SIEVE WALD AND QLR INFERENCES ON
SEMI/NONPARAMETRIC CONDITIONAL MOMENT MODELS

By

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This paper considers inference on functionals of semi/nonparametric conditional moment restrictions with possibly nonsmooth generalized residuals, which include all of the (nonlinear) nonparametric instrumental variables (IV) as special cases. These models are often ill-posed and hence it is difficult to verify whether a (possibly nonlinear) functional is root-$n$ estimable or not. We provide computationally simple, unified inference procedures that are asymptotically valid regardless of whether a functional is root-$n$ estimable or not. We establish the following new useful results: (1) the asymptotic normality of a plug-in penalized sieve minimum distance (PSMD) estimator of a (possibly nonlinear) functional; (2) the consistency of simple sieve variance estimators for the plug-in PSMD estimator, and hence the asymptotic chi-square distribution of the sieve Wald statistic; (3) the asymptotic chi-square distribution of an optimally weighted sieve quasi likelihood ratio (QLR) test under the null hypothesis; (4) the asymptotic tight distribution of a non-optimally weighted sieve QLR statistic under the null; (5) the consistency of generalized residual bootstrap sieve Wald and QLR tests; (6) local power properties of sieve Wald and QLR tests and of their bootstrap versions; (7) asymptotic properties of sieve Wald and SQLR for functionals of increasing dimension. Simulation studies and an empirical illustration of a nonparametric quantile IV regression are presented.

**KEYWORDS:** Nonlinear nonparametric instrumental variables; Penalized sieve minimum distance; Irregular functional; Sieve variance estimators; Sieve Wald; Sieve quasi likelihood ratio; Generalized residual bootstrap; Local power; Wilks phenomenon.

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1 Earlier versions, some entitled “On PSMD inference of functionals of nonparametric conditional moment restrictions”, were presented in April 2009 at the Banff conference on seminonparametrics, in June 2009 at the Cemmap conference on quantile regression, in July 2009 at the SITE conference on nonparametrics, in September 2009 at the Stats in the Chateau/France, in June 2010 at the Cemmep workshop on recent developments in nonparametric instrumental variable methods, in August 2010 at the Beijing international conference on statistics and society, and econometric workshops in numerous universities. We thank a co-editor, two referees, Don Andrews, Peter Bickel, Gary Chamberlain, Tim Christensen, Michael Jansson, Jim Powell and especially Andres Santos for helpful comments. We thank Yinjia Qiu for excellent research assistant in simulations using R. Chen acknowledges financial support from National Science Foundation grant SES-0838161 and Cowles Foundation. Any errors are the responsibility of the authors.
This paper is about inference on functionals of the unknown true parameters \( \alpha_0 \equiv (\theta_0', h_0) \) satisfying the semi/nonparametric conditional moment restrictions

\[
E[\rho(Y, X; \theta_0, h_0)|X] = 0 \quad \text{a.s. - } X,
\]

where \( Y \) is a vector of endogenous variables and \( X \) is a vector of conditioning (or instrumental) variables. The conditional distribution of \( Y \) given \( X \), \( F_{Y|X} \), is not specified beyond that it satisfies (1.1). \( \rho(\cdot; \theta_0, h_0) \) is a \( d_\rho \times 1 \)-vector of generalized residual functions whose functional forms are known up to the unknown parameters \( \alpha_0 \equiv (\theta_0', h_0) \in \Theta \times H \), with \( \theta_0 \equiv (\theta_{01}, ..., \theta_{0d_\theta})' \in \Theta \) being a \( d_\theta \times 1 \)-vector of finite dimensional parameters and \( h_0 \equiv (h_{01}(\cdot), ..., h_{0q}(\cdot)) \in H \) being a \( 1 \times d_q \)-vector valued function. The arguments of each unknown function \( h_\ell(\cdot) \) may differ across \( \ell = 1, ..., q \), may depend on \( \theta, h_{\ell'}(\cdot), \ell' \neq \ell, X \) and \( Y \). The residual function \( \rho(\cdot; \alpha) \) could be nonlinear and pointwise non-smooth in the parameters \( \alpha \equiv (\theta', h) \in \Theta \times H \).

The general framework (1.1) nests many widely used nonparametric and semiparametric models in economics and finance. Well known examples include nonparametric mean instrumental variables regressions (NPIV): \( E[Y_1 - h_0(Y_2)|X] = 0 \) (e.g., Hall and Horowitz (2005), Carrasco et al. (2007), Blundell et al. (2007), Darolles et al. (2011), Horowitz (2011)); nonparametric quantile instrumental variables regressions (NPQIV): \( E[1\{Y_1 \leq h_0(Y_2)\} - \gamma|X] = 0 \) (e.g., Chernozhukov and Hansen (2005), Chernozhukov et al. (2007), Horowitz and Lee (2007), Chen and Pouzo (2012a), Gagliardini and Scaillet (2012)); semi/nonparametric demand models with endogeneity (e.g., Blundell et al. (2007), Chen and Pouzo (2009), Souza-Rodrigues (2012)); semi/nonparametric random coefficient panel data regressions (e.g., Chamberlain (1992), Graham and
Powell (2012)); semi/nonparametric spatial models with endogeneity (e.g., Pinkse et al. (2002), Merlo and de Paula (2013)); semi/nonparametric asset pricing models (e.g., Hansen and Richard (1987), Gallant and Tauchen (1989), Chen and Ludvigson (2009), Chen et al. (2013), Penaranda and Sentana (2013)); semi/nonparametric static and dynamic game models (e.g., Bajari et al. (2011)); nonparametric optimal endogenous contract models (e.g., Bontemps and Martimort (2013)). Additional examples of the general model (1.1) can be found in Chamberlain (1992), Newey and Powell (2003), Ai and Chen (2003), Chen and Pouzo (2012a), Chen et al. (2014) and the references therein. In fact, model (1.1) includes all of the (nonlinear) semi/nonparametric IV regressions when the unknown functions \( h_0 \) depend on the endogenous variables \( Y \):

\[
(1.2) \quad E[\rho(Y_1; \theta_0, h_0(Y_2))|X] = 0 \quad a.s. - X,
\]

which could lead to difficult (nonlinear) nonparametric ill-posed inverse problems with unknown operators.

Let \( \{Z_i = (Y_i', X_i')\}_{i=1}^n \) be a random sample from the distribution of \( Z \equiv (Y', X') \) that satisfies the conditional moment restrictions (1.1) with a unique \( \alpha_0 \equiv (\theta_0', h_0) \). Let \( \phi : \Theta \times \mathcal{H} \rightarrow \mathbb{R}^{d_\phi} \) be a (possibly nonlinear) functional with a finite \( d_\phi \geq 1 \). Typical linear functionals include an Euclidean functional \( \phi(\alpha) = \theta \), a point evaluation functional \( \phi(\alpha) = h(y_2) \) (for \( y_2 \in \text{supp}(Y_2) \)), a weighted derivative functional \( \phi(h) = \int w(y_2) \nabla h(y_2) dy_2 \) and many others. Typical nonlinear functionals include a quadratic functional \( \int w(y_2) |h(y_2)|^2 dy_2 \), a quadratic derivative functional \( \int w(y_2) |\nabla h(y_2)|^2 dy_2 \), a consumer surplus or an average consumer surplus functional of an endogenous demand function \( h \). We are interested in computationally simple, valid inferences on any \( \phi(\alpha_0) \) of the general model (1.1) with i.i.d. data.

\(^1\)See our Cowles Foundation Discussion Paper No. 1897 for general theory allowing for weakly dependent data.
Although some functionals of the model (1.1), such as the (point) e-
valuation functional, are known \textit{a priori} to be estimated at slower than
root-\(n\) rates, others, such as the weighted derivative functional, are far
less clear without a stare at their semiparametric efficiency bound ex-
pressions. This is because a non-singular efficiency bound is a necessary
condition for \(\phi(\alpha_0)\) to be estimated at a root-\(n\) rate. Unfortunately, as
pointed out in Chamberlain (1992) and Ai and Chen (2012), there is gen-
erally no closed form solution for the efficiency bound of \(\phi(\alpha_0)\) (including
\(\theta_0\)) of model (1.1), especially so when \(\rho(\cdot; \theta_0, h_0)\) contains several unknown
functions and/or when the unknown functions \(h_0\) of endogenous variables
enter \(\rho(\cdot; \theta_0, h_0)\) nonlinerly. It is thus difficult to verify whether the effi-
ciency bound for \(\phi(\alpha_0)\) is singular or not. Therefore, it is highly desirable
for applied researchers to be able to conduct simple valid inferences on
\(\phi(\alpha_0)\) regardless of whether it is root-\(n\) estimable or not. This is the main
goal of our paper.

