_i^H - (1, X_i')\theta, (1, X_i')\theta - W_i^L)'$. The following gives a general discussion of these assumptions, with references to the interval regression model as an example. I then state primitive sufficient conditions in the interval regression model that imply these assumptions with $\gamma = 2$. Section A of the appendix gives primitive conditions in additional settings.

The assumptions used here are similar to the conditions used in Armstrong (2015) to derive the asymptotic distribution and local power function of a KS statistic with bounded weights. In particular, Assumption 4.1 corresponds to the version of Assumption 3.1 in Armstrong (2015) used in Section 5 of that paper, in which part (ii) is replaced by Assumption 5.1 in Armstrong (2015). Part (i) strengthens the version used in Armstrong (2015) by extending it to a neighborhood of θ_0, and part (v) is an additional condition on the derivative with respect to θ. These additional conditions are used to derive local power, and are similar to Assumption 7.1 in Armstrong (2015).

Assumption 4.1 is the main substantive condition that gives rise to the local power results derived in this paper. It states that the conditional mean of the moment conditions

is equal to zero only at a finite number of points. In the context of the interval regression model, this holds for θ_0 on the boundary of the identified set when the regression line $x'\theta_0$ is tangent to $E(W_i^H|X_i = x)$ or $E(W_i^H|X_i = x)$ at a finite number of points. In general, a sufficient condition for this in the case where X_i has compact support is that $\bar{m}_j(\theta, x)$ takes its minimum on the interior of the support of X_i and $\bar{m}_j(\theta, x)$ is twice continuously differentiable with a positive definite second derivative matrix at any point where it takes a minimum (see Section A.1 in the appendix).

The most natural case where this does not hold is where $E(W_i^H|X_i = x)$ or $E(W_i^L|X_i = x)$ is linear and equal to $(1, x')\theta$ on a nondegenerate interval (the other possibility is for $E(W_i^H|X_i = x) - (1, x')\theta_0$ to be zero on a set with infinitely many elements, but with zero probability, such as with the function $\sin(1/x)$). This holds in the point identified case where $P(W_i^H = W_i^L|X_i) = 1$ for X_i on a nondegenerate interval (and, in particular, in the special case where $W_i^H = W_i^L$ with probability one, leading to the usual linear regression model). However, when θ is set identified, this is a knife-edge case: even if $E(W_i^H|X_i) = (1, X_i')\theta_0$ for X_i on a nondegenerate interval for a given θ_0 on the boundary of the identified set, we will typically have $E(W_i^H|X_i = x) = (1, x')\tilde{\theta}_0$ only on a finite set for $\tilde{\theta}_0$ close to θ_0.

This is illustrated by Figures 1 and 2, which are taken directly from Section 2.2 of Armstrong (2015). Each figure shows the conditional mean $E(W_i^H|X_i = x)$ for some dgp along with regression lines corresponding to particular parameter values θ (the lower conditional mean $E(W_i^L|X_i = x)$ can be taken to be below the area shown in each figure). In Figure 1, the regression line $(1, x')\theta = \theta_1 + \theta_2 x$ is tangent to the conditional mean at a single point, and Assumption 4.1 holds for the parameter θ. In Figure 2, the regression line $\theta_{a,2} + \theta_{a,2}x$ corresponding to the parameter θ_a is equal to $E(W_i^H|X_i = x)$ on a nondegenerate interval, so that Assumption 4.1 does not hold. However, at nearby parameter values such as θ_b, the regression line is equal to $E(W_i^H|X_i = x)$ at a single point and Assumption 4.1 holds. See Section 2.2 of Armstrong (2015) for further discussion.

In the case where $\bar{m}(\theta_0, x)$ is twice continuously differentiable in x, part (ii) of Assumption 4.1 follows from a second order Taylor expansion at x_k, so long as the second derivative matrix is positive definite. In this case, Assumption 4.1 holds with $\gamma = 2$ and $\psi_{j,k}(u) = u'V_j(x_k)u/2$, where $V_j(x_k)$ is the second derivative matrix of $x \mapsto \bar{m}(\theta_0, x)$ at x_k. In the interval regression model, the second derivative of $m_1(\theta_0, x)$ is equal to the second derivative of $E(W_i^H|X_i = x)$ (and similarly for $m_2(\theta_0, x)$ and $-E(W_i^L|X_i = x)$), so this translates directly to an assumption of a positive definite second derivative matrix of $E(W_i^H|X_i = x)$. In the case where $\bar{m}(\theta_0, x)$ is Lipschitz continuous, part (ii) of Assumption 4.1 will hold with

$\gamma = 1$ if we place additional regularity conditions on the one-sided directional derivative of $\bar{m}(\theta_0, x)$. The parameter θ in Figure 1 illustrates a case where Assumption 4.1 holds with $\gamma = 2$, while the parameter θ_b in Figure 2 illustrates a case where Assumption 4.1 holds with $\gamma = 1$. See Theorem A.1 in Section A.2 of the appendix for a formal statement in the interval regression model.

The remaining assumptions are regularity conditions that translate easily to primitive objects in the case of interval regression. For part (v), note that $\bar{m}_{\theta,1}(\theta, x) = -(1, x')$ and $\bar{m}_{\theta,2}(\theta, x) = (1, x')$, which are clearly continuous, so this assumption holds without further conditions on the dgp.

The following gives a formal statement of primitive conditions for the interval regression model in the case where the conditional means are twice differentiable. The proof of this result uses the ideas in the discussion above, and is given in Section A.2 of the appendix.

Theorem 4.1. *Suppose that the following conditions hold.*

i.) The conditional means $E(W_i^H|X_i = x)$ and $E(W_i^L|X_i = x)$ are twice differentiable with continuous second derivatives, X_i has a continuous density and compact support, and W_i^H and W_i^L are bounded from above and below by finite constants.

ii.) For any point $\tilde{x}$ such that $E(W_i^H|X_i = \tilde{x}) = (1, \tilde{x}')\theta_0$, $\tilde{x}$ is in the interior of the support of X_i, $var(W_i^H|X_i = x)$ is positive and continuous at $\tilde{x}$ and $E(W_i^H|X_i = x)$ has a positive definite second derivative matrix at $\tilde{x}$. The same holds for $E(W_i^L|X_i = x)$ with "positive definite" replaced by "negative definite."

Then Assumptions 4.1, and 4.2 hold, with $\gamma = 2$ in Assumption 4.1.

4.2 Comparison with Conditions Leading to Parametric Rates

Under Assumption 4.1, the conditional mean $\bar{m}_j(\theta_0, x) = E(m(W_i, \theta_0)|X_i = x)$ is minimized on a finite set, and behaves like $\|x - x_k\|^\gamma$ for x_k in this set and nearby x. As shown in Section 5 below, this leads to power against alternatives that approach the identified set at a slower than $\sqrt{n}$ rate. As suggested by the intuitive description of these results in Section 2, this arises because, as θ_n approaches the identified set, the conditional moment inequalities are violated on a set with vanishing probability. This is similar to the case of nonparametric kernel estimation, in which bias-variance tradeoffs and the level of smoothness determine the rate of convergence (see, e.g., Wasserman, 2007).

In contrast, Andrews and Shi (2013), Kim (2008) and Lee et al. (2013) consider the case where $\bar{m}_j(\theta_0, x)$ is minimized on a nondegenerate interval. In this case, the portion of the support of X_i on which the inequality is violated does not vanish as θ_n approaches the boundary of the identified set. This leads to nontrivial power at alternatives that approach the null at a $1/\sqrt{n}$ rate. As discussed above, the latter case is typical under point identification and holds by construction with moment equalities, but it corresponds to a knife-edge case under set identification.

To understand these issues, it is helpful to make a comparison to the case of nonparametric regression, where kernel estimators can converge at a faster rate if certain derivatives are equal to zero. For example, local linear estimators converge at a $n^{2/5}$ rate when the conditional mean is twice differentiable with nonzero derivative and a bandwidth is used that decreases like $n^{-1/5}$, but a faster rate can be obtained when the second derivative is zero, using a bandwidth sequence that converges more slowly. As discussed in the introduction, such issues are related to the problem of "uniformity in the underlying distribution," and the typical approach to formalizing the notion that the optimal rate under a second derivative condition is $n^{2/5}$ is to use a minimax criterion, in which one requires good performance uniformly over all dgps with a certain bound on the second derivative (see Fan, 1993, for a formulation of this approach for local linear estimators).

In the present setting, the results in this paper show that, even though $\sqrt{n}$ local power is possible in certain special cases, the minimax (worst-case) power is slower than $\sqrt{n}$ when one only places bounds on derivatives of certain objects. In particular, while a bound on the second derivative of $E(W_i^H|X_i = x)$ and $E(W_i^L|X_i = x)$ does not imply Assumption 4.1 in the interval regression model, one can construct a dgp such that Assumption 4.1 holds with $\gamma = 2$ for any nonzero bound on the second derivative. Thus, the minimax rates of local power for CvM statistics under a bound on the second derivative are at least as slow as the rates derived in this paper, which are slower than $\sqrt{n}$. Since the results in Armstrong (2014b) show that the corresponding KS statistics achieve a better rate for local alternatives uniformly over dgps with a bound on the second derivative (and additional regularity conditions), this means that the KS statistic is preferred to the CvM statistic under a minimax criterion in this class. See Section A.5 in the appendix for formal statements.

5 Local Power Results

In this section, I derive local power results for CvM test statistics under the conditions given in Section 4. Sections 5.1, 5.2 and 5.3 give the power results. Section 5.4 provides an analysis of the quantities that determine local asymptotic power.

5.1 Instrument Based CvM Statistics with Bounded Weights

To describe the power results, we need some additional notation. Define

$$\begin{aligned}&\lambda_{\text{bdd}}(a,j,k,p) = \lambda_{\text{bdd}}(a, \bar{m}_{\theta,j}(\theta_0,x_k), \psi_{j,k}, f_X(x_k), f_\mu(x_k,0), p)\\ &\equiv \int\int \left| \int \left[\|x\|^\gamma \psi_{j,k}\left(\frac{x}{\|x\|}\right) + \bar{m}_{\theta,j}(\theta_0,x_k)a \right] k((x-\tilde{x})/h) f_X(x_k)\, dx \right|_-^p f_\mu(x_k,0)\, d\tilde{x}\, dh.\end{aligned}$$

Theorem 5.1. *Let*

$$a_n = an^{-\gamma/\{2[d_X+\gamma+(d_X+1)/p]\}}$$

for some vector a. Under Assumptions 3.3, 3.4, 4.1, and 4.2,

$$n^{1/2} T_{n,p,1,\mu}(\theta_0 + a_n) \xrightarrow{p} \left(\sum_{k=1}^{|\mathcal{X}_0|} \sum_{j \in \tilde{J}(k)} \lambda_{bdd}(a,j,k,p) \right)^{1/p} \equiv r_{bdd}(a)$$

where $r_{bdd}(a) \to 0$ as $a \to 0$.

Theorem 5.1 has immediate consequences for the power of tests based on CvM statistics with bounded weightings.

Theorem 5.2. *If, in addition to the conditions of Theorem 5.1, Assumption 3.1 holds, the power*

$$E\phi_{n,p,1,\mu}(\theta_0 + a_n)$$

of the CvM test with bounded weights will converge to zero for $r_{bdd}(a) < c$. If a is close enough to zero, $r_{bdd}(a)$ will be less than c so that the power will converge to zero under $\theta_0 + a_n$. If, in addition, Assumption 3.2 holds, the power under $\theta_0 + a_n$ given by the above display will converge to 1 for $r_{bdd}(a) > c$.

The $n^{-\gamma/\{2[d_X+\gamma+(d_X+1)/p]\}}$ rate for instrument based CvM statistics with bounded weights is slower than the $n^{-\gamma/\{2[d_X+\gamma]\}}$ rate derived for the corresponding KS test in Theorem 14 of Armstrong (2015) (for $\gamma = 2$) and Theorem 5.1 of Armstrong (2014b) (α from that paper plays the role of γ here). Note also that local power increases as p increases, and becomes aribrarily close to the rate for the KS test as p increases.

Theorem 5.2 shows that asymptotic power depends on the dgp through the quantity λ_{bdd} defined above, which in turn depends on other quantities such as $\bar{m}_{\theta,j}(\theta_0, x_k)a$. As will be seen in the sections below, the asymptotic power results for other statistics considered here will depend on similar quantities. We provide a detailed analysis of the quantities that determine asymptotic power in Section 5.4 below.

5.2 Instrument Based CvM Statistics with Variance Weights

Define

$$\begin{aligned}&\lambda_{\text{var}}(a, j, k, p)\\ &\equiv \int\int \left| \int \left[\|x\|^\gamma \psi_{j,k}\left(\frac{x}{\|x\|}\right) + \bar{m}_{\theta,j}(\theta_0, x_k)a \right] w_j(x_k) h^{-d_X/2} k((x-\tilde{x})/h) f_X(x_k)\, dx \right|_-^p \\ &f_\mu(x_k, 0)\, d\tilde{x}\, dh\end{aligned}$$

where $w_j(x_k) \equiv (s_j^2(x_k, \theta_0) f_X(x_k) \int k(u)^2\, du)^{-1/2}$.

Theorem 5.3. *Let*

$$a_n = an^{-\gamma/\{2[d_X/2+\gamma+(d_X+1)/p]\}}.$$

Suppose that $\sigma_n(n/\log n)^{1/2} \to \infty$ and Assumptions 3.3, 3.4, 4.1, and 4.2 hold. Then

$$n^{1/2} T_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta_0 + a_n) \leq \left(\sum_{k=1}^{|\mathcal{X}_0|} \sum_{j\in J(k)} \lambda_{var}(a, j, k, p) \right)^{1/p} + o_p(1) \equiv r_{var}(a) + o_p(1)$$

where $r_{var}(a) \to 0$ as $a \to 0$. If, in addition, $\sigma_n n^{d_X/\{4[d_X/2+\gamma+(d_X+1)/p]\}} \to 0$, the above display will hold with the inequality replaced by equality.

The result has immediate consequences for the power of tests based on CvM statistics with truncated variance weightings.

Theorem 5.4. *Let a_n be defined as in Theorem 5.3 and suppose that the conditions of that theorem and Assumption 3.1 hold. The power of the test based on the CvM statistic with truncated variance weights*

$$E\phi_{n,p,(\sigma\vee\sigma_n)^{-1},\mu}(\theta_0 + a_n)$$

will converge to zero for $r_{var}(a) < c$. For a close enough to 0, $r_{var}(a)$ will be less than c so that the asymptotic power under $\theta_0 + a_n$ will be 0. If, in addition, Assumption 3.2 holds and $\sigma_n n^{d_X/\{4[d_X/2+\gamma+(d_X+1)/p]\}} \to 0$, the power function under $\theta_0 + a_n$ given by the above display will converge to 1 for $r_{var}(a) > c$.

As with bounded weighting functions, the rate for detecting local alternatives with CvM statistics with variance weights is slower than the rate for the corresponding KS test. The $n^{-\gamma/\{2[d_X/2+\gamma+(d_X+1)/p]\}}$ rate for variance weighted CvM statistics derived above contrasts with the $(n/\log n)^{-\gamma/[2(d_X/2+\gamma)]}$ rate for the corresponding KS test derived in Armstrong and Chan (2016) and Armstrong (2014b) (the results from the latter paper on rates of convergence of confidence regions in the Hausdorff metric imply these local power results). The rate for CvM statistics approaches the rate for KS statistics as $p \to \infty$.

