AFRIAT FROM MAXMIN

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Abstract

Afriat's original method of proof is restored by using the minmax theorem.

Kewords: Afriat's Theorem, Revealed Preference, GARP

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1 Introduction

Afriat's Theorem brilliantly characterizes the observable implications of utility maximization. Consider a finite set of observable price-consumption data $\{(p^1, x^1), ..., (p^n, x^n)\}$. What are the testable implications of the hypothesis that all the consumption bundles x^i were chosen by maximizing the same unobservable utility function u over budget constraints determined by the corresponding prices p^i?

To answer this question, Afriat [1967] defined the observable net expenditure matrix $A_{ij} = p^i \cdot (x^j - x^i)$ and the unobservable net utility matrix $\Phi_{ij} = u(x^i) - u(x^j)$. If utility were observable, we could deduce the ranking of the consumption bundles by the sign of each Φ_{ij}. Lacking that information directly, the observable data nevertheless indirectly reveal that $u(x^i) \geq u(x^j)$ whenever $A_{ij} \leq 0$, since in that case x^j must have been affordable when x^i was chosen. If u is known to be monotonic, then $A_{ij} < 0$ reveals $u(x^i) > u(x^j)$.

It follows that if the data is derived from maximization of a monotonic utility, then there can be no cycle in the A matrix containing a negative element but no positive element. This property is called the generalized axiom of revealed preference or GARP.[1] Afriat's Theorem asserts that GARP is the only observable implication of utility maximization, even if one restricts attention to concave and monotonic utilities.

Afriat gave an extremely interesting, though complicated, inductive/combinatorial proof of his theorem, but his argument was incomplete, because it failed to deal with the cases in which some $A_{ij} = 0$. In the proof he introduced another important property of matrices, which we shall call additive GARP, or AGARP, which requires that the sum of the entries of any cyclic subset of the matrix must

[1] For the origins of this name, and the distinction between GARP and SARP (the Strong Axiom of Revealed Preference), see Fostel, Scarf, and Todd [2004].

be nonnegative. With this concept in mind, his proof can be divided into three steps. First, he argued that if A satisfies GARP, then the prices in the given data can be rescaled, replacing each p^i with $\lambda_i p^i$, creating a new data set with net expenditure matrix ΛA that satisfies AGARP as well as GARP. This is the most interesting part of his proof, but also the part that is incomplete. Second he shows that for any matrix a satisfying AGARP, like $a = \Lambda A$, there are hypothetical utility levels $(\varphi_1, ..., \varphi_n)$ and a hypothetical net utility matrix $\Phi_{ij} = \varphi_i - \varphi_j$ such that $a + \Phi \geq 0$. Third, he shows that then there must be a concave and monotonic utility u, with $u(x^i) = \varphi_i$, such that $x^i \in \arg\max_{x \in R^L}\{u(x) - p^i \cdot x\}$ for all i.

Several later authors sought simpler and complete proofs, along different lines from Afriat. Varian [1982] gave a different inductive/combinatorial proof, using step three of Afriat but combining the first two steps. Fostel, Scarf, and Todd [2004] did the same in a shorter proof. Diewert [1973] observed that Afriat's theorem could be looked at as a problem in linear programming, and in their second proof, Fostel, Scarf, and Todd succeeded in giving a duality theorem proof of Afriat's theorem, again combining steps one and two.

I return to Afriat's original approach and prove steps one and two separately, both from the maxmin theorem of two person zero sum games. In step one, the "Afriat" player chooses scalar multiples of the prices, while the other player chooses a cycle. The maxmin theorem allows me to complete Afriat's first step, and to strengthen it. I show that the "Afriat" player can find scalar multiples such that no matter what nonzero cycle the other player chooses, the sum of the net expenditures over the cycle will be strictly positive. I call this strict additive GARP, or SAGARP. This strict conclusion allows me to fill the small gap in Afriat's original logic, albeit using a maxmin rather than combinatorial method. In my second lemma, the "Afriat" player chooses the hypothetical utilities, and the other player chooses an entry ij. Using another maxmin argument I show that the "Afriat" player can guarantee that every $a_{ij} + \Phi_{ij} \geq 0$.

I present my proof because I believe each of the first two parts of Afriat's argument are worthy of proof on their own, and to help illuminate the power of his approach. His method of proof contains more information than his theorem. My strengthened version of step one does not follow from Afriat's theorem itself, and it allows me to derive his theorem correctly. The separation of the two steps allows one to instantly derive the theorem of Brown and Calsamiglia [2008] that the only observable implication of utility maximization and constant marginal utility is that A satisfies AGARP. Finally, some readers might agree that a game theoretic proof is the most straightforward. It uses familiar concepts, and it does not require the introduction of any artificial auxiliary variables. Naturally any linear programming proof, such as the one obtained by Fostel, Scarf, and Todd, can be reinterpreted as a maxmin proof. But my proof follows a different logic (for example, by separating steps 1 and 2).