In this paper, for the general model (1.1) that could be nonlinearly
ill-posed and for any \(\phi(\alpha_0)\) that may or may not be root-\(n\) estimable,
we first establish the asymptotic normality of the plug-in penalized sieve
minimum distance (PSMD) estimator \(\phi(\hat{\alpha}_n)\) of \(\phi(\alpha_0)\). For the model (1.1)
with (pointwise) smooth residuals \(\rho(Z; \alpha)\) in \(\alpha_0\), we propose two simple
consistent sieve variance estimators for possibly slower than root-\(n\) esti-
mator \(\phi(\hat{\alpha}_n)\), which immediately leads to the asymptotic chi-square dis-
bution of the sieve Wald statistic. However, there is no simple variance
estimator for \(\phi(\hat{\alpha}_n)\) when \(\rho(Z, \alpha)\) is not pointwise smooth in \(\alpha_0\) (with-
out estimating an extra unknown nuisance function or using numerical
derivatives). We then consider a PSMD criterion based test of the null
hypothesis \(\phi(\alpha_0) = \phi_0\). We show that an optimally weighted sieve quasi
likelihood ratio (SQLR) statistic is asymptotically chi-square distributed
under the null hypothesis. This allows us to construct confidence sets for
\( \phi(\alpha_0) \) by inverting the optimally weighted SQLR statistic, without the need to compute a variance estimator for \( \phi(\hat{\alpha}_n) \). Nevertheless, in complicated real data analysis applied researchers might like to use simple but possibly non-optimally weighed PSMD procedures for estimation of and inference on \( \phi(\alpha_0) \). We show that the non-optimally weighted SQLR statistic still has a tight limiting distribution under the null regardless of whether \( \phi(\alpha_0) \) is root-\( n \) estimable or not. In addition, we establish the consistency of the generalized residual bootstrap (possibly non-optimally weighted) SQLR and sieve Wald tests under virtually the same conditions as those used to derive the limiting distributions of the original-sample statistics. The bootstrap SQLR would then lead to alternative confidence sets construction for \( \phi(\alpha_0) \) without the need to compute a variance estimator for \( \phi(\hat{\alpha}_n) \). To ease notation burden, we present the above listed theoretical results for a scalar-valued functional in the main text. In Appendix A we present the asymptotic properties of sieve Wald and SQLR for functionals of increasing dimension (i.e., \( d(\phi) = \text{dim}(\phi) \) could grow with sample size \( n \)). We also provide the local power properties of sieve Wald and SQLR tests as well as their bootstrap versions in Appendix A. Regardless of whether a possibly nonlinear functional \( \phi(\alpha_0) \) is root-\( n \) estimable or not, we show that the optimally weighted SQLR is more powerful than the non-optimally weighed SQLR, and that the SQLR and the sieve Wald using the same weighting matrix have the same local power in terms of first order asymptotic theory.

To the best of our knowledge, our paper is the first to provide a unified theory about sieve Wald and SQLR inferences on (possibly nonlinear) \( \phi(\alpha_0) \) satisfying the general semi/nonparametric model (1.1) with possibly non-smooth residuals.\(^2\) Our results allow applied researchers to

\(^2\)We also provide asymptotic properties of sieve score and bootstrap sieve score statistics in the online Appendix D.
obtain limiting distribution of the plug-in PSMD estimator \( \phi(\hat{\alpha}_n) \) and to
construct confidence sets for any \( \phi(\alpha_0) \) regardless of whether it is root-\( n \)
estimable or not. Our paper is also the first to provide local power properties
of sieve Wald and SQLR tests and their bootstrap versions of general nonlinear
hypotheses for the model (1.1).

Roughly speaking, our results extend the classical theories on Wald and
QLR tests of nonlinear hypothesis based on root-\( n \) consistent paramet-
ric minimum distance estimator \( \hat{\alpha}_n \) to those based on slower than root-\( n \)
consistent nonparametric minimum distance estimator \( \hat{\alpha}_n \equiv (\hat{\theta}_n', \hat{h}_n) \) of
\( \alpha_0 \equiv (\theta_0', h_0) \) satisfying the model (1.1). The implementations of the sieve
Wald and SQLR also resemble the classical Wald and QLR based on
parametric extreme estimators and hence are computationally attractive.
For example, our sieve t (Wald) test on a general nonlinear hypothesis
\( \phi(h_0) = \phi_0 \) of the NPIV model \( E[Y_1 - h_0(Y_2)|X] = 0 \) can be implemented
as a standard t (Wald) test for a parametric linear IV model using two
stage least squares (see Subsection 2.2). The proof techniques are quite
different, however, because one is no longer able to rely on the root-\( n \)
asymptotic normality of \( \hat{\alpha}_n \) and then a standard “delta-method” to es-
tablish the asymptotic normality of \( \sqrt{n}(\phi(\hat{\alpha}_n) - \phi(\alpha_0)) \). In our framework
(1.2), \( \sqrt{n}(\phi(\hat{\alpha}_n) - \phi(\alpha_0)) \) could diverge to infinity under the combined ef-
ects of (i) slower convergence rate of \( \hat{\alpha}_n \) to \( \alpha_0 \) due to the ill-posed inverse
problem and (ii) nonlinearity of either the functional \( \phi() \) or the residual
function \( \rho() \) in \( h \). Our proof strategy relies on the convergence rates of
the PSMD estimator \( \hat{\alpha}_n \) to \( \alpha_0 \) in both weak and strong metrics, and then
the local curvatures of the functional \( \phi() \) and the criterion function under
these two metrics. The weak metric is intrinsic to the variance of the linear
approximation to \( \phi(\hat{\alpha}_n) - \phi(\alpha_0) \), while the strong metric controls the non-
linearity (in \( \alpha \)) of the functional \( \phi() \) and of the conditional mean function
\( m(\cdot, \alpha) = E[\rho(Y, X; \alpha)|X = \cdot] \). Unfortunately the convergence rate in the
strong metric could be very slow due to the illposed inverse problem. This explains why it is difficult to establish the asymptotic normality of \( \phi(\hat{\alpha}_n) \) for a nonlinear functional \( \phi() \) even in the NPIV model. Our paper builds upon the recent results on convergence rates in Chen and Pouzo (2012a) and others. In particular, under virtually the same conditions as those in Chen and Pouzo (2012a), we show that our generalized residual bootstrap PSMD estimator of \( \alpha_0 \) is consistent and achieves the same convergence rates as that of the original-sample PSMD estimator \( \hat{\alpha}_n \). This result is then used to establish the consistency of the bootstrap sieve Wald and the bootstrap SQLR statistics under virtually the same conditions as those used to derive the limiting distributions of the original-sample statistics.

There are some published work about estimation of and inference on a particular linear functional, the Euclidean parameter \( \phi(\alpha) = \theta \), of the general model (1.1) when \( \theta_0 \) is assumed to be root-\( n \) estimable; see Ai and Chen (2003), Chen and Pouzo (2009), Otsu (2011) and others. None of the existing work allows for \( \theta_0 \) being irregular (i.e., slower than root-\( n \) estimable) however. When specializing our general theory to inference on \( \theta_0 \) of the model (1.1), we not only recover the results of Ai and Chen (2003) and Chen and Pouzo (2009), but also provide local power properties of sieve Wald and SQLR as well as valid bootstrap (possibly non-optimally weighted) SQLR inference. Moreover, our results remain valid even when \( \theta_0 \) is irregular.

When specializing our theory to inference on a particular irregular lin-

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3The convergence rate of the bootstrap PSMD estimator is also very useful for the consistency of the bootstrap Wald statistic for semiparametric two-step GMM estimation of Euclidean parameters when the first-step unknown functions are estimated via a PSMD procedure. See e.g., Chen et al. (2003).