5.3 Statistics Based on Kernel Estimates

To describe the local asymptotic power functions, define

$$\lambda_{\text{kern}}(a,h,j,k,p) \equiv \int \left| \int \left[\|x\|^\gamma \psi_{j,k}\left(\frac{x}{\|x\|}\right) + \bar{m}_{\theta,j}(\theta_0,x_k)a \right] h^{-d_X} k((x-\tilde{x})/h)\omega_j(\theta_0,x_k)\,dx \right|_-^p d\tilde{x}.$$

and

$$\tilde{\lambda}_{\text{kern}}(a,j,k,p) \equiv \int \left| \left[[\|v\|^\gamma \psi_{j,k}\left(\frac{v}{\|v\|}\right) + \bar{m}_{\theta,j}(\theta_0,x_k)a \right] \omega_j(\theta_0,x_k) \right|_-^p dv.$$

Theorem 5.5. *Suppose that Assumptions 3.4, 4.1, 4.2 and 4.3 hold, and that the kernel function k satisfies Assumption 3.3. In addition, suppose that the bandwidth h satisfies $h/n^{-s} \to c_h$ for some $0 < s < 1/d_X$ and $c_h > 0$, the kernel function k satisfies $\int k(u)\,du = 1$ and that the functions $\psi_{j,k}$ in Assumption 4.1 are continuous. Let $a_n = an^{-q}$ for some*

$a \in \mathbb{R}^{d_\theta}$ *where*

$$q = \begin{cases} s\gamma & \text{if } s < 1/[2(\gamma + d_X/p + d_X/2)] \\ (1 - sd_X)/[2(1 + d_X/(p\gamma))] & \text{if } s \geq 1/[2(\gamma + d_X/p + d_X/2)] \end{cases}$$

and let $\theta_n = \theta_0 + a_n$. *If* $s > 1/[2(\gamma + d_X/p + d_X/2)]$, *then*

$$(nh^{d_X})^{1/2} T_{n,p,kern}(\theta_n) \xrightarrow{p} c_h^{d_X/2} \left(\sum_{k=1}^{|\mathcal{X}_0|} \sum_{j \in J(k)} \tilde{\lambda}_{kern}(a, j, k, p) \right)^{1/p} \equiv \tilde{r}_{kern}(a).$$

If $s = 1/[2(\gamma + d_X/p + d_X/2)]$, *then*

$$(nh^{d_X})^{1/2} T_{n,p,kern}(\theta_n) \xrightarrow{p} c_h^{d_X/2} \left(\sum_{k=1}^{|\mathcal{X}_0|} \sum_{j \in J(k)} \lambda_{kern}(a, c_h, j, k, p) \right)^{1/p} \equiv r_{kern}(a, c_h).$$

If $s < 1/[2(\gamma + d_X/p + d_X/2)]$, *then*

$$(nh^{d_X})^{1/2} T_{n,p,kern}(\theta_n)$$

will converge in probability to 0 *if*

$$\left(\sum_{k=1}^{|\mathcal{X}_0|} \sum_{j \in J(k)} \lambda_{kern}(a, c_h, j, k, p) \right)^{1/p}$$

is 0 *in a neighborhood of* (a, c_h), *and will converge to* ∞ *if this expression is strictly positive.*

The result has immediate implications for the power of tests based on kernel CvM statistics.

Theorem 5.6. *Let* a_n *be defined as in Theorem 5.5 and suppose that the conditions of that theorem and Assumption 3.1 hold. If* $s > 1/[2(\gamma + d_X/p + d_X/2)]$, *the power of the test based on the kernel CvM statistic*

$$E\phi_{n,p,kern}(\theta_0 + a_n)$$

will converge to zero for $\tilde{r}_{kern}(a) < c$. *If* $s = 1/[2(\gamma + d_X/p + d_X/2)]$, *the power given by the above display will converge to zero for* $\tilde{r}_{kern}(a, c_h) < c$. *If* $s < 1/[2(\gamma + d_X/p + d_X/2)]$, *the*

power given by the above display will converge to zero if $\tilde{r}_{kern}(a, c_h) = 0$ *in a neighborhood of* (a, c_h)*. If, in addition, Assumption 3.2 holds, the power given by the above display will converge to* 1 *if* $\tilde{r}_{kern}(a) > c$*,* $r_{kern}(a, c_h) > c$*, or* $r_{kern}(a, c_h) > 0$ *in the cases where* s *is greater than, equal to, or less than* $1/[2(\gamma + d_X/p + d_X/2)]$ *respectively.*

As with instrument based statistics, the rate for detecting local alternatives with the kernel CvM test is slower than the rate for the corresponding KS statistic. The rate derived in Theorem 5.5 can be written as $\max\{(nh^{d_X})^{-1/[2(1+d_X/(p\gamma))]}, h^\gamma\}$, which is slower than the $\max\left\{(nh^{d_X}/\log n)^{-1/2}, h^\gamma\right\}$ rate for kernel based KS statistics derived in Armstrong (2014b). As with the instrument based statistics, the CvM test is more powerful for p larger, and the rate approaches the rate for the KS test as p goes to ∞.

Theorem 5.5 can be used to choose the optimal bandwidth in this setting. The rate $a_n = an^{-q}$ is best when $s = 1/[2(\gamma + d_X/p + d_X/2)]$, which gives an exponent in the rate of

$$q = \frac{\gamma}{2(\gamma + d_X/p + d_X/2)} = \frac{1 - sd_X}{2(1 + d_X/(p\gamma))} = s\gamma.$$

Note that this rate is faster than the $n^{-\gamma/[2(d_X/2+\gamma+(d_X+1)/p))]}$ rate that can be obtained with instrument based CvM tests with variance weights. Thus, restricting the class of instruments using prior knowledge of the data generating process leads to a faster rate with CvM statistics. In contrast, instrument based KS statistics with variance weights can achieve the same rate as kernel KS statistics that use prior knowledge of the data generating process to choose the bandwidth optimally (cf. Armstrong, 2014b; Armstrong and Chan, 2016; Chetverikov, 2012).

5.4 Analysis of Asymptotic Power Functions

The asymptotic power results in this paper apply to a sequence $\theta_0 + a_n$ where $a_n = an^\rho$ for some rate exponent ρ and vector $a \in \mathbb{R}^{d_\theta}$. To understand how power varies with distance from the identified set, let us consider local power for a vector $a = t \cdot \tilde{a}$, where $\tilde{a} \in \mathbb{R}^{d_\theta}$ and $t > 0$ is a scalar. That is, we fix a direction $\tilde{a} \in \mathbb{R}^{d_\theta}$ and ask how power varies as we move in this direction away from θ_0 on the boundary of the identified set Θ_0.

For the instrument based CvM test with bounded weights, power will converge to zero if $r_{\text{bdd}}(t \cdot a) < c$, and will converge to one if $r_{\text{bdd}}(t \cdot a) > c$. Similar statements hold for the other tests with r_{bdd} replaced by r_{var}, $\tilde{r}_{\text{kern}}$ or r_{kern}. Thus, if we plot asymptotic power as a function of t, we obtain a step function where asymptotic power is either 0 or 1 depending

on whether t is above a certain value. This arises from the fact that these test statistics are degenerate under the null distributions considered here (see the discussion in Section 3).

The quantities r_{bdd}, r_{var}, $\tilde{r}_{\text{kern}}$ and r_{kern} which determine local power depend on the quantities λ_{bdd}, λ_{var}, λ_{kern} and $\tilde{\lambda}_{\text{kern}}$. Note that $\lambda_{\text{bdd}}(t \cdot \tilde{a}, j, k, p)$, $\lambda_{\text{var}}(t \cdot \tilde{a}, j, k, p)$ $\lambda_{\text{kern}}(t \cdot \tilde{a}, j, k, p)$ and $\tilde{\lambda}_{\text{kern}}(t \cdot \tilde{a}, j, k, p)$ are all zero when $\bar{m}_{j,\theta}(\theta_0, x_k)\tilde{a} \geq 0$, and are (for t large enough) strictly positive when $\bar{m}_{\theta,j}(\theta_0, x_k)\tilde{a} < 0$. The condition $\bar{m}_{\theta,j}(\theta_0, x_k)\tilde{a} < 0$ simply means that moving in the direction $\tilde{a}$ away from θ_0 moves the conditional mean $\bar{m}_j(\theta, x)$ downward at a point x_k where the inequality is binding (thereby ensuring that the null is indeed violated at $\theta_0 + t \cdot \tilde{a} \cdot n^{\rho}$), and that this shows up in the first derivative approximation. On the other hand, if $\bar{m}_{j,\theta}(\theta_0, x_k)\tilde{a} > 0$ at all points x_k where the inequality binds, then moving in the direction $\tilde{a}$ moves the conditional moments upward, so that the null still holds under θ_n.

To further analyze these quantities, note that, using similar arguments to the derivations in Section 2,

$$
\begin{aligned}
&\lambda_{\text{bdd}}(t \cdot \tilde{a}, j, k, p) \\
&= \int \int \left| \int \left[\|x\|^{\gamma} \psi_{j,k}\left(\frac{x}{\|x\|}\right) + t \cdot \bar{m}_{\theta,j}(\theta_0, x_k)\tilde{a} \right] k((x - \tilde{x})/h) f_X(x_k)\, dx \right|_{-}^{p} f_{\mu}(x_k, 0)\, d\tilde{x}\, dh \\
&= \int \int \left| \int \left[t\|u\|^{\gamma} \psi_{j,k}\left(\frac{u}{\|u\|}\right) + t \cdot \bar{m}_{\theta,j}(\theta_0, x_k)\tilde{a} \right] \right. \\
&\qquad \left. \cdot\, k((u - v)/\tilde{h}) f_X(x_k)\, t^{d_X/\gamma} du \right|_{-}^{p} f_{\mu}(x_k, 0)\, t^{d_X/\gamma} dv\, t^{1/\gamma} d\tilde{h} \\
&= t^{(1+d_X/\gamma)p + d_X/\gamma + 1/\gamma} \lambda_{\text{bdd}}(\tilde{a}, j, k, p)
\end{aligned}
$$

where we use the change of variables $u = x/t^{1/\gamma}$, $v = \tilde{x}/t^{1/\gamma}$ and $\tilde{h} = h/t^{1/\gamma}$. Thus,

$$
\begin{aligned}
r_{\text{bdd}}(t \cdot \tilde{a}) &= \left(\sum_{k=1}^{|\mathcal{X}_0|} \sum_{j \in \tilde{J}(k)} \lambda_{\text{bdd}}(t \cdot \tilde{a}, j, k, p) \right)^{1/p} \\
&= \left(t^{(1+d_X/\gamma)p + d_X/\gamma + 1/\gamma} \sum_{k=1}^{|\mathcal{X}_0|} \sum_{j \in \tilde{J}(k)} \lambda_{\text{bdd}}(\tilde{a}, j, k, p) \right)^{1/p} = t^{(1+d_X/\gamma) + (d_X+1)/(\gamma p)} r_{\text{bdd}}(\tilde{a}).
\end{aligned}
$$

Thus, when $r_{\text{bdd}}(\tilde{a}) > 0$ (which, as discussed above, will typically hold when the sequence of local alternatives θ_n is outside of the identified set), we obtain asymptotic power one when $t^{(1+d_X/\gamma)+(d_X+1)/(\gamma p)} r_{\text{bdd}}(\tilde{a})$ is greater than the asymptotic critical value c, and we obtain asymptotic power zero when $t^{(1+d_X/\gamma)+(d_X+1)/(\gamma p)} r_{\text{bdd}}(\tilde{a}) < c$. A similar argument using the

same change of variables shows that $r_{\text{var}}(t \cdot \tilde{a}) = t^{1+d_X/(2\gamma)+(d_X+1)/(\gamma p)} r_{\text{var}}(\tilde{a})$, so that the same analysis goes through for the variance weighted CvM statistic (with a different exponent for t).

For kernel based CvM statistics, local asymptotic power depends on $\tilde{r}_{\text{kern}}$ or r_{kern} depending on how quickly the bandwidth converges to zero. For the case where the bandwidth converges to zero quickly enough ($s > 1/[2(\gamma + d_X/p + d_X/2)]$), local power power is determined by $\tilde{r}_{\text{kern}}$. Note that

$$\begin{aligned}
\tilde{\lambda}_{\text{kern}}(t \cdot \tilde{a}, j, k, p) &= \int \left| \left[[\|v\|^\gamma \psi_{j,k}\left(\frac{v}{\|v\|}\right) + t \cdot \bar{m}_{\theta,j}(\theta_0, x_k)\tilde{a}\right] \omega_j(\theta_0, x_k) \right|_-^p dv \\
&= \int \left| \left[[t\|u\|^\gamma \psi_{j,k}\left(\frac{u}{\|u\|}\right) + t \cdot \bar{m}_{\theta,j}(\theta_0, x_k)\tilde{a}\right] \omega_j(\theta_0, x_k) \right|_-^p t^{d_X/\gamma} du \\
&= t^{p+d_X/\gamma} \tilde{\lambda}_{\text{kern}}(\tilde{a}, j, k, p)
\end{aligned}$$

where we use the change of variables $u = v/t^{1/\gamma}$. Thus, $\tilde{r}_{\text{kern}}(t \cdot \tilde{a}) = t^{1+d_X/(p\gamma)} \tilde{r}_{\text{kern}}(\tilde{a})$, and we have power approaching one or zero depending on whether t is large enough so that $t^{1+d_X/(p\gamma)} \tilde{r}_{\text{kern}}(\tilde{a})$ is greater than the limit c of the critical value.

For the case where the bandwidth decreases more slowly ($s \leq 1/[2(\gamma + d_X/p + d_X/2)]$), local power is determined by $r_{\text{kern}}(t \cdot \tilde{a}, c_h)$. For t small enough, this will be equal to zero so that power for θ_n will converge to 0. As t increases, $r_{\text{kern}}(t \cdot \tilde{a}, c_h)$ will be positive once t is large enough so long as $\bar{m}_{\theta,j}(\theta_0, x_k)\tilde{a} > 0$ for some j, k, and increases without bound as t increases in this case. However, unlike the other cases, $r_{\text{kern}}(t \cdot \tilde{a}, c_h)$ does not appear to have a simple form that is separable in t.

6 Monte Carlo

This section reports the results of a Monte Carlo study of the finite sample properties of the statistics considered in this paper. I perform Monte Carlos based on a median regression model with potentially endogenously missing data. I use the same data generating processes as for the Monte Carlos for variance weighted KS statistics in Armstrong and Chan (2016). A description of the model and data generating processes is repeated here for convenience.

The latent variable W_i^* follows a linear median regression model given the observed covariate X_i: $q_{1/2}(W_i^*|X_i) = \theta_1 + \theta_2 X_i$ where $q_{1/2}(W_i^*|X_i)$ is the conditional median of W_i^* given X_i. Define $W_i^H = W_i^*$ when W_i^* is observed and $W_i^H = \infty$ otherwise. This gives the conditional moment inequality $E[I(\theta_1 + \theta_2 X_i \leq W_i^H) - 1/2|X_i] \geq 0$ a.s. (a similar inequality

can be formed with the lower bound W_i^L defined analogously, but with $W_i^L = -\infty$ when W_i^* is unobserved, which would give the interval quantile regression setup of Section A.3 of the appendix; the Monte Carlos focus on the inequality corresponding to W_i^H for simplicity). This model allows for arbitrary correlation between the "missingness" process and (W_i^*, X_i), so that the resulting bounds can be used to assess sensitivity to missingness at random assumptions that would point identify the model.

Each design uses data from the true model $W_i^* = \theta_1^* + \theta_2^* X_i + u_i$, where $(\theta_1^*, \theta_2^*) = (0, 0)$ and u_i is independent of X_i with $u_i \sim \text{unif}(-1, 1)$. The outcome variable W_i^* is then set to be missing independently of W_i^* with probability $p(X_i)$ (note that, while the data are generated according to a missingness at random assumption and a particular parameter value, the tests are robust to failure of this assumption, which leads to a lack of point identification), where $p(x)$ is varied in each of three designs:

$$\begin{aligned}
&\text{Design 1:} \quad p(x) = .1 \\
&\text{Design 2:} \quad p(x) = .02 + 2 \cdot .98 \cdot |x - .5| \\
&\text{Design 3:} \quad p(x) = .02 + 4 \cdot .98 \cdot (x - .5)^2.
\end{aligned}$$

This leads to the identified set $\Theta_0 = \{(\theta_1, \theta_2)' | \theta_1 + \theta_2 x \leq q_{1/2}(W_i^H | X_i = x) \text{ all } x \in [0, 1]\}$ where $q_{1/2}(W_i^H | X_i = x)$ can be calculated for each design as $q_{1/2}(W_i^H | X_i = x) = 1/(1 - p(x)) - 1$. For each design, the Monte Carlo power of each test is reported for $\theta = (\overline{\theta}_1 + a, 0)$ where $\overline{\theta}_1 = \sup\{\theta_1 | (\theta_1, 0) \in \Theta_0\}$ and a varies over the set $\{.1, .2., .3, .4, .5\}$. This leads to local alternatives that satisfy the conditions of this paper with $\gamma = 1$ for Design 2 and $\gamma = 2$ for Design 3. Design 1 leads to a flat conditional mean for which asymptotic theory predicts the following rates (for the instrument functions used here): $n^{-1/2}$ for kernel and instrument based CvM and unweighted instrument based KS statistics, $(n/\log n)^{-1/2}$ for variance weighted instrument KS statistics and $(nh/\log n)^{-1/2}$ for kernel KS statistics (see Andrews and Shi, 2013; Armstrong, 2014b; Chernozhukov et al., 2013; Lee et al., 2013).