In Sections 2 and 3, I recapitulate the definitions of GARP, AGARP, and SAGARP, and in Section 4, I prove Afriat's Theorem from the maxmin theorem. I present the Brown-Calsamiglia Theorem as a Corollary.

2 Generalized Axiom of Revealed Preference

Definition: The pair (p, x) is a price-consumption datum if $p \in R^L_+ \backslash \{0\}$, $x \in R^L_+$.

Consider a fixed finite set of of price-consumption data $\{(p^1, x^1), ..., (p^n, x^n)\}$.

Definition: The utility function $u : R^L_+ \to R$ rationalizes the price-consumption data $\{(p^1, x^1), ..., (p^n, x^n)\}$ iff for every i,

$$x^i \in \arg\max_{x \in R^L_+} \{u(x) : p^i \cdot (x - x^i) \leq 0\}.$$

Define the net expenditure matrix A by $A_{ij} = p^i \cdot (x^j - x^i)$.

If u rationalizes the data, then $A_{ij} \leq 0$ implies $u(x^i) \geq u(x^j)$, since x^j is affordable at price p^i and x^i was chosen. We say that x^i is **revealed preferred** to x^j.

We say that u is monotonic iff $u(y) > u(x)$ whenever $y >> x$.[2] If u is monotonic and rationalizes the data, then $A_ij < 0$ implies that $u(x^i) > u(x^j)$, since if $p^i \cdot x^j < p^i \cdot x^i$, then there is also $y >> x^j$ (and therefore $u(y) > u(x^j)$) with $p^i \cdot y < p^i \cdot x^i$. We say then that x^i is **revealed strictly preferred** to x^j. It follows from the transitivity of utility maximization choices, that if a monotonic u rationalizes the data, then there can be no cycle in the consumption data of revealed preference including a strict revealed preference. More precisely, consider the following definitions.

Definition: A **cycle** c on $N = \{1, ..., n\}$ is a sequence of distinct integers $(i_1, i_2, ..., i_k)$ with each $i_j \in N$. The cycle c defines a one-to-one function $c : N \to N$ by $c(i_j) = i_{j+1}$ if $1 \leq j \leq k-1$, $c(i_k) = i_1$, and $c(i) = i$ if i is not in the sequence. Note that if for some $i \in N$, $c(i) \neq i$, then for all $t \geq 1$, $c^{t+1}(i) \neq c^t(i)$.

Definition: Given an $n \times n$ matrix a with zeroes on the diagonal, any cycle c defines a **cyclic subset** $a^c = \{a_{ic(i)} : i \in N\}$. Call a^c **nonzero** if some element of it is nonzero.

Definition: An $n \times n$ matrix a with zeroes on the diagonal satisfies **GARP** iff every nonzero cyclic subset a^c contains a positive element.

We have just argued that if a monotonic u rationalizes the data $\{(p^1, x^1), ..., (p^n, x^n)\}$, then the associated net expenditure matrix A must satisfy GARP.

3 Additive GARP (AGARP and SAGARP)

Definition: An $n \times n$ matrix a with zeroes on the diagonal satisfies **additive GARP** or **AGARP** iff for every cycle c, $sum(a^c) = \sum_{i=1}^n a_{ic(i)} \geq 0$.

Definition: An $n \times n$ matrix a with zeroes on the diagonal satisfies **strictly additive GARP** or **SAGARP** iff for every nonzero cycle c, $sum(a^c) = \sum_{i=1}^n a_{ic(i)} > 0$.

Clearly SAGARP is stronger than AGARP which is stronger than GARP. We shall see that if a data set satisfies GARP, then by rescaling the prices, replacing each p^i with $\lambda_i p^i$, we can create a new data set with net expenditure matrix ΛA that satisfies AGARP and even SAGARP, as well as GARP.

[2] the notation $y >> x$ means that $y_i > x_i$ for all i.

4 Afriat's Theorem

Afriat's Theorem: The price-consumption data $\{(p^1, x^1), ..., (p^n, x^n)\}$ can be rationalized by a continuous, concave, and monotonic function $u : R^L_+ \to R$ if and only if the matrix A defined by $A_{ij} = p^i \cdot (x^j - x^i)$ satisfies GARP.

Lemma 1: Suppose that a is an $n \times n$ matrix with zeroes on the diagonal satisfying GARP. Then there is an $n \times n$ diagonal matrix Λ with strictly positive diagonal elements such that Λa satisfies SAGARP.