4It is known that \( \theta_0 \) could have singular semiparametric efficiency bound and could not be root-\( n \) estimable; see Chamberlain (2010), Kahn and Tamer (2010), Graham and Powell (2012) and the references therein. Following Kahn and Tamer (2010) and Graham and Powell (2012) we call such a \( \theta_0 \) irregular. Many applied papers on complicated semi/nonparametric models simply assume that \( \theta_0 \) is root-\( n \) estimable.
ear functional, the point evaluation functional $\phi(\alpha) = h(\gamma_2)$, of the semi/nonparametric IV model (1.2), we automatically obtain the pointwise asymptotic normality of the PSMD estimator of $h_0(\gamma_2)$ and different ways to construct its confidence set. These results are directly applicable to the NPIV example with $\rho(Y_1; \theta_0, h_0(Y_2)) = Y_1 - h_0(Y_2)$ and to the NPQIV example with $\rho(Y_1; \theta_0, h_0(Y_2)) = 1\{Y_1 \leq h_0(Y_2)\} - \gamma$. Previously, Horowitz (2007) and Gagliardini and Scaillet (2012) established the pointwise asymptotic normality of their kernel based function space Tikhonov regularization estimators of $h_0(\gamma_2)$ for the NPIV and the NPQIV examples respectively. Immediately after our paper was first presented in April 2009 Banff/Canada conference on semiparametrics, the authors of Horowitz and Lee (2012) informed us that they were concurrently working on confidence bands for $h_0$ using a particular SMD estimator of the NPIV example. To the best of our knowledge, there is no inference results, in the existing literature, on any nonlinear functional of $h_0$ even for the NPIV and NPQIV examples. Our paper is the first to provide simple sieve Wald and SQLR tests for (possibly) nonlinear functionals satisfying the general semi/nonparametric IV model (1.2).

The rest of the paper is organized as follows. Section 2 presents the plug-in PSMD estimator $\phi(\hat{\alpha}_n)$ of a (possibly nonlinear) functional $\phi$ evaluated at $\alpha_0 \equiv (\theta'_0, h_0)$ satisfying the model (1.1). It also provides an overview of the main asymptotic results that will be established in the subsequent sections, and illustrates the applications through a point evaluation functional $\phi(\alpha) = h(\gamma_2)$, a weighted derivative functional $\phi(h) = \int w(y_2)\nabla h(y_2)dy_2$, and a quadratic functional $\phi(\alpha) = \int w(y_2)|h(y_2)|^2dy_2$ of the NPIV and NPQIV examples. Section 3 states the basic regularity conditions. Section 4 provides the asymptotic properties of sieve t (Wald) and sieve QLR statistics. Section 5 establishes the consistency of the bootstrap sieve t (Wald) and the bootstrap SQLR statistics. Section 6 verifies
the key regularity conditions for the asymptotic theories via the three functionals of the NPIV and NPQIV examples presented in Section 2. Section 7 presents simulation studies and an empirical illustration. Section 8 briefly concludes. Appendix A consists of several subsections, presenting (1) further results on sieve Riesz representation of a functional of interest; (2) the convergence rates of the bootstrap PSMD estimator \( \hat{\alpha}_n^B \) for model (1.1); (3) the local power properties of sieve Wald and SQLR tests and of their bootstrap versions; (4) asymptotic properties of sieve Wald and SQLR for functionals of increasing dimension; (5) low level sufficient conditions with a series least squares (LS) estimated conditional mean function \( m(\cdot, \alpha) = E[p(Y, X; \alpha)|X = \cdot] \); and (6) additional useful lemmas with series LS estimated \( m(\cdot, \alpha) \). Online supplemental materials consist of Appendices B, C and D. Appendix B contains additional theoretical results (including other consistent variance estimators and other bootstrap sieve Wald tests) and proofs of all the results stated in the main text. Appendix C contains proofs of all the results stated in Appendix A. The online Appendix D provides computationally attractive sieve score test and sieve score bootstrap.

**Notation.** We use “\( \equiv \)” to implicitly define a term or introduce a notation. For any column vector \( A \), we let \( A' \) denote its transpose and \( ||A||_e \) its Euclidean norm (i.e., \( ||A||_e \equiv \sqrt{A'A} \), although sometimes we use \( |A| = ||A||_e \) for simplicity). Let \( ||A||_W^2 \equiv A'WA \) for a positive definite weighting matrix \( W \). Let \( \lambda_{\text{max}}(W) \) and \( \lambda_{\text{min}}(W) \) denote the maximal and minimal eigenvalues of \( W \) respectively. All random variables \( Z \equiv (Y', X')', Z_i \equiv (Y_i', X_i')' \) are defined on a complete probability space \((Z, \mathcal{B}_Z, P_Z)\), where \( P_Z \) is the joint probability distribution of \((Y', X')\). We define \((Z^\infty, \mathcal{B}_Z^\infty, P_{Z^\infty})\) as the probability space of the sequences \((Z_1, Z_2, \ldots)\). For simplicity we assume that \( Y \) and \( X \) are continuous random variables. Let \( f_X (F_X) \) be the marginal density (cdf) of \( X \) with support \( X \), and \( f_{Y|X} \)
(F_{Y|X}) be the conditional density (cdf) of Y given X. Let \( E_P[\cdot] \) denote the expectation with respect to a measure P. Sometimes we use \( P \) for \( P_{Z^\infty} \) and \( E[\cdot] \) for \( E_{P_{Z^\infty}}[\cdot] \). Denote \( L^p(\Omega,d\mu), 1 \leq p < \infty \), as a space of measurable functions with \( ||g||_{L^p(\Omega,d\mu)} \equiv \{ \int_{\Omega} |g(t)|^p d\mu(t) \}^{1/p} < \infty \), where \( \Omega \) is the support of the sigma-finite positive measure \( d\mu \) (sometimes \( L^p(d\mu) \) and \( ||g||_{L^p(d\mu)} \) are used). For any (possibly random) positive sequences \( \{a_n\}_{n=1}^\infty \) and \( \{b_n\}_{n=1}^\infty \), \( a_n = O_P(b_n) \) means that \( \lim_{n \to \infty} \limsup_n \Pr (a_n/b_n > c) = 0 \); \( a_n = o_P(b_n) \) means that for all \( \varepsilon > 0 \), \( \lim_{n \to \infty} \Pr (a_n/b_n > \varepsilon) = 0 \); and \( a_n \prec b_n \) means that there exist two constants \( 0 < c_1 \leq c_2 < \infty \) such that \( c_1 a_n \leq b_n \leq c_2 a_n \). Also, we use “wpa1-\( P_{Z^\infty} \)” (or simply wpa1) for an event \( A_n \), to denote that \( P_{Z^\infty}(A_n) \to 1 \) as \( n \to \infty \). We use \( \mathcal{A}_n \equiv \mathcal{A}_{k(n)} \) and \( \mathcal{H}_n \equiv \mathcal{H}_{k(n)} \) for various sieve spaces. We assume \( \dim(\mathcal{A}_{k(n)}) \prec \dim(\mathcal{H}_{k(n)}) \times k(n) \) for simplicity, all of which grow to infinity with the sample size \( n \). We use \( \text{const.}, c \) or \( C \) to mean a positive finite constant that is independent of sample size but can take different values at different places. For sequences, \( \{a_n\}_n \), we sometimes use \( a_n \nrightarrow a \) (\( a_n \searrow a \)) to denote, that the sequence converges to \( a \) and that is increasing (decreasing) sequence. For any mapping \( F: \mathcal{H}_1 \to \mathcal{H}_2 \) between two generic Banach spaces, \( \frac{dF(a_0)}{da}[v] \equiv \left. \frac{dF(a_0 + \tau v)}{d\tau} \right|_{\tau=0} \) is the pathwise (or Gateaux) derivative at \( a_0 \) in the direction \( v \in \mathcal{H}_1 \). And \( \frac{dF(a_0)}{da}[\mathbf{v}] \equiv \left( \frac{dF(a_0)}{da}[v_1], \cdots, \frac{dF(a_0)}{da}[v_k] \right) \) for \( \mathbf{v} = (v_1, \cdots, v_k) \) with \( v_j \in \mathcal{H}_1 \) for all \( j = 1, \ldots, k \).

