DYNAMIC REVENUE MAXIMIZATION:
A CONTINUOUS TIME APPROACH

By

Dirk Bergemann and Philipp Strack

July 2014

COWLES FOUNDATION DISCUSSION PAPER NO. 1953

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

Dynamic Revenue Maximization: A Continuous Time Approach*

Dirk Bergemann† Philipp Strack ‡

July 4, 2014

Abstract

We characterize the profit-maximizing mechanism for repeatedly selling a non-durable good in continuous time. The valuation of each agent is private information and changes over time. At the time of contracting every agent privately observes his initial type which influences the evolution of his valuation process. In the profit-maximizing mechanism the allocation is distorted in favor of agents with high initial types.

We derive the optimal mechanism in closed form, which enables us to compare the distortion in various examples. The case where the valuation of the agents follows an arithmetic/geometric Brownian motion, Ornstein-Uhlenbeck process, or is derived from a Bayesian learning model are discussed. We show that depending on the nature of the private information and the valuation process the distortion might increase or decrease over time.

Keywords: Mechanism Design, Dynamic Auctions, Repeated Sales

JEL Classification: D44, D82, D83.

*The first authors acknowledges financial support through NSF Grants SES 0851200 and ICES 1215808. We would like to thank seminar audiences at the University of Chicago and Microsoft Research for many helpful comments.

†Department of Economics, Yale University, New Haven, CT 06511, dirk.bergemann@yale.edu

‡Microsoft Research New England & UC Berkeley, One Memorial Drive, 02142 Cambridge, MA, philipp.strack@gmail.com

1 Introduction

1.1 Motivation

We analyze the nature of the optimal, revenue-maximizing, contract, in a dynamic environment with private information at the time of contracting and in all future periods. In contrast to almost all of the received literature, we consider a setting in continuous, rather than discrete, time. Within the continuous time setting, we are mostly concerned with environments where the uncertainty, and in particular the private information of the agent can be described by a Brownian motion. Throughout, we restrict our attention to allocation problems that are time separable, i.e. allocation problems where the current allocation choice does not restrict future allocation choices. This restriction is sufficiently mild to include most, if not all of the allocation problems explicitly analyzed in the literature so far, for the example the optimal quantity provision by the monopolist as in Battaglini (2005) or the separable environments in Pavan, Segal, and Toikka (2014). But the focus on time separable allocation problems is restrictive in that it excludes problems such as the optimal timing of a sale of a durable good, where the present decision, say a sale, naturally preempts certain future decision, say a sale, again.

We shall show that the continuous time setting with Brownian motions presents us with at least three advantages over the discrete time setting. The first advantage of the continuous time and Brownian motion setting is that it allows to restrict attention to a small class of deviations, deviations that we call *consistent*. The consistent deviations, by themselves only necessary conditions, nonetheless allow us to completely describe the indirect utility of the agent in any incentive compatible mechanism. More precisely, at time zero the initial shock of the agent is drawn and the initial shock determines the probability measure of the entire future valuation process. An important innovation of the paper is that it does not rely on any backward induction arguments. In fact, backward induction is replaced with a direct calculation of expected payoffs of deviations. If the agent deviates he changes the probability measure of the reported valuation process. To avoid working with the change in measures directly we restrict attention to consistent deviations. We call the deviation consistent if, after his initial misreport, say b instead of a, the agent reports his valuation as if it would follow the same Brownian motion as the one which drives his true valuation. As there is a true initial shock, namely b, which could have made these subsequent reports, the principal can not detect such a deviation and is forced to assign the allocation and transfer process of the imitated shock b. In particular, this allows us to evaluate the payoffs of the truthful and the consistently deviating agent with respect to the same expectation operator. Now, as we assume the initial shock to be one dimensional and given that all deviations are parametrized over the time zero shock, standard mechanism design arguments deliver the smoothness of the value function of the agent.

The large class of time separable allocation policies then allows us to rewrite the sufficiency conditions exclusively in terms of the flow virtual utilities, which by our assumption on the valuation process only depends on the current valuation and the initial shock of the agent. By using the class of consistent deviations and allowing for time separable allocation policies, we can completely avoid the verification of the incentive compatibility conditions via backward induction methods which was the basic instrument to establish the sufficient conditions used in all of the preceding literature with dynamic adverse selection.

The second advantage of the continuous time approach is that we can explicitly derive the optimal dynamic allocation process and the associated transfers. We can therefore describe the nature of the optimal policy in much greater detail than it has been possible in discrete time environments. We consider in some detail a number of well-known stochastic processes, in particular the arithmetic and the geometric Brownian motion. The natural starting point here is to consider the case in which the private information of the agent is the current state of the process, in particular the initial state of the Brownian motion is private information, but where the drift and volatility of the process are publicly known. In the models with discrete time, to be discussed below, this corresponds to the case where the private information represents the current state of the Markov process, but where the Markov transition matrix itself is publicly known. But our technique also allows us to consider the case when either the drift or the variance of the Brownian motion constitutes the initial private information. Subsequently, the state of the process will also be private information, but at the beginning, the starting point of the process is assumed to be commonly known to remain within a one-dimensional model of private information at each point in time. In particular, we can allow the variance rather than the mean of the stochastic process to form the private information, and yet display transparent sufficient conditions for optimality. In much of the earlier literature, the types had to be assumed to be ordered according to first-order stochastic dominance in order to give rise to sufficient conditions for optimality. We should also emphasize that with the exception of a recent paper by Boleslavksy and Said (2013), the earlier contributions with an infinite horizon did not allow for the possibility that the very structure of the stochastic process may constitute the private information.

The third advantage of the continuous time Brownian motion approach is that we can relate our model of dynamic adverse selection to a number of continuous time models with dynamic moral hazard. For example, in DeMarzo and Sannikov (2006), (2008), the state of the process describes the current cash flow of the project and the moral hazard problem is that the entrepreneur, the agent, shares the unobservable cash flow with the investor, the principal. In turn, we can restate the moral hazard problem as an adverse selection problem, where the principal has to induce the agent to tell the truth about the state of the process, and can achieve truthtelling through appropriate transfer payments, i.e. sharing rules of the cash flow. In contrast to the literature, which assumes that there is no private information prior to the contract, we can solve for the optimal contract in the presence of private information before and after the contract is signed.

In the first part of the paper, we restrict attention to the case where the current valuation of the agent is a function of the initial state, the current state, and time only, and importantly does not depend on the entire realized path of the Brownian motion. With this restriction, the flow virtual utility depends only on the initial shock and the valuation. In the second part of the paper, we generalize the analysis and allow the valuation of the agent to be a function of the entire realized path of the process, yet retain the property that the virtual utility itself depends only on the initial shock and the valuation. This will allow us to include in our analysis the Ornstein-Uhlenbeck process and the Bayesian learning process about an unknown and normally distributed drift of a Brownian motion.

1.2 Related Literature

The analysis of the revenue maximizing contract in an environment where the private information may change over time appears first in a seminal paper of Baron and Besanko (1984). They considered a two period model of a regulator facing a monopolist with unknown, but in every period, constant marginal cost. Besanko (1985) offers an extension to a finite horizon environment with a general cost function, where the unknown parameter is either distributed independently and identically over time, or follows a first-order autoregressive process. Since these early contributions, the literature has developed rapidly. Courty and Li (2000) consider the revenue maximizing contract in a sequential screening problem, where the preferences of the buyer may change over time. They only considered a terminal allocation problem in the second period, where Baron and Besanko (1984) considered a sequence of allocation problems, but their model extends the analysis to environments where the private information, the state of the world, is ordered according to either first or second order stochastic dominance. Battaglini (2005) considered a quantity discriminating monopolist who provides a menu of choices to a consumer whose valuation can change over time according to a commonly known Markov process. In contrast to the earlier work, he explicitly considers an infinite time horizon and showed that the distortion due to the initial private information vanishes over time. Eső and Szentes (2007) rephrased the two period sequential screening problem by showing that the additional signal arriving in period two can always be represented by a different signal with the property that the signal in period one and two are orthogonal to each other. The orthogonal representation of the signals allows them to think of the period two information as incremental relative to period one. And in particular, it allows them to establish the level of the information rent that arises solely from the initial private information, the period one information. Pavan, Segal, and Toikka (2014) consider a general environment in an infinite horizon setting and allowing for general allocation problems, encompassing the earlier literature (with continuous type spaces). They obtain general necessary conditions for incentive compatibility and present a variety of sufficient conditions for revenue maximizing contracts for specific classes of environments.

A feature common to almost of these contributions is that the private information of the agent is represented by the current state of a Markov process, and that the new information that the agent receives is controlled by the current state, and in turn, leads to a new state of the Markov process. By contrast, Boleslavksy and Said (2013) let the initial private information of the agent be the nature of the Markov process itself, for example the parameter describing the persistence of the state, and then take the initial state of the process as commonly known. Interestingly, this dramatically changes the impact that the initial private information has on the future allocations. In particular, the distortions in the future allocation may now rise over time rather than decline as in the earlier literature. Finally, Kakade, Lobel, and Nazerzadeh (2013) consider a class of dynamic allocation problems, a suitable generalization of the single unit allocation problem and impose a separability condition (additive or multiplicative) on the interaction of the initial private information and all subsequent signals. The separability condition allows them to obtain an explicit characterization of the revenue maximizing contract and derive transparent sufficient conditions for the optimal contract.

2 Dynamic Sales: An Example

One of the simplest economic situations that gives rise to a dynamic mechanism design problem is a repeated sales problem where the buyer is unsure about his future valuation for the good. Examples of such situations are gym membership and phone contracts. At any given point in time the buyer knows how much he values making a call or going to the gym, but he might only have a probabilistic assessment on how much he values the service tomorrow or a year in the future. Usually, it is harder for the buyer to asses how much he values the good at times that are further in the future. Mathematically this uncertainty about future valuations can be captured by modelling the buyers valuation as a stochastic process.

From the monopolistic sellers point of view the question arises whether the uncertainty of the buyer can be used to increase profits by using a dynamic contract. In reality a variety of dynamic contracts is used (for example for gym memberships and mobile phone contracts):

1. Flatrates where the buyer only pays a fixed fee regardless of his consumption.
2. Two part tariffs where the buyer selects from a menu a fixed fee and a price of consumption. He pays the fixed fee independent of his level of consumption. In addition the buyer has to pay for his consumption. Tariffs with higher fixed fees feature lower prices of consumption.
3. Two part tariffs where the buyer selects from a menu a fixed fee and an amount of free consumption. He pays the fixed fee independent of his consumption. In addition the buyer has to pay for his consumption if it exceeds a threshold. Tariffs with higher fixed fees feature higher amounts of free consumption.

While those dynamic contracts can be observed in a wide range of situations their theoretical properties are unclear. Using a dynamic mechanism design perspective we can explain why and under what circumstances those dynamic contracts are used.

Assume the valuation $(v_t)_{t\in\mathbb{R}_+}$ of the buyer is a geometric Brownian motion which is shifted upwards by $\underline{v} \geq 0$, i.e.

$$dv_t = (v_t - \underline{v})dW_t \,, \tag{1}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a Brownian motion. The initial valuation of the buyer $v_0 \in (\underline{v}, \infty)$, is distributed according to the absolutely continuous distribution function $F : (\underline{v}, \infty) \to [0, 1]$ with density $f = F'$. We assume that F is such that $v \mapsto \frac{1-F(v)}{f(v)\,v}$ is non-increasing. The choice of the shifted geometric Brownian motion as a valuation process ensures that the valuation v_t for the good will be greater than $\underline{v}$ at every point in time t. Furthermore, the valuation at time t is the agent's best estimate of his valuation at later times $s > t$, i.e.

$$v_t = \mathbb{E}[v_s \,|\, v_t] \,.$$

At every point in time t the buyer chooses an amount of consumption $x_t \in X \subseteq \mathbb{R}_+$ and pays p_t such that his overall utility equals

$$\mathbb{E}\left[\int_0^\infty e^{-rt}(v_t \cdot x_t - p_t)\mathrm{d}t\right] \,.$$

In the following section we describe the revenue maximizing dynamic contract offered by a monopolistic seller. In general, dynamic contracts could have complicated features as the payments at time t could depend on all the past consumption decisions and messages sent by the agent. However, as we will show in the next section offering a menu of simple static contracts is sufficient to maximize the expected intertemporal revenue.