Proof: Let C be the (finite) set of all cycles c on N for which a^c is nonzero. We suppose C is nonempty, for otherwise the lemma is trivially true.

Consider the two person zero sum game in which the Afriat player chooses any row $i \in N$ and the Cycle player chooses any nonzero cycle $c \in C$. Cycle pays Afriat $a_{ic(i)}$, which is well-defined since each cyclic subset a^c contains exactly one element from each row. By GARP, each nonzero cyclic subset contains a positive element, so Afriat could trivially assure himself a positive payoff if he moved second. We show he can do so even if he moves first, with the correct mixed strategy.

Denote the set of mixed strategies of Afriat by $\Delta^{n-1} = \{\lambda = (\lambda_1, ..., \lambda_n) \in R^n_+ : \sum \lambda_i = 1\}$. Denote the set of mixed strategies of Cycle by $\Delta^{\#C-1} = \{\pi = (\pi(c)_{c\in C} : \pi(c) \geq 0$ for all $c \in C$ and $\sum_{c\in C} \pi(c) = 1\}$. The payoff to Afriat from a mixed strategy pair $(\lambda, \pi) \in \Delta^{n-1} \times \Delta^{\#C-1}$ is

$$\sum_{c\in C} \pi(c) \sum_{i=1}^{n} \lambda_i a_{ic(i)}$$

By von Neumann's minmax theorem the game has a minmax solution $(\lambda^*, \pi^*) \in \Delta^{n-1} \times \Delta^{\#C-1}$ with payoff to Afriat of $v = \sum_{c\in C} \pi^*(c) \sum_{i=1}^{n} \lambda^*_i a_{ic(i)}$.

We shall prove that $v > 0$ by showing that if $v \leq 0$, then there must be a way of splicing together cycles in C to create another cycle that violates GARP. If $v \leq 0$, then no pure strategy of Afriat pays more than 0, hence

$$\sum_{c\in C} \pi^*(c) a_{ic(i)} \leq 0 \text{ for all } i = 1, ..., n$$

Take any cycle that has positive π^* weight. By GARP it contains a positive element, say in row i_1. From the i_1th inequality above, there must be another cycle $c \in C$ that has positive π^* weight with $a_{i_1 c(i_1)} < 0$. Let $i_2 = c(i_1)$. Proceed in cycle c, setting $i_{k+1} = c(i_k)$ as long as $a_{i_k c(i_k)} \leq 0$. If i_k is reached for which $a_{i_k c(i_k)} > 0$, then from the i_kth inequality above, there must be another cycle $d \in C$ that has positive π^* weight with $a_{i_k d(i_k)} < 0$. In that case let $i_{k+1} = d(i_k)$. In this manner of splicing cycles an unlimited sequence $(i_1, i_2, ..., i_k, ...)$ is generated with all $a_{i_t i_{t+1}} \leq 0$. Let i_ℓ be the first entry that repeats an earlier entry, say i_j. The cyclic set a^{c^*} derived from $c^* = (i_j, i_{j+1}, ..., i_{\ell-1})$ violates GARP, because all its entries are nonpositive, and because it must include the (negative) entry point of the cycle that generated $i_{\ell-1} i_\ell = i_{\ell-1} i_j$. This contradiction proves $v > 0$.

From the definition of minmax solution, no pure strategy of Cycle gives a better payoff for him than v, hence

$$\sum_{i=1}^{n} \lambda_i^* a_{ic(i)} \geq v > 0 \text{ for all } c \in C$$

Since the number of cycles is finite, we can perturb the λ^* slightly to make them all strictly positive without changing the fact that $sum(\Lambda a)^c > 0$ for every nonzero cycle.[3] Take $\Lambda_{ii} = \lambda_i$ for all i. **QED**

Lemma 2: Let a be an $n \times n$ matrix with zeroes on the diagonal satisfying AGARP. Then there is $\varphi^* \in R^n$ such that $\min_{i,j}[a_{ij} + \varphi_i^* - \varphi_j^*] \geq 0$.