2. PSMD ESTIMATION AND INFERENCE: AN OVERVIEW

2.1. The Penalized Sieve Minimum Distance Estimator

Let \( m(X, \alpha) \equiv E \left[ \rho(Y, X; \alpha) | X \right] = \int \rho(y, X; \alpha) dF_{Y|X}(y) \) be a \( d_\rho \times 1 \) vector valued conditional mean function, \( \Sigma(X) \) be a \( d_\rho \times d_\rho \) positive definite (a.s. \( - X \)) weighting matrix, and

\[
Q(\alpha) \equiv E \left[ m(X, \alpha)' \Sigma(X)^{-1} m(X, \alpha) \right] \equiv E \left[ ||m(X, \alpha)||^2_{\Sigma^{-1}} \right]
\]
be the population minimum distance (MD) criterion function. Then the semi/nonparametric conditional moment model (1.1) can be equivalently expressed as \( m(X, \alpha) = 0 \) a.s. \(-X\), where \( \alpha_0 \equiv (\theta_0, h_0) \in A \equiv \Theta \times \mathcal{H} \), or as

\[
\inf_{\alpha \in A} Q(\alpha) = Q(\alpha_0) = 0.
\]

Let \( \Sigma_0(X) \equiv \text{Var}(\rho(Y;X;\alpha_0)|X) \) be positive definite for almost all \( X \). In this paper as well as in most applications \( \Sigma(X) \) is chosen to be either \( \text{I}d \) (identity) or \( \Sigma_0(X) \) for almost all \( X \). We call \( Q^0(\alpha) \equiv E \left[ ||m(X,\alpha)||_{\Sigma_0}^2 \right] \) the population optimally weighted MD criterion function.

Let \( \phi : A \to \mathbb{R}^{d} \) be a functional with a finite \( d \geq 1 \). We are interested in inference on \( \phi(\alpha_0) \). Let

\[
(2.1) \quad \hat{Q}_n(\alpha) \equiv \frac{1}{n} \sum_{i=1}^{n} \hat{m}(X_i, \alpha)\hat{\Sigma}(X_i)^{-1}\hat{m}(X_i, \alpha)
\]

be a sample estimate of \( Q(\alpha) \), where \( \hat{m}(X, \alpha) \) and \( \hat{\Sigma}(X) \) are any consistent estimators of \( m(X, \alpha) \) and \( \Sigma(X) \) respectively. When \( \hat{\Sigma}(X) = \hat{\Sigma}_0(X) \) is a consistent estimator of the optimal weighting matrix \( \Sigma_0(X) \), we call the corresponding \( \hat{Q}_n(\alpha) \) the sample optimally weighted MD criterion \( \hat{Q}_n^0(\alpha) \).

We estimate \( \phi(\alpha_0) \) by \( \phi(\hat{\alpha}_n) \), where \( \hat{\alpha}_n \equiv (\hat{\theta}_n, \hat{h}_n) \) is an approximate penalized sieve minimum distance (PSMD) estimator of \( \alpha_0 \equiv (\theta_0', h_0) \), defined as

\[
(2.2) \quad \hat{Q}_n(\hat{\alpha}_n) + \lambda_n \text{Pen}(\hat{h}_n) \leq \inf_{\alpha \in A_k(n)} \left\{ \hat{Q}_n(\alpha) + \lambda_n \text{Pen}(h) \right\} + o_{P_{\infty}}(n^{-1}),
\]

where \( \lambda_n \text{Pen}(h) \geq 0 \) is a penalty term such that \( \lambda_n = o(1) \); and \( A_k(n) \equiv \Theta \times \mathcal{H}_{k(n)} \) is a finite dimensional sieve for \( A \equiv \Theta \times \mathcal{H} \), more precisely, \( \mathcal{H}_{k(n)} \) is a finite dimensional linear sieve for \( \mathcal{H} \):

\[
(2.3) \quad \mathcal{H}_{k(n)} = \left\{ h \in \mathcal{H} : h(\cdot) = \sum_{k=1}^{k(n)} \beta_{k} q_k(\cdot) = \beta' q^{k(n)}(\cdot) \right\},
\]
where \( \{ q_k \}_{k=1}^\infty \) is a sequence of known basis functions of a Banach space \( (\mathcal{H}, \| \cdot \|_H) \) such as wavelets, splines, Fourier series, Hermite polynomial series, etc. And \( k(n) \to \infty \) as \( n \to \infty \).

For the purely nonparametric conditional moment models \( E [ \rho(Y, X; h_0) | X ] = 0 \), Chen and Pouzo (2012a) proposed more general approximate PSMD estimators of \( h_0 \) by allowing for possibly infinite dimensional sieves (i.e., \( \dim(\mathcal{H}_{k(n)}) = k(n) \leq \infty \)). Nevertheless, both the theoretical properties and Monte Carlo simulations in Chen and Pouzo (2012a) recommend the use of the PSMD procedures with slowly growing finite-dimensional linear sieves with a tiny penalty (i.e., \( k(n) \to \infty, k(n)/n \to 0 \) as \( n \to \infty \) with a very small \( \lambda_n = o(n^{-1}) \), and hence the main smoothing parameter is the sieve dimension \( k(n) \)). This class of PSMD estimators include the original SMD estimators of Newey and Powell (2003) and Ai and Chen (2003) as special cases, and has been used in recent empirical estimation of semiparametric structural models in microeconomics and asset pricing with endogeneity. See, e.g., Blundell et al. (2007), Horowitz (2011), Chen and Pouzo (2009), Bajari et al. (2011), Souza-Rodrigues (2012), Pinkse et al. (2002), Merlo and de Paula (2013), Bontemps and Martimort (2013), Chen and Ludvigson (2009), Chen et al. (2013), Penaranda and Sentana (2013) and others.

In this paper we shall develop inferential theory for \( \phi(\alpha_0) \) based on the PSMD procedures with slowly growing finite-dimensional sieves \( \mathcal{A}_{k(n)} = \Theta \times \mathcal{H}_{k(n)} \). We first establish the large sample theories under a high level “local quadratic approximation” (LQA) condition, which allows for any consistent nonparametric estimator \( \hat{m}(x, \alpha) \) that is linear in \( \rho(Z, \alpha) \):

\[
\hat{m}(x, \alpha) \equiv \sum_{i=1}^{n} \rho(Z_i, \alpha) A_n(X_i, x)
\]

where \( A_n(X_i, x) \) is a known measurable function of \( \{X_j\}_{j=1}^{n} \) for all \( x \), whose expression varies according to different nonparametric procedures.
such as kernel, local linear regression, series and nearest neighbors. In Appendix A we provide lower level sufficient conditions for this LQA assumption when \( \hat{m}(x, \alpha) \) is the series least squares (LS) estimator \( \hat{\alpha} \):

\[
\hat{m}(x, \alpha) = \left( \sum_{i=1}^{n} \rho(Z_i, \alpha)p^{J_n}(X_i) \right)'(P'P)^{-1}p^{J_n}(x),
\]

which is a linear nonparametric estimator \( \hat{\phi} \) with \( A_n(X_i, x) = p^{J_n}(X_i)'(P'P)^{-1}p^{J_n}(x) \), where \( \{p_j\}_{j=1}^{\infty} \) is a sequence of known basis functions that can approximate any square integrable functions of \( X \) well, \( p^{J_n}(X) = (p_1(X), \ldots, p_{J_n}(X))' \), \( P = (p^{J_n}(X_1), \ldots, p^{J_n}(X_n))' \), and \( (P'P)^{-1} \) is the generalized inverse of the matrix \( P'P \). Following Blundell et al. (2007) and Chen and Pouzo (2009), we let \( p^{J_n}(X) \) be a tensor-product linear sieve basis, and \( J_n \) be the dimension of \( p^{J_n}(X) \) such that \( J_n \geq d+ k(n) \rightarrow \infty \) and \( \frac{J_n}{n} \rightarrow 0 \) as \( n \rightarrow \infty \).