To evaluate dynamic contracts from the sellers perspective, we assume that the seller faces continuous, non-decreasing production cost $c : X \to \mathbb{R}_+$, such that his overall payoff equals

$$\mathbb{E}\left[\int_0^\infty e^{-rt}(p_t - c(x_t))\mathrm{d}t\right] \,.$$

The results derived later in this paper will prove that (under some regularity conditions on F) an optimal contract (indirect mechanism) for the seller is of the following form: At time zero the seller offers a menu of static contracts each consisting of a time independent fixed membership fee $m \geq 0$, and a consumption dependent payment:

$$q(m, x_t) = A(m)c(x_t) - [A(m) - 1]\,\underline{v}x_t.$$

The consumption dependent payment q consists of a price of consumption of $A(m) \geq 1$ and a linear consumption discount $(A(m) - 1)\underline{v}x_t$. If the buyer accepts a contract he has to pay the

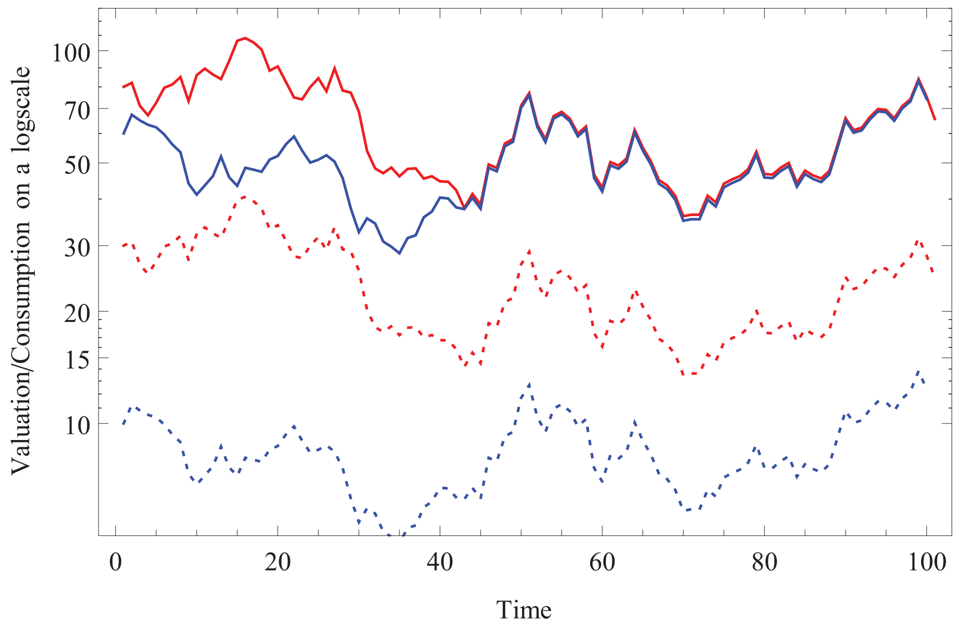

Figure 1: The initial valuation v_0 is exponentially distributed with mean 50 and the valuation evolves as a geometric Brownian motion without drift. The solid lines are two paths of the valuation starting at an initial valuation of 60 (red) and 80 (blue) which coincide after time $t = 45$. The dashed lines are the consumption levels in the revenue maximizing contract if the cost of production is quadratic $c(x) = x^2/2$. As the optimal consumption is linear in the valuation they are parallel on a logarithmic scale. Note, that even after the valuations coincide the consumption levels of the agents with different initial valuations differ and the optimal consumption level react with differing intensity to changes in the valuations. The consumption of the agent in the welfare maximizing contract would exactly equal his valuation.

membership fee $m \geq 0$ independent of his consumption. At the same time he has to pay $q(m, x_t)$ depending on his consumption x_t in period t such that his overall payment at time t equals

$$p_t = m + q(m, x_t) = m + A(m)c(x_t) - [(A(m) - 1]\, \underline{v} x_t\, . \tag{2}$$

The optimal fixed fee $m(v_0)$ chosen by the agent depends on the agent's initial valuation v_0 will be such that $A(m(v_0)) = \frac{v_0}{J(v_0)}$, where $J(v) = v - \frac{1-F(v)}{f(v)}$ is the virtual valuation.

2.1 Flat Rate Contracts

In a flat rate contract the payment p_t is constant over time and independent of the buyers consumption. As the buyers utility increases in the consumption level he will always consume the good at the maximum possible intensity.

Assume the production cost c is constant and equal to zero, the set of possible allocations is given by $X = [0, 1]$, and the minimal valuation $\underline{v}$ equals zero. A direct consequence of the transfers described in (2) is the following result characterizing an optimal mechanism with zero (marginal) cost of production: The optimal mechanism is a flat rate where every agent who accepts the contract at time zero, makes a constant flow payment, independent of his consumption, and consumes the maximal possible amount: $x_t = 1$.

While the buyer enjoys a utility of v_t from consuming the good he dislikes the payments p and he will suffer from a negative flow utility $v_t - p$ if $v_t < p$. If his current valuation v_t is below the flat rate price p, not only is his current flow of utility negative, but also his expected continuation utility of the contract:

$$\mathbb{E}\left[\int_t^\infty e^{-rs}(v_s - p_s)\mathrm{d}t \,|\, v_t\right] = \frac{v_t - p}{r}\, . \tag{3}$$

However, as the agent is (legally) bound to the contract he is forced to make the payments. Hence a flat rate contract makes use of the fact that the agent can commit himself to future payments and consumption before he learns his valuation.

As a consequence of condition (3) only the agents with an initial valuation $v_0 \geq p$ accept the contract. All agents with an initial valuation $v_0 < p$ reject the contract and never consume the good no matter how high the consumption utility is at times $t > 0$.

2.2 Two-Part Tariffs

Having seen that zero marginal cost lead to flat rate tariffs, the next section describes the optimal contract for convex costs. Assume that the minimal valuation $\underline{v}$ equals zero and the cost function c is convex. By condition (2) a two-part tariff where the agent pays m independent of his consumption and $A(m)c(x)$ depending on his is a revenue maximizing contract for the principal. It is worth noting that a simple menu of static two part tariffs can hence maximize the revenue of the principal.

Example 1. *Let $c(x) = x^2/2$ and the initial valuation be exponentially distributed with mean μ, i.e. $F(v_0) = 1 - \exp(\frac{v_0}{\mu})$ and $\underline{v} = 0$. The optimal contract sets for every fixed fee $m \in (0, \infty)$ a price of consumption x_t equal to:*

$$A(m) = \frac{x_t^2}{2}\left[1 - \exp\left(\frac{-mr2(r-\sigma)}{\mu}\right)\right]^{-1}.$$

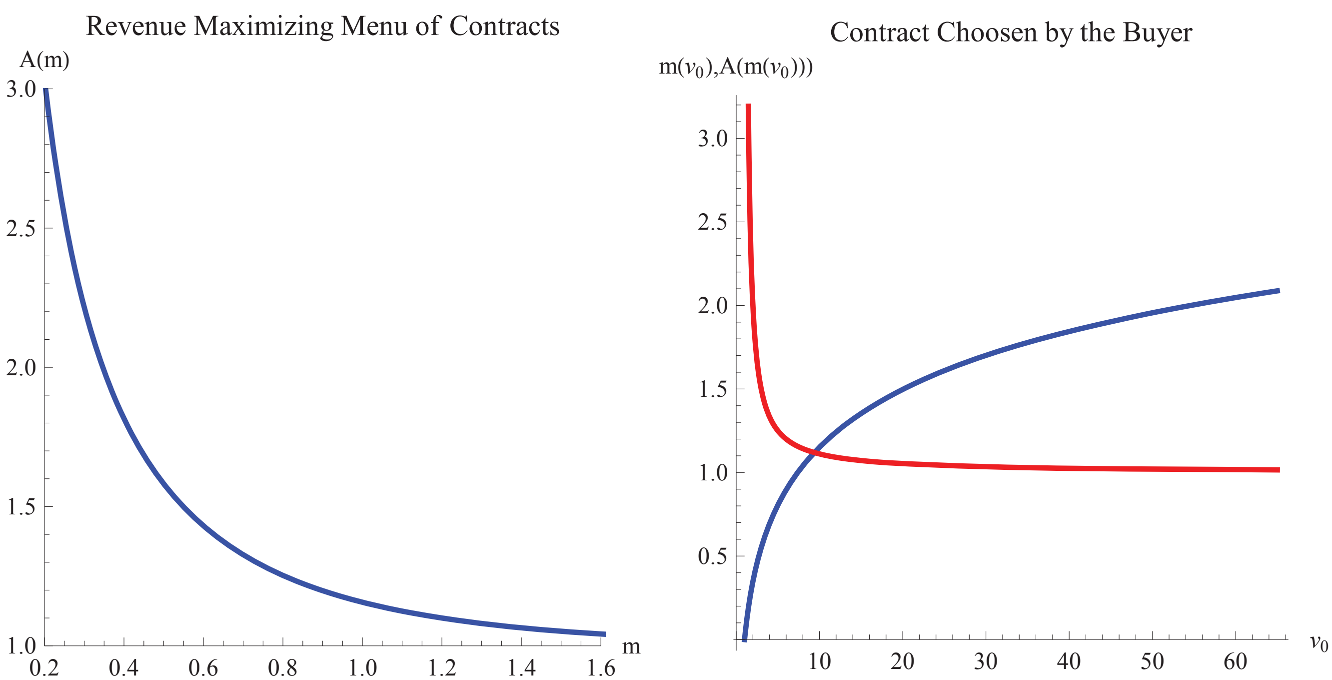

Figure 2: Illustration of Example 1: $\underline{v} = 0, c(x) = x^2/2$, v_0 exponentially distributed with mean $\mu = 1$ and a constant discount factor of $r = 1$: On the left the optimal menu of contracts. On the right the contract $(m(v_0), A(m(v_0)))$ chosen by the consumer depending on his initial valuation v_0.

Figure 2.2 illustrates that the consumption how the consumption at time t depends on the time zero valuation in the context of Example 1.

2.3 Free Minute Contract

Throughout this section we assume that the minimal valuation $\underline{v}$ is strictly positive and that the density at the minimal valuation is large enough $f(\underline{v}) > 1/\underline{v}$. In addition we assume that the marginal cost of providing the good vanish for small quantities, i.e. $c'(0) = 0$. When the agent decides how much to consume at time t he solves the maximization problem

$$\max_x \{xv_t - (z + A(m)c(x) - (A(m) - 1)\underline{v}x)\} .$$

This leads to the first order condition

$$0 = v_t - A(m)c'(x) + (A(m) - 1)\underline{v} \;\Rightarrow\; c'(x) = \underline{v} + \frac{(v_t - \underline{v})}{A(m)} .$$

As the marginal cost of providing the good vanish if the quantity goes to zero it follows that the consumption of the agent is bounded from below at every point in time by $c'^{-1}(\underline{v})$. Hence we can interpret the amount $c'^{-1}(\underline{v})$ as a quantity provided to the agent for free. This is a feature which could be observed for example in mobile phone contracts. In such a contract the agent can consume a certain number of minutes for free and only has to pay for the consumption exceeding this amount.

Interestingly, the allocation of the agent with the lowest possible valuation $v_t = \underline{v}$ is not distorted, while the allocation of agents with higher valuations is distorted downwards compared to the socially efficient allocation. This is surprising as in static mechanisms, it is usually only the agent with highest possible valuation that receives an undistorted allocation. In this model the agent with the highest time zero valuation receives an undistorted allocation. This example illustrates that this no-distortion at the top result only applies to the valuation at the time of contracting, but not to later valuations.

3 The General Model

There are n agents indexed by $i \in \{1, \ldots, n\} = N$. Time is continuous and indexed by $t \in [0, T]$, where the time horizon T can be finite or infinite, and if the time horizon is infinite, then we assume a discount factor $r \in \mathbb{R}_+$ which is strictly positive, $r > 0$.

The flow preferences of agent i are represented by a quasilinear utility function:

$$v_t^i \cdot u^i(t, x_t^i) - p_t^i. \tag{4}$$

The function $u : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, \overline{u}]$ is continuous and strictly increasing in x, decreasing in t and satisfies $u(t, 0) = 0$ for all $t \in \mathbb{R}_+$. We refer to $u(t, x_t^i)$ as the *valuation* of $x_t^i \in [0, \overline{x}] \subset \mathbb{R}_+$ with $0 \leq \overline{x} < \infty$. The allocation x_t^i can be interpreted as either the quantity or quality of a good that is allocated to agent i at time t. The *type* of agent i in period t is given by $v_t^i \in \mathbb{R}_+$ and the flow utility in period t is given by the product of the type and the valuation. The payment in period t is denoted by $p_t^i \in \mathbb{R}$.

The type v_t^i of agent i at time t depends on his *initial shock* θ^i at time $t = 0$ and the contemporaneous shock W_t^i at time t:

$$v_t^i \triangleq \phi^i(t, \theta^i, W_t^i) \,. \tag{5}$$

Note, that the initial private information θ need not to be the initial valuation, but might be any other characteristic determining the probability measure over paths of the valuation $(v_t)_{t \in \mathbb{R}_+}$. In case of the Brownian this might be the initial value, the drift, or the variance, in case of a mean reverting process this might be the mean reversion speed or the long run-average. At time zero each agent privately learns his initial shock $\theta^i \in (\underline{\theta}, \bar{\theta}) = \Theta \subseteq \mathbb{R}$, which is drawn from a common prior distribution $F^i : \mathbb{R} \to [0, 1]$, independently across agents.

The distribution F^i has a strictly positive density $f^i > 0$ with decreasing inverse hazard rate $(1 - F^i) / f^i$. The contemporaneous shock is given by a random process $(W_t^i)_{t \in \mathbb{R}_+}$ of agent i that changes over time as a consequence of a sequence of incremental shocks and W_t^i is assumed to be independent of W_t^j for every $j \neq i$. The function $\phi^i : \mathbb{R}_+ \times \Theta \times \mathbb{R} \to \mathbb{R}$ aggregates the initial shock θ^i and the contemporaneous W_t^i of agent i into his type v_t^i. In Section 2 and, later in Section 7, the valuation function $u^i\,(t, x_t^i)$ is simply a linear function $u^i\,(t, x_t^i) = x_t^i$ and the type v_t^i can then be directly interpreted as the *marginal willingness to pay* of agent i. We shall sometimes use this interpretation even without a linear valuation function $u^i\,(t, x_t^i)$.