Proof: Let $v \equiv \sup_{\varphi \in R^n} \min_{i,j}[a_{ij} + \varphi_i - \varphi_j]$. Observe (by taking $\varphi = 0$) that $v \geq -||a||_\infty = -\max_{i,j}\{|a_{ij}|\}$. Therefore we can confine the sup search to φ with $|\varphi_i - \varphi_j| \leq 2||a||_\infty$. Clearly adding a constant to each φ_i does not change anything, so WLOG we can also restrict attention to φ with $\sum \varphi_i = 0$. Let $S = \{\varphi \in R^n : ||\varphi||_\infty \leq 2||a||_\infty, \text{ and } \sum \varphi_i = 0\}$. Since S is compact, there must be some $\varphi^* \in S$ with

$$v \equiv \sup_{\varphi \in R^n} \min_{i,j}[a_{ij} + \varphi_i - \varphi_j] = \max_{\varphi \in S} \min_{i,j}[a_{ij} + \varphi_i - \varphi_j] = \min_{i,j}[a_{ij} + \varphi_i^* - \varphi_j^*]$$

For the same reasons we may suppose that φ^* is one of the maximizers, over all $\varphi \in R^n$, with the fewest number of ij for which $v = [a_{ij} + \varphi_i^* - \varphi_j^*]$. It follows that if there is some ij, $i \neq j$, at which v is achieved, $v = [a_{ij} + \varphi_i^* - \varphi_j^*]$, then v must also be achieved at some jk with $j \neq k$. Otherwise, by subtracting a small constant from φ_j^* we could find another $\varphi \in R^n$ which either increases v or reduces the number of ij at which v is achieved.

Define the $n \times n$ matrix Φ by $\Phi_{ij} = \varphi_i^* - \varphi_j^*$. From the last paragraph we see that by starting from ij and jk we can construct a cycle c such that each element of $[a + \Phi]^c$, is equal to v. Since a satisfies AGARP, and since the sum over any cycle of Φ must be 0, $A + \Phi$ must also satisfy AGARP. So $sum[a + \Phi]^c = (\#c)v \geq 0$, hence $v \geq 0$. **QED**

Lemma 3: Suppose the price-consumption data $\{(p^1, x^1), ..., (p^n, x^n)\}$ generates a net expenditure matrix a satisfying AGARP, and suppose there are hypothetical utility levels $\varphi \in R^n$ generating a net utility matrix Φ with $a + \Phi \geq 0$. Then there is a concave, monotonic, and continuous utility function $u : R^L \to R$ with $u(x^i) = \varphi_i$, such that $x^i \in \arg\max_{x \in R^L}\{u(x) - p^i \cdot x\}$ for all i.

Proof: For all $x \in R^L$ define

$$u(x) = \min_{1 \leq i \leq n} [\varphi_i + p^i \cdot (x - x^i)]$$

As the minimum of linear functions, u is concave and continuous. Since each p^i is nonnegative and nonzero, $p^i \cdot z > 0$ for any $z >> 0$, hence u is monotonic.

[3] Note that we were able to deduce that $\lambda >> 0$ by proving first that Λ satisfies SAGARP rather than just AGARP.

Since rearranging terms in $a_{ji} + \Phi_{ji} \geq 0$ gives $\varphi_j + p^j(x^i - x^j) \geq \varphi_i$ for all i, j, we conclude that $u(x^i) = \varphi_i$ for all i. Clearly $u(x) - p^i \cdot x = \min_{1 \leq k \leq n}[\varphi_k + p^k \cdot (x - x^k)] - p^i \cdot x \leq [\varphi_i + p^i \cdot (x - x^i)] - p^i \cdot x = \varphi_i - p^i \cdot x^i = u(x^i) - p^i \cdot x^i$. **QED**

Proof of Theorem: Given data generating a net expenditure matrix A satisfying GARP, follow lemmas 1-3, yielding strictly positive multipliers λ_i and continuous, concave, monotonic utility u such that $x^i \in \arg\max_{x \in R^L}\{u(x) - \lambda_i p^i \cdot x\}$ for all i. It follows that if for some $x \in R^L$, $p \cdot x \leq p \cdot x^i$, then $u(x^i) \geq u(x)$.

Conversely, if the data is rationalized by any monotonic utility, it is trivial that A satisfies GARP. **QED**

Corollary: The price-consumption data $\{(p^1, x^1), ..., (p^n, x^n)\}$ generates a net expenditure matrix $A_{ij} = p^i \cdot (x^j - x^i)$ that satisfies AGARP if and only if there is a continuous, concave, and monotonic function $u : R^L_+ \to R$ such that $x^i \in \arg\max_{x \in R^L}\{u(x) - p^i \cdot x\}$ for all i.

Proof: Assuming A satisfies AGARP, apply the same proof given for Afriat, skipping lemma 1 by taking Λ to be the identity matrix. To argue in the opposite direction, the utility maximization condition immediately gives $A_{ij} + \Phi_{ij} = A_{ij} + u(x^i) - u(x^j) \geq 0$. Hence $A + \Phi$ trivially satisfies AGARP. But any cyclic subset of Φ sums to 0. So A satisfies AGARP. **QED**

5 References

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