### 2.2. Preview of the Main Results for Inference

For simplicity we let \( \phi : \mathbb{R}^{d\theta} \times \mathcal{H} \rightarrow \mathbb{R} \) be a real-valued functional. Let \( \hat{\phi}_n \equiv \phi(\hat{\alpha}_n) \) be the plug-in PSMD estimator of \( \phi(\alpha_0) \) for \( \alpha_0 = (\theta_0', h_0) \in \text{int}(\Theta) \times \mathcal{H} \).

#### Sieve t (or Wald) statistic

Regardless of whether \( \phi(\alpha_0) \) is \( \sqrt{n} \) estimable or not, Theorem 4.1 shows that \( \frac{\sqrt{n}(\phi(\hat{\alpha}_n) - \phi(\alpha_0))}{||v^*_n||_{sd}} \) is asymptotically standard normal, and the sieve variance \( ||v^*_n||_{sd}^2 \) has a closed form expression resembling the “delta-method” variance for a parametric MD problem:

\[
||v^*_n||_{sd}^2 = \left( \frac{d\phi(\alpha_0)}{d\alpha} [q^{k(n)}(\cdot)]' D_n^{-1} \mathcal{U}_n D_n^{-1} \left( \frac{d\phi(\alpha_0)}{d\alpha} [q^{k(n)}(\cdot)]' \right) \right),
\]

where \( [q^{k(n)}(\cdot)]' = \left( 1_{d_0}', q^{k(n)}(\cdot)' \right)' \) is a \( (d_0 + k(n)) \times 1 \) vector with \( 1_{d_0} \) a
$d_\theta \times 1$ vector of 1’s,

\begin{equation}
\left(2.7\right)
\frac{d\phi(\alpha_0)}{d\alpha} [q^{k(n)}(.)] \equiv \frac{\partial \phi(\theta_0 + \theta, h_0 + \beta'q^{k(n)}(.))}{\partial \gamma'} |_{\gamma=0} = \left( \frac{\partial \phi(\alpha_0)}{\partial \theta'}, \frac{d\phi(\alpha_0)}{dh} [q^{k(n)}(.)'] \right)'
\end{equation}

and $\gamma \equiv (\theta', \beta')'$ are $(d_\theta + k(n)) \times 1$ vectors, $
\frac{d\phi(\alpha_0)}{dh} [q^{k(n)}(.)'] \equiv \frac{\partial \phi(\theta_0, h_0 + \beta'q^{k(n)}(.))}{\partial \beta} |_{\beta=0},$

and

\begin{equation}
\left(2.8\right)
D_n = E \left[ \left( \frac{dm(X, \alpha_0)}{d\alpha} [q^{k(n)}(.)'] \right)' \Sigma(X)^{-1} \left( \frac{dm(X, \alpha_0)}{d\alpha} [q^{k(n)}(.)'] \right) \right],
\end{equation}

\begin{equation}
\left(2.9\right)
\hat{\delta}_n = E \left[ \left( \frac{dm(X, \alpha_0)}{d\alpha} [q^{k(n)}(.)'] \right)' \Sigma(X)^{-1} \rho(Z, \alpha_0) \Sigma(X)^{-1} \left( \frac{dm(X, \alpha_0)}{d\alpha} [q^{k(n)}(.)'] \right) \right],
\end{equation}

where

\begin{equation}
\frac{dm(X, \alpha_0)}{d\alpha} [q^{k(n)}(.)'] \equiv \frac{\partial E[\rho(Z, \theta_0 + \theta, h_0 + \beta'q^{k(n)}(.))|X]}{\partial \gamma} |_{\gamma=0} \text{ is a } d_\rho \times (d_\theta + k(n)) \text{ matrix.}
\end{equation}

The closed form expression of $||v_n^*||^2_{sd}$ immediately leads to simple consistent plug-in sieve variance estimators; one of which is

\begin{equation}
\left(2.10\right)
||\hat{v}_n^*||^2_{sd} = \hat{V}_1 = \left( \frac{d\phi(\hat{\alpha}_n)}{d\alpha} [q^{k(n)}(.)] \right)' \hat{D}_n \hat{\delta}_n \hat{D}_n' \left( \frac{d\phi(\hat{\alpha}_n)}{d\alpha} [q^{k(n)}(.)] \right),
\end{equation}

where

\begin{equation}
\frac{d\phi(\hat{\alpha}_n)}{d\alpha} [q^{k(n)}(.)] \equiv \frac{\partial \phi(\hat{\theta}_n + \theta, \hat{h}_n + \beta'q^{k(n)}(.))}{\partial \gamma} |_{\gamma=0} \text{ and}
\end{equation}

\begin{equation}
\left(2.11\right)
\hat{D}_n = \frac{1}{n} \sum_{i=1}^{n} \left[ \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{d\alpha} [q^{k(n)}(.)'] \right)' \tilde{\Sigma}(X_i)^{-1} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{d\alpha} [q^{k(n)}(.)'] \right) \right],
\end{equation}

\begin{equation}
\left(2.12\right)
\end{equation}
\[ \hat{\varnothing}_n = \frac{1}{n} \sum_{i=1}^{n} \left[ \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{d\alpha} \hat{T}^k(\cdot) \right) \right] \cdot \left. \right| \hat{M}(X_i) \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{d\alpha} \hat{T}^k(\cdot) \right) \right| . \]

where \( \hat{M}(X_i) \equiv \hat{\Sigma}(X_i)^{-1} \rho(Z_i, \hat{\alpha}_n) \rho(Z_i, \hat{\alpha}_n) \hat{\Sigma}(X_i)^{-1} \). Theorem 4.2 then presents the asymptotic normality of the sieve (Student’s) t statistic:

\[ \tilde{W}_n \equiv \sqrt{n} \frac{\phi(\hat{\alpha}_n) - \phi(\alpha_0)}{\|v_n^*\|_{n,sd}} \Rightarrow N(0, 1). \]

**Sieve QLR statistic.** In addition to the sieve t (or sieve Wald) statistic, we could also use sieve quasi likelihood ratio for constructing confidence set of \( \phi(\alpha_0) \) and for hypothesis testing of \( H_0 : \phi(\alpha_0) = \phi_0 \) against \( H_1 : \phi(\alpha_0) \neq \phi_0 \). Denote

\[ (2.13) \quad \hat{QLR}_n(\phi_0) \equiv n \left( \inf_{\alpha \in A_k(n) : \phi(\alpha) = \phi_0} \hat{Q}_n(\alpha) - \hat{Q}_n(\hat{\alpha}_n) \right) \]

as the **sieve quasi likelihood ratio** (SQLR) statistic. It becomes an **optimally weighted SQLR** statistic, \( \hat{QLR}_n^0(\phi_0) \), when \( \hat{Q}_n(\alpha) \) is the optimally weighted MD criterion \( \hat{Q}_n^0(\alpha) \). Regardless of whether \( \phi(\alpha_0) \) is \( \sqrt{n} \) estimable or not, Theorems 4.3(2) and 4.4 show that \( \hat{QLR}_n^0(\phi_0) \) is asymptotically chi-square distributed under the null \( H_0 \), and diverges to infinity under the fixed alternatives \( H_1 \). Theorem A.1 in Appendix A states that \( \hat{QLR}_n(\phi_0) \) is asymptotically noncentral chi-square distributed under local alternatives. One could compute 100(1 - \( \tau \))% confidence set for \( \phi(\alpha_0) \) as

\[ \left\{ r \in \mathbb{R} : \hat{QLR}_n(r) \leq c_{\chi^2_1}(1 - \tau) \right\}, \]

where \( c_{\chi^2_1}(1 - \tau) \) is the \( (1 - \tau) \)-th quantile of the \( \chi^2_1 \) distribution.

**Bootstrap sieve QLR statistic.** Regardless of whether \( \phi(\alpha_0) \) is \( \sqrt{n} \) estimable or not, Theorems 4.3(1) and 4.4 establish that the possibly non-optimally weighted SQLR statistic \( \hat{QLR}_n(\phi_0) \) is stochastically bounded under the null \( H_0 \) and diverges to infinity under the fixed alternatives \( H_1 \).