For the instrument based statistics, I use the class of functions $\{x \mapsto I(s < x < s + t) | 0 \leq s \leq s + t \leq 1\}$ and the the Lebesgue measure on $\{(s, t) | 0 \leq s \leq s + t \leq 1\}$ for μ for the instrument based CvM statistics. This corresponds to the multiscale kernel instruments in Assumption 3.3 with the uniform kernel. For the kernel based statistics, the uniform kernel is used, and the supremum or integral is taken over the set $[h/2, 1 - h/2]$, so that the support of the kernel function is always contained in the support of X_i. For the CvM statistics, the simulations use the test with L_p exponent $p = 1$. For each test statistic, the critical value is taken from the least favorable null distribution, calculated exactly (up to Monte Carlo

error) using the distribution under $(\overline{\theta}_1, 0)$ under Design 1. For the kernel estimators, the bandwidths $n^{-1/5}$, $n^{-1/3}$ and $n^{-1/2}$ are used, and, for the truncated variance weighted CvM statistics, the values $n^{-1/5}/4$, $n^{-1/3}/4$ and $n^{-1/2}/4$ are used for the truncation parameter σ_n^2 (this corresponds to truncating the variance of functions $I(s < x < s + t)$ with t less than $n^{-1/5}$, $n^{-1/3}$ and $n^{-1/2}$). For comparison, results for the variance weighted instrument KS statistic, which corresponds to the multiscale statistic of Armstrong and Chan (2016), are reported as well (taken directly from that paper).

Overall, the Monte Carlo results support the claim that, for the data generating processes and classes of instrument functions considered in the theoretical results in this paper, KS statistics perform better than CvM statistics. For Design 2 and Design 3, which follow the conditions of this paper with $\gamma = 1$ and $\gamma = 2$ respectively, the instrument based KS statistic has more power than the instrument based CvM statistic in basically all cases. For the kernel statistics, the KS test performs better unless the bandwidth is chosen to be much too small. For example, for Design 3, the optimal bandwidth for the kernel statistic is of order $n^{-1/5}$, and the kernel KS statistic performs better than the kernel CvM statistic with this bandwidth. However, the kernel statistic performs worse for smaller bandwidths when the sample size is not too large (although the KS statistic does almost as well or better with 1000 observations, suggesting that the asymptotics of Theorem 5.5 have started to kick in at this point).

Note also that power in the Monte Carlos is very sensitive to the design, with greater power for Design 3 than Design 2. This is to be expected given the asymptotic results. Under Design 3, the assumptions of this paper hold with $\gamma = 2$, while, under Design 2, the assumptions hold with $\gamma = 1$. The results of Section 5 show that asymptotic power is increasing in γ (the rate at which local alternatives may approach the null with nontrivial power is faster for larger γ) for each of the test statistics considered.

For Design 1, asymptotic results from elsewhere in the literature predict that the instrument based statistics with the instruments used here perform about the same (in terms of the rate for detecting local alternatives) for KS and CvM statistics, although the variance weighted KS statistic performs slightly worse (by a $\log n$ factor). For kernel statistics, asymptotic theory predicts that KS statistics will perform worse than CvM statistics in this case (the latter can achieve a $n^{-1/2}$ rate, while the former cannot if the bandwidth goes to zero). All of these predictions are borne out in the Monte Carlos: instrument based statistics all perform well with the weighted KS statistics performing slightly worse, while CvM version is better for kernel statistics.

The Monte Carlo results also fit well with the prescription of the weighted instrument KS or "multiscale" statistic of Armstrong (2011), Armstrong (2014b), Armstrong and Chan (2016) and Chetverikov (2012) as the only test among the ones considered here that comes close to having the best power among these test statistics for all three Monte Carlo designs (according to asymptotic approximations, the weighted instrument KS test achieves the best rate to at least within a $\log n$ factor in all three cases, while each of the other statistics considered here performs worse by a polynomial factor in at least one case). While other statistics perform slightly better in certain cases, they perform much worse in others (e.g. the kernel KS statistic performs slightly better in Design 3 with the optimal bandwidth, $n^{-1/5}$, but performs much worse when other bandwidths are chosen, or with any bandwidth choice in Design 1).

7 Conclusion

This paper derives local power results for tests for conditional moment inequality models based on several forms of CvM statistics in the set identified case. The power comparisons hold under conditions that arise naturally in the set identified case, and determine the minimax rate. Combined with results for KS statistics, these results can be used to decide on the test statistic, weighting function, class of instruments and critical value to maximize power in these models. The results show that KS tests are preferred to CvM statistics and that variance weightings are preferred to bounded weightings, and allow the researcher to choose the bandwidth optimally when a kernel based approach is used. In addition, these results show that, while choosing the critical value based on moment selection procedures or restricting the class of instrument functions has relatively little effect on power with variance weighted KS statistics, these choices can have a large effect on power with CvM statistics or unweighted KS statistics.

A Primitive Conditions and Minimax Bounds

This appendix gives primitive conditions for the assumptions used in this paper, and shows how the (pointwise in the underlying distribution) results for local alternatives considered in the paper can be used to bound the minimax power of CvM tests in classes of underlying distributions where the conditional mean is constrained only by smoothness assumptions. Since the corresponding KS statistic has a faster rate in these classes, this justifies the claim

that the CvM tests considered here perform worse in these models under a minimax criterion. Section A.1 gives general primitive conditions for the assumption that the contact set $\mathcal{X}_0$ in Assumption 4.1 is finite. Sections A.2, A.3 and A.4 provide primitive conditions for the assumptions used in this paper in various settings. Section A.5 uses the results in the body of this paper to give conditions under which the CvM statistics considered in this paper do not achieve the optimal rate minimax rate, and verifies these conditions for the interval regression model.

A.1 Primitive Conditions for Finite Contact Set

If we assume that the support of X_i is compact, and that the minimizing set $\{x|\bar{m}_j(\theta, x) = 0\}$ is contained on the interior of the support of X_i, then the minimizing set will be finite so long as $\bar{m}_j(\theta, x)$ is twice continuously differentiable with strictly positive definite second derivative matrix at any minimum. This follows from the proof of Lemma B.1 in the supplementary appendix of Armstrong (2015), and we state the result here for convenience. (Note that the lemma in Armstrong (2015) assumes a third derivative, since a third derivative is used for other results in that paper. However, a inspection of the proof shows that a continuous second derivative suffices.)

Lemma A.1. *Let $h : \mathcal{X} \to \mathbb{R}$ be twice continuously differentiable on the compact set $\mathcal{X} \subseteq \mathbb{R}^k$. Suppose that, for any minimizer $\tilde{x}$ of $h(x)$, $\tilde{x}$ is on the interior of $\mathcal{X}$, and that the second derivative matrix of h is strictly positive definite at $\tilde{x}$. Then the set of minimizers of $h(x)$ over $\mathcal{X}$ is finite.*

Proof. The result follows from the proof of Lemma B.1 in the supplementary appendix of Armstrong (2015). □

A.2 Interval Regression

This section gives primitive conditions for the interval regression model described in the Introduction, which falls into the setup of this paper with $W_i = (X_i, W_i^L, W_i^H)$ and $m(W_i, \theta) = (W_i^H - (1, X_i')\theta, (1, X_i')\theta - W_i^L)'$. First, I prove Theorem 4.1. Then, I give conditions under which the assumptions in the main text hold with $\gamma = 1$.

Proof of Theorem 4.1. First, note that the set of x such that $\bar{m}_j(\theta, x) = 0$ for some j is finite by Lemma A.1. Part (ii) of Assumption 4.1 follows from a second order Taylor expansion, and part (i) follows by compactness of the support of X_i and continuity of the first two

derivatives of the conditional means. Part (iv) is immediate from part (ii) of the conditions of the theorem and the fact that the conditional variance is constant in θ for this model. For part (v), note that $\frac{d}{d\theta}\bar{m}_1(\theta, x) = -\frac{d}{d\theta}\bar{m}_2(\theta, x) = (1, x')$, which is clearly continuous in (θ, x). Assumption 4.2 is immediate from the bounds on W_i^H and W_i^L. □

For the Lipschitz case ($\gamma = 1$), we can replace the assumption of two derivatives with a condition on the directional one-sided first derivatives. Here, we make the assumption of finiteness of the set where the conditional moments bind directly, since arguments involving second derivatives do not apply. In the following, $\mathbb{S}^{d_X-1}$ denotes the unit sphere $\{u \in \mathbb{R}^{d_X} | \|u\| = 1\}$.

Assumption A.1. *i.) The conditional means $E(W_i^H|X_i = x)$ and $E(W_i^L|X_i = x)$ are Lipschitz continuous, X_i has a continuous density and compact support, and W_i^H and W_i^L are bounded from above and below by finite constants.*

ii.) The set $\mathcal{X}_0 \equiv \{x|E(W_i^H|X_i = x) = (1, x')\theta_0\}$ is finite, and, for any point $\tilde{x} \in \mathcal{X}_0$, $\tilde{x}$ is in the interior of the support of X_i, $var(W_i^H|X_i = x)$ is positive and continuous at $\tilde{x}$ and the one-sided directional derivative $\frac{d}{dt_+}[E(W_i^H|X_i = \tilde{x}+tu) - (1, (\tilde{x}+tu)')\theta_0]$ is bounded from below away from zero at $t = 0$ and is right continuous at $t = 0$ uniformly over $u \in \mathbb{S}^{d_X-1}$. The same holds for $E(W_i^L|X_i = x)$ with "positive" replaced by "negative" in the last statement.

Theorem A.1. *Under Assumption A.1, Assumptions 4.1 and 4.2 hold, with $\gamma = 1$ in Assumption 4.1.*

Proof. Part (ii) of Assumption 4.1 follows from a first order Taylor expansion, and part (i) follows by compactness of the support of X_i and the continuity and lower bound on the directional derivatives. The verification of the remaining conditions is the same as in the twice differentiable case. □

A.3 Interval Quantile Regression

For the interval quantile regression model, the latent variable W_i^* follows a linear quantile regression model $q_\tau(W_i^*|X_i) = (1, X_i')\theta$, where τ is given and $q_\tau(U|V)$ denotes the τth conditional quantile of U given V for random variables U and V. As with interval mean regression, we observe (X_i, W_i^L, W_i^H) where $[W_i^L, W_i^H]$ is known to contain W_i^*. This falls into our setup with $m(W_i, \theta) = (\tau - I(W_i^H \leq (1, X_i')\theta), I(W_i^L \leq (1, X_i')\theta) - \tau)'$.

For the interval quantile regression model, one can use essentially the same assumptions as for the interval mean regression model considered above, but with conditional means replaced by conditional quantiles. In the interest of space, we consider only the case where the conditional quantile function has two derivatives ($\gamma = 2$).

Assumption A.2. *i.) The conditional quantiles $q_\tau(W_i^H|X_i = x)$ and $q_\tau(W_i^L|X_i = x)$ are twice differentiable with continuous second derivatives and X_i has a continuous density and compact support.*

ii.) For any $\tilde{x}$ such that $q_\tau(W_i^H|X_i = \tilde{x}) = (1, \tilde{x}')\theta_0$, $\tilde{x}$ is in the interior of the support of X_i and $q_\tau(W_i^H|X_i = x)$ has a positive definite second derivative matrix at $\tilde{x}$. The same holds for $q_\tau(W_i^L|X_i = x)$ with "positive definite" replaced by "negative definite."

In addition, we will also require an assumption on the conditional densities of W_i^H and W_i^L given X_i.

Assumption A.3. *For some $\eta > 0$, $W_i^H|X_i$ and $W_i^L|X_i$ have conditional densities $f_{W_i^H|X_i}(w|x)$ and $f_{W_i^L|X_i}(w|x)$ on $\{(x,w)|q_{\tau,P}(W_i^H|X_i = x) - \eta \leq w \leq q_{\tau,P}(W_i^H|X_i = x) + \eta\}$ and $\{(x,w)|q_{\tau,P}(W_i^L|X_i = x) - \eta \leq w \leq q_{\tau,P}(W_i^L|X_i = x) + \eta\}$ respectively that are continuous as a function of (x,w) and bounded away from zero on these sets.*

Assumption A.3 is similar to Assumption B.3 in Armstrong (2014b). As discussed in Armstrong (2014b), this type of condition will hold, for example, when (X_i, W_i^*) has a smooth joint density, and W_i^* is either missing (in which case $W_i^L = -\infty$ and $W_i^H = \infty$) or fully observed (in which case $W_i^L = W_i^H = W_i^*$), so long as the probability that W_i^* is missing conditional on $(X_i, W_i^*) = (x,w)$ is smooth as a function of (x,w).

Theorem A.2. *Suppose that Assumptions A.2 and A.3 hold. Then Assumptions 4.1 and 4.2 hold, with $\gamma = 2$ in Assumption 4.1.*

Proof. Let $\theta_0 \in \Theta_0$ satisfy the conditions of the theorem and let $\tilde{x}$ be such that $q_\tau(W_i^H|X_i = \tilde{x}) = (1, \tilde{x}')\theta_0$. Let $V(x)$ denote the second derivative matrix of $x \mapsto q_\tau(W_i^H|X_i = x)$. Then, for δ small enough and $\|x - \tilde{x}\| \leq \delta$,

$$\bar{m}_1(\theta, x) = \tau - P(W_i^H \leq (1, X_i')\theta_0|X_i = x) = \int_{(1,x')\theta_0}^{q_\tau(W_i^H|X_i=x)} f_{W_i^H|X_i}(w|x)\, dw$$
$$= \int_{(1,x')\theta_0}^{(1,x')\theta_0 + (x-\tilde{x})'V(\tilde{x})(x-\tilde{x}) + r(x)} f_{W_i^H|X_i}(w|x)\, dw$$

where $\lim_{x\to\tilde{x}} r(x) = 0$ and the last step follows from a second order Taylor expansion. This expression is bounded from above by $\overline{f}(\delta)\cdot[(x-\tilde{x})'V(\tilde{x})(x-\tilde{x})+\overline{r}(\delta)]$ and from below by $\underline{f}(\delta)\cdot[(x-\tilde{x})'V(\tilde{x})(x-\tilde{x})+\underline{r}(\delta)]$ where $\overline{f}(\delta)$ and $\overline{r}(\delta)$ are upper bounds for $f_{W_i^H|X_i}(w|x)$ and $r(x)$ on $\{(x,w)|\|x-\tilde{x}\|\leq\delta, (1,x')\theta_0 \leq w \leq q_\tau(W_i^H|X_i=x)\}$ and $\underline{f}(\delta)$ and $\underline{r}(\delta)$ are lower bounds. As $\delta\to 0$, $\overline{f}(\delta)$ and $\underline{f}(\delta)$ converge to $f_{W_i^H|X_i}((1,\tilde{x}')\theta_0|\tilde{x})$ and $\overline{r}(\delta)$ and $\underline{r}(\delta)$ converge to 0, so that

$$\sup_{\|x-\tilde{x}\|\leq\delta}\left\|\frac{\tau - P(W_i^H\leq(1,X_i')\theta_0|X_i=x)}{\|x-\tilde{x}\|^2} - \frac{(x-\tilde{x})'}{\|x-\tilde{x}\|}V(\tilde{x})\frac{(x-\tilde{x})'}{\|x-\tilde{x}\|}\cdot f_{W_i^H|X_i}((1,\tilde{x}')\theta_0|\tilde{x})\right\| \overset{\delta\to 0}{\to} 0.$$

Applying this argument to the finite set of values $\tilde{x}$ such that $\tau - P(W_i^H \leq (1,X_i')\theta_0|X_i = x) = 0$ and a symmetric argument for W_i^L, it follows that part (ii) of Assumption 4.1 holds with $\gamma = 2$.

To verify part (i) of Assumption 4.1 first note that the set $\mathcal{X}_0 = \{x|q_\tau(W_i^H|X_i=x) = (1,x')\theta\}$ is finite by Lemma A.1. Using this and similar arguments to those used in the proof of Theorem 4.1, there exists $\varepsilon > 0$ and $\delta > 0$ such that $q_\tau(W_i^H|X_i=x) - (1,x)'\theta$ is bounded away from zero for $\|\theta-\theta_0\| < \varepsilon$ and x such that, for all $\tilde{x}\in\mathcal{X}_0$, $\|x-\tilde{x}\|\geq\delta$. It then follows from Assumption A.3 that $\tau - P(W_i^H \leq (1,X_i')\theta_0|X_i=x)$ is bounded away from zero on such a set. Part (i) of Assumption 4.1 follows from this and a similar argument for W_i^L.

For part (iv) of Assumption 4.1, note that the conditional variance of the moment function corresponding to W_i^H is $P(W_i^H\leq(1,x')\theta|X_i=x)[1-P(W_i^H\leq(1,x')\theta|X_i=x)]$, so it suffices to show that $P(W_i^H\leq(1,x')\theta|X_i=x)$ is in the set $(0,1)$ and is continuous in (θ,x) at each $(\theta_0,\tilde{x})$ such that $\bar{m}_1(\theta,x) = P(W_i^H\leq(1,\tilde{x}')\theta_0|X_i=\tilde{x}) = \tau$. This follows since, by Assumption A.3, W_i^H has a continuous conditional density in a neighborhood of $(1,\tilde{x}')\theta_0$.