The function ϕ is twice differentiable in every direction and in the following we use a small annotation for partial derivatives, i.e.

$$\phi_\theta(t, \theta, w) \triangleq \frac{\partial \phi(t, \theta, w)}{\partial \theta}.$$

If θ is the initial value of the process v_0 the derivative ϕ_θ is know in the mathematical literature on stochastic processes as the *stochastic flow* or *generalized stochastic flow* if θ determines the evolution of a diffusion by influencing the drift or variance term (see for example Kunita 1997). The stochastic flow process $(\phi_\theta(t, \theta, W_t^i))_{t \in \mathbb{R}_+}$ is the analogue of the impulse response functions described in the discrete time dynamic mechanism design literature (see Pavan, Segal, and Toikka 2014, Definition 3). As we will see in the examples presented later the stochastic flow is of a very simple form for many classical continuous time diffusion processes, like the Brownian motion etc.

We assume that for every agent i a higher initial shock θ^i leads to a higher type, i.e. $\phi_\theta^i(t, \theta, w) \geq 0$ and an agent i who observed a higher value of the process W_t^i has a higher type, i.e. $\phi_w^i(t, \theta, w) \geq 0$ for every $(t, \theta, w) \in \mathbb{R}_+ \times \Theta \times \mathbb{R}$.

Assumption 1 (Decreasing Influence of Initial Shock)**.**
The relative impact of the initial shock on the type:

$$\frac{\phi_\theta(t, \theta, w)}{\phi(t, \theta, w)} \tag{6}$$

is non-increasing in w for every $(t, \theta, w) \in \mathbb{R}_+ \times \Theta \times \mathbb{R}$.

Assumption 2 (Decreasing Influence of Initial vs Contemporaneous Shock)**.**
The ratio of the marginal impact of initial and contemporaneous shocks:

$$\frac{\phi_\theta(t, \theta, w)}{\phi_w(t, \theta, w)} \tag{7}$$

is non-increasing in θ for every $(t, \theta, w) \in \mathbb{R}_+ \times \Theta \times \mathbb{R}$.

The last assumption implies that the type with a large initial shock is influenced less by the contemporaneous shocks that arrive after time zero.

Assumption 3 (Finite Expected Impact of the Initial Shock)**.**
The expected influence of the initial shock on the type grows at most exponentially, i.e. there exists two constants $C \in \mathbb{R}_+, q \in (0, r)$ such that $\mathbb{E}\left[\phi_\theta(t, \theta^i, W_t^i)\right] \leq Ce^{qt}$ for all $t \in \mathbb{R}_+$ and $\theta \in \Theta$.

At every point in time t the principal chooses an allocation $x_t \in X$ from a compact, convex set $X \subset \mathbb{R}^n_+$, where x^i_t can be interpreted as the quantity or quality of a good that is allocated to agent i at time t. We assume that it is always possible to allocate zero to an agent, i.e.

$$x \in X \Rightarrow (x^1, \dots, x^{i-1}, 0, x^{i+1}, \dots, x^n) \in X\,.$$

To ensure that the problem is well posed we assume that for every feasible allocation process $x^i = (x^i_t)$ gives finite expected utility to agent i, i.e.

$$\mathbb{E}\left[\int_0^T e^{-rt}\mathbf{1}_{\{v^i_t \geq 0\}} v^i_t u(t, x^i_t)\mathrm{d}t \,|\, \theta^i\right] < \infty,$$

for every θ^i in the support of F. The principal receives the sum of discounted flow payments $\sum_{i \in N} p^i_t$ minus the production costs $c(x_t)$:[1]

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(\sum_{i\in N} p^i_t - c\left(x_t\right)\right)\mathrm{d}t\right]\,. \tag{8}$$

The cost

$$c : X \to \mathbb{R}_+$$

is continuous and increasing in every component with $c\left(0\right) = 0$.

Definition 1 (Value Function)**.**
The indirect utility, or value function, $V^i(\theta^i)$ *of agent* i *given his initial shock* θ^i*, his consumption process* $(x^i_t)_{t\in\mathbb{R}_+}$ *and his payment process* $(p^i_t)_{t\in\mathbb{R}_+}$ *is*

$$V^i(\theta^i) = \mathbb{E}\left[\int_0^T e^{-rt}\left(u^i(t, x^i_t)v^i_t - p^i_t\right)\mathrm{d}t \,|\, \theta^i\right]\,. \tag{9}$$

A contract specifies an allocation process $(x_t)_{t\in\mathbb{R}_+}$ and a payment process $(p_t)_{t\in\mathbb{R}_+}$. The allocation x_t and the payment p_t can depend on all types reported $(v^i_s)_{s\leq t, i\in N}$ by the agents prior to time t. We assume that the agent has an outside option of zero and thus require the following definition:

[1]The restriction to flow payments is without loss of generality as the agent can commit to future payments. More precisely if a contract requires the agent to make a lump sum payment at time t the contract can instead require the agent to make flow payments from time t to time T without changing the agents incentives or the expected discounted revenue of the principal.

Definition 2 (Incentive Compatibility)**.** *A contract* $(x_t, p_t)_{t\in\mathbb{R}_+}$ *is incentive compatible if for every agent* i *it is individually rational to accept the contract*

$$V^i(\theta^i) \geq 0 \text{ for all } \theta^i \in \Theta\,,$$

and it is optimal to report his type $(v^i_t)_{t\in\mathbb{R}_+}$ *truthfully at every point in time* $t \in \mathbb{R}_+$*.*

4 Welfare Maximization

This section first derives the social welfare maximizing allocation in the complete information set-up and later shows how it can be implemented using dynamic Vickrey-Clarke-Groves payments.

Let us first assume that the social planner observes the valuations v_t of all agents at every point in time directly. Given the transferable utility, we define the flow welfare function $s : \mathbb{R}_+ \times \mathbb{R}^n \times X \to \mathbb{R}$ that maps an allocation $x \in X$ and a vector of valuations $v \in \mathbb{R}^n$ into the associated flow of welfare

$$s(t, v, x) = \sum_{i\in N} v^i u(t, x^i) - c(x)\,. \tag{10}$$

The social value of the allocation process $(x_t)_{t\in[0,T]}$ aggregates the discounted flow of social welfare over time and is given by:

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(\sum_{i\in N} v^i_t u^i(t, x^i_t) - c\,(x_t)\right)\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T e^{-rt} s(t, v_t, x_t)\mathrm{d}t\right]\,. \tag{11}$$

As the allocation x_t at time t does not influence the future evolution of valuations or the set of possible future allocations the problem of finding a socially efficient allocation is time-separable. We define the optimal allocation function $x^\dagger : \mathbb{R}_+ \times \mathbb{R}^n \to \mathcal{P}(X)$ that maps a point in time t and a vector of valuations v into the set of optimal allocations

$$X^\dagger(t, v) = \arg\max_{x\in X} s(t, v, x)\,. \tag{12}$$

An allocation process $(x_t)_{t\in[0,T]}$ is welfare maximizing if and only if $x_t \in X^\dagger(t, v_t)$ almost surely for every $t \in [0, T]$. The following theorem establishes the existence of a welfare maximizing mechanism:[2]

[2]While Theorem 1 shows that there exists a welfare maximizing allocation that can be implemented there nevertheless might exist welfare maximizing allocation processes that can not be implemented. Such a situation can arise if the set $X^\dagger(t, v)$ contains more than one element and the selection made by the welfare maximizing allocation process conditions on past valuations.

Theorem 1 (Welfare Maxizing Mechanism)**.** *Any welfare maximizing allocation* $x^{\dagger} : \mathbb{R}_+ \times \mathbb{R}^n \to X^{\dagger}(t, v)$ *that at time* t *depends only on the vector of valuations* v_t *can be implemented in ex post equilibrium via the static Vickrey-Clarke-Groves payments*

$$p_t^{\dagger\, i} \triangleq p^{\dagger\, i}(t, v_t) = \max_{x \in X} \sum_{j \neq i} \left[u(t, x) - u(t, x^{\dagger}(t, v_t)) \right] v_t^j - c(x) + c(x^{\dagger}(t, v_t))\,. \tag{13}$$

Proof. As the allocation and the payment at time t depends only on the vector of valuations v_t agents do not need to report their types θ. As the allocation $x^{\dagger}$ and the payments $p^{\dagger}$ at time t depend only on the vector of time t valuations v_t the reporting problem is time separable and it is optimal for the agent to report his valuation truthful if and only if for all $t \in \mathbb{R}_+$ and all $v, \hat{v}^i$:

$$v^i u(t, x^{\dagger}(t, v)) - p^{\dagger}(t, v) \geq v^i u(t, x^{\dagger}(t, (\hat{v}^i, v^{-i}))) - p^{\dagger}(t, (\hat{v}^i, v^{-i}))\,.$$

If follows from the static VCG argument that reporting $\hat{v}_t^i$ instead of his true type v_t^i is not a profitable deviation for agent i at time t

$$\begin{aligned} & v_t^i u(t, x^{\dagger}(t, (\hat{v}_t^i, v_t^{-i}))) - \max_{x \in X} \sum_{j \neq i} \left[u(t, x) - u(t, x^{\dagger}(t, (\hat{v}_t^i, v_t^{-i}))) \right] v_t^j - c(x) + c(x^{\dagger}(t, v_t)) \\ &= \sum_{j \in N} v_t^j u(t, x^{\dagger}(t, (\hat{v}_t^i, v_t^{-i}))) - c(x^{\dagger}(t, (\hat{v}_t^i, v_t^{-i}))) - \max_{x \in X} \sum_{j \neq i} u(t, x) v_t^j + c(x) \\ &\leq \sum_{j \in N} v_t^i u(t, x^{\dagger}(t, v_t)) - c(x^{\dagger}(t, v_t)) - \max_{x \in X} \sum_{j \neq i} u(t, x) v_t^j + c(x) = v_t^i u(t, x(t, v_t)) - p^i(t, v_t)\,. \square \end{aligned}$$

5 Revenue Maximization

In this section we derive a revenue maximizing direct mechanism. Without loss of generality we restrict attention to direct mechanisms, where every agent i report his type v_t^i truthfully. To do so we first proof a revenue equivalence result for incentive compatible mechanisms.

5.1 Necessity

We begin by establishing that the value function of the agent if he reports truthfully is Lipschitz continuous. As ϕ^i is strictly increasing in w we can implicitly define the function $\omega : \mathbb{R}_+ \times \Theta \times \mathbb{R} \to \mathbb{R}$ by

$$v^i = \phi^i(t, \theta, \omega(t, \theta^i, v^i)) \text{ for all } (t, \theta^i) \in \mathbb{R}_+ \times \Theta\,. \tag{14}$$

We derive a necessary condition for incentive compatibility that is based only on the robustness of the mechanism to a small class of deviations, which we refer to as *consistent deviations*.

Definition 3 (Consistent Deviation)**.**
A deviation by agent i is referred to as a consistent deviation if an agent with type $v_0 = \phi(0, a, W_0)$ (and associated initial shock $a \in \Theta$) misreports $\widehat{v}_0 = \phi(0, b, W_0)$ (and associated initial shock $b \in \Theta$) at $t = 0$ continues to report:

$$\widehat{v}_t^i = \phi(t, b, \omega(t, a, v_t^i)), \tag{15}$$

instead of his true type v_t^i at all future dates $t \in \mathbb{R}_+$.

Thus, an agent who misreports with a consistent deviation, continues to misreport his type v_t^i in all future periods. More precisely, agent i's reported type $\hat{v}_t^i = \phi(t, b, W_t^i)$ equals the type he would have had if his initial shock would have been b instead of a. We note that the consistent misreport has the property that the principal could infer from the misreport the true realized path of contemporaneous shocks W_t^i. Now, since the allocation depends on the type v_t^i rather than the path of contemporaneous shocks W_t^i, the (inferred) truthfulness in the shocks is not of immediate use for the principal. We now show that this, one-dimensional, class of consistent deviations is sufficient to uniquely pin down the value function of the agent in any incentive compatible mechanism at time $t = 0$. We should emphasize that in contrast to the deviations analyzed in the literature, in particular with respect to the necessary conditions, the class of consistent deviations we consider here are not local deviations at one point in time, but rather represent a global deviation in the sense that the agent changes his reports at every point in time.