---

5See Theorems 5.2 and A.4 for properties of bootstrap sieve t statistics.
We then consider a bootstrap version of the SQLR statistic. Let \( \hat{QLR}_n^B \) denote a bootstrap SQLR statistic:

\[
\hat{QLR}_n^B(\hat{\phi}_n) \equiv n \left( \inf_{\alpha \in A_k(\alpha) : \phi(\alpha) = \hat{\phi}_n} \hat{Q}_n^B(\alpha) - \inf_{\alpha \in A_k(\alpha)} \hat{Q}_n^B(\alpha) \right),
\]

where \( \hat{\phi}_n \equiv \phi(\hat{\alpha}_n) \), and \( \hat{Q}_n^B(\alpha) \) is a bootstrap version of \( \hat{Q}_n(\alpha) \):

\[
\hat{Q}_n^B(\alpha) \equiv \frac{1}{n} \sum_{i=1}^n \hat{m}^B(X_i, \alpha)' \Sigma(X_i)^{-1} \hat{m}^B(X_i, \alpha),
\]

where \( \hat{m}^B(x, \alpha) \) is a bootstrap version of \( \hat{m}(x, \alpha) \), which is computed in the same way as that of \( \hat{m}(x, \alpha) \) except that we use \( \omega_{i,n} \rho(Z_i, \alpha) \) instead of \( \rho(Z_i, \alpha) \). Here \( \{\omega_{i,n} \geq 0\}_{i=1}^n \) is a sequence of bootstrap weights that has mean 1 and is independent of the original data \( \{Z_i\}_{i=1}^n \). Typical weights include an i.i.d. weight \( \{\omega_i \geq 0\}_{i=1}^n \) with \( E[\omega_i] = 1, E[|\omega_i - 1|^2] = 1 \) and \( E[|\omega_i - 1|^{2+\epsilon}] < \infty \) for some \( \epsilon > 0 \), or a multinomial weight (i.e., \( (\omega_1, ..., \omega_n) \sim \text{Multinomial}(n; n^{-1}, ..., n^{-1}) \)). For example, if \( \hat{m}(x, \alpha) \) is a series LS estimator (2.5) of \( m(x, \alpha) \), then \( \hat{m}^B(x, \alpha) \) is a bootstrap series LS estimator of \( m(x, \alpha) \), defined as:

\[
\hat{m}^B(x, \alpha) \equiv \left( \sum_{i=1}^n \omega_{i,n} \rho(Z_i, \alpha)p^{J_n}(X_i)' \right) (P'P)^{-1} p^{J_n}(x).
\]

We sometimes call our bootstrap procedure "generalized residual bootstrap" since it is based on randomly perturbing the generalized residual function \( \rho(Z, \alpha) \); see Section 5 for details. Theorems 5.3 and A.2 establish that under the null \( H_0 \), the fixed alternatives \( H_1 \) or the local alternatives \( \hat{\phi}_0 \) the conditional distribution of \( \hat{QLR}_n^B(\hat{\phi}_n) \) (given the data) always converges to the asymptotic null distribution of \( \hat{QLR}_n(\phi_0) \). Let \( \hat{c}_n(a) \) be the \( a - \) th quantile of the distribution of \( \hat{QLR}_n(\phi_0) \) (conditional on the data

\[ \text{See Section A.3 for definition of the local alternatives and the behaviors of } \hat{QLR}_n(\phi_0) \text{ and } \hat{QLR}_n^B(\hat{\phi}_n) \text{ under the local alternatives.} \]
\(\{Z_i\}_{i=1}^n\). Then for any \(\tau \in (0, 1)\), we have \(\lim_{n \to \infty} \Pr\{\hat{QLR}_n(\phi_0) > \hat{c}_n(1-\tau)\} = \tau\) under the null \(H_0\), \(\lim_{n \to \infty} \Pr\{\hat{QLR}_n(\phi_0) > \hat{c}_n(1-\tau)\} = 1\) under the fixed alternatives \(H_1\), and \(\lim_{n \to \infty} \Pr\{\hat{QLR}_n(\phi_0) > \hat{c}_n(1-\tau)\} > \tau\) under the local alternatives. We could also construct a 100(1-\(\tau\))\% confidence set using the bootstrap critical values:

\[
\left\{ r \in \mathbb{R} : \hat{QLR}_n(r) \leq \hat{c}_n(1-\tau) \right\}.
\]

The bootstrap consistency holds for possibly non-optimally weighted SQL-R statistic and possibly irregular functionals, without the need to compute standard errors.

**Which method to use?** When sieve Wald and SQLR tests are computed using the same weighting matrix \(\hat{\Sigma}\), there is no local power difference in terms of first order asymptotic theories; see Appendix A. As will be demonstrated in simulation Section 7 while SQLR and bootstrap SQL-R tests are useful for models \((1.1)\) with (pointwise) non-smooth \(\rho(Z; \alpha)\), sieve Wald (or t) statistic is computationally attractive for models with smooth \(\rho(Z; \alpha)\). Empirical researchers could apply either inference method depending on whether the residual function \(\rho(Z; \alpha)\) in their specific application is pointwise differentiable with respect to \(\alpha\) or not.

### 2.2.1. Applications to NPIV and NPQIV models

**An illustration via the NPIV model.** Blundell et al. (2007) and Chen and Reiß (2011) established the convergence rate of the identity weighted (i.e., \(\hat{\Sigma} = \Sigma = 1\)) PSMD estimator \(\hat{h}_n \in \mathcal{H}_{k(n)}\) of the NPIV model:

\[
Y_1 = h_0(Y_2) + U, \quad E(U|X) = 0.
\]

By Theorem 4.1

\[
\sqrt{n} \frac{\phi(\hat{h}_n) - \phi(h_0)}{||v_n^*||_{sd}} \Rightarrow N(0, 1)
\]
with \( ||v_n^\ast||^2_{sd} = \frac{d\phi(h_0)}{dh} [q^{k(n)}(\cdot)]' D_n^{-1} D_n^{-1} \frac{d\phi(h_0)}{dh} [q^{k(n)}(\cdot)] \),

\begin{equation}
D_n = E \left( E[q^{k(n)}(Y_2)|X] E[q^{k(n)}(Y_2)|X]' \right),
\end{equation}

\begin{equation}
\mathcal{D}_n = E \left( E[q^{k(n)}(Y_2)|X] U^2 E[q^{k(n)}(Y_2)|X]' \right)
\end{equation}

and \( \frac{d\phi(h_0)}{dh} [q^{k(n)}(\cdot)] \equiv \frac{\partial \phi(h_0 + \beta q^{k(n)}(\cdot))}{\partial \beta} \big|_{\beta = 0} \). For example, for a functional \( \phi(h) = h(\bar{y}_2) \), or \( \int w(y) \nabla h(y) dy \) or \( \int w(y) |h(y)|^2 dy \), we have \( \frac{d\phi(h_0)}{dh} [q^{k(n)}(\cdot)] = q^{k(n)}(\bar{y}_2) \), or \( \int w(y) \nabla q^{k(n)}(y) dy \) or \( \int h_0(y) w(y) q^{k(n)}(y) dy \).