For part (v) of Assumption 4.1, note that, for (x,θ) such that W_i^H has a conditional density given $X_i = x$ at $(1,x')\theta$,

$$\bar{m}_{\theta,1}(\theta,x) = -\frac{d}{d\theta'}P(W_i^H\leq(1,x')\theta|X_i=x) = -f_{W_i^H|X_i=x}((1,x')\theta|x)(1,x').$$

This is continuous in (θ,x) in a small enough neighborhood of any $(\theta_0,\tilde{x})$ with $\bar{m}_{\theta,1}(\theta_0,\tilde{x}) = 0$, since $f_{W_i^H|X_i=x}(w|x)$ is continuous for w, x in a neighborhood of at $x=\tilde{x}$ and $w=(1,\tilde{x}')\theta_0$ for any such θ_0 and $\tilde{x}$ by Assumption A.3.

□

A.4 Selection Model

The interval regression model contains, as a special case, an approach to selection models based on bounds suggested in Manski (1990). In particular, consider a selection model in which we are interested in the mean of Y_i^*, which is not always observed. Suppose that Y_i^* is known to take values in $[\underline{Y}, \overline{Y}]$ for some fixed $\underline{Y}$ and $\overline{Y}$, and a variable X_i is available such that $E(Y_i^*|X_i) = E(Y_i^*)$ (i.e. Y_i^* is mean independent of X_i), and such that X_i shifts the conditional probability of observing Y_i^*. For example, we may be interested in the offer wage Y_i^*, which is typically only observed when individual i actually works. In this case, the variable X_i can be taken to be anything that shifts labor force participation through the opportunity cost of working (such as income from other sources such as family or government benefits) while being independent of the distribution of offer wages.

Let D_i denote an indicator variable that is 1 when Y_i^* is observed and 0 otherwise. We observe (X_i, Y_i, D_i) where $Y_i = D_i \cdot Y_i^*$. Following Manski (1990), note that, letting $W_i^L = Y_i \cdot D_i + \underline{Y} \cdot (1 - D_i)$ and $W_i^L = Y_i \cdot D_i + \overline{Y} \cdot (1 - D_i)$, we have $W_i^L \leq Y_i^* \leq W_i^H$ with probability one. Letting $\theta = E(Y_i^*)$ and using the fact that $E(Y_i^*) = E(Y_i^*|X_i)$ a.s., we obtain our setup with $m(W_i, X_i, \theta) = (W_i^H - \theta, \theta - W_i^L)'$. This is a special case of the interval regression model of Section A.2, with $(\theta, 0_{1\times d_X})$ playing the role of θ. That is, we have the interval regression model with the slope parameter constrained to be zero. Thus, if we consider a null value θ_0 and a sequence of alternatives in the interval regression model for which the slope parameter is zero, the results of Section A.2 apply immediately to give primitive conditions for Assumption 4.1 (here Assumption 4.2 holds by construction and the assumption that Y_i^* is bounded).

Note that $E(W_i^H|X_i = x) = E(Y_i^* D_i|X_i = x) + \overline{Y} \cdot [1 - P(D_i = 1|X_i = x)]$. Thus, a sufficient condition for $E(W_i^H|X_i = x)$ to be twice differentiable (or Lipschitz) is for $P(D_i = 1|X_i = x)$ and $E(Y_i^* D_i|X_i = x)$ to be twice differentiable (or Lipschitz). It is also worth noting that cases where $E(W_i^H|X_i = x)$ is minimized at the (possibly infinite) boundary of the support of X_i are often of interest, and arise naturally in this setting (see, e.g., Andrews and Schafgans 1998 and Heckman 1990). While Assumption 4.1 formally precludes the possibility that the minimum of $E(W_i^H|X_i = x)$ is taken at the boundary of the support of X_i, such cases can be handled for certain forms of instrument based statistics by transforming the support of X_i (see Section B.3 of Armstrong 2014b for an example of this type of argument applied to instrument based KS statistics). We leave this extension for future research.

A.5 Minimax Rates

The power results in this paper hold under conditions that are arguably common in practice in the set identified case. However, there are certainly cases (data generating processes, points on the boundary of the identified set and directions for the local alternative) for which other conditions will be appropriate. The purpose of this section is to show that, if the underlying distribution is constrained only by smoothness conditions and other regularity conditions, there will always exist a possible underlying distribution and sequence of local alternatives that satisfy these properties, with γ governed by the smoothness conditions imposed. Thus, any test that achieves good uniform power in these classes against alternatives that are closer than the pointwise rates derived here for CvM statistics will be preferred under a minimax criterion. By results in Armstrong (2014b), it follows that, for certain classes of alternatives defined by smoothness conditions, the variance weighted KS statistic of Armstrong (2014b), Armstrong and Chan (2016) and Chetverikov (2012) is preferred to the CvM statistics considered in this paper under a minimax criterion.

To formalize these ideas, the rest of this section considers classes $\mathcal{P}$ of underlying distributions and uses the notation E_P and $\Theta_0(P)$ to denote expectations and the identified set under a distribution P. In the results below, $d(\theta, \tilde{\theta})$ denotes the Euclidean distance $\|\theta - \tilde{\theta}\|$.

Theorem A.3. *Let $\phi_{CvM}(\theta)$ be one of the CvM tests defined in (11) or (12) with the critical value satisfying Assumption 3.1, the class $\mathcal{G}$ or kernel function k satisfying Assumption 3.3, and the measure μ satisfying Assumption 3.4 for the instrument case and the weighting satisfying Assumption 4.3 for the kernel case. Let $\mathcal{P}$ be any class of distributions such that, for some $P^* \in \mathcal{P}$ and θ_0^* on the boundary of $\Theta_0(P^*)$, Assumptions 4.1 and 4.2 hold, and either (a) θ_0^* is on the boundary of the convex hull of $\Theta_0(P^*)$ or (b) for some $a \in \mathbb{R}^{d_\theta}$ and a constant K, $d(\theta_0^*, \theta_0^* + ar) \leq K \cdot d(\theta_0, \theta_0^* + ar)$ for all $\theta_0 \in \Theta_0(P^*)$ and r small enough. Then, for a small enough constant $C_* > 0$,*

$$\limsup_{n\to\infty} \inf_{P\in\mathcal{P}} \inf_{\theta \text{ s.t. } d(\theta,\theta_0)\geq C_* r_n} \inf_{\text{all } \theta_0\in\Theta_0(P)} E_P \phi_{CvM}(\theta) = 0,$$

where r_n depends on the test and is given in Table 1 with γ given in Assumption 4.1.

Proof. Under condition (b), the result is immediate from the results in the main text, since the quantity in the display in the theorem is less than $\limsup_{n\to\infty} E_{P^*}\phi_{CvM}(\theta_0^* + aC_* r_n K/\|a\|)$ for P^*, θ_0^* and a given in the theorem. The result follows since condition (a) implies condition (b) with $K = 1$. To see this, note that, by the supporting hyperplane theorem, there exists a

vector a with $\|a\| = 1$ such that $a'\tilde{\theta}_0 \leq a'\theta_0^*$ for all $\tilde{\theta}_0$ in the convex hull of $\Theta_0(P^*)$. For this a and any scalar $r > 0$ and $\tilde{\theta}_0 \in \Theta_0(P^*)$, $d(\theta_0^*+ar, \tilde{\theta}_0)^2 - d(\theta_0^*+ar, \theta_0)^2 = \|\theta_0^*+ar-\tilde{\theta}_0\|^2 - r^2a'a = \|\theta_0^* - \tilde{\theta}_0\|^2 + 2ra'(\theta_0^* - \tilde{\theta}_0) + r^2a'a - r^2a'a \geq \|\theta_0^* - \tilde{\theta}_0\|^2 \geq 0$. □

A class $\mathcal{P}$ of underlying distributions will typically contain a P^* satisfying these conditions so long as it is sufficiently unrestricted (e.g. if the only restrictions are smoothness conditions, etc.). Theorems A.5 and A.6 below give primitive conditions for this in the interval regression model.

Under additional regularity conditions on $\mathcal{P}$, the inverse variance weighted KS statistic of Armstrong (2014b), Armstrong and Chan (2016) and Chetverikov (2012) achieves a strictly better minimax rate than the upper bounds for CvM statistics given in Theorem A.3. This is stated in the next theorem, which follows immediately from results in Armstrong (2014b) (the results in Armstrong, 2014b consider a stronger notion of coverage and power).

For concreteness, let us consider a specific version of the inverse variance weighted KS statistic considered in Armstrong (2014b). Let $T_{n,\infty,(\sigma\vee\sigma_n)^{-1}}(\theta)$ be given by (8) with $\mathcal{G} = \{x \mapsto I(\|x-\tilde{x}\| \leq h) | \tilde{x} \in \mathbb{R}^{d_X}, h \in [0,\infty)\}$ and $\omega_j(\theta, g) = \{\hat{\sigma}_j(\theta, g) \vee [(\log n)^2/n]\}^{-1}$. Let $\phi_{n,\infty,(\sigma\vee\sigma_n)^{-1}}(\theta)$ be given by (13) with this definition of $T_{n,\infty,(\sigma\vee\sigma_n)^{-1}}(\theta)$ and with $\hat{c}_{n,\infty,(\sigma\vee\sigma_n)^{-1}}$ given by the constant K in Theorem 3.1 in Armstrong (2014b). In the interest of concreteness, the above formulation uses certain conservative constants and tuning parameters in defining the test $\phi_{n,\infty,(\sigma\vee\sigma_n)^{-1}}(\theta)$. Less conservative and data driven methods for choosing these constants have been considered by Armstrong and Chan (2016) and Chetverikov (2012).

Theorem A.4. *Suppose that $\mathcal{P}$ satisfies Assumptions 4.1, 4.3, 4.4 and 4.5 in Armstrong (2014b), with γ taking the place of α in that paper. Then $\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} \sup_{\theta_0\in\Theta_0(P)} E_P\phi_{n,\infty,(\sigma\vee\sigma_n)^{-1}}(\theta_0) = 0$ and, for a large enough constant C^*,*

$$\liminf_{n\to\infty} \inf_{P\in\mathcal{P}} \inf_{\theta \text{ s.t. } d(\theta,\theta_0)\geq C^*[(\log n)/n]^{\gamma/(d_X+2\gamma)} \text{ all } \theta_0\in\Theta_0(P)} E_P\phi_{n,\infty,(\sigma\vee\sigma_n)^{-1}}(\theta) = 1.$$

Proof. Since Assumptions 3.1-3.3 in Armstrong (2014b) follow by definition of the statistic, the result follows from Theorem 4.2 in that paper, with Assumption 4.2(i) in Armstrong (2014b) following from Theorem 4.3 in that paper (since Assumption 4.6 and 4.2(ii) in that paper hold by construction). For $\mathcal{C}_n$ the setwise confidence set constructed from

$\phi_{n,\infty,(\sigma\vee\sigma_n)^{-1}}(\theta)$ in Armstrong (2014b),

$$\begin{aligned}
&\inf_{P\in\mathcal{P}} \inf_{\theta \text{ s.t. } d(\theta,\theta_0)\geq C^*[(\log n)/n]^{\gamma/(d_X+2\gamma)} \text{ all } \theta_0\in\Theta_0(P)} E_P\phi_{n,\infty,(\sigma\vee\sigma_n)^{-1}}(\theta) \\
&= \inf_{P\in\mathcal{P}} \inf_{\theta \text{ s.t. } d(\theta,\theta_0)\geq C^*[(\log n)/n]^{\gamma/(d_X+2\gamma)} \text{ all } \theta_0\in\Theta_0(P)} P(\theta\notin\mathcal{C}_n) \\
&\geq \inf_{P\in\mathcal{P}} P(\theta\notin\mathcal{C}_n \text{ all } \theta \text{ s.t. } d(\theta,\theta_0)\geq C^*[(\log n)/n]^{\gamma/(d_X+2\gamma)} \text{ all } \theta_0\in\Theta_0(P)) \\
&\geq \inf_{P\in\mathcal{P}} P(d_H(\Theta_0(P),\mathcal{C}_n) < C^*[(\log n)/n]^{\gamma/(d_X+2\gamma)})
\end{aligned}$$

where $d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\}$ is the Hausdorff distance. This converges to 1 for large enough C^* by Theorem 4.2 in Armstrong (2014b). □

The classes $\mathcal{P}$ used in Theorem A.4 impose smoothness conditions on the conditional mean along with a condition on the derivative of the conditional mean with respect to θ (cases where the latter condition fails appear to favor KS statistics over CvM statistics as well; see Section A.4 of Armstrong, 2014b). Note that the rate given above for the weighted KS statistic $\phi_{n,\infty,(\sigma\vee\sigma_n)^{-1}}$ corresponds to the minimax L_∞ rate for nonparametric testing problems (Lepski and Tsybakov, 2000) and to the minimax rate for estimating a conditional mean (Stone, 1982; see Menzel, 2010 for related results for estimating the identified set in a setting similar to the one considered here). The results here show that the CvM statistics considered here do not achieve this rate, and in fact have a minimax rate that is worse by at least a polynomial amount.

I now turn to the interval regression model and consider primitive conditions. The next two theorems show that certain classes of underlying distributions for the interval regression model will always contain a distribution with a sequence of local alternatives that satisfy the conditions of this paper. The conclusion of Theorem A.3 then follows immediately, since the identified set is convex in the interval regression model. Theorem A.5 considers the case where the constraints on the conditional mean embodied in $\mathcal{P}$ essentially only restrict the conditional means of W_i^H and W_i^L to a Lipschitz smoothness class. Theorem A.6 considers the smoother case where a bound is placed on the second derivative. For primitive conditions for the conditions of Theorem A.4 in the interval regression model for the case where $d_X = 1$ and $\gamma = 1$ or 2, see Armstrong (2014b), Section 6.2.

Theorem A.5. *Let $\mathcal{P}$ be any class of underlying distributions for (X_i, W_i^H, W_i^L) in the interval regression model such that, for all $P\in\mathcal{P}$, W_i^H and W_i^L are bounded and X_i has a continuous density on its support $\mathcal{X}_P$. Suppose that, for some set $\mathcal{X}\subseteq\mathbb{R}^{d_X}$ and some*

interval $[a, b]$*, the following holds: for any function* $f : \mathcal{X} \to [a, b]$ *such that*

$$|f(x) - f(\tilde{x})| \leq K\|x - \tilde{x}\|,$$

there exists a $P \in \mathcal{P}$ *such that* $E_P(W_i^H|X_i) = f(X_i)$ *and* $E_P(W_i^L|X_i) \leq a$ *almost surely, and* $\mathcal{X}_P = \mathcal{X}$*. Then there exists a* $P^* \in \mathcal{P}$ *and* $\theta_0^* \in \Theta_0(P^*)$ *that satisfies the conditions of Theorem A.3, with* $\gamma = 1$ *and* $\psi_{j,k}(u) = K$ *in Assumption 4.1.*

Proof. Under these assumptions, there exists a distribution $P \in \mathcal{P}$ such that $E_P(W_i^H|X_i = x) = b - K[(\varepsilon - \|x - x_0\|) \vee 0]$ for some $\varepsilon > 0$ and x_0 on the interior of the support of X_i, and $E_P(W_i^L|X_i = x)$ is bounded from above away from $b - 2\varepsilon$. For $\theta = (b - K\varepsilon, 0)$, this satisfies the conditions of Theorem A.1. □

Theorem A.6. *Let* $\mathcal{P}$ *be any class of underlying distributions for* (X_i, W_i^H, W_i^L) *in the interval regression model such that, for all* $P \in \mathcal{P}$*,* W_i^H *and* W_i^L *are bounded and* X_i *has a continuous density on its support* $\mathcal{X}_P$*. Suppose that, for some set* $\mathcal{X} \subseteq \mathbb{R}^{d_X}$ *and some interval* $[a, b]$*, for any function* $f : \mathcal{X} \to [a, b]$ *such that*

$$\left| \frac{d^2}{dt^2} f(x + tu) \right| \leq K$$

for all $u \in \mathbb{R}^{d_X}$ *with* $\|u\| = 1$*, there exists a* $P \in \mathcal{P}$ *such that* $E_P(W_i^H|X_i) = f(X_i)$ *and* $E_P(W_i^L|X_i) \leq a$ *almost surely, and* $\mathcal{X}_P = \mathcal{X}$*. Then there exists a* $P^* \in \mathcal{P}$ *and* $\theta_0^* \in \Theta_0(P^*)$ *that satisfies the conditions of Theorem A.3, with* $\gamma = 2$ *and* $\psi_{j,k}(u) = K/2$ *in Assumption 4.1.*

Proof. The result follows by similar arguments to Theorem A.5 since a function can be constructed for $E_P(W_i^H|X_i = x)$ that has a unique interior minimum with second derivative matrix KI at its minimum and takes values between, say, $(a + b)/2$ and b. □

B Proofs and Auxiliary Results

Section B.1 contains auxiliary results used in the rest of this appendix. These results are restatements or simple extensions of well known results on uniform convergence, and do not constitute part of the main novel contribution of the paper. Section B.2 of this appendix derives critical values for CvM statistics with variance weights. Section B.3 contains proofs of the results in the body of the paper.