As $\phi(0, \theta, W_0)$ is strictly increasing in θ, it is convenient to describe the initial report directly in terms of the true initial shock a and the reported initial shock b. We thus define $V(a, b)$ to be the indirect utility of an agent with initial shock a but who reports shock b and misreports his type consistently as $\hat{v}_t^i = \phi(t, b, \omega(t, a, v_t^i))$. Note that by construction $W_t^i = \omega(t, a, v_t^i)$. Consequently the allocation agent i gets by consistently reporting b is the same allocation $x_t^i(b)$ an agent of initial shock b gets if he reports truthfully. Hence $V^i(a, b)$ is the indirect utility of an agent who has the initial shock a but reports initial shock b and misreports his type consistently and is given by:

$$V^i(a, b) = \mathbb{E}\left[\int_0^T e^{-rt}\left(u(t, x_t^i(b))\phi(t, a, W_t^i) - p_t^i(b)\right) \mathrm{d}t\right].$$

Note, that when restricted to consistent deviations the mechanism design problem turns into a standard one-dimensional problem, and the Envelope theorem yields the derivative of the indirect utility function of the agent:

Proposition 1 (Regularity of Value Function)**.**
The indirect utility function V^i of every agent $i \in N$ in any incentive compatible mechanism is absolutely continuous and has the weak derivative

$$V^i_\theta(\theta) = \mathbb{E}\left[\int_0^T e^{-rt} u(t, x^i_t(\theta))\phi_\theta(t, \theta^i, W^i_t)\mathrm{d}t\right] \ a.e.\,. \tag{16}$$

Proof. As the agent can always use consistent deviations, a necessary condition for incentive compatibility is $V(a,a) = \sup_b V(a,b)$. As ϕ is differentiable the derivative of V with respect to the first variable is given by

$$\begin{aligned} V_a(a,b) &= \frac{\partial}{\partial a}\mathbb{E}\left[\int_0^T e^{-rt}\left(u(t,x^i_t(b))\phi(t,a,W^i_t) - p^i_t(b)\right)\mathrm{d}t\right] \\ &= \mathbb{E}\left[\int_0^T e^{-rt}\left(u(t,x^i_t(b))\phi_\theta(t,a,W^i_t)\right)\mathrm{d}t\right] \le \overline{u}\,\mathbb{E}\left[\int_0^T e^{-rt}\phi_\theta(t,a,W^i_t)\mathrm{d}t\right], \end{aligned}$$

which is bounded by a constant by Assumption 3. By the Envelope theorem (see Milgrom and Segal (2002), Theorem 1 and Theorem 2) we have that $V^i(\theta) = V^i(\theta,\theta)$ is absolutely continuous an the (weak) derivative is given by (16).

□

We introduce the virtual valuation function $J : \mathbb{R}_+ \times \Theta \times \mathbb{R} \to \mathbb{R}$ as:

$$J(t,\theta^i,v^i) = v^i - \frac{1-F(\theta^i)}{f(\theta^i)}\phi_\theta(t,\theta^i,w(t,\theta^i,v^i))\,. \tag{17}$$

The properties of the virtual valuation are summarized in the following proposition:

Proposition 2 (Monotonicity of the Virtual Valuation)**.**
If the virtual valuation $J(t,\theta^i,v^i)$ is positive then it is non-decreasing in θ^i and v^i.

Proof. As there is no risk of confusing agents we drop the upper indices in the proof and denote by (θ, v) the type and the valuation of agent i. Assume that the virtual valuation is positive $J(t,\theta,v) > 0$. We first prove the monotonicity in v and than in θ.

<u>Part 1: $J(t,\theta,v) > 0 \Rightarrow J_v(t,\theta,v) \geq 0$:</u>

Note that

$$J(t,\theta,v) = v - \frac{1-F(\theta)}{f(\theta)}\phi_\theta(t,\theta,w(t,\theta,v)) = v\left(1 - \frac{1-F(\theta)}{f(\theta)}\frac{\phi_\theta(t,\theta,w(t,\theta,v))}{\phi(t,\theta,w(t,\theta,v))}\right).$$

As $\phi_\theta > 0$ it follows that $J(t,\theta,v) \leq v$ and hence $v \geq 0$. Consequently the second term needs to be positive as well. Clearly, $v \mapsto v$ is non-decreasing. As ϕ_θ/ϕ is non-increasing in w by (6) and $w(t,\theta,v)$ is increasing in v, so the second term is increasing in v.

<u>Part 2: $J(t,\theta,v) > 0 \Rightarrow J_\theta(t,\theta,v) \geq 0$:</u>

It remains to prove that the virtual valuation $J(t,\theta,v) = v - \frac{1-F(\theta)}{f(\theta)}\phi_\theta(t,\theta,w(t,\theta,v))$ is non-decreasing in θ. First, note that $\frac{1-F(\theta)}{f(\theta)}$ is non-increasing in θ by assumption. Second, note that $0 = \phi_\theta + \phi_w w_\theta$ and hence

$$\begin{aligned}\frac{\partial}{\partial\theta}\phi_\theta(t,\theta,w(t,\theta,v)) &= \phi_{\theta\theta}(t,\theta,w(t,\theta,v)) + \phi_{\theta w}(t,\theta,w(t,\theta,v))w_\theta(t,\theta,v)\\ &= \phi_{\theta\theta}(t,\theta,w(t,\theta,v)) - \phi_{\theta w}(t,\theta,w(t,\theta,v))\frac{\phi_\theta(t,\theta,w(t,\theta,v))}{\phi_w(t,\theta,w(t,\theta,v))}.\end{aligned}$$

Now we replace $w(t,\theta,v)$ by w and prove that the derivative is negative for any $w \in \mathbb{R}$:

$$\begin{aligned}&= \phi_\theta(t,\theta,w)\left(\frac{\phi_{\theta\theta}(t,\theta,w)}{\phi_\theta(t,\theta,w)} - \frac{\phi_{\theta w}(t,\theta,w)}{\phi_w(t,\theta,w)}\right)\\ &= \phi_\theta(t,\theta,w)\left(\frac{\partial}{\partial\theta}\log(\phi_\theta(t,\theta,w)) - \frac{\partial}{\partial\theta}\log(\phi_w(t,\theta,w)\right)\\ &= \phi_\theta(t,\theta,w)\frac{\partial}{\partial\theta}\log\left(\frac{\phi_\theta(t,\theta,w)}{\phi_w(t,\theta,w)}\right)\\ &\leq 0\,.\end{aligned}$$

The last step follows as $\frac{\phi_\theta(t,\theta,w)}{\phi_w(t,\theta,w)}$ is decreasing in θ by (7), and so the logarithm is decreasing as well. □

We observe that Proposition 2 establishes the monotonicity of the virtual valuation only for the case that the virtual valuation is positive. In fact, our assumptions are not strong enough to ensure the monotonicity of the virtual valuation independent of its sign. The reason not to impose stronger monotonicity conditions is that for many important examples discussed later (for example the geometric Brownian motion with unknown initial value) the virtual valuation is only monotone if positive.

We can now establish a revenue equivalence result that describes the revenue of the principal in any incentive compatible mechanism solely in terms of the allocation process $x = (x_t)_{t\in\mathbb{R}_+}$ and the expected time zero value the lowest type derives from the contract $V^i(\underline{\theta})$.

Theorem 2 (Revenue Equivalence)**.**
For any incentive compatible direct mechanism the expected payoff of the principal depends only on the allocation process $(x_t)_{t\in\mathbb{R}_+}$ and is given by the virtual value:

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(\sum_{i\in N} p_t^i - c(x_t)\right)\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T e^{-rt}\left(\sum_{i\in N} J(t,\theta_t^i,v_t^i)u(t,x_t^i) - c(x_t)\right)\mathrm{d}t\right] - \sum_{i\in N} V^i(\underline{\theta}). \quad (18)$$

Proof. Partial integration gives that in any incentive compatible mechanism (x, p) the expected transfer received by the principal from agent i equals the expected virtual valuation of agent i :

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{-rt}p_t^i\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^T e^{-rt}u(t,x_t^i)v_t^i\mathrm{d}t\right] - \int_{\underline{\theta}}^{\overline{\theta}} f(\theta^i)V^i(\theta^i)\mathrm{d}\theta^{\mathrm{i}} \\ &= \mathbb{E}\left[\int_0^T e^{-rt}u(t,x_t^i)v_t^i\mathrm{d}t\right] - \int_{\underline{\theta}}^{\overline{\theta}} f(\theta^i)\frac{1-F(\theta^i)}{f(\theta^i)}V_\theta^i(\theta^i)\mathrm{d}\theta^i - V^i(\underline{\theta}) \\ &= \mathbb{E}\left[\int_0^T e^{-rt}u(t,x_t^i)\left(v_t^i - \frac{1-F(\theta^i)}{f(\theta^i)}\phi_\theta(t,\theta^i,W_t^i)\right)\mathrm{d}t\right] - V^i(\underline{\theta}).\end{aligned}$$

Summing up the transfers of all agents and subtracting the cost gives the result. □

As Theorem 2 provides a necessary condition for incentive compatibility it follows that if there exists an incentive compatible contract (x, p) such that the allocation process x maximizes the expected virtual valuation given by (18) it maximizes the principal's surplus. Clearly, to maximize the virtual surplus it is optimal to set the transfer to the lowest initial shock equal to zero: $V^i(\underline{\theta}) = 0$ for all agents $i \in N$. The revenue of the principal defined by (18) equals the expected welfare when true valuations are replaced with virtual valuations:

$$\mathbb{E}\left[\int_0^T e^{-rt}s(t,J(t,\theta_t,v_t),x_t)\mathrm{d}t\right]. \quad (19)$$

In the next step we establish that there exists a direct mechanism that maximizes the expected virtual value defined in (18). To do so let us first state the following lemma which ensure that there exists a time separable allocation that maximizes the virtual value:

Proposition 3 (Virtual Value Maximizing Allocation)**.**
There exists an allocation function $x^\star : \mathbb{R}_+ \times \Theta \times \mathbb{R}^n \to X$ *such that the process*

$$x_t^\star \triangleq x_t^\star(t, \theta, v_t)$$

maximizes the expected virtual valuation of the principal defined in (17)*. Furthermore, the allocation* ${x^\star}^i(t, \theta, v_t)$ *of agent* i *is non-decreasing in his valuation* v_t^i *and his initial type* θ^i*.*

Proof. For every t, θ, v_t there exists a non-empty set of allocations which maximize the flow of virtual values[3]

$$X^\star(t, \theta, v_t) = \arg\max_{x \in X} s(t, J(t, \theta, v_t), x) = \arg\max_{x \in X} \sum_{j \in N} J(t, \theta^j, v_t^j) u(t, x^j) - c(x).$$

As u and c are increasing in x^i is is optimal to set the consumption of agent i to zero $x^i = 0$ if his virtual valuation $J(t, \theta^i, v_t^i)$ is negative. As u is increasing in x and J is increasing in θ^i and v^i by Proposition 2 it follows that the objective function of the principal $\sum_{i \in N} \max\{0, J(t, \theta^i, v_t^i)\} u(t, x^i) - c(x)$ is super-modular in (θ^i, x^i) and (v_t^i, x^i). By Topkis' theorem, there exists a quantity $x^\star(t, \theta, v_t) \in X^\star(t, \theta, v_t)$ that maximizes the flow virtual value such that the allocation ${x^\star}^i(t, \theta, v_t)$ of agent i is non-decreasing in θ^i and v_t^i. As the virtual value of the principal at time t depends only on t, the initial reports θ, and the type v_t, this flow allocation that conditions only on (t, θ, v_t) is an optimal allocation process:

$$\sup_{(x_t)} \mathbb{E}\left[\int_0^T e^{-rt} s(t, J(t, \theta_t^i, v_t^i), x_t) \mathrm{d}t\right] = \mathbb{E}\left[\int_0^T e^{-rt} \sup_{x \in X} s(t, J(t, \theta_t^i, v_t^i), x_t) \mathrm{d}t\right]. \qquad \square$$

5.2 Sufficiency

To prove incentive compatibility of the optimal allocation process let us first establish a version of a classic result in static mechanism design.

Proposition 4 (Static Implementation)**.**
Let $y \subset \mathbb{R}$ *and let* $q : Y \times Y \to \mathbb{R}$ *be absolutely continuous in the first variable with weak derivative* $q_1 : Y \times Y \to \mathbb{R}_+$ *and let* q_1 *be increasing in the second variable. Then the payment*

$$p(y) = q(y, y) - \int_0^y q_1(z, z) \mathrm{d}z \,.$$

ensures that truth-telling is optimal, i.e. $q(y, y) - p(y) \geq q(y, \hat{y}) - p(\hat{y})$ *for all* $y, \hat{y} \in Y$ *.*

[3] We denote by $J(t, \theta, v_t) \in \mathbb{R}^n$ the vector of virtual valuations, i.e. $J(t, \theta, v_t)^i = J(t, \theta^i, v_t^i)$.

Proof. We have that

$$q(y,\hat{y}) - p(\hat{y}) = q(y,\hat{y}) - q(\hat{y},\hat{y}) + \int_0^{\hat{y}} q_1(z,z)\mathrm{d}z = \int_{\hat{y}}^{y} q_1(z,\hat{y})\mathrm{d}z + \int_0^{\hat{y}} q_1(z,z)\mathrm{d}z$$
$$= \int_{\hat{y}}^{y} q_1(z,\hat{y}) - q_1(z,z)\mathrm{d}z + \int_0^{y} q_1(z,z)\mathrm{d}z \le \int_0^{y} q_1(z,z)\mathrm{d}z = q(y,y) - p(y)\,. \quad \square$$

In the first step we construct flow payments that make truthful reporting of valuations optimal (on and off the equilibrium path) if the virtual valuation maximizing allocation process $x^\star$ is implemented. Define the payment process $q_t \triangleq q(t,\theta,v_t)$ where the flow payment $q^i : t \times \Theta \times \mathbb{R}^n \to \mathbb{R}$ of agent i is given by:

$$q^i(t,\theta,v_t) \triangleq v_t^i\, u(t, x^{\star i}(t,\theta,v_t)) - \int_0^{v_t^i} u(t, x^{\star i}(t,\theta,(v_t^{-i}, z)))\mathrm{d}z\,.$$

Proposition 5 (Incentive Compatible Transfers)**.**
In the contract $(x^\star, q)$ it is optimal for every agent at every point in time $t > 0$ to report her valuation v_t^i truthfully, irrespective of the reported types θ and reported prior valuations $(v_s)_{s<t}$.