If \( 0 < \inf_x \Sigma_0(x) \leq \sup_x \Sigma_0(x) < \infty \) then

\[ ||v_n^\ast||^2_{sd} \times \frac{d\phi(h_0)}{dh} [q^{k(n)}(\cdot)]' D_n^{-1} \frac{d\phi(h_0)}{dh} [q^{k(n)}(\cdot)] \]

Without endogeneity (say \( Y_2 = X \)) the model becomes the nonparametric LS regression

\[ Y_1 = h_0(Y_2) + U, \quad E(U|Y_2) = 0, \]

and the variance satisfies \( ||v_n^\ast||^2_{sd,ex} \times \frac{d\phi(h_0)}{dh} [q^{k(n)}(\cdot)]' D_n^{-1} \frac{d\phi(h_0)}{dh} [q^{k(n)}(\cdot)] \),

\( D_{n,ex} = E[q^{k(n)}(Y_2)|X] q^{k(n)}(Y_2)' \). Since the conditional expectation \( E[q^{k(n)}(Y_2)|X] \)

is a contraction, \( D_n \leq D_{n,ex} \) and \( ||v_n^\ast||^2_{sd} \geq const. ||v_n^\ast||^2_{sd,ex} \). Under mild conditions (see, e.g., Newey and Powell (2003), Blundell et al. (2007), Darolles et al. (2011), Horowitz (2011)), the minimal eigenvalue of \( D_n \), \( \lambda_{\min}(D_n) \), goes to zero while \( \lambda_{\min}(D_{n,ex}) \) stays strictly positive as \( k(n) \to \infty \). In fact, \( D_{n,ex} = I_{k(n)} \) and \( \lambda_{\min}(D_{n,ex}) = 1 \) if \( \{q_j\}_{j=1}^\infty \) is an orthonormal basis of \( L^2(f_{Y_2}) \), while \( \lambda_{\min}(D_n) \propto \exp(-k(n)) \) if the conditional density of \( Y_2 \) given \( X \) is normal. Therefore, while \( \lim_{k(n)\to\infty} ||v_n^\ast||^2_{sd,ex} = \infty \) always implies \( \lim_{k(n)\to\infty} ||v_n^\ast||^2_{sd} = \infty \), it is possible that \( \lim_{k(n)\to\infty} ||v_n^\ast||^2_{sd,ex} < \infty \) but \( \lim_{k(n)\to\infty} ||v_n^\ast||^2_{sd} = \infty \). For example, the point evaluation functional \( \phi(h) = h(\bar{y}_2) \) is known to be irregular for the nonparametric LS regression and hence for the NPIV (2.18) as well. Under mild conditions on the weight \( w() \) and the smoothness of \( h_0 \), the weighted derivative functional \( (\phi(h) = \int w(y) \nabla h(y) dy) \) and the quadratic functional
\(
\phi(h) = \int w(y) |h(y)|^2 dy
\)

of the nonparametric LS regression are typically root-\(n\) estimable, but they could be irregular for the NPIV (2.18). See Section 6 for details.

Regardless of whether \(\lim_{k(n) \to \infty} ||v_n^*||_{sd}^2\) is finite or infinite, Theorem 4.2 shows that the sieve variance \(||v_n^*||_{sd}^2\) can be consistently estimated by a plug-in sieve variance estimator \(||\hat{v}_n^*||_{sd}^2\), and that \(\sqrt{n} \frac{\phi(\hat{h}_n) - \phi(h_0)}{||\hat{v}_n^*||_{sd}} \Rightarrow N(0, 1)\).

When the conditional mean function \(m(x, h)\) is estimated by the series LS estimator (2.5) as in Newey and Powell (2003), Ai and Chen (2003) and Blundell et al. (2007), with \(\hat{U}_i = Y_{1i} - \hat{h}_n(Y_{2i})\), the sieve variance estimator \(||\hat{v}_n^*||_{sd}^2\) given in (2.10) has a more explicit expression:

\[
||\hat{v}_n^*||_{sd}^2 = \hat{V}_1 = \left( \frac{d\phi(\hat{h}_n)}{dh} [q^{k(n)}(\cdot)] \right)' \hat{D}_n \hat{\Omega}_n \hat{D}_n' \left( \frac{d\phi(\hat{h}_n)}{dh} [q^{k(n)}(\cdot)] \right),
\]

where

\[
\frac{d\phi(\hat{h}_n)}{dh} [q^{k(n)}(\cdot)] \equiv \frac{\partial \phi(\hat{h}_n + \beta' q^{k(n)}(\cdot))}{\partial \beta} |_{\beta = 0}
\]

and

\[
\hat{D}_n = \frac{1}{n} \hat{C}_n (P'P)^{-1}(\hat{C}_n)', \quad \hat{\Omega}_n = \sum_{j=1}^{n} q^{k(n)}(Y_{2j}) p^{Jn}(X_j)',
\]

(2.21) \(\hat{\Theta}_n = \frac{1}{n} \hat{C}_n (P'P)^{-1} \left( \sum_{i=1}^{n} p^{Jn}(X_i) \hat{U}_i^2 p^{Jn}(X_i)' \right) (P'P)^{-1}(\hat{C}_n)'.\)

Interestingly, this sieve variance estimator becomes the one computed via the two stage least squares (2SLS) as if the NPIV model (2.18) were a parametric IV regression\(^7\) \(Y_1 = q^{k(n)}(Y_{2j})' \beta_{0n} + U, E[q^{k(n)}(Y_2)U] \neq 0, E[p^{Jn}(X)U] = 0\) and \(E[p^{Jn}(X)q^{k(n)}(Y_2)']\) has a column rank \(k(n) \leq J_n\).

See Subsection 7.1 for simulation studies of finite sample performances of this sieve variance estimator \(\hat{V}_1\) for both a linear and a nonlinear functional \(\phi(h)\).

\(^7\)This confirms a conjecture of Newey (2013) for the NPIV model (2.18).
An illustration via the NPQIV model. As an application of their general theory, Chen and Pouzo (2012a) presented the consistency and the rate of convergence of the PSMD estimator \( \hat{h}_n \in H_{k(n)} \) of the NPQIV model:

\[
(2.22) \quad Y_1 = h_0(Y_2) + U, \quad \Pr(U \leq 0|X) = \gamma.
\]

In this example we have \( \Sigma(X) = \gamma(1 - \gamma) \). So we could use \( \hat{\Sigma}(X) = \gamma(1 - \gamma) \) and \( \hat{Q}_n(\alpha) \) given in (2.1) becomes the optimally weighted MD criterion.

By Theorem \[1\]

\[
\sqrt{n} \frac{\phi(\hat{h}_n) - \phi(h_0)}{||v_n^*||_{sd}} \Rightarrow N(0, 1)
\]

with \( ||v_n^*||_{sd}^2 = \left( \frac{d\phi(h_0)}{dh} [q^{k(n)}(\cdot)] \right)' D_n^- \left( \frac{d\phi(h_0)}{dh} [q^{k(n)}(\cdot)] \right) \) and

\[
(2.23) \quad D_n = \frac{1}{\gamma(1 - \gamma)} E \left( E[f_{U|Y_2,X}(0)q^{k(n)}(Y_2)|X]E[f_{U|Y_2,X}(0)q^{k(n)}(Y_2)|X]' \right).
\]

Without endogeneity (say \( Y_2 = X \)), the model becomes the nonparametric quantile regression

\[
Y_1 = h_0(Y_2) + U, \quad \Pr(U \leq 0|Y_2) = \gamma,
\]

and the sieve variance becomes \( ||v_n^*||_{sd,ex}^2 = \left( \frac{d\phi(h_0)}{dh} [q^{k(n)}(\cdot)] \right)' D_{n,ex}^- \left( \frac{d\phi(h_0)}{dh} [q^{k(n)}(\cdot)] \right) \) with \( D_{n,ex} = \frac{1}{\gamma(1 - \gamma)} E \left( \{f_{U|Y_2}(0)\}^2 \{q^{k(n)}(Y_2)\} \{q^{k(n)}(Y_2)\}' \right) \). Again \( D_n \leq D_{n,ex} \) and \( ||v_n^*||_{sd,ex}^2 \geq ||v_n^*||_{sd}^2 \). Under mild conditions (see, e.g., Chen and Pouzo (2012a), Chen et al. (2014)), \( \lambda_{\min}(D_n) \to 0 \) while \( \lambda_{\min}(D_{n,ex}) \) stays strictly positive as \( k(n) \to \infty \). All of the above discussions for a functional \( \phi(h) \) of the NPIV (2.18) now apply to the functional of the NPQIV (2.22). In particular, a functional \( \phi(h) \) could be root-\( n \) estimable for the nonparametric quantile regression (\( \lim_{k(n)\to\infty} ||v_n^*||_{sd,ex}^2 < \infty \) but
irregular for the NPQIV (2.22) \(\lim_{k(n)\to\infty} \|v^*_n\|_s^2 = \infty\). See Section 6 for details.