B.1 Auxiliary Results

We state some results on uniform convergence that will be used in the proofs of the main results. The results in this section are essentially restatements of results used in Armstrong (2014b), which are in turn minor extensions of results in Pollard (1984). Throughout this section, we consider iid observations $Z_1, \ldots, Z_n$ and a sequence of classes of functions $\mathcal{F}_n$ on the sample space. Let $\sigma(f)^2 = Ef(Z_i)^2 - (Ef(Z_i))^2$ and let $\hat{\sigma}(f)^2 = E_n f(Z_i)^2 - (E_n f(Z_i))^2$.

Lemma B.1. *Suppose that $|f(Z_i)| \leq \overline{f}$ a.s. and that*

$$\sup_{n \in \mathbb{N}} \sup_{Q} N(\varepsilon, \mathcal{F}_n, L_1(Q)) \leq A\varepsilon^{-W}$$

for some A and W, where N is the covering number defined in Pollard (1984) and the supremum over Q is over all probability measures. Let σ_n be a sequence of constants with $\sigma_n \sqrt{n/\log n} \to \infty$. Then, for some constant C,

$$\frac{\sqrt{n}}{\sqrt{\log n}} \sup_{f \in \mathcal{F}_n} \left| \frac{(E_n - E)f(Z_i)}{\sigma(f) \vee \sigma_n} \right| \leq C$$

with probability approaching one and

$$\sup_{f \in \mathcal{F}_n} \left| \frac{(E_n - E)f(Z_i)}{\sigma(f)^2 \vee \sigma_n^2} \right| \xrightarrow{p} 0.$$

Proof. The first display follows by applying Lemma A.1 in Armstrong (2014b) to the sequence of classes of functions $\{f - E_P f(Z_i) | f \in \mathcal{F}_n\}$, which satisfies the conditions of that lemma by Lemma A.5 in Armstrong (2014b). The second display follows from the first display since

$$\sup_{f \in \mathcal{F}_n} \left| \frac{(E_n - E)f(Z_i)}{\sigma(f)^2 \vee \sigma_n^2} \right| \leq \frac{1}{\sigma_n} \sup_{f \in \mathcal{F}_n} \left| \frac{(E_n - E)f(Z_i)}{\sigma(f) \vee \sigma_n} \right| = \frac{\sqrt{\log n}}{\sigma_n \sqrt{n}} \frac{\sqrt{n}}{\sqrt{\log n}} \sup_{f \in \mathcal{F}_n} \left| \frac{(E_n - E)f(Z_i)}{\sigma(f) \vee \sigma_n} \right|$$

and $\sqrt{\log n}/(\sigma_n \sqrt{n}) \to 0$. □

Lemma B.2. *Under the conditions of Lemma B.1,*

$$\sup_{f \in \mathcal{F}_n} \left| \frac{\hat{\sigma}(f) \vee \sigma_n}{\sigma(f) \vee \sigma_n} - 1 \right| \xrightarrow{p} 0.$$

Proof. By continuity of $t \mapsto \sqrt{t}$ at 1, it suffices to prove that $\sup_{f \in \mathcal{F}_n} \left| \frac{\hat{\sigma}(f)^2 \vee \sigma_n^2}{\sigma(f)^2 \vee \sigma_n^2} - 1 \right| \xrightarrow{p} 0$. We

have

$$\sup_{f\in\mathcal{F}_n}\left|\frac{\hat{\sigma}(f)^2\vee\sigma_n^2}{\sigma(f)^2\vee\sigma_n^2}-1\right|=\sup_{f\in\mathcal{F}_n}\left|\frac{\hat{\sigma}(f)^2\vee\sigma_n^2-\sigma(f)^2\vee\sigma_n^2}{\sigma(f)^2\vee\sigma_n^2}\right|\leq\sup_{f\in\mathcal{F}_n}\left|\frac{\hat{\sigma}(f)^2-\sigma(f)^2}{\sigma(f)^2\vee\sigma_n^2}\right|.$$

Note that

$$\hat{\sigma}(f)^2-\sigma(f)^2=(E_n-E)[f(Z_i)-Ef(Z_i)]^2-[(E_n-E)f(Z_i)]^2. \tag{14}$$

Since $\sigma[(f-Ef(Z_i))^2]^2\leq E[f(Z_i)-Ef(Z_i)]^4\leq 4\overline{f}^2\sigma(f)^2$, we have

$$\sup_{f\in\mathcal{F}_n}\frac{|(E_n-E)[f(Z_i)-Ef(Z_i)]^2|}{\sigma(f)^2\vee\sigma_n^2}\leq\sup_{f\in\mathcal{F}_n}\frac{|(E_n-E)[f(Z_i)-Ef(Z_i)]^2|}{\sigma[(f-Ef(Z_i))^2]^2\vee\sigma_n^2}\cdot(4\overline{f}^2)\vee 1$$

which converges in probability to zero by Lemma B.1 (using Lemma A.5 in Armstrong, 2014b to verify that the sequence of classes of functions $\{[f-Ef(Z_i)]^2|f\in\mathcal{F}_n\}$ satisfies the conditions of the lemma). Since

$$\frac{[(E_n-E)f(Z_i)]^2}{\sigma(f)^2\vee\sigma_n^2}\xrightarrow{p}0$$

by Lemma B.1, the result now follows from this and the triangle inequality applied to (14). □

Lemma B.3. *Suppose that $|f(Z_i)|\leq\overline{f}$ and that $\sigma_n\sqrt{n}\geq 1$. Then*

$$E\left|\frac{\sqrt{n}(E_n-E)f(Z_i)}{\sigma(f)\vee\sigma_n}\right|^p\leq C_{p,\overline{f}}$$

for a constant $C_{p,\overline{f}}$ that depends only on p and $\overline{f}$.

Proof. By Bernstein's inequality,

$$P\left(\left|\frac{\sqrt{n}(E_n-E)f(Z_i)}{\sigma(f)\vee\sigma_n}\right|>t\right)\leq\exp\left(-\frac{1}{2}\frac{n[\sigma(f)\vee\sigma_n]^2t^2}{n\sigma^2(f)+\frac{1}{3}\cdot 2\overline{f}\cdot\sqrt{n}[\sigma(f)\vee\sigma_n]t}\right)$$
$$\leq\exp\left(-\frac{1}{2}\frac{t^2}{1+\frac{1}{3}\cdot 2\overline{f}\cdot\frac{t}{\sqrt{n}[\sigma(f)\vee\sigma_n]}}\right)\leq\exp\left(-\frac{1}{2}\frac{t^2}{1+\frac{1}{3}\cdot 2\overline{f}\cdot t}\right)\leq\exp\left(-\frac{1}{2}\frac{t^2}{1+\frac{1}{3}\cdot 2\overline{f}\cdot t}\right).$$

For $t \geq 1$, this is bounded by $\exp\left(-\frac{t}{2+\frac{2}{3}\cdot 2\overline{f}}\right)$. Thus,

$$E\left|\frac{\sqrt{n}(E_n - E)f(Z_i)}{\sigma(f) \vee \sigma_n}\right|^p = \int_{t=0}^{\infty} P\left(\left|\frac{\sqrt{n}(E_n - E)f(Z_i)}{\sigma(f) \vee \sigma_n}\right|^p > t\right) dt$$
$$\leq 1 + \int_{t=1}^{\infty} \exp\left(-\frac{t^{1/p}}{2 + \frac{2}{3} \cdot 2\overline{f}}\right) dt$$

which is finite and depends only on p and $\overline{f}$ as claimed. □

B.2 Critical Values for CvM Statistics with Variance Weights

For bounded choices of ω (which corresponds to σ_n bounded away from zero when a truncated variance weighting is used), Kim (2008) and Andrews and Shi (2013) derive a $\sqrt{n}$ rate of convergence to an asymptotic distribution that may be degenerate. Armstrong (2014b) shows that letting σ_n go to zero generally decreases the rate of convergence to $\sqrt{n/\log n}$ for the KS statistic $T_{n,\infty,\omega}$. In contrast to the KS case, CvM statistics do not behave much differently if the variance is allowed to go to zero, although some additional arguments are needed to show this.

To deal with the behavior of the CvM statistic for small variances, I place the following condition on the measure over which the sample means are integrated.

Assumption B.1. *$\mu(\{g|\sigma_j(\theta, g) \leq \delta\}) \to 0$ as $\delta \to 0$ for all j.*

This condition will hold for the choices of $\mathcal{G}$ and μ used in the body of the paper, and also allow for more general choices of $\mathcal{G}$ and μ. I also make the following assumption on the complexity of the class of functions $\mathcal{G}$, which is also satisfied by the class used in the paper.

Assumption B.2. *For some constants A and ε, the covering number $N(\varepsilon, \mathcal{G}, L_1(Q))$ defined in Pollard (1984) satisfies*

$$\sup_Q N(\varepsilon, \mathcal{G}, L_1(Q)) \leq A\varepsilon^{-W},$$

whre the supremum is over all probability measures.

The following condition imposes a bounded distribution of the function m.

Assumption B.3. *For some nonrandom constant $\overline{Y}$, $|m_j(W_i, \theta)| \leq \overline{Y}$ for each j with probability one.*

Theorem B.1. *Suppose that $\sigma_n\sqrt{n/\log n} \to \infty$ and that Assumptions B.1, B.2 and B.3 hold. Then, for $\theta \in \Theta_0$,*

$$n^{1/2}T_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta) \leq \left[\int \sum_{j=1}^{d_Y} \left|\frac{\sqrt{n}(E_n - E)m_j(W_i,\theta)g(X_i)}{\hat{\sigma}_j(\theta,g)\vee\sigma_n}\right|_-^p d\mu(g)\right]^{1/p}$$
$$\xrightarrow{d} \left[\int \sum_{j=1}^{d_Y} |\mathbb{G}_j(g,\theta)/\sigma_j(\theta,g)|_-^p \, d\mu(g)\right]^{1/p}$$

where $\mathbb{G}(g,\theta)$ is a vector of Gaussian processes with covariance function

$$\rho(g,\tilde{g}) = E[m(W_i,\theta)g(X_i) - Em(W_i,\theta)g(X_i)][m(W_i,\theta)\tilde{g}(X_i) - Em(W_i,\theta)\tilde{g}(X_i)]'.$$

Proof. The result with the integral truncated over $\{\sigma_j(\theta,g) \leq \delta | \text{all } j\}$ follows immediately from standard arguments using functional central limit theorems. This, along with Lemma B.4 below gives, letting $Z_n(\delta)$ be the integral truncated at $\{\sigma_j(\theta,g) \leq \delta | \text{all } j\}$ and $Z(\delta)$ be the limiting variable with this truncation,

$$P(Z_n(\delta) - \varepsilon \leq t) - \varepsilon \leq P(n^{1/2}T_{n,p,\omega,\mu}(\theta) \leq t) \leq P(Z_n(\delta) \leq t)$$

for large enough n for any $\varepsilon > 0$. The lim inf of the left hand size is greater than $P(Z(\delta) \leq t - 2\varepsilon) - 2\varepsilon$, and the lim sup of the right hand side is less than $P(Z(\delta) \leq t + \varepsilon) + \varepsilon$. We can bound $P(Z(\delta) \leq t - 2\varepsilon) - 2\varepsilon$ from below by $P(Z \leq t - 2\varepsilon) - 2\varepsilon$, and we can bound $P(Z(\delta) \leq t + \varepsilon) + \varepsilon$ from above by $P(Z \leq t + 2\varepsilon) + 2\varepsilon$ by making δ small enough by a version of Lemma B.4 for the limiting process. Since ε was arbitrary, this gives the result. □

The proof of the theorem above uses the following auxiliary lemma, which shows that functions g with low enough variance have little effect on the integral asymptotically.

Lemma B.4. *Fix j and suppose that Assumptions B.1, B.2 and B.3 hold, and that the null hypothesis holds under θ. Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$P\left(\sqrt{n}\left[\int_{\sigma_j(\theta,g)\leq\delta} |E_n m_j(W_i,\theta)g(X_i)/(\hat{\sigma}_j(\theta,g)\vee\sigma_n)|_-^p \, d\mu(g)\right]^{1/p} > \varepsilon\right) \leq \varepsilon.$$

Proof. We have

$$
\begin{aligned}
&E\int_{\sigma_j(\theta,g)\leq\delta} |\sqrt{n}E_n m_j(W_i,\theta)g(X_i)/(\sigma_j(\theta,g)\vee\sigma_n)|_-^p \, d\mu(g) \\
&= \int_{\sigma_j(\theta,g)\leq\delta} E|\sqrt{n}E_n m_j(W_i,\theta)g(X_i)/(\sigma_j(\theta,g)\vee\sigma_n)|_-^p \, d\mu(g) \\
&\leq \int_{\sigma_j(\theta,g)\leq\delta} E|\sqrt{n}(E_n-E) m_j(W_i,\theta)g(X_i)/(\sigma_j(\theta,g)\vee\sigma_n)|^p \, d\mu(g) \leq \mu\left(\{g|\sigma_j(\theta,g)\leq\delta\}\right)\cdot C_{p,\overline{Y}}
\end{aligned}
$$

for $C_{p,\overline{Y}}$ given in Lemma B.3. Applying Markov's inequality and using Assumption B.1, it follows that, for any $\varepsilon>0$, there exists a δ such that

$$
P\left(\sqrt{n}\left[\int_{\sigma_j(\theta,g)\leq\delta} |E_n m_j(W_i,\theta)g(X_i)/(\sigma_j(\theta,g)\vee\sigma_n)|_-^p \, d\mu(g)\right]^{1/p} > \varepsilon/2\right) \leq \varepsilon/2.
$$

The result follows since

$$
\begin{aligned}
&\sqrt{n}\left[\int_{\sigma_j(\theta,g)\leq\delta} |E_n m_j(W_i,\theta)g(X_i)/(\hat{\sigma}_j(\theta,g)\vee\sigma_n)|_-^p \, d\mu(g)\right]^{1/p} \\
&\leq \sqrt{n}\left[\int_{\sigma_j(\theta,g)\leq\delta} |E_n m_j(W_i,\theta)g(X_i)/(\sigma_j(\theta,g)\vee\sigma_n)|_-^p \, d\mu(g)\right]^{1/p} \cdot \sup_g (\sigma_j(\theta,g)\vee\sigma_n)/(\hat{\sigma}_j(\theta,g)\vee\sigma_n)
\end{aligned}
$$

and $\sup_g(\sigma_j(\theta,g)\vee\sigma_n)/(\hat{\sigma}_j(\theta,g)\vee\sigma_n)\leq 2$ with probability approaching one by Lemma B.2. □

B.3 Proofs

This section contains proofs of the results in the body of the paper. The proofs use a number of auxiliary lemmas, which are stated and proved first. In the following, θ_n is always assumed to be a sequence converging to θ_0.

Lemma B.5. *Under the assumptions of Theorem 5.5, there exists a constant C such that*

$$
\sup_{x\in\mathbb{R}^{d_X}} \frac{\sqrt{n}}{\sqrt{h^{d_X}\log n}} \left|(E_n-E)m(W_i,\theta_n)k((X_i-x)/h)\right| \leq C
$$

and

$$\sup_{x\in\mathbb{R}^{d_X}} \frac{\sqrt{n}}{\sqrt{h^{d_X}\log n}} |(E_n - E)k((X_i - x)/h)| \leq C$$

with probability approaching one. In addition,

$$\sup_{\{x|\omega_j(\theta_n,x)>0 \text{ some } j\}} \left| \frac{E_n k((X_i - h)/h)}{Ek((X_i - h)/h)} - 1 \right| \xrightarrow{p} 0.$$

Proof. The first two displays follow from Lemma B.1 after noting that

$$var(m(W_i, \theta_n)k((X_i - x)/h)) \leq \overline{Y}^2\overline{k}^2\overline{f}_X B^{d_X} h^{d_X}$$

where $\overline{k}$ and $\overline{f}_X$ are bounds for k and f_X, and B is such that $k(u) = 0$ whenever $\max_{1\leq j\leq d_X} |u_j| > B/2$, and similarly for $var(k((X_i - x)/h))$, and that $\sqrt{h^{d_X}}\sqrt{n}/\sqrt{\log n} \to \infty$ under these assumptions.