Proof. As the allocation $x^\star(t,\theta,v_t)$ and the payment $q(t,\theta,v_t)$ at time t are independent of all past reported valuations $(v_s)_{s<t}$ the reporting problem of the agent is time-separable. As u is increasing in x, and $x^\star$ is increasing in v^i by Proposition 2, we can apply Proposition 4 to

$$(v^i, \hat{v}^i) \mapsto v^i u(t, x^\star(t,\theta,(\hat{v}^i, v^{-i})))\,,$$

and so guarantee that the payment scheme $q(t,\theta,v)$ makes truthful reporting of valuations optimal for all $t, \theta, v, \hat{v}^i$. $\square$

It remains to augment the payments from Proposition 5 with additional payments that make it optimal for the agents to report their initial types θ truthfully. Note, that as the payments from Proposition 5 ensure truthful reporting of valuations even after initial misreports, we know how agents will behave even after an initial deviation. This insight transforms the time zero reporting problem into a static design problem in which the payments from Proposition 4 can be used to provide incentives.

Define the payment process

$$p_t^\star \triangleq q(t,\theta,v_t) + m(\theta)$$

where the fixed flow payment $m^i : \Theta \to \mathbb{R}$ of agent i is given by:

$$m^i(\theta) = \mathbb{E}\Big[\int_0^T \frac{re^{-rt}}{1-e^{-rT}}\big[v_t^i\, u(t, x^{\star,i}(t,(\hat\theta^i,\theta^{-i}),v)) - \int_{\underline{\theta}}^{\theta^i} \phi_\theta(t,z,W_t^i) u(t, x^{\star\,i}(t,z,\phi(t,z,W_t^i)))\mathrm{d}z\big]\mathrm{d}t\Big].$$

Theorem 3 (Revenue Maximizing Contract)**.**

In the contract $(x^\star, p^\star)$ it is optimal for every agent at every point in time $t > 0$ to report her type and valuation v_t^i truthfully, irrespective of the reported types θ and reported prior valuations $(v_s)_{s<t}$.

Proof. Start with the flow payments q of Proposition 5. By construction of the payments each agent reports his type truthfully independent of his initial report θ. Let $\hat V(\theta^i, \hat\theta^i)$ be the agent's value if she is of initial type θ^i reports $\hat\theta^i$ and reports truthful after time zero (not that this is not! a consistent, but an optimal deviation after time zero)

$$\hat V(\theta^i, \hat\theta^i) = \mathbb{E}\left[\int_0^T e^{-rt}\left[v_t^i\, u(t, x^{\star\,i}(t,(\hat\theta^i,\theta^{-i}),v_t^i)) - q(t,(\hat\theta^i,\theta^{-i}),v_t)\right]\mathrm{d}t\right].$$

As it is optimal to report v^i truthfully we have that

$$\frac{\partial}{\partial v_t^i}\left(v_t^i\, u(t, x^{\star\,i}(t,(\hat\theta^i,\theta^{-i}),v_t^i)) - q(t,(\hat\theta^i,\theta^{-i}),v_t)\right) = u(t, x^\star(t,(\hat\theta^i,\theta^{-i}),v_t^i)).$$

Thus, the derivative of agent i's value with respect to his initial type is given by

$$\hat V_{\theta^i}(\theta^i, \hat\theta^i) = \mathbb{E}\left[\int_0^T e^{-rt}\left[\phi_\theta(t,\theta^i,W_t^i)\, u(t, x^{\star\,i}(t,(\hat\theta^i,\theta^{-i}),v_t^i))\right]\mathrm{d}t\right]. \tag{20}$$

As ϕ_θ is positive, u is increasing in x, and $x^{\star\,i}$ is increasing in $\hat\theta^i$ by Proposition 2, Proposition 4 implies that truthful reporting of θ^i is optimal for agent i if he has to make a payment of $m^i(\theta)(1-e^{-rT})/r$ at time zero. As the agent can commit to payments we can transform this payment into a constant flow payment with the same discounted present value by multiplying with $r/(1-e^{-rT})$. Note, that as the payment does not depend on the valuations it is optimal for the agent to report his valuations truthfully in the contract $(x^\star, p^\star)$ where $p_t^\star \triangleq q(t,\theta,v_t) + m(\theta)$.

□

5.3 Return to the Initial Example

Let us now return to the example discussed in Section 2. In this section we discussed that in the single agent case where the valuation follows a (shifted) geometric Brownian Motion a menu over static two part tariffs is revenue maximizing. Thus, let v be a shifted geometric Brownian motion, i.e. $\mathrm{d}v_t = (v_t - \underline{v})\mathrm{d}t$. Let the initial value of the process be the private information of the agent $\theta = v_0$ distributed according to $F : [\underline{v}, \overline{v}] \to [0,1]$ such that $f(\underline{v}) \geq 1/\underline{v}$.

Proposition 6.
An indirect, revenue maximizing mechanism is given by a menu of prices (parametrized over z) of the form

$$p_t \equiv p(z, x_t) = z + A(z)c(x_t) - (A(z) - 1)\underline{v}$$

Note, that the optimality of two part tariffs in the single agent case does not rely on the assumptions made in Proposition 6 and we have the following general result:

Proposition 7 (Two Part Tariffs)**.**
There exist a revenue maximizing two part tariff, where at time zero the agent communicates θ truthfully and then at every point in time t chooses his consumption x_t and pays $\tilde{p}(t, \theta, x_t)$.

Proof. Define the set valuations such that a given allocation x is optimal at time t

$$V^\star(t, \theta, x) = \{v \in \mathbb{R} : x = X^\star(t, \theta, v)\} .$$

We can define the payment as ∞ if an allocation is never optimal, i.e. $V^\star = \emptyset$. For every allocation x such that $V^\star(t, \theta, x) \neq \emptyset$ there exists at least one valuation v such that the agent would receive this allocation x if he reports v in the direct mechanism of Theorem 3. The payment of the mechanism described in Theorem 3 depends only on the allocation, but not on the valuation v. Thus, we have that the following payment implements the virtual valuation maximizing allocation in an indirect mechanism:

$$\tilde{p}(t, \theta, x) = \begin{cases} \inf\{p(t, \theta, v) : v \in V^\star(t, \theta, x)\}, & \text{if } V^\star(t, \theta, x) \neq \emptyset; \\ \infty, & \text{else .} \end{cases}$$

□

5.4 A Closer Look at the Related Literature

Our model can be understood as a generalization of the setup analyzed Eső and Szentes (2007). Eső and Szentes consider a two period model in which a single allocative decision is made in the second period. They show that in every two period model, where one signal arrives at the beginning of every period, one can represent the type after the arrival of the second period signal as a function of the first period and an independent second period signal[4]. In our setup we assume an identical signal structure to exist in continuous time. The type at every point in time v_t can be represented as a function ϕ of the initial shock θ^i and an independent time t signal W_t, i.e. $v_t = \phi(t, \theta^i, W_t^i)$. Note, that while Eső and Szentes prove the existence of such a signal decomposition we need to assume it as even in a three period model the valuation might depend on all the signals that arrive after time zero.

Our Assumptions 1 and 2 are similar to the Assumptions 1 and 2 made in Eső and Szentes. More precisely in Lemma 2 Eső and Szentes show that their Assumption 1 is equivalent to (in our notation)

$$\phi_{\theta w}(t, \theta, w) \leq 0, \tag{A}$$

and their Assumption 2 is equivalent to (in our notation)

$$\frac{\phi_{\theta\theta}(t, \theta, w)}{\phi_\theta(t, \theta, w)} \leq \frac{\phi_{\theta w}(t, \theta, w)}{\phi_w(t, \theta, w)}. \tag{B}$$

As

$$\frac{\partial}{\partial w} \frac{\phi_\theta}{\phi} = \frac{\phi_{\theta w}\phi - \phi_\theta \phi_w}{\phi^2},$$

Assumption 1 of Eső and Szentes implies our Assumption 1 and is thus stronger. As

$$\frac{\partial}{\partial \theta} \frac{\phi_\theta}{\phi_w} = \frac{\phi_{\theta\theta}\phi_w - \phi_\theta \phi_{\theta w}}{\phi_w^2} = \frac{\phi_\theta}{\phi_w} \left(\frac{\phi_{\theta\theta}}{\phi_\theta} - \frac{\phi_{\theta w}}{\phi_w} \right)$$

Assumption 2 of our setup is exactly equivalent to Assumption 2 in Eső and Szentes. Hence, the basic conditions on the payoffs and the shocks extend the conditions in Eső and Szentes directly to an environment with many periods and many (flow) allocation decisions.

However, the proof strategy here departs substantially from the one pursued in Eső and Szentes and offers an arguably more direct route to the derivation of the revenue maximizing contract. A

[4] In Eső and Szentes the function ϕ is called u_i, the first period signal θ^i is called v_i and the independent signal W_t^i in period $t = 2$ is called s_i.

direct comparison might be informative for the reader. The derivation of the optimal mechanism in Eső and Szentes proceeds in two steps.

First, they derive the optimal mechanism when the second period signal is directly observed by the principal, so that the private information of the agents consists only of the first period signal. This establishes an upper bound on the revenue that the principal can achieve when all signals are private information to the agent, their Proposition 1. We omit this step completely and directly establish the revenue equivalence result by means of the consistent deviations.

Second, they establish that the upper bound derived in Proposition 1 can be achieved, i.e. there are transfers that make the allocation incentive compatible. To characterize the incentive constraints in the first period, Eső and Szentes integrate over the second period continuation payoffs. This approach relies on a backward induction argument that requires recursively integrating over all future period payoffs, which makes it difficult to extend the argument beyond the two period setting. In Lemma 3, they derive restrictions on the payoffs if the agent reported truthfully in first period and is induced to be report thruthfully in the second period. In Lemma 4, it is established that in any incentive compatible mechanism, conditional on a deviation in the first period, in the second period it is optimal for the agent to misreport "to correct for his lie" in terms of the final valuation for the object.[5] With the determination of the optimal strategy in the second period, Lemma 5 then establishes the value of a deviation in the first period in any incentive compatible mechanism. And finally, Lemma 6 establishes the revenue equivalence result from the point of view of the agent's expected utility. Theorem 1 then recovers the revenue of the principal and shows that it achieves the upper bound established in Proposition 1.

By contrast, we aggregate the shocks, the initial shock and all subsequent shocks, in a single type, a sufficient statistic for the private information of the agent. We then establish the revenue equivalence result directly with a single class of deviations, the consistent deviations, by first establishing (Lipschitz) continuity of the value function of the agent, our Proposition 1, and then directly establishing revenue equivalence from the principal's point of view, our Theorem 2. The consistent deviations have the crucial property that their expectation (beyond the initial shock) can

[5] A methodological drawback of requiring the agent to correct his lie is that it requires a full support assumption to ensure that after every history there exists a signal that leads to any type. While a correcting report does not exist if the full support assumption is not satisfied, by contrast a consistent report is always well defined.

be computed by the same probability measure as the true type, and hence when we compute the value function of the agent (and obtain the revenue equivalence result) we do not have to appeal to a recursive argument at all, but rather form a single expectation conditioning only on the initial shock. As we establish the revenue equivalence (and the optimal mechanism) by using only a small class of deviations, we then have to verify the incentive compatibility with respect to all possible deviations to complete the argument. Now, the allocation of the optimal mechanism can be shown to depend only on the initial type and the current type, our Proposition 3. Moreover, the incentive constraints of the agent, conditional on the allocation plan, depends only on his current type. Thus, by standard arguments of static incentive compatibility, our Proposition 5, we can find flow payments such that the agent reports truthfully irrespective of the initial report, our Proposition 6. Finally, Theorem 3, using a simple supermodularity argument shows that we can complement the flow transfers with initial payments to get truthful initial reports, which in turn lead to the value function.

Similar to the handicap auction of Esö and Szentes the revenue maximizing mechanism in our setting only discriminates based on the initial shock.

Pavan, Segal, and Toikka (2014) observed in the context of a discrete time environment that time-separability of the allocation plus monotonicity of the virtual valuation in θ^i and v_t^i is sufficient to ensure strong monotonicity of the virtual valuation maximizing allocation (monotonicity in θ^i and v_t^i after every history). Furthermore, they show that strong monotonicity is sufficient for the implementability of the virtual valuation maximizing allocation (Corollary 1).

As the allocation at time t does not change the set of possible allocations at later times our environment is time-separable. Our assumptions are similar to the assumptions made in the section discussing separable environments in Pavan, Segal, and Toikka in the sense that they ensure strong monotonicity which in turn implies implementability of the virtual value maximizing allocation. However, our assumption on the stochastic process, are weaker than the assumptions made on the primitives in Proposition 1 in Pavan, Segal, and Toikka to allow for the geometric Brownian motion. The reason we can establish sufficiency under weaker assumptions on the stochastic process lies in the multiplicative separable structure we assume between the valuation and the allocation.