For the last display, note that, for x such that $\omega_j(\theta_n, x) > 0$ for some j, $Ek((X_i - x)/h) \geq \underline{f}_X h^{d_X} \int k(u)\, du$ for large enough n, where $\underline{f}_X$ is a lower bound for the density of X_i (which can be taken to be ε in Assumption 4.3). Thus,

$$\begin{aligned}
&\sup_{\{x|\omega_j(\theta_n,x)>0 \text{ some } j\}} \left| \frac{E_n k((X_i - h)/h)}{Ek((X_i - h)/h)} - 1 \right| \leq \sup_{x\in\mathbb{R}^{d_X}} \left| \frac{(E_n - E)k((X_i - h)/h)}{\underline{f}_X h^{d_X} \int k(u)\, du} \right| \\
&= \sup_{x\in\mathbb{R}^{d_X}} \frac{\sqrt{n}}{\sqrt{h^{d_X}\log n}} |(E_n - E)k((X_i - h)/h)| \cdot \frac{\sqrt{h^{d_X}\log n}}{\sqrt{n}\underline{f}_X h^{d_X} \int k(u)\, du}.
\end{aligned}$$

The result then follows from the second display, since $\frac{\sqrt{\log n}}{\sqrt{nh^{d_X}}} \to 0$. □

Let

$$\tilde{T}_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta) = \left[\int_{h>0} \int_x \sum_{j=1}^{d_Y} \left| \frac{E_n m(W_i, \theta)k((X_i - x)/h)}{\sigma_j(\theta, x, h) \vee \sigma_n} \right|_-^p f_\mu(x, h)\, dx\, dh \right]^{1/p}$$

and let

$$\tilde{T}_{n,p,\text{kern}}(\theta) = \left[\int_x \sum_{j=1}^{d_Y} \left| \frac{E_n m(W_i, \theta)k((X_i - x)/h)}{Ek((X_i - x)/h)} \right|_-^p \omega_j(\theta, x)\, dx\, dh \right]^{1/p}.$$

The notation $\sigma_j(\theta, \tilde{x}, h)$ is used to denote $\sigma_j(\theta, g)$ where $g(x) = k((x - \tilde{x})/h)$.

Lemma B.6. *Under Assumptions 3.3, 3.4, 4.1 and 4.2,*

$$\sqrt{n}T_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta_n) = \sqrt{n}\tilde{T}_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta_n)(1 + o_P(1))$$

for any sequence $\theta_n \to \theta_0$. If Assumption 4.3 holds as well, then

$$(nh^{d_X})^{1/2}T_{n,p,kern}(\theta_n) = (nh^{d_X})^{1/2}\tilde{T}_{n,p,kern}(\theta_n)(1 + o_P(1))$$

for any sequence $\theta_n \to \theta_0$.

Proof. We have

$$|\sqrt{n}T_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta_n) - \sqrt{n}\tilde{T}_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta_n)| \leq \sqrt{n}\tilde{T}_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta) \cdot \sup_{x,j}\left|\frac{\sigma_j(\theta_n, x, h)\vee\sigma_n}{\hat{\sigma}_j(\theta_n, x, h)\vee\sigma_n} - 1\right|.$$

Thus, the first display follows from Lemma B.2.

Similarly, for the second display,

$$\begin{aligned}&|(nh^{d_X})^{1/2}T_{n,p,\text{kern}}(\theta_n) - (nh^{d_X})^{1/2}\tilde{T}_{n,p,\text{kern}}(\theta_n)| \\ &\leq (nh^{d_X})^{1/2}\tilde{T}_{n,p,\text{kern}}(\theta_n) \cdot \sup_{\{x|\omega_j(\theta,x)>0 \text{ some } j\}}\left|\frac{Ek((X_i - x)/h)}{E_n k((X_i - x)/h)} - 1\right|,\end{aligned}$$

and the result follows from Lemma B.5. □

Let

$$\tilde{\tilde{T}}_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta) = \left[\int_{h>0}\int_x \sum_{j=1}^{d_Y}\left|\frac{Em(W_i, \theta)k((X_i - x)/h)}{\sigma_j(\theta, x, h)\vee\sigma_n}\right|_-^p f_\mu(x, h)\,dx\,dh\right]^{1/p}$$

and let

$$\tilde{\tilde{T}}_{n,p,\text{kern}}(\theta) = \left[\int_x \sum_{j=1}^{d_Y}\left|\frac{Em(W_i, \theta)k((X_i - x)/h)}{Ek((X_i - x)/h)}\right|_-^p \omega_j(\theta, x)\,dx\,dh\right]^{1/p}.$$

Also define

$$\tilde{\tilde{T}}_{n,p,1,\mu}(\theta) = \left[\int_{h>0} \int_x \sum_{j=1}^{d_Y} |Em(W_i,\theta)k((X_i-x)/h)|_-^p \, f_\mu(x,h)\, dx\, dh \right]^{1/p}.$$

Lemma B.7. *Under Assumptions 3.3, 3.4, 4.1 and 4.2,*

$$\sqrt{n}\tilde{T}_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta_n) = \sqrt{n}\tilde{\tilde{T}}_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta_n) + o_P(1).$$

and

$$\sqrt{n}T_{n,p,1,\mu}(\theta_n) = \sqrt{n}\tilde{\tilde{T}}_{n,p,1,\mu}(\theta_n) + o_P(1).$$

Proof. Let $\tilde{\sigma}_n \to 0$ be such that $\tilde{\sigma}_n\sqrt{n/\log n} \to \infty$ and $\tilde{\sigma}_n/\sigma_n \to 0$ (i.e. $\tilde{\sigma}_n$ is chosen to be much smaller than σ_n, but such that the assumptions still hold for $\tilde{\sigma}_n$). Note that

$$\begin{aligned}
&\sqrt{n}|\tilde{\tilde{T}}_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta_n) - \tilde{T}_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta_n)| \\
&\leq \left[\int\int_{(x,h)\in\hat{\mathcal{G}}} \sum_{j=1}^{d_Y} \left| \sqrt{n}\frac{(E_n - E)m(W_i,\theta_n)k((X_i-x)/h)}{\sigma_j(\theta,x,h)\vee\sigma_n} \right|^p f_\mu(x,h)\, dx\, dh \right]^{1/p}
\end{aligned}$$

where $\hat{\mathcal{G}} = \{(x,h)|Em(W_i,\theta_n)k((X_i-x)/h) < 0 \text{ or } E_n(W_i,\theta_n)k((X_i-x)/h) < 0\}$.

For any $\varepsilon > 0$, there exists an $\eta > 0$ such that, for $h > \varepsilon$ and large enough n,

$$Em_j(W_i,\theta_n)k((X_i-x)/h) \geq \eta Ek((X_i-x)/h) \geq \eta \cdot var[m_j(W_i,\theta_n)k((X_i-x)/h)] \cdot \frac{1}{\overline{kY}^2}$$

where the second inequality follows since

$$var[m_j(W_i,\theta_n)k((X_i-x)/h)] \leq \overline{Y}^2 E[k((X_i-x)/h)^2] \leq \overline{Y}^2\overline{k}Ek((X_i-x)/h).$$

Thus, for large enough n we will have

$$\begin{aligned}
&E_n m_j(W_i,\theta_n)k((X_i-x)/h) \\
&\geq (E_n - E)m_j(W_i,\theta_n)k((X_i-x)/h) + var[m_j(W_i,\theta_n)k((X_i-x)/h)] \cdot \frac{\eta}{\overline{kY}^2},
\end{aligned}$$

and the last line is positive for all (x,h) with $\sigma_j(\theta_n,x,h) \geq \tilde{\sigma}_n$ with probability approaching one by Lemma B.1.

From this and the fact that $Em(W_i,\theta_n)k((X_i-x)/h) \geq 0$ for all $h > \varepsilon$ for large enough n, it follows that $\hat{\mathcal{G}} \subseteq \{(x,h)|h \leq \varepsilon \text{ or } \sigma_j(\theta_n,x,h) < \tilde{\sigma}_n\}$ with probability approaching one. Note that

$$\begin{aligned} &E\int\int_{\{(x,h)|h\leq\varepsilon\}} \sum_{j=1}^{d_Y} \left|\frac{\sqrt{n}(E_n-E)m(W_i,\theta_n)k((X_i-x)/h)}{\sigma_j(\theta,x,h)\vee\sigma_n}\right|^p f_\mu(x,h)\,dx\,dh \\ &= \int\int_{\{(x,h)|h\leq\varepsilon\}} \sum_{j=1}^{d_Y} E\left|\frac{\sqrt{n}(E_n-E)m(W_i,\theta_n)k((X_i-x)/h)}{\sigma_j(\theta,x,h)\vee\sigma_n}\right|^p f_\mu(x,h)\,dx\,dh \end{aligned}$$

by Fubini's theorem, and this can be made arbitrarily small by making ε small by Lemma B.3 and Assumption 3.4. Similarly,

$$\begin{aligned} &E\int\int_{\{(x,h)|\sigma_j(\theta_n,x,h)<\tilde{\sigma}_n \text{ some } j\}} \sum_{j=1}^{d_Y} \left|\frac{\sqrt{n}(E_n-E)m(W_i,\theta_n)k((X_i-x)/h)}{\sigma_j(\theta,x,h)\vee\sigma_n}\right|^p f_\mu(x,h)\,dx\,dh \\ &\leq \mu(\mathbb{R}^{d_X}\times[0,\infty)) \cdot \sup_{\{(x,h,j)|\sigma_j(\theta_n,x,h)<\tilde{\sigma}_n\}} E\left|\frac{\sqrt{n}(E_n-E)m(W_i,\theta_n)k((X_i-x)/h)}{\sigma_j(\theta,x,h)\vee\sigma_n}\right|^p \\ &= \mu(\mathbb{R}^{d_X}\times[0,\infty)) \cdot \sup_{\{(x,h,j)|\sigma_j(\theta_n,x,h)<\tilde{\sigma}_n\}} E\left|\frac{\sqrt{n}(E_n-E)m(W_i,\theta_n)k((X_i-x)/h)}{\sigma_j(\theta,x,h)\vee\tilde{\sigma}_n}\right|^p \frac{\tilde{\sigma}_n}{\sigma_n}, \end{aligned}$$

which converges to zero by Lemma B.3. Using this and Markov's inequality, it follows that $\sqrt{n}|\tilde{\tilde{T}}_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta) - \tilde{T}_{n,p,(\hat{\sigma}\vee\sigma_n)^{-1},\mu}(\theta)|$ can be made arbitrarily small with probability approaching one by making ε small. This gives the first display of the lemma.

The second display follows by the same argument with σ_n set to the supremum of $\sigma_j(\theta,x,h)$ over x, h on the support of μ, θ in a neighborhood of θ_0 and all j. □

Lemma B.8. *Under Assumptions 3.3, 3.4, 4.1, 4.2 and 4.3,*

$$(nh^{d_X})^{1/2}\tilde{T}_{n,p,kern}(\theta_n) = (nh^{d_X})^{1/2}\tilde{\tilde{T}}_{n,p,kern}(\theta_n) + o_P(1).$$

Proof. For any $\varepsilon > 0$, there is an $\eta > 0$ such that $Em_j(W_i,\theta_n)k((X_i-x)/h) > \eta Ek((X_i-x)/h)$ for all $x \in \bar{\mathcal{X}}(\varepsilon)$ where $\bar{\mathcal{X}}(\varepsilon)$ is the set of x with $\|x-x_k\| \geq \varepsilon$ for all $k = 1,\ldots,\ell$ and $\omega_j(\theta_n,x) > 0$ for some j. Thus, arguing as in Lemma B.7 and using Lemma B.5, it follows

that, with probability approaching one,

$$
\begin{aligned}
&(nh^{d_X})^{1/2}|\tilde{T}_{n,p,\text{kern}}(\theta_n) - \tilde{\tilde{T}}_{n,p,\text{kern}}(\theta_n)| \\
&\leq \left[\int_{x\notin\bar{\mathcal{X}}(\varepsilon)} \sum_{j=1}^{d_Y} \left|\frac{\sqrt{nh^{d_X}}(E_n - E)m_j(W_i,\theta_n)k((X_i-x)/h)}{Ek((X_i-x)/h)}\right|^p \omega_j(\theta_n,x)\,dx\right]^{1/p}.
\end{aligned}
$$

Using Markov's inequality and Fubini's theorem along with the fact that $\int_{x\notin\bar{\mathcal{X}}(\varepsilon)} w_j(\theta_n x)\,dx$ can be made arbitrarily small by making ε small, the result follows so long as

$$
E\left|\frac{\sqrt{nh^{d_X}}(E_n - E)m_j(W_i,\theta_n)k((X_i-x)/h)}{Ek((X_i-x)/h)}\right|^p
$$

can be bounded uniformly over x such that $\omega_j(\theta_n,x) > 0$. But this follows from Lemma B.3, since, by Assumptions 3.3 and 4.3, for some $\delta > 0$, $Ek((X_i-x)/h) \geq \delta h^{d_X}$ for all x with $\omega_j(\theta_n,x) > 0$. □

For the following lemma, recall that $w_j(x_k) = (s_j^2(x_k,\theta_0)f_X(x_k)\int k(u)^2\,du)^{-1/2}$ and $s_j^2(x,\theta) = var(m(W_i,\theta)|X_i = x)$.

Lemma B.9. *Under Assumptions 3.3, 3.4, 4.1 and 4.2, for $k = 1,\ldots,\ell$*

$$
\sup_{\|(x,h)-(x_k,0)\|\leq\varepsilon_n} \left|h^{-d_X/2}\sigma_j(\theta_n,x,h) - w_j(x_k)^{-1}\right| \to 0.
$$

for any sequences $\varepsilon_n \to 0$ and $\theta_n \to \theta_0$.

Proof. By differentiability of the square root function at $w_j^{-2}(x_k)$, it suffices to show that $\sup_{\|(x,h)-(x_k,0)\|\leq\varepsilon_n} \left|h^{-d_X}\sigma_j^2(\theta_n,x,h) - w_j^{-2}(x_k)\right| \to 0$. Note that

$$
\begin{aligned}
&h^{-d_X}\sigma_j^2(\theta_n,x,h) = h^{-d_X}E[m(W_i,\theta_n)^2k((X_i-x)/h)^2] - h^{-d_X}\{E[m(W_i,\theta_n)k((X_i-x)/h)]\}^2 \\
&= h^{-d_X}\int s_j^2(\tilde{x},\theta_n)k((\tilde{x}-x)/h)^2 f_X(\tilde{x})\,d\tilde{x} \\
&+ h^{-d_X}\int E[m(W_i,\theta_n)|X_i=\tilde{x}]^2k((\tilde{x}-x)/h)^2 f_X(\tilde{x})\,d\tilde{x} \\
&- h^{-d_X}\left\{\int E[m(W_i,\theta_n)|X_i=\tilde{x}]k((\tilde{x}-x)/h)f_X(\tilde{x})\,d\tilde{x}\right\}^2.
\end{aligned}
$$

By Assumption 3.3 and part (iii) of Assumption 4.1, the second term is bounded by a constant times $\sup_{\|(x,h)-(x_k,0)\|\leq\varepsilon_n} E[m(W_i,\theta_n)|X_i = x]^2$, which converges to zero by continuity of

$E[m(W_i,\theta)|X_i = x]$ at (θ_0, x_k). By Assumptions 3.3 and 4.1, the third term is bounded by a constant times $h^{-d_X} \cdot h^{2d_X} \leq \varepsilon_n^{d_X}$ uniformly over (x,h) with $\|(x,h)-(x_k,0)\| \leq \varepsilon_n$. Using a change of variables, the first term can be written as $\int s_j^2(x+uh,\theta_n)k(u)^2 f_X(x+uh)\,du$, which converges to $w_j^{-2}(x_k)$ uniformly over $\|(x,h)-(x_k,0)\| \leq \varepsilon_n$ by continuity of s_j and f_X, and by Assumption 3.3. □

Lemma B.10. *Suppose that Assumptions 3.3, 3.4, 4.1, 4.2 and 4.3 hold, and that* $\int k(u)\,du = 1$. *Then*

$$\sup_{\|x-x_k\|\leq\varepsilon} |h^{-d_X} Ek((X_i - x)/h) - f_X(x_k)| \to 0$$

as $h \to 0$ *and* $\varepsilon \to 0$ *for* $k = 1, \ldots, \ell$.