Note, that the revenue maximizing mechanism proposed in this paper is a menu over static contracts. This means that payments and allocations at time t depend only on the time t valuations and the time zero types. In Pavan, Segal, and Toikka allocations and payments depend on the complete history of valuations.

A major advantage of the continuous time model is that it allows us to easily obtain closed form expressions for the revenue maximizing mechanism. This enables us to analyze the distortion introduced due to revenue maximization under different informational assumptions in the next section.

6 Long-run Behavior of the Distortion

In this section we analyze how the distortion behaves in the long-run. We are interested in the expected social welfare generated by the revenue maximizing allocation compared to the expected welfare generated by the socially optimal allocation.

6.1 Necessary and Sufficient Conditions

We make the following definition:

Definition 4 (Vanishing Distortion)**.**
The distortion vanishes over time if the social welfare generated by the revenue maximizing allocation converges to the social welfare generated by the socially optimal allocation:

$$\lim_{t\to\infty} \mathbb{E}\left[s\big(t, v_t, x(t, v_t)\big) - s\big(t, v_t, x(t, J(t, \theta, v_t))\big)\right] \,.$$

Proposition 8 (Long-run Behaviour of the Distortion)**.**
The following two statements characterize the long-run behavior of the distortion:

(a) The distortion vanishes in the long run if the expected valuation of any type converges to the expected valuation of the lowest type, i.e.

$$\lim_{t\to\infty} \mathbb{E}\left[v_t \,|\, \theta^i = x\right] - \mathbb{E}\left[v_t \,|\, \theta^i = \underline{\theta}\right] \to 0\,. \tag{21}$$

(b) If $n = 1$, $u(t, x) = x$, c is twice continuously differentiable, strictly convex with $0 < c'' \leq D$ and the expected valuation of any type does not converges to the expected valuation of the lowest type (i.e. (21) *is not satisfied) then the distortion does not vanish.*

Proof. First note that the difference in the expected type to the lowest initial shock equals

$$\begin{aligned}
\mathbb{E}\left[v_t \,|\, \theta^i = x\right] - \mathbb{E}\left[v_t \,|\, \theta^i = \underline{\theta}\right] &= \mathbb{E}\left[\phi(t,\theta^i,W_t^i) - \phi(t,\underline{\theta},W_t^i)\right] \\
&= \mathbb{E}\left[[(1-F(z))\phi(t,z,W_t^i)]_{z=\underline{\theta}}^{z=\overline{\theta}} + \int_{\underline{\theta}}^{\overline{\theta}} f(z)\phi(t,z,W_t^i)\mathrm{d}z\right] \\
&= \mathbb{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \frac{1-F(z)}{f(z)}\phi_\theta(t,z,W_t^i)f(z)\mathrm{d}z\right] \\
&= \mathbb{E}\left[\frac{1-F(\theta^i)}{f(\theta^i)}\phi_\theta(t,\theta^i,W_t^i)\right] .
\end{aligned}$$

Part (a): We prove that the distortion vanishes if $\lim_{t\to\infty}\mathbb{E}\left[\frac{1-F(\theta^i)}{f(\theta^i)}\phi_\theta(t,\theta,W_t)\right] = 0$. We first show that the welfare loss at a fixed point in time can be bounded by the difference between virtual valuation $J \in \mathbb{R}^n$ and valuation $v \in \mathbb{R}^n$

$$\begin{aligned}
&s(t,v,x^\star(t,v)) - s(t,v,x^\star(t,J)) \\
&\quad= \left(\sum_{i\in N} v^i u(t,x^{\star i}(t,v)) - c(x^\star(t,v))\right) - \left(\sum_{i\in N} v^i u(t,x^{\star i}(t,J)) - c(x^\star(t,J))\right) \\
&\quad= \left(\sum_{i\in N} v^i u(t,x^{\star i}(t,v)) - c(x^\star(t,v))\right) - \left(\sum_{i\in N} J^i u(t,x^{\star i}(t,J)) - c(x^\star(t,J))\right) \\
&\qquad\quad - \sum_{i\in N}(v_i - J^i)u(t,x^{\star i}(t,J)) \\
&\quad\le \left(\sum_{i\in N} v^i u(t,x^{\star i}(t,v)) - c(x^\star(t,v))\right) - \left(\sum_{i\in N} J^i u(t,x^{\star i}(t,v)) - c(x^\star(t,v))\right) \\
&\qquad\quad - \sum_{i\in N}(v_i - J^i)u(t,x^{\star i}(t,J)) \\
&\quad= \sum_{i\in N}(v^i - J^i)(u(t,x^{\star i}(t,v)) - u(t,x^{\star i}(t,J)).
\end{aligned}$$

As the set of possible allocations X is compact and u is continuous there exists a constant $C > 0$ such that

$$\sum_{i\in N}(v^i - J^i)(u(t,x^{\star i}(t,v)) - u(t,x^{\star i}(t,J)) \le C\sum_{i\in N}(v^i - J^i)\,.$$

Hence the welfare loss resulting from the revenue maximizing allocation resulting from the revenue maximizing allocation is linearly bounded by the difference between virtual and real valuation. As the difference between v_t^i and J_t^i equals $\frac{1-F(\theta^i)}{f(\theta^i)}\phi_\theta(t,\theta^i,W_t^i)$ it follows that

$$\begin{aligned}\mathbb{E}\left[s(t,v_t,x^\star(t,v_t)) - s(t,v_t,x^\star(t,J_t))\right] &\leq C\,\mathbb{E}\left[\sum_{i\in N}(v^i - J^i)\right]\\ &= C\,\mathbb{E}\left[\sum_{i\in N}\frac{1-F(\theta^i)}{f(\theta^i)}\phi_\theta(t,\theta^i,W_t^i)\right]\\ &= C\left(\mathbb{E}\left[v_t\,|\,\theta^i = x\right] - \mathbb{E}\left[v_t\,|\,\theta^i = \underline{\theta}\right]\right)\,.\end{aligned}$$

Taking the limit $t\to\infty$ gives the result.

Part (b): We prove that the distortion does not vanish in the long run if the expected type of any initial shock does not converge to the expected type of the lowest initial shock. First, we prove that the distortion changes the allocation. As $u(t,x)=x$ is linear and c is convex this implies that the function $x\mapsto vx - c(x)$ is concave and has an interior maximizer for every (t,v). This implies that for every point in time t and every valuation v

$$0 = v - c'(x^\star(t,v))\,.$$

By the implicit function theorem

$$x_v^\star(t,v) = \frac{1}{c''(x^\star(t,v))} \geq \frac{1}{D}\,.$$

Intuitively this means that the allocation is responsive to the type v. We calculate the change in social welfare induced by the type v and the virtual valuation J

$$\begin{aligned}s(t,v,x^\star(t,v)) - s(t,v,x^\star(t,J)) &= [vx^\star(t,v) - c(x^\star(t,v))] - [vx^\star(t,J) - c(x^\star(t,J))]\\ &= \int_J^v x^\star(t,z)\mathrm{d}z - (v-J)x^\star(t,J)\\ &= \int_J^v x^\star(t,z) - x^\star(t,J)\mathrm{d}z\\ &\geq \frac{1}{D}\int_J^v (z-J)\mathrm{d}z = \frac{(v-J)^2}{2D}.\end{aligned}$$

As the difference between valuation and virtual valuation is given by $\frac{1-F(\theta^i)}{f(\theta^i)}\phi_\theta(t,\theta^i,W_t^i)$ taking expectations yields

$$\begin{aligned}\mathbb{E}\left[s(t,v,x(v))-s(t,v,x(J))\right] &\geq \frac{1}{2D}\mathbb{E}\left[\left(\frac{1-F(\theta^i)}{f(\theta^i)}\phi_\theta(t,\theta^i,W_t^i)\right)^2\right] \\ &\geq \frac{1}{2D}\mathbb{E}\left[\left(\frac{1-F(\theta^i)}{f(\theta^i)}\phi_\theta(t,\theta^i,W_t^i)\right)\right]^2 \\ &= \frac{\mathbb{E}\left[v_t \mid \theta^i = x\right]-\mathbb{E}\left[v_t \mid \theta^i\right]}{2D},\end{aligned}$$

where the middle step follows from Jensen's inequality.

□

7 Sequential Auctions and Distortions

We now illustrate some of our general results within the context of a sequential auction model. The allocation problem is as follows. At every point in time t, the owner of a single unit of a, possibly divisible, object wishes to allocate it among the competing bidders, $i = 1, ..., n$. The allocation space is given in every period t by $x_t^i \in [0,1]$ and $\sum_{i=1}^N x_t^i \leq 1$. The marginal cost of providing the object is constant and normalized to zero within the constraint of a single unit. The flow utility of each agent i is described by

$$v_t^i \cdot x_t^i - p_t^i.$$

Thus, we assume that $u_i\left(t, x_t^i\right) = x_t^i$ for all i and t_i, and hence v_t^i immediately represents the willingness to pay of the agent in period t. We can interpret the allocation process as a process of intertemporal licensing where the current use of the object is determined on the basis of the past and current reports of the agents, and in particular, the assignment of the object can move back and forth between the competing agents. Alternatively, the description of the valuation could be rephrased as a description of the marginal cost of producing a single good, and the associated allocation process is the solution to a long-term procurement contract with competing producers. As in the static theory of optimal procurement, the virtual valuation would then be replaced by the virtual cost, but the structure of the allocation process would remain intact.

7.1 Arithmetic Brownian Motion

In the opening example we represented the valuation process by a geometric Brownian motion, now we represent the valuation process by the arithmetic Brownian motion, thus indicating the versatility of the current approach. We are particularly interested in discussing how the nature of the private information rent, as captured by the virtual utility, changes over time, and influences the allocation process. We also derive an associated payment process p_t^i which guarantees that the interim participation constraint of all agents and all types is maintained throughout the dynamic mechanism.

The arithmetic Brownian motion v_t^i is completely described by its initial value v_0^i and the drift μ and the variance σ of the diffusion process W_t. The willingness to pay of agent i therefore evolves according to:

$$\mathrm{d}v_t^i = \mu \mathrm{d}t + \sigma \mathrm{d}W_t^i,$$

so that the type of agent i, his willingness to pay, can be represented as:

$$v_t^i = v_0^i + \mu t + \sigma W_t^i. \tag{22}$$

Interestingly, we can analyze the incentive problem when either one of the three determinants of the Brownian motion, the initial value, the drift or the variance is unknown, whereas the remaining two are commonly known. Below, we begin the analysis with the case of an unknown initial value, then consider the case of unknown drift, and finally the case of unknown variance. Surprisingly, we find that even though we consider the same stochastic process, the nature of the private information, i.e. about which aspect of the process the agent is privately informed, has a substantial impact on the optimal allocation. In particular, we find that the distortion is either constant, increasing or random (and increasing in expectation) depending on the precise nature of the private information.

Unknown Initial Value We begin with the case where the initial value of the Brownian motion, $v_0^i = \theta^i$, is private information to agent i, as are all future realizations of the Brownian motion, v_t^i. In contrast, the drift μ and the variance σ of the Brownian motion are assumed to be commonly known. Given the above representation of the Brownian motion, we have

$$v_t = \phi(t, \theta^i, W_t^i) = \theta^i + \mu t + \sigma W_t^i. \tag{23}$$

We can immediately verify that the partial derivative of ϕ with respect to θ is given by $\phi_\theta = 1$. It follows that the virtual valuation is given by:

$$J^i(t, \theta^i, v_t^i) = v_t^i - \frac{1 - F(\theta^i)}{f(\theta^i)}, \tag{24}$$

and we find that the distortion imposed by the revenue maximizing mechanism is constant over time. In every period, the object is allocated to the agent i_t^* with the highest virtual utility, provided that the valuation is positive: Thus, the allocation proceeds by finding the bidder with the highest valuation, after taking into account a handicap, that is determined once and for all through the report of the initial shock.

Earlier, we gave a general description of the payments decomposed into an annualized up-front payment z and a flow payment p_t. In the present auction environment, we can give an explicit description of the flow payments in terms of the virtual utility of the agents. The associated flow transfer of the bidders, p_t^i, which also follows directly from the logic of the second price auction are given by:

$$p_t^i = \begin{cases} \max_{j \neq i} \left\{ v_t^j - \frac{1-F(\theta^j)}{f(\theta^j)} \right\} + \frac{1-F(\theta^i)}{f(\theta^i)}, & \text{if} \quad i = i_t^*; \\ 0, & \text{if} \quad i \neq i_t^*. \end{cases} \tag{25}$$

Thus, it is only the winning bidder who incurs a flow payment. By rewriting (25), we find that the winning bidder has to pay his valuation, but receives a discount, namely his information rent, which is exactly equal to the difference in the virtual utility between the winning bidder and the next highest bidder, i.e.

$$p_t^{i^*} = v_t^{i^*} - \left(J^{i^*}(t, \theta^{i^*}, v_t^{i^*}) - \max_{j \neq i^*} \left\{ J^j(t, \theta^j, v_t^j) \right\} \right). \tag{26}$$

By construction of the transfer function, the flow net utility of the bidder is positive whenever he is assigned the object, as

$$v_t^{i^*} \geq v_t^j - \frac{1 - F(\theta^j)}{f(\theta^j)} + \frac{1 - F(\theta^{i^*})}{f(\theta^{i^*})}, \tag{27}$$

and thus, the flow allocation proceeds as a "handicap" second price auction, where the price of the winner is determined by the current value of the second highest bidder, as measured by the virtual utility, and the "handicap" is computed as the difference between the constant handicap of the current winner and the current second highest bidder. The above version of the handicap auction also appears in Eső and Szentes (2007) in a discrete time, two period model of a single unit auction. Similarly, Board (2007) develops a handicap auction in a discrete time, infinite horizon model, but where the object is allocated only once, at an optimal stopping time. There, the handicap is represented as here, by the constant terms, $(1 - F(\theta^j))/f(\theta^j)$ and $(1 - F(\theta^i))/f(\theta^i)$, but the second highest value is computed as the continuation value of the remaining bidders, as in Bergemann and Välimäki (2010).

Unknown Drift We now consider the case where the initial private information of the agent is with respect to the drift of the Brownian motion. Let $v_t^i \in \mathbb{R}_+$ be an arithmetic Brownian motion with drift θ and known variance σ and known initial value, v_0^i:

$$v_t = \phi(t, \theta^i, W_t^i) = v_0^i + \mu t + \sigma W_t^i. \tag{28}$$

The derivative of the valuation ϕ with respect to the initial private information θ, which now represents the drift of the Brownian motion, is given by $\phi_\theta = t$. Thus the virtual valuation is now given by:

$$J(t, \theta^i, v_t^i) = v_t^i - \frac{1 - F(\theta^i)}{f(\theta^i)} t. \tag{29}$$

The flow payment can again be determined as before and is of exactly the same form as (26), and the virtual utility function is given by (29). The distortion is still formed on the basis of the handicap, by the inverse hazard rate $(1 - F(\theta^i))/f(\theta^i)$, but interestingly, we now find that the handicap is increasing linearly in time. It follows that in the contrast to the above case of the unknown starting value, the distortion is growing deterministically over time, and thus certainly not vanishing over time. Since v_t^i might be growing as well, the deterministic increase in the distortion however does not allow us to conclude that the assignment of the object is terminated with probability one at some finite time T, a conclusion that we will arrive at later when we consider the geometric Brownian motion.

Unknown Variance We conclude the analysis of the arithmetic Brownian motion with the case of unknown variance. The valuation v_t^i then evolves according to:

$$v_t = \phi(t, \theta^i, W_t^i) = v_0 + \mu t + \theta W_t^i. \tag{30}$$

Now, the initial private information θ represents the volatility of the Brownian motion, another structural parameter of the stochastic process, and thus the volatility of the valuation process. The derivative of the valuation ϕ with respect to the initial private information θ now takes the form:

$$\phi_\theta = \frac{\phi - v_0 - \mu t}{\theta}$$

In consequence the virtual valuation of agent i can be expressed as:

$$\begin{aligned} J(t,\theta^i,v_t^i) &= v_t^i - \frac{1-F(\theta^i)}{f(\theta^i)} \frac{v_t^i - v_0 - \mu t}{\theta^i} \\ &= v_t^i \left(1 - \frac{1-F(\theta^i)}{f(\theta^i)\theta^i}\right) + \frac{1-F(\theta^i)}{f(\theta^i)\theta^i}(v_0 + \mu t). \end{aligned} \tag{31}$$

The variance of the Brownian motion does not lend itself to an ordering of first order stochastic dominance, rather it is ordered to second order stochastic dominance. Formally, in the case of unknown variance ϕ does not satisfy the assumptions $\phi_\theta \geq 0$ and Assumption 2. Those assumptions are only used to establish that the virtual valuation is increasing in θ, v if it takes positive values, which we ensure here by assuming that $\mu, v_0 \leq 0$.

The basic idea is to use the convexity of the objective function to guarantee that an increase in variance leads to an increase in the expected (virtual) valuation. After all, if the virtual valuation turns negative, the seller does not want to assign the object to the buyer, thus the revenue is flat and equal to zero. It therefore follows that the revenue of the seller naturally has a convex like property. But in contrast to the utility of the buyer, which is linear in v_t, and hence strictly convex if truncated below by zero, the virtual valuation of the seller has additional terms, as displayed by (31) which need to be controlled to guarantee the monotonicity of the virtual utility. From the expression of the virtual utility function we can immediately derive sufficient conditions for the monotonicity. Thus if we assume that the initial value v_0 is negative, $v_0 \leq 0$, and the arithmetic Brownian motion has a negative drift $\mu \leq 0$, then we are guaranteed that the convexity argument is sufficiently strong.

Formally, let $\hat{\theta}$ be the solution to $\hat{\theta} - \frac{1-F(\hat{\theta})}{f(\hat{\theta})} = 0$. As

$$J(t,\theta^i,v_t^i) \leq v_t^i \left(1 - \frac{1-F(\theta^i)}{f(\theta^i)\theta^i}\right)$$

the virtual valuation $J(t,\theta^i,v_t^i)$ is only positive if the valuation v_t^i is negative, for all $\theta^i < \hat{\theta}$. We first show that this implies that the expected discounted payment of the agent is negative if $\theta^i < \hat{\theta}$:

$$\mathbb{E}\left[\int_0^T e^{-rt} p_t^i \mathrm{d}t\right] = \mathbb{E}\left[\int_0^T e^{-rt} J(t, \theta^i, v_t^i) u(t, x_t^i) \mathrm{d}t\right] \leq \left(1 - \frac{1 - F(\theta^i)}{f(\theta^i)\theta^i}\right) \mathbb{E}\left[\int_0^T e^{-rt} v_t^i u(t, x_t^i) \mathrm{d}t\right]$$
$$\leq \left(1 - \frac{1 - F(\theta^i)}{f(\theta^i)\theta^i}\right) \mathbb{E}\left[\int_0^T e^{-rt} p_t^i \mathrm{d}t\right] \Rightarrow \mathbb{E}\left[\int_0^T e^{-rt} p_t^i \mathrm{d}t\right] \leq 0\,.$$

Hence, it can never be optimal to allocate to an agent with variance $\theta^i < \hat{\theta}$. Thus, we ignore agents with low variance $\theta^i < \theta$ and never allocate the object to them. As $\frac{1-F(\theta)}{f(\theta)}$ is non-increasing we have that $1 - \frac{1-F(\theta)}{f(\theta)\theta} > 0$ for all $\theta > \hat{\theta}$ and hence $J(t, \theta^i, v_t^i)$ is increasing in v_t^i and θ^i for all $v_t^i > 0\,, \theta^i > \hat{\theta}$. Hence, by the argument of Proposition 5, there exists a payment such that truthful reporting of valuations becomes optimal irrespective of the reported types. As the virtual valuation

$$J(t, \theta^i, \phi(t, \theta^i, W_t^i)) = W_t^i(\theta - \frac{1 - F(\theta)}{f(\theta)}) + \mu t + v_0$$

is increasing in θ whenever $W_t^i > 0$ and decreasing whenever $W_t^i < 0$ it follows that the product

$$W_t^i u(t, x^{\star\, i}(t, (\hat{\theta}^i, \theta^{-i}), v_t^i))$$

is increasing in the reported type $\hat{\theta}^i$. The derivative of the agents utility with respect to his initial type (Equation (20)) simplifies to

$$\mathbb{E}\left[\int_0^T e^{-rt} \left[W_t^i\, u(t, x^{\star\, i}(t, (\hat{\theta}^i, \theta^{-i}), v_t^i))\right] \mathrm{d}t\right]$$

and thus, by the argument of Theorem 3, the virtual valuation maximizing allocation for the types $\theta > \hat{\theta}$ is incentive compatible.

The last two examples emphasize that our approach can accommodate not only private information about the initial state of a random process, but also private information about the structural parameters of the stochastic process per se, such as the mean or the variance of the process. Importantly, and in contrast to much of the previous literature, we can also accommodate private information about a state variable, such as the variance of the Brownian motion, that cannot be ordered in the sense of first order stochastic dominance.

7.2 Geometric Brownian Motion

The second class of private information processes that we analyze describe the willingness to pay v_t^i of agent i by a geometric Brownian motion, the class of process that we investigated earlier in some detail in Section 2. The geometric Brownian motion v_t^i is completely described by its initial value v_0^i, the drift μ and the variance σ of the diffusion process W_t. The willingness to pay of agent i evolves according to:

$$\mathrm{d}v_t^i = v_t^i \left(\mu \mathrm{d}t + \sigma \mathrm{d}W_t^i\right),$$

so that the valuation of agent i can be represented as

$$v_t^i = v_0^i \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t^i\right). \tag{32}$$

Note that the geometric Brownian motion is always positive which makes it particularly suitable to describe a valuation process.

Unknown Initial Value First we analyze the case where the valuation process is a geometric Brownian motion and the initial valuation is the private information, i.e. $v_0^i = \theta^i$. Given the above representation of the geometric Brownian motion, we have

$$v_t^i = \phi(t, \theta^i, W_t^i) = \theta^i \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t^i\right). \tag{33}$$

We verify that the partial derivative of ϕ with respect to the private information θ^i is given by $\phi_\theta = \frac{\phi}{\theta}$ and thus the expression of the virtual valuation is given by:

$$J^i(t, \theta^i, v_t^i) = v_t^i \left(1 - \frac{1 - F(\theta^i)}{f(\theta^i)\theta^i}\right).$$

It follows that the distortion imposed by the revenue maximizing mechanism is now linear in the valuation at every point in time. The resulting revenue maximizing allocation simply multiplies the valuation of every agent by a constant and then chooses the one with the highest product. Importantly, each agent i is now either included or excluded from the market forever. If the second term in the virtual utility, the (inverse) weighted hazard rate is negative, then agent i never receives the object. Conversely, if the hazard rate term is positive, then agent i has a positive probability of receiving the object at every time t. The associated flow payments p_t^i are similar to those of a second prize auction:

$$p_t^i = \begin{cases} \max\left(0, \max_{j \neq i} v_t^j \frac{1 - \frac{1 - F(\theta^j)}{f(\theta^j)}}{1 - \frac{1 - F(\theta^i)}{f(\theta^i)}}\right), & \text{if } i_t^* = i; \\ 0, & \text{if } i_t^* \neq i. \end{cases}$$

Unknown Drift We now consider the case where the initial private information of the agent constitutes the drift of the geometric Brownian motion. The valuation v_t^i then evolves according to:

$$v_t^i = \phi(t, \theta^i, W_t^i) = v_0^i \exp\left((\theta - \frac{\sigma^2}{2})t + \sigma W_t^i\right), \tag{34}$$

and the derivative of ϕ with respect to θ which is given by $\phi_\theta = \phi t$. Thus the virtual valuation is given by:

$$J(t, \theta^i, v_t^i) = v_t^i \left(1 - \frac{1 - F(\theta^i)}{f(\theta^i)} t\right). \tag{35}$$

Interestingly, the distortion is still formed on the basis of a multiplicative handicap, but now the handicap factor is increasing linearly in time. It follows that in contrast to the above case of an unknown initial value, the distortion is now growing over time. As v_t is positive, it follows that the virtual valuation is positive before a deterministic time

$$T = \frac{f(\theta^i)}{1 - F(\theta^i)},$$

and negative afterwards. Thus, the allocation of the object to agent i ends with probability one at time $T = \frac{f(\theta^i)}{1-F(\theta^i)}$.

In every period, the object is allocated to the agent i_t^* with the highest virtual utility, provided that it is positive and the associated flow payments p_t^i are again similar to those of a second prize auction:

$$p_t^i = \begin{cases} \max\left(0, \max_{j \neq i} v_t^j \frac{1 - \frac{1-F(\theta^j)}{f(\theta^j)} t}{1 - \frac{1-F(\theta^i)}{f(\theta^i)} t}\right), & \text{if } i_t^* = i; \\ 0, & \text{if } i_t^* \neq i. \end{cases}$$

In a recent paper, Boleslavksy and Said (2013) derive the revenue maximizing contract in a discrete time setting where the private information of a single agent is the uptick probability of a multiplicative random walk. As it is well known, the geometric Brownian motion can be viewed as the continuous time limit of the discrete time multiplicative random walk stochastic process. Thus, it is naturally of interest to compare their results to the implications following our analysis. In terms of the private information of the agent, the unknown drift in the geometric Brownian motion here, represents the unknown uptick probability analyzed in Boleslavksy and Said (2013). As the general convergence result of the stochastic process itself would suggest, we can also establish, see the appendix for the details, that the continuous time limit of the virtual valuation derived in Boleslavksy and Said (2013) is the virtual valuation derived above by (35). However in the continuous time limit the expression for the virtual valuation, see (35) above, becomes notably easier to express and to interpret. The analysis in Boleslavksy and Said (2013) explicitly verifies the validity of the incentive constraints in the case of a single agent. With the general approach taken here, we obtain the revenue optimal allocation in the presence of many agents.