Proof. We have

$$h^{-d_X} Ek((X_i - x)/h) = h^{-d_X} \int k((\tilde{x} - x)/h) f_X(\tilde{x})\,d\tilde{x} = \int k(u) f_X(x+uh)\,du,$$

and $\int k(u)\,du = 1$ and $f_X(x+uh)$ converges to $f_X(x_k)$ uniformly over $\|x - x_k\| \leq \varepsilon$ and u in the support of k as $\varepsilon \to 0$ and $h \to 0$. □

For notational convenience in the following lemmas, define, for (j,k) with $j \in J(k)$,

$$\tilde{\psi}_{j,k}(x - x_k) = \frac{\bar{m}_j(\theta_0, x) - \bar{m}_j(\theta_0, x_k)}{\|x - x_k\|^{\gamma(j,k)}}$$

so that

$$\sup_{\|x-x_k\|<\delta} \left| \tilde{\psi}_{j,k}(x - x_k) - \psi_{j,k}\left(\frac{x - x_k}{\|x - x_k\|}\right) \right| \to 0$$

under Assumption 4.1.

Lemma B.11. *Under Assumptions 3.3, 3.4, 4.1 and 4.2, for any* $a \in \mathbb{R}^{d_\theta}$,

$$r^{-[d_X + p(d_X+\gamma)+1]/\gamma} \int \int \sum_{j=1}^{d_Y} |Em_j(W_i, \theta_0 + ra)k((X_i - \tilde{x})/h)|_-^p f_\mu(\tilde{x}, h)\,d\tilde{x}\,dh$$

$$\overset{r\to 0}{\to} \sum_{k=1}^{\mathcal{X}_0} \sum_{j\in\tilde{J}(k)} \lambda_{bdd}(a, j, k, p).$$

Proof. For simplicity, assume that $\gamma(j,k) = \gamma$ for all j,k. The general result follows from applying the same arguments to show that areas of (x,h) near (j,k) with $\gamma(j,k) < \gamma$ do not matter asymptotically.

For C large enough, the integrand will be zero unless $\max\{\|\tilde{x} - x_k\|, h\} < Cr^{1/\gamma}$ for some k with $j \in J(k)$. Thus, it suffices to prove the lemma for, fixing (j,k) with $j \in J(k)$,

$$\begin{aligned}
&\int\int |Em_j(W_i, \theta_0 + ra)k((X_i - \tilde{x})/h)|_-^p f_\mu(\tilde{x}, h)\, d\tilde{x}\, dh \\
&= \int\int \left| \int \bar{m}_j(\theta_0 + ra, x)k((x - \tilde{x})/h) f_X(x)\, dx \right|_-^p f_\mu(\tilde{x}, h)\, d\tilde{x}\, dh \\
&= \int\int \left| \int [\|x - x_k\|^\gamma \tilde{\psi}_{j,k}(x - x_k) + \bar{m}_{\theta,j}(\theta^*(r), x)ra]k((x - \tilde{x})/h) f_X(x)\, dx \right|_-^p f_\mu(\tilde{x}, h)\, d\tilde{x}\, dh
\end{aligned}$$

where the integrals are taken over $\|\tilde{x} - x_k\| < Cr^{1/\gamma}, h < Cr^{1/\gamma}$ and $\theta^*(r)$ is between θ_0 and $\theta_0 + ra$ (we suppress the dependence of $\theta^*(r)$ on x in the notation). Using the change of variables $u = (x - x_k)/r^{1/\gamma}$, $v = (x - x_k)/r^{1/\gamma}$, $\tilde{h} = h/r^{1/\gamma}$, this is equal to

$$\begin{aligned}
&\int\int \left| \int [\|r^{1/\gamma}u\|^\gamma \tilde{\psi}_{j,k}(r^{1/\gamma}u) + \bar{m}_{\theta,j}(\theta^*(r), x_k + r^{1/\gamma}u)ra]k((u - v)/\tilde{h}) f_X(x_k + r^{1/\gamma}u) r^{d_X/\gamma}\, du \right|_-^p \\
&f_\mu(x_k + r^{1/\gamma}v, r^{1/\gamma}\tilde{h}) r^{d_X/\gamma}\, dv r^{1/\gamma}\, d\tilde{h} \\
&= r^{[d_X + 1 + p(\gamma + d_X)]/\gamma} \int\int \left| \int [\|u\|^\gamma \tilde{\psi}_{j,k}(r^{1/\gamma}u) + \bar{m}_{\theta,j}(\theta^*(r), x_k + r^{1/\gamma}u)a]k((u - v)/\tilde{h}) f_X(x_k + r^{1/\gamma}u)\, du \right|_-^p \\
&f_\mu(x_k + r^{1/\gamma}v, r^{1/\gamma}\tilde{h})\, dv\, d\tilde{h}
\end{aligned}$$

where the integrals are taken over $\|v\| < C, \tilde{h} < C$. The result now follows from the dominated convergence theorem (here, and in subsequent results involving sequences of the form $\int |\int g_n(z,w)\, d\mu(z)|_-^p\, d\nu(w)$, the dominated convergence theorem is applied to the inner integral for each w, and again to the outer integral).

□

Lemma B.12. *Under the conditions of Theorem 5.3, for any $a \in \mathbb{R}^{d_\theta}$,*

$$\begin{aligned}
&r^{-[d_X + p(d_X/2 + \gamma) + 1]/\gamma} \int\int \sum_{j=1}^{d_Y} |Em_j(W_i, \theta_0 + ra)k((X_i - \tilde{x})/h)/(\sigma_j(\theta_0 + ra, \tilde{x}, h) \vee \sigma_n)|_-^p f_\mu(\tilde{x}, h)\, d\tilde{x}\, dh \\
&\leq \sum_{k=1}^{\mathcal{X}_0} \sum_{j \in \tilde{J}(k)} \lambda_{var}(a, j, k, p) + o(1)
\end{aligned}$$

for any $r = r_n \to 0$. If, in addition, $\sigma_n r_n^{-d_X/(2\gamma)} \to 0$, the above display will hold with the inequality replaced by equality.

Proof. As in the previous lemma, the following argument assumes, for simplicity, that $\gamma(j,k) = \gamma$ for all (j,k) with $j \in J(k)$. Let $\tilde{s}_j(r,\tilde{x},h) = \sigma_j(\theta_0 + ra, \tilde{x}, h)/h^{d_X/2}$. As before, for large enough C, the integrand will be zero unless $\max\{\|\tilde{x} - x_k\|, h\} < Cr^{1/\gamma}$ for some k with $j \in J(k)$. Thus, it suffices to prove the result for, fixing (j,k) with $j \in J(k)$,

$$\begin{aligned}
&\int\int |Em_j(W_i, \theta_0 + ra)k((X_i - \tilde{x})/h)(h^{-d_X/2}\tilde{s}_j^{-1}(r,\tilde{x},h) \wedge \sigma_n^{-1})|_-^p f_\mu(\tilde{x},h)\, d\tilde{x}\, dh \\
&= \int\int \left| \int [\|x - x_k\|^\gamma \tilde{\psi}_{j,k}(x - x_k) + \bar{m}_{\theta,j}(\theta^*(r), x) ra] \right. \\
&\left. k((x - \tilde{x})/h)(h^{-d_X/2}\tilde{s}_j^{-1}(r,\tilde{x},h) \wedge \sigma_n^{-1}) f_X(x)\, dx \right|_-^p f_\mu(\tilde{x},h)\, d\tilde{x}\, dh
\end{aligned}$$

where the integral is taken over $\|\tilde{x} - x_k\| < Cr^{1/\gamma}$, $h < Cr^{1/\gamma}$ and $\theta^*(r)$ is between θ_0 and $\theta_0 + ra$. Using the change of variables $u = (x - x_k)/r^{1/\gamma}$, $v = (\tilde{x} - x_k)/r^{1/\gamma}$,$\tilde{h} = h/r^{1/\gamma}$, this is equal to

$$\begin{aligned}
&\int\int \left| \int r[\|u\|^\gamma \tilde{\psi}_{j,k}(r^{1/\gamma}u) + \bar{m}_{\theta,j}(\theta^*(r), x_k + ur^{1/\gamma})a] k((u - v)/\tilde{h}) \right. \\
&\left. (((r^{1/\gamma}\tilde{h})^{-d_X/2}\tilde{s}_j^{-1}(r, x_k + vr^{1/\gamma}, r^{1/\gamma}\tilde{h})) \wedge \sigma_n^{-1}) f_X(x_k + ur^{1/\gamma}) r^{d_X/\gamma}\, du \right|_-^p \\
&f_\mu(x_k + vr^{1/\gamma}, r^{1/\gamma}\tilde{h}) r^{d_X/\gamma}\, dv r^{1/\gamma}\, d\tilde{h} \\
&= r^{[p(\gamma + d_X/2) + d_X + 1]/\gamma} \int\int \left| \int [\|u\|^\gamma \tilde{\psi}_{j,k}(r^{1/\gamma}u) + \bar{m}_{\theta,j}(\theta^*(r), x_k + ur^{1/\gamma})a] k((u - v)/\tilde{h}) \right. \\
&\left. ((\tilde{h}^{-d_X/2}\tilde{s}_j^{-1}(r, x_k + vr^{1/\gamma}, r^{1/\gamma}\tilde{h})) \wedge (r^{d_X/(2\gamma)}\sigma_n^{-1})) f_X(x_k + ur^{1/\gamma})\, du \right|_-^p f_\mu(x_k + vr^{1/\gamma}, r^{1/\gamma}\tilde{h})\, dv\, d\tilde{h}.
\end{aligned}$$

where the integral is taken over $\|v\| < C$, $h < C$. By Lemma B.9 and the dominated convergence theorem, this converges to $\lambda_{var}(a,j,k,p)$ if $\sigma_n r_n^{-d_X/(2\gamma)} \to 0$. If $\sigma_n r_n^{-d_X/(2\gamma)}$ does not converge to zero, the above display is bounded from above by the same expression with σ_n^{-1} replaced by ∞.

□

Lemma B.13. *Under the conditions of Theorem 5.5, for any $a \in \mathbb{R}^{d_\theta}$,*

$$r^{-(\gamma p + d_X)/\gamma} \int \sum_{j=1}^{d_Y} |[Em_j(W_i, \theta_0 + ra)k((X_i - x)/h)/Ek((X_i - x)/h)]\omega_j(\theta_0 + ra, x)|_-^p \, dx$$
$$\to \sum_{k=1}^{|\mathcal{X}_0|} \sum_{j \in J(k)} \lambda_{kern}(a, c_{h,r}, j, k, p)$$

as $r \to 0$ with $h/r^{1/\gamma} \to c_{h,r}$ for $c_{h,r} > 0$. If the limit is zero for $(a, c_{h,r})$ in a neighborhood of the given values, the sequence will be exactly equal to zero for large enough r.

If $h/r^{1/\gamma} \to 0$, then, as $r \to 0$,

$$r^{-(\gamma p + d_X)/\gamma} \int \sum_{j=1}^{d_Y} |[Em_j(W_i, \theta_0 + ra)k((X_i - x)/h)/Ek((X_i - x)/h)]\omega_j(\theta_0 + ra, x)|_-^p \, dx$$
$$\to \sum_{k=1}^{|\mathcal{X}_0|} \sum_{j \in J(k)} \tilde{\lambda}_{kern}(a, j, k, p).$$

Proof. As before, this proof treats the case where $J(k) = \tilde{J}(k)$ for ease of exposition. As with the proofs of Lemmas B.11 and B.12, it suffices to prove the result for, fixing (j, k) with $j \in J(k)$,

$$\int |[Em_j(W_i, \theta_0 + ra)k((X_i - \tilde{x})/h)/Ek((X_i - \tilde{x})/h)]\omega_j(\theta_0 + ra, \tilde{x})|_-^p \, d\tilde{x}$$
$$= \int \left| \int [\|x - x_k\|^\gamma \tilde{\psi}_{j,k}(x - x_k) + \bar{m}_{\theta,j}(\theta^*(r), x)ra]k((x - \tilde{x})/h) f_X(x) \, dx h^{-d_X} b(\tilde{x}) \omega_j(\theta_0 + ra, \tilde{x}) \right|_-^p d\tilde{x}$$

where the integral is over $\|\tilde{x} - x_k\| < Cr^{1/\gamma}$ and $b(\tilde{x}) \equiv h^{d_X}/Ek((X_i - \tilde{x})/h)$ converges to $(f_X(x_k))^{-1}$ uniformly over $\tilde{x}$ in any shrinking neighborhood of x_k by Lemma B.10. Let $\tilde{h} = h/r^{1/\gamma}$. By the change of variables $u = (x - x_k)/r^{1/\gamma}$, $v = (\tilde{x} - x_k)/r^{1/\gamma}$, the above

display is equal to

$$\int \left| \int [\|ur^{1/\gamma}\|^\gamma \tilde{\psi}_{j,k}(ur^{1/\gamma}) + \bar{m}_{\theta,j}(\theta^*(r), x_k + ur^{1/\gamma})ra] k((u-v)/\tilde{h}) f_X(x_k + ur^{1/\gamma}) r^{d_X/\gamma}\, du \right.$$
$$\left. (r^{1/\gamma}\tilde{h})^{-d_X} b(x_k + vr^{1/\gamma}) \omega_j(\theta_0 + ra, x_k + r^{1/\gamma}v) \right|_-^p r^{d_X/\gamma}\, dv$$
$$= r^{p+d_X/\gamma} \int \left| \int [\|u\|^\gamma \tilde{\psi}_{j,k}(ur^{1/\gamma}) + \bar{m}_{\theta,j}(\theta^*(r), x_k + ur^{1/\gamma})a] k((u-v)/\tilde{h}) f_X(x_k + ur^{1/\gamma})\, du \right.$$
$$\left. \tilde{h}^{-d_X} b(x_k + vr^{1/\gamma}) \omega_j(\theta_0 + ra, x_k + r^{1/\gamma}v) \right|_-^p dv \tag{15}$$

where the integral is over $v < C$. The first display of the lemma (the case where $h/r^{1/\gamma} \to c_{h,r}$ for $c_{h,r} > 0$) follows from this and the dominated convergence theorem.

To show that the sequence is exactly zero for small enough r when the limit is zero in a neighborhood of $(a, c_{h,r})$, note, that, if the limit is zero in a neighborhood of $(a, c_{h,r})$, we will have, for all $(\tilde{a}, \tilde{c}_{h,r})$ in this neighborhood and any v,

$$\int \left[\|u\|^\gamma \psi_{j,k}\left(\frac{u}{\|u\|}\right) + \bar{m}_{\theta,j}(\theta_0, x_k)\tilde{a} \right] k((u-v)/\tilde{c}_{h,r})\, du$$
$$= \int \left[\tilde{c}_{h,r}^\gamma \|\tilde{u}\|^\gamma \psi_{j,k}\left(\frac{u}{\|u\|}\right) + \bar{m}_{\theta,j}(\theta_0, x_k)\tilde{a} \right] k(\tilde{u} - \tilde{v})\, \tilde{c}_{h,r}^{d_X} d\tilde{u} \geq 0.$$

Evaluating this at $(\tilde{c}_{r,h}, \tilde{a})$ such that $\tilde{c}_{h,r}^\gamma \leq c_{h,r}^\gamma(1-\varepsilon)$ and (for the case where $\bar{m}_{\theta,j}(\theta_0, x_k)a$ is negative) $\bar{m}_{\theta,j}(\theta_0, x_k)\tilde{a} \leq (\bar{m}_{\theta,j}(\theta_0, x_k)a)(1+\varepsilon)$ shows that

$$\int \left[c_{h,r}^\gamma \|\tilde{u}\|^\gamma \psi_{j,k}\left(\frac{u}{\|u\|}\right) \cdot (1-\varepsilon) + (\bar{m}_{\theta,j}(\theta_0, x_k)a)(1+\varepsilon) \right] k(\tilde{u} - \tilde{v})\, d\tilde{u} \geq 0$$

for all v for some $\varepsilon > 0$. The above display is, for small enough r, a lower bound for the inner integral in (15) times a constant that does not depend on r, so that, for small enough r, the inner integral in (15) will be nonnegative for all v and (15) will eventually be equal to zero.

For the case where $\tilde{h} = h/r^{1/\gamma} \to 0$, multiplying (15) by $r^{-(p+d_X/\gamma)}$ gives, after the change of variables $\tilde{u} = (u-v)/\tilde{h}$,

$$\int \left| \int [\|\tilde{h}\tilde{u} + v\|^\gamma \tilde{\psi}_{j,k}((\tilde{h}\tilde{u} + v)r^{1/\gamma}) + \bar{m}_{\theta,j}(\theta^*(r), x_k + (\tilde{h}\tilde{u} + v)r^{1/\gamma})a] k(\tilde{u}) f_X(x_k + (\tilde{u}\tilde{h} + v)r^{1/\gamma})\, d\tilde{u} \right.$$
$$\left. b(x_k + vr^{1/\gamma}) \omega_j(\theta_0 + ra, x_k + r^{1/\gamma}v) \right|_-^p dv$$

which converges to

$$\int |[\|v\|^{\gamma}\psi_{j,k}(v/\|v\|) + \bar{m}_{\theta,j}(\theta_0, x_k)a]\omega_j(\theta_0, x_k)|_-^p \, dv$$

by the dominated convergence theorem, as required.

□

We are now ready for the proofs of the main results.

proof of Theorem 5.1. The result follows immediately from Lemmas B.7 and B.11 since $(n^{-\gamma/\{2[d_X+\gamma+(d_X+1)/p]\}})^{-[d_X+p(d_X+\gamma)+1]/(\gamma p)} = n^{1/2}$. □

proof of Theorem 5.3. The result follows immediately from Lemmas B.6, B.7 and B.12 since $(n^{-\gamma/\{2[d_X/2+\gamma+(d_X+1)/p]\}})^{-[d_X+p(d_X/2+\gamma)+1]/(\gamma p)} = n^{1/2}$. □

proof of Theorem 5.5. The result follows from Lemmas B.6, B.8 and B.13. Note that $(nh^{d_X})^{p/2}/(n^{1-d_X s})^{p/2}$ $c_h^{d_X p/2}$, and that, for the case where $s \geq 1/[2(\gamma + d_X/p + d_X/2)$,

$$(n^{-q})^{-(\gamma p+d_X)/(\gamma p)} = (n^{-(1-sd_X)/[2(1+d_X/(p\gamma))]})^{-(\gamma p+d_X)/(\gamma p)} = n^{(1-sd_X)/2}.$$

For the case where $s < 1/[2(\gamma + d_X/p + d_X/2)]$, it follows from Lemmas B.6, B.8 and B.13 that

$$n^{q(\gamma p+d_X)/(\gamma p)} T_n(\theta_0 + a_n) \xrightarrow{p} \left(\sum_{k=1}^{|\mathcal{X}_0|} \sum_{j \in J(k)} \lambda_{\text{kern}}(a, c_h, j, k, p) \right)^{1/p}$$

so that $(nh^{d_X})^{1/2} T_n(\theta_0 + a_n)$ will converge to ∞ in this case if the limit in the above display is strictly positive. If the limit in the above display is zero in a neighborhood of (a, c_h), it follows from Lemmas B.6 and B.8 that $(nh^{d_X})^{1/2} T_n(\theta_0 + a_n)$ is, up to $o_p(1)$, equal to a term that is zero for large enough n by Lemma B.13.

□

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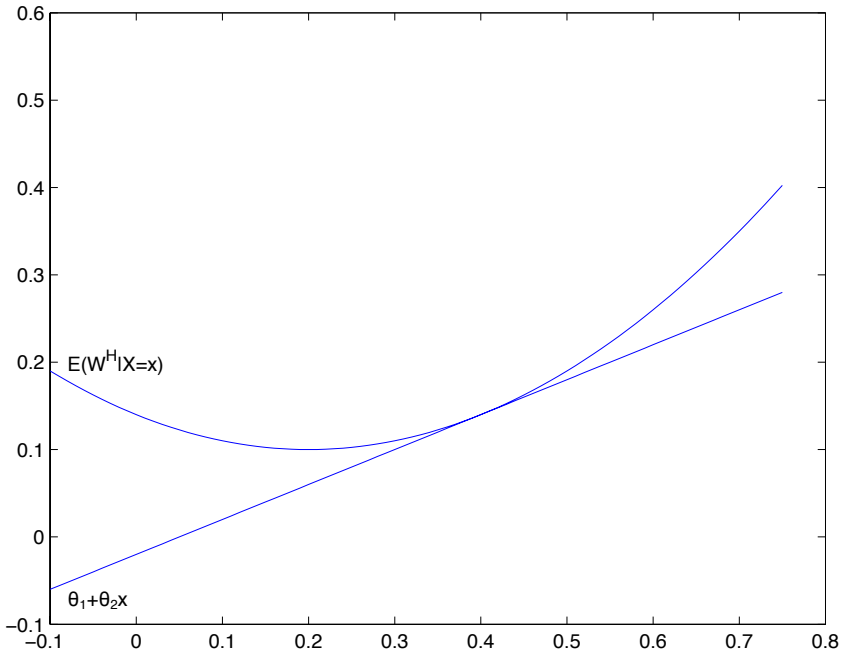

Figure 1: Case where Assumption 4.1 holds with $\gamma = 2$

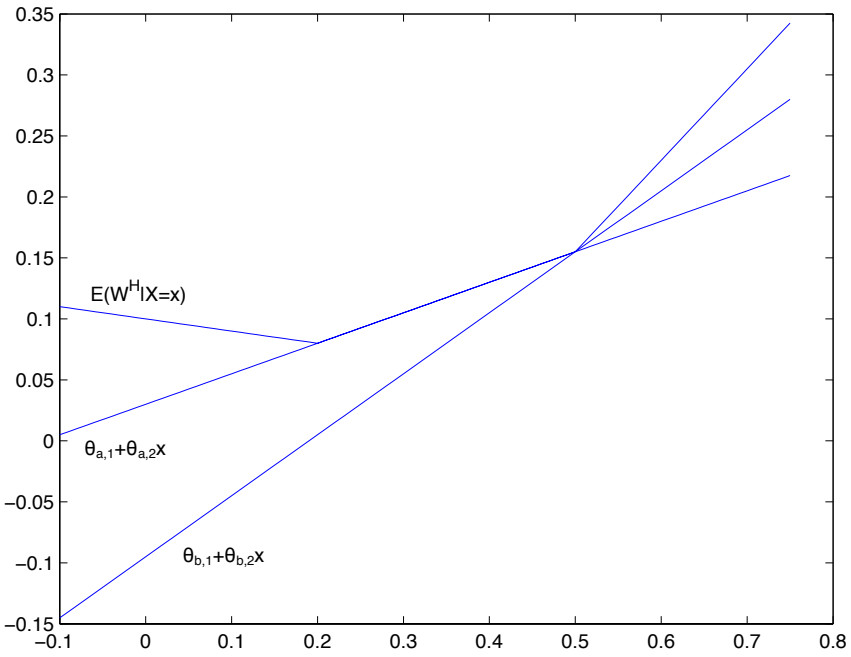

Figure 2: Case where Assumption 4.1 does not hold (θ_a) and case where Assumption 4.1 holds with $\gamma = 1$ (θ_b)

$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
0.1	0.196	0.593	0.818
0.2	0.458	0.973	1
0.3	0.775	1	1
0.4	0.952	1	1
0.5	0.995	1	1

Table 3: Power for Unweighted Instrument CvM Test under Design 1

$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
0.1	0.166	0.644	0.835
0.2	0.442	0.989	1
0.3	0.781	1	1
0.4	0.957	1	1
0.5	0.994	1	1

Table 4: Power for Unweighted Instrument KS Test under Design 1

σ_n^2	$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
$\frac{1}{4}n^{-1/5}$	0.1	0.198	0.567	0.859
	0.2	0.49	0.977	1
	0.3	0.77	1	1
	0.4	0.955	1	1
	0.5	0.997	1	1
$\frac{1}{4}n^{-1/3}$	0.1	0.208	0.62	0.851
	0.2	0.475	0.983	1
	0.3	0.808	1	1
	0.4	0.958	1	1
	0.5	0.994	1	1
$\frac{1}{4}n^{-1/2}$	0.1	0.203	0.591	0.822
	0.2	0.474	0.981	1
	0.3	0.804	1	1
	0.4	0.946	1	1
	0.5	0.996	1	1

Table 5: Power for Weighted Instrument CvM Test under Design 1

t_n	$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
$n^{-1/5}$	0.1	0.207	0.503	0.729
	0.2	0.48	0.954	1
	0.3	0.759	1	1
	0.4	0.956	1	1
	0.5	0.997	1	1
$n^{-1/3}$	0.1	0.144	0.453	0.63
	0.2	0.378	0.939	0.998
	0.3	0.691	1	1
	0.4	0.886	1	1
	0.5	0.982	1	1
$n^{-1/2}$	0.1	0.156	0.358	0.502
	0.2	0.348	0.898	0.991
	0.3	0.649	0.999	1
	0.4	0.862	1	1
	0.5	0.974	1	1

Table 6: Power for Weighted Instrument KS Test under Design 1 (from Armstrong and Chan (2016))

h_n	$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
$n^{-1/5}$	0.1	0.186	0.547	0.858
	0.2	0.453	0.97	1
	0.3	0.729	1	1
	0.4	0.934	1	1
	0.5	0.994	1	1
$n^{-1/3}$	0.1	0.188	0.663	0.843
	0.2	0.452	0.987	1
	0.3	0.794	1	1
	0.4	0.947	1	1
	0.5	0.997	1	1
$n^{-1/2}$	0.1	0.185	0.582	0.848
	0.2	0.443	0.977	1
	0.3	0.78	1	1
	0.4	0.942	1	1
	0.5	0.997	1	1

Table 7: Power for Kernel CvM Test under Design 1

h_n	$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
$n^{-1/5}$	0.1	0.16	0.439	0.625
	0.2	0.343	0.92	0.997
	0.3	0.62	0.999	1
	0.4	0.883	1	1
	0.5	0.975	1	1
$n^{-1/3}$	0.1	0.095	0.266	0.481
	0.2	0.201	0.715	0.929
	0.3	0.382	0.976	1
	0.4	0.606	0.999	1
	0.5	0.809	1	1
$n^{-1/2}$	0.1	0	0.094	0.138
	0.2	0	0.255	0.404
	0.3	0	0.508	0.773
	0.4	0	0.812	0.982
	0.5	0	0.976	1

Table 8: Power for Kernel KS Test under Design 1

$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
0.1	0	0	0
0.2	0.001	0	0
0.3	0.005	0	0
0.4	0.008	0.001	0.004
0.5	0.023	0.054	0.119

Table 9: Power for Unweighted Instrument CvM Test under Design 2

$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
0.1	0	0	0
0.2	0.003	0.002	0.001
0.3	0.007	0.022	0.037
0.4	0.01	0.145	0.412
0.5	0.039	0.596	0.884

Table 10: Power for Unweighted Instrument KS Test under Design 2

σ_n^2	$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
$\frac{1}{4}n^{-1/5}$	0.1	0	0	0
	0.2	0	0	0
	0.3	0.003	0	0
	0.4	0.007	0.006	0.013
	0.5	0.04	0.118	0.294
$\frac{1}{4}n^{-1/3}$	0.1	0	0	0
	0.2	0	0	0
	0.3	0.001	0.001	0
	0.4	0.011	0.009	0.016
	0.5	0.032	0.139	0.371
$\frac{1}{4}n^{-1/2}$	0.1	0	0	0
	0.2	0.001	0	0
	0.3	0.003	0	0
	0.4	0.009	0.003	0.014
	0.5	0.034	0.114	0.288

Table 11: Power for Weighted Instrument CvM Test under Design 2

t_n	$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
$n^{-1/5}$	0.1	0	0	0
	0.2	0.006	0.016	0.032
	0.3	0.026	0.138	0.295
	0.4	0.064	0.449	0.831
	0.5	0.175	0.848	0.995
$n^{-1/3}$	0.1	0.007	0.012	0.005
	0.2	0.016	0.062	0.1
	0.3	0.041	0.215	0.456
	0.4	0.119	0.604	0.876
	0.5	0.21	0.902	0.996
$n^{-1/2}$	0.1	0.006	0.014	0.01
	0.2	0.023	0.057	0.086
	0.3	0.038	0.229	0.389
	0.4	0.119	0.532	0.791
	0.5	0.203	0.85	0.982

Table 12: Power for Weighted Instrument KS Test under Design 2 (from Armstrong and Chan (2016))

h_n	$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
$n^{-1/5}$	0.1	0	0	0
	0.2	0.001	0.002	0
	0.3	0.008	0.007	0.024
	0.4	0.012	0.108	0.369
	0.5	0.074	0.484	0.923
$n^{-1/3}$	0.1	0	0.001	0
	0.2	0.001	0	0
	0.3	0.003	0.009	0.011
	0.4	0.023	0.126	0.273
	0.5	0.062	0.519	0.848
$n^{-1/2}$	0.1	0	0	0
	0.2	0.001	0	0
	0.3	0.001	0	0
	0.4	0.005	0.007	0.023
	0.5	0.023	0.089	0.308

Table 13: Power for Kernel CvM Test under Design 2

h_n	$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
$n^{-1/5}$	0.1	0.001	0.001	0.001
	0.2	0.009	0.029	0.049
	0.3	0.044	0.185	0.386
	0.4	0.082	0.524	0.867
	0.5	0.18	0.879	0.997
$n^{-1/3}$	0.1	0.007	0.015	0.014
	0.2	0.015	0.067	0.129
	0.3	0.029	0.18	0.454
	0.4	0.087	0.525	0.856
	0.5	0.167	0.825	0.98
$n^{-1/2}$	0.1	0	0.014	0.006
	0.2	0	0.025	0.032
	0.3	0	0.057	0.123
	0.4	0	0.163	0.286
	0.5	0	0.321	0.604

Table 14: Power for Kernel KS Test under Design 2

$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
0.1	0.005	0	0.001
0.2	0.031	0.046	0.058
0.3	0.131	0.454	0.743
0.4	0.359	0.914	0.997
0.5	0.619	0.999	1

Table 15: Power for Unweighted Instrument CvM Test under Design 3

$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
0.1	0.006	0.015	0.013
0.2	0.027	0.231	0.402
0.3	0.117	0.737	0.959
0.4	0.34	0.982	1
0.5	0.568	1	1

Table 16: Power for Unweighted Instrument KS Test under Design 3

σ_n^2	$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
$\frac{1}{4}n^{-1/5}$	0.1	0.006	0	0.001
	0.2	0.037	0.079	0.136
	0.3	0.133	0.515	0.837
	0.4	0.341	0.941	1
	0.5	0.636	1	1
$\frac{1}{4}n^{-1/3}$	0.1	0.006	0.003	0.001
	0.2	0.029	0.065	0.173
	0.3	0.143	0.514	0.872
	0.4	0.375	0.961	1
	0.5	0.642	1	1
$\frac{1}{4}n^{-1/2}$	0.1	0.006	0.003	0
	0.2	0.043	0.059	0.101
	0.3	0.161	0.52	0.845
	0.4	0.335	0.935	0.999
	0.5	0.63	0.999	1

Table 17: Power for Weighted Instrument CvM Test under Design 3

t_n	$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
$n^{-1/5}$	0.1	0.034	0.064	0.12
	0.2	0.093	0.466	0.704
	0.3	0.272	0.869	0.99
	0.4	0.501	0.994	1
	0.5	0.767	1	1
$n^{-1/3}$	0.1	0.039	0.104	0.116
	0.2	0.112	0.429	0.64
	0.3	0.257	0.838	0.979
	0.4	0.463	0.994	1
	0.5	0.717	1	1
$n^{-1/2}$	0.1	0.03	0.083	0.087
	0.2	0.121	0.325	0.523
	0.3	0.24	0.762	0.967
	0.4	0.397	0.984	1
	0.5	0.669	1	1

Table 18: Power for Weighted Instrument KS Test under Design 3 (from Armstrong and Chan (2016))

h_n	$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
$n^{-1/5}$	0.1	0.013	0.017	0.018
	0.2	0.05	0.229	0.446
	0.3	0.187	0.757	0.965
	0.4	0.411	0.98	1
	0.5	0.698	1	1
$n^{-1/3}$	0.1	0.007	0.012	0.01
	0.2	0.044	0.167	0.323
	0.3	0.173	0.676	0.932
	0.4	0.377	0.986	1
	0.5	0.657	1	1
$n^{-1/2}$	0.1	0.002	0.001	0
	0.2	0.029	0.03	0.049
	0.3	0.082	0.326	0.654
	0.4	0.21	0.866	0.991
	0.5	0.47	0.996	1

Table 19: Power for Kernel CvM Test under Design 3

h_n	$\theta_1 - \overline{\theta}_1$	$n = 100$	$n = 500$	$n = 1000$
$n^{-1/5}$	0.1	0.043	0.087	0.161
	0.2	0.099	0.487	0.722
	0.3	0.261	0.876	0.99
	0.4	0.48	0.995	1
	0.5	0.746	1	1
$n^{-1/3}$	0.1	0.037	0.086	0.122
	0.2	0.079	0.297	0.528
	0.3	0.164	0.646	0.912
	0.4	0.296	0.937	0.999
	0.5	0.507	0.996	1
$n^{-1/2}$	0.1	0	0.035	0.026
	0.2	0	0.087	0.118
	0.3	0	0.195	0.385
	0.4	0	0.427	0.703
	0.5	0	0.716	0.952

Table 20: Power for Kernel KS Test under Design 3