7.3 Ornstein-Uhlenbeck process

Next we describe the implications for the revenue maximizing allocation if the stochastic process is given by the Ornstein-Uhlenbeck process, which is the continuous-time analogue of the discrete-time AR(1) process. The Ornstein Uhlenbeck process v_t^i is completely described by its initial value v_0^i, the mean reversion level μ, the mean reversion speed $m \geq 0$ and the variance $\sigma \geq 0$ of the diffusion process B_t. The willingness to pay of agent i evolves according to the stochastic differential equation:

$$\mathrm{d}v_t^i = m(\mu - v_t)\mathrm{d}t + \sigma \mathrm{d}B_t^i \,,$$

where B_t is a standard Brownian motion. The Ornstein-Uhlenbeck process can be represented using a distinct Brownian motion $\tilde{B}$ as:

$$v_t = v_0 e^{-mt} + \mu(1 - e^{-mt}) + \frac{\sigma e^{-mt}}{\sqrt{2m}} \tilde{B}_{2^{mt}-1} \,. \tag{36}$$

Hence we can define the process W as a time-changed Brownian Motion by

$$W_t^m = \frac{e^{-mt}}{\sqrt{2m}} \tilde{B}_{2^{mt}-1} \,.$$

Using W we can represent the valuation of the agent as

$$v_t = v_0 e^{-mt} + \mu(1 - e^{-mt}) + \sigma W_t^m \,.$$

Unknown Initial Value First we analyze the case where the valuation process is an Ornstein Uhlenbeck process and the initial valuation is private information, i.e. $v_0^i = \theta^i$. Given the representation (36) it follows that

$$\frac{\partial v_t^i}{\partial \theta^i} = e^{-mt} \text{ and } \frac{\partial v_t^i}{\partial W_t^i} = \sigma \,.$$

Thus, Assumption 1 and 2 are satisfied. The virtual valuation J equals

$$J(t, \theta^i, v_t^i) = v_t^i - \frac{1 - F(\theta^i)}{f(\theta^i)} e^{-mt} .$$

Hence the optimal mechanism is a handicap mechanism with a deterministic handicap that is exponentially decreasing over time. As the Ornstein-Uhlenbeck process in the long run converges to a stationary distribution which is independent of the starting value θ^i, Proposition 8 applies and the distortion vanishes in the long run. Intuitively the initial valuation does not change the expected valuation in the long run.

Unknown Long Run Average Similarly to the earlier analysis of the arithmetic or geometric Brownian motion, we can also take the structural parameter of the stochastic process to be the private information of the agent, that is we can take the expected long run average of the process to be the private information of agent i, i.e. $\mu = \theta^i$. Given the representation (36) it follows that

$$\frac{\partial v_t^i}{\partial \theta^i} = 1 - e^{-mt} \text{ and } \frac{\partial v_t^i}{\partial W_t^i} = \sigma .$$

Thus, Assumption 1 and 2 are satisfied. The virtual valuation J equals

$$J(t, \theta^i, v_t^i) = v_t^i - \frac{1 - F(\theta^i)}{f(\theta^i)} (1 - e^{-mt}) .$$

Hence the optimal mechanism is a handicap mechanism with a deterministic handicap that is increasing over time. As the Ornstein-Uhlenbeck process converges in the long run to a stationary distribution which depends on the long run average θ^i the distortion increases in the long run. Intuitively the expected valuation converges to the long run average θ^i, and so does the virtual valuation, it converges to the long rune average as well.

7.4 A Bandit Model of Learning

Finally, we consider the allocations and distortions that arise in a multi-armed bandit model of learning. Here, each agent is assumed to have an unknown valuation $\eta^i \in \mathbb{R}_+$ for the object. While the valuation is initially unknown to the agent he observes a signal z_t^i about his valuation for the object. The evolution of the signal z^i given η^i follows the stochastic differential equation:

$$\mathrm{d}z_t^i = \eta^i \mathrm{d}t + \sigma \mathrm{d}W_t^i .$$

Let us denote by $\mathcal{F}^{z,i}$ the filtration generated by the signal z^i. As his true valuation is unknown to the agent he uses his signal to calculate his expected valuation $v_t = \mathbb{E}(\eta^i \,|\, \mathcal{F}_t^{z,i})$. Let us denote by F the distribution of v_0, i.e. $v_0 \sim F$. It is known, see Liptser and Shiryayev (1977), Theorem 10.1, that the expected valuation v^i of agent i follows the differential equation:

$$\mathrm{d}v_t = \frac{1}{1+\sigma^2 t}\mathrm{d}W_t^{z,i},$$

where $W_t^{z,i} = \int_0^t \mathrm{d}z_s - v_s \mathrm{d}s$ is a Brownian motion with respect to $\mathcal{F}^z$. Thus the valuation v_t^i at time t equals:

$$v_t^i = v_0^i + \int_0^t \frac{1}{1+\sigma^2 t}\mathrm{d}W_t^{z,i}.$$

If the initial valuation $v_0^i = \theta^i$ is the private information of the agent at time zero, then it follows that his virtual valuation is given by:

$$J(t,\theta^i,v_t^i) = v_t^i - \frac{1-F(\theta^i)}{f(\theta^i)}.$$

Clearly, J is increasing in v_t^i. If we assume further that $\frac{1-F(\theta^i)}{f(\theta^i)}$ is decreasing, then J is also increasing in θ^i. We then find that as $\frac{1-F(\theta^i)}{f(\theta^i)}$ is constant over time that the distortion in the allocation does not vanish in the long-run.

8 Conclusion

We analyzed a class of dynamic allocation problems with private information in continuous time. In contrast to much of the received literature in dynamic mechanism design, the private information of each agent was not restricted to the current state of the Markov process. In particular, the private information was allowed to pertain to structural parameters of the stochastic process such as the drift of the arithmetic or geometric Brownian motion, or the long-run average of the mean-reverting process. By allowing for a richer class of private information structures, we gained a better understanding about the nature of the distortion due the private information. In contrast to the Markovian settings, where the distortions induced by the revenue maximizing allocation are typically vanishing over time, we have shown that the distortion can be constant, increasing or decreasing over time. The analysis of the private information in terms of the stochastic flow, the equivalent of the impulse response functions in continuous time, allowed us directly the nature of the private information to the nature of the intertemporal distortion.

A distinct advantage of the continous time approach taken here is that we could offer explicit solutions, in terms of the optimal allocation, the level of distortion and the transfer payments. We highlighted this advantage in the introductory example in which we gave complete, explicit and surprisingly simple solutions to a class of sales/licensing problems. In particular, we showed that we can implement the dynamic optimal contract by means of an essentially static contract, a membership contract, that displayed such common empirical features as flat rates, free consumption units and two part tariffs.

Appendix

Proof of Proposition 6. Note that a strong solution for the geometric Brownian motion is given by

$$v_t = v_0 \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right) + \underline{v}.$$

By (17) the virtual valuation equals

$$J(t, v_0, v_t) = v_t\left(1 - \frac{1 - F(v_0)}{f(v_0)v_0}\right) + \frac{1 - F(v_0)}{f(v_0)v_0}\underline{v}\,. \tag{37}$$

As shown in Theorem 2 the seller aims at maximizing

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(J_t x_t - c(x_t)\right)\right].$$

Define $A(v_0) = \left(1 - \frac{1-F(v_0)}{f(v_0)v_0}\right)^{-1}$. At every point in time t the seller aims at choosing the consumption level x_t that maximizes the virtual valuation

$$\begin{aligned} J(t, v_0, v_t)x_t - c(x) &= \left(v_t\left(1 - \frac{1 - F(v_0)}{f(v_0)v_0}\right) + \frac{1 - F(v_0)}{f(v_0)v_0}\underline{v}\right)x - c(x) \\ &= A(v_0)^{-1}\left(v_t x - A(v_0)c(x) + (A(v_0) - 1)x\underline{v}\right) \end{aligned}$$

Consequently a payment of $p_t = A(v_0)c(x) - (A(v_0) - 1)x\underline{v}$ perfectly aligns the interest of the buyer and the seller at every point in time $t > 0$. It remains to prove that it is incentive compatible for the buyer to report his time zero valuation truthfully.

Let us first deal with the case where $\underline{v} = 0$. Note that in this case Assumption 1 and 2 are satisfied and thus Proposition 2 yields the monotonicity of the virtual valuation $J(t, v_0, v_t)$ in v_0 and v_t conditional on $J_t \geq 0$. If $\underline{v}$ is greater zero it follows from $f(\underline{v}) > 1/\underline{v}$ and the monotonicity of $\frac{1-F(v_0)}{f(v_0)v_0}$ that for all $v_0 \geq \underline{v}$

$$1 - \frac{1 - F(v_0)}{f(v_0)v_0} > 0\,.$$

Hence, the virtual valuation defined in (37) is increasing in v_t and v_0. The proof of Theorem 3 show that this is sufficient for for the existence of a payment that makes it incentive compatible to report the time zero valuation truthfully.

Consider now the special case of quadratic costs, $c(x) = x^2/2$ and let the initial valuation v_0 be exponentially distributed with mean $\hat{v}$:

$$\mathbb{P}[v_0 \leq x] = 1 - \exp\left(-v_0/\hat{v}\right)\,.$$

Consider the situation where the agent decided on a contract $(m, A(m))$ and the consumption tariff $A(m)$ is fixed. The optimal consumption of the agent at time t is given by

$$\{x_t\} = \arg\max_{x \geq 0} \left(x\, v_t - A(m) \frac{x^2}{2} \right) = \frac{v_t}{A(m)} .$$

Hence, the agents expected time zero utility from the contract is

$$\begin{aligned} \max_{(x_t)_{t \in \mathbb{R}_+}} \mathbb{E} \left[\int_0^\infty e^{-rt} (v_t x_t - m - A(m) c(x_t)) \right] &= \mathbb{E} \left[\int_0^\infty e^{-rt} \left(\frac{v_t^2}{2A(m)} - m \right) \right] \\ &= \frac{v_0}{2A(m)(r - \sigma)} - \frac{m}{r} . \end{aligned} \tag{38}$$

Hence, if the agent will choose his optimal contract he will maximize (38) over m select a contract $(m, A(m))$ only based on his time zero valuation v_0. Let us denote by $m(v_0)$ the fixed fee chosen by the agent of initial valuation v_0. In the optimal contract

$$A(m(v_0)) = \begin{cases} \frac{v_0}{v_0 - \mu} & \text{if } v_0 \geq \mu \\ \infty & \text{else} . \end{cases}$$

Hence all buyers who initially have a valuation below the average time zero valuation μ will be excluded and never consume the good no matter how high their future valuation is. □

Relationship to Boleslavksy and Said (2013)

We briefly establish the relationship between the multiplicative random walk in the discrete time environment of Boleslavksy and Said (2013) and the geometric Brownian motion analyzed here. Let $(X_k)_{k \in \mathbb{N}}$ be a multiplicative random walk, i.e.

$$X_{k+1} = \begin{cases} u\, X_k, & \text{with probability } \theta, \\ d\, X_k, & \text{with probability } 1 - \theta ; \end{cases}$$

for some $d < 1 < u$ and let the uptick probability $\theta \in (0, 1)$ be the private information. Boleslavksy and Said (2013) show, see page 11, Eq. (7), that the virtual valuation in period k equals[6]

$$v_k^i \left(1 - \sum_{s \leq k} \mathbf{1}_{\{X_s = dX_{s-1}\}} \frac{u - d}{d(1 - \theta)} \frac{1 - F(\theta)}{f(\theta)} \right) .$$

[6] For convenience we translated their result into our notation. We use k for the period to clearly differentiate between periods and physical time.

In the next step we let the period length Δ go to zero. To do so let $d \equiv d^{\Delta}, u \equiv u^{\Delta}$ and $t \equiv \Delta k \in \mathbb{N}$. The virtual valuation at the physical time t thus equals

$$v_t^i \left(1 - \sum_{s \leq \frac{t}{\Delta}} \mathbf{1}_{\{X_s = dX_{s-1}\}} \left((\frac{u}{d})^{\Delta} - 1 \right) \frac{1-F(\theta)}{f(\theta)(1-\theta)} \right) .$$

Note that $\sum_{s \leq \frac{t}{\Delta}} \mathbf{1}_{\{X_s = dX_{s-1}\}}$ is Binomial distributed and converges to its expectation for $\Delta \to 0$, i.e.

$$\lim_{\Delta \to 0} \sum_{s \leq \frac{t}{\Delta}} \mathbf{1}_{\{X_s = dX_{s-1}\}} = \mathbb{E} \left[\sum_{s \leq \frac{t}{\Delta}} \mathbf{1}_{\{X_s = dX_{s-1}\}} \right] = (1-\theta) \frac{t}{\Delta}.$$

As $\lim_{\Delta \to 0} \frac{1}{\Delta} \left((\frac{u}{d})^{\Delta} - 1 \right) = 1$ we have that the virtual valuation goes to:

$$\begin{aligned} v_t^i \left(1 - (1-\theta) \frac{t}{\Delta} \left((\frac{u}{d})^{\Delta} - 1 \right) \frac{1-F(\theta)}{f(\theta)(1-\theta)} \right) &= v_t^i \left(1 - t \frac{1}{\Delta} \left((\frac{u}{d})^{\Delta} - 1 \right) \frac{1-F(\theta)}{f(\theta)} \right) \\ &= v_t^i \left(1 - \frac{1-F(\theta)}{f(\theta)} t \right), \end{aligned}$$

which establishes the convergence to the virtual valuation derived earlier in (35).

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