By Theorems 4.3(2) and 4.4 the optimally weighted SQLR statistic \(\hat{QLR}_n^0(\phi_0) \Rightarrow \chi^2_1\) under the null of \(\phi(h_0) = \phi_0\), and diverges to infinity under the alternative of \(\phi(h_0) \neq \phi_0\). We can compute confidence set for a functional \(\phi(h)\), such as an evaluation or a weighted derivative functional, as \(\{r \in \mathbb{R}: \hat{QLR}_n^0(r) \leq c_{\chi^2_1}(\tau)\}\). See Subsection 7.2 for an empirical illustration of this result to the NPQIV Engel curve regression using the British Family Survey data set that was first used in Blundell et al. (2007). Instead of using the asymptotic critical values, we could also construct a confidence set using the bootstrap critical values as in (2.17).

3. BASIC REGULARITY CONDITIONS

Before we establish asymptotic properties of sieve t (Wald) and SQLR statistics, we need to present three sets of basic regularity conditions. The first set of assumptions allows us to establish the convergence rates of the PSMD estimator \(\hat{\alpha}_n\) to the true parameter value \(\alpha_0\) in both weak and strong metrics, which in turn allows us to concentrate on some shrinking neighborhood of \(\alpha_0\) in the semi/nonparametric model (1.1). The second and third regularity conditions are respectively about the local curvatures of the functional \(\phi()\) and of the criterion function under these two metrics. The weak metric \(\|\cdot\|\) is closely related to the variance of the linear approximation to \(\phi(\hat{\alpha}_n) - \phi(\alpha_0)\), while the strong metric \(\|\cdot\|_s\) is used to control the nonlinearity (in \(\alpha\)) of the functional \(\phi()\) and of the conditional mean function \(m(x, \alpha)\). This section is mostly technical and applied researchers could skip this and directly go to the subsequent sections on the asymptotic properties of sieve Wald and SQLR statistics.
3.1. A brief discussion on the convergence rate of the PSMD estimator

For the purely nonparametric conditional moment model \( E[\rho(Y, X; h_0(\cdot))|X] = 0 \), Chen and Pouzo (2012a) established the consistency and the convergence rates of their various PSMD estimators of \( h_0 \). Their results can be trivially extended to establish the corresponding properties of our PSMD estimator \( \hat{\alpha}_n \equiv (\hat{\theta}_n, \hat{h}_n) \) defined in (2.2). For the sake of easy reference and to introduce basic assumptions and notation, we present some sufficient conditions for consistency and the convergence rate here. These conditions are also needed to establish the consistency and the convergence rate of bootstrap PSMD estimators (see Lemma A.1). We first impose three conditions on identification, sieve spaces, penalty functions and sample criterion function. We equip the parameter space \( A \equiv \Theta \times H \) with a (strong) norm \( \|\alpha\|_s \equiv \|\theta\|_e + \|h\|_H \).

Assumption 3.1 (Identification, sieves, criterion)  
(i) \( E[\rho(Y, X; \alpha)|X] = 0 \) if and only if \( \alpha \in (A, \|\cdot\|_s) \) with \( \|\alpha - \alpha_0\|_s = 0 \); (ii) For all \( k \geq 1 \), \( A_k \equiv \Theta \times H_k \), \( \Theta \) is a compact subset in \( \mathbb{R}^{d\theta} \) with a non-empty interior, \( \{H_k : k \geq 1\} \) is a non-decreasing sequence of non-empty closed linear subsets of a Banach space \( (H, \|\cdot\|_H) \) such that \( H = \text{cl}(\bigcup_k H_k) \), and there is \( \Pi_n h_0 \in H_k(n) \) with \( \|\Pi_n h_0 - h_0\|_H = o(1) \); (iii) \( Q : (A, \|\cdot\|_s) \to [0, \infty) \) is lower semicontinuous\(^8\)  
(iv) \( \Sigma(x) \) and \( \Sigma_0(x) \) are positive definite, and their smallest and largest eigenvalues are finite and positive uniformly in \( x \in \mathcal{X} \).

Assumption 3.2 (Penalty)  
(i) \( \lambda_n > 0 \), \( Q(\Pi_n \alpha_0) + o(n^{-1}) = O(\lambda_n) = o(1) \); (ii) \( |\text{Pen}(\Pi_n h_0) - \text{Pen}(h_0)| = O(1) \) with \( \text{Pen}(h_0) < \infty \); (iii) \( \text{Pen} : (H, \|\cdot\|_H) \to [0, \infty) \) is lower semicompact\(^9\)

---

\(^8\)A function \( Q \) is lower semicontinuous at a point \( \alpha_o \in A \) iff \( \lim_{\|\alpha - \alpha_o\| \to 0} Q(\alpha) \geq Q(\alpha_o) \); it is lower semicontinuous if it is lower semicontinuous at any point in \( A \).

\(^9\)A function \( \text{Pen} \) is lower semicompact iff for all \( M \), \( \{h \in H: \text{Pen}(h) \leq M\} \) is a compact subset in \( (H, \|\cdot\|_H) \).
Let \( \Pi_n \alpha \equiv (\theta', \Pi_n h) \in \mathcal{A}_{k(n)} \equiv \Theta \times \mathcal{H}_{k(n)} \). Let \( \mathcal{A}_{k(n)}^{M_0} \equiv \Theta \times \mathcal{H}_{k(n)}^{M_0} \equiv \{ \alpha = (\theta', h) \in \mathcal{A}_{k(n)} : \lambda_n \text{Pen}(h) \leq \lambda_n M_0 \} \) for a large but finite \( M_0 \) such that \( \Pi_n \alpha_0 \in \mathcal{A}_{k(n)}^{M_0} \) and that \( \hat{\alpha}_n \in \mathcal{A}_{k(n)}^{M_0} \) with probability arbitrarily close to one for all large \( n \). Let \( \{ \bar{\delta}_{m,n}^2 \}_{n=1}^{\infty} \) be a sequence of positive real values that decrease to zero as \( n \to \infty \).

**Assumption 3.3 (Sample Criterion)** (i) \( \hat{Q}_n(\Pi_n \alpha_0) \leq c_0 Q(\Pi_n \alpha_0) + o_{P_Z}(n^{-1}) \) for a finite constant \( c_0 > 0 \); (ii) \( \hat{Q}_n(\alpha) \geq c Q(\alpha) - O_{P_Z}(\bar{\delta}_{m,n}^2) \) uniformly over \( \mathcal{A}_{k(n)}^{M_0} \) for some \( \bar{\delta}_{m,n}^2 = o(1) \) and a finite constant \( c > 0 \).

The following result is a minor modification of Theorem 3.2 of Chen and Pouzo (2012a).

**Lemma 3.1** Let \( \hat{\alpha}_n \) be the PSMD estimator defined in (2.2), and Assumptions 3.1, 3.2 and 3.3 hold. Then: \( || \hat{\alpha}_n - \alpha_0 ||_s = o_{P_Z}(1) \) and \( \text{Pen}(\hat{h}_n) = O_{P_Z}(1) \).

Given the consistency result, the PSMD estimator belongs to any \( || \cdot ||_s \)–neighborhood around \( \alpha_0 \) wpa1. We can restrict our attention to a convex, \( || \cdot ||_s \)–neighborhood around \( \alpha_0 \), denoted as \( \mathcal{A}_{os} \) such that

\[
\mathcal{A}_{os} \subset \{ \alpha \in \mathcal{A} : || \alpha - \alpha_0 ||_s < M_0, \lambda_n \text{Pen}(h) < \lambda_n M_0 \}
\]

for a positive finite constant \( M_0 \) (the existence of a convex \( \mathcal{A}_{os} \) is implied by the convexity of \( \mathcal{A} \) and quasi-convexity of \( \text{Pen}(\cdot) \)). For any \( \alpha \in \mathcal{A}_{os} \) we define a pathwise derivative as

\[
\frac{dn(X, \alpha_0)}{d\alpha} [\alpha - \alpha_0] \equiv \left. \frac{dE[\rho(Z, (1 - \tau)\alpha_0 + \tau \alpha)|X]}{d\tau} \right|_{\tau = 0} \quad \text{a.s. } X
\]

\[
= \left. \frac{dE[\rho(Z, \alpha_0)|X]}{d\theta'} (\theta - \theta_0) \right|_{\theta = \theta_0} + \left. \frac{dE[\rho(Z, \alpha_0)|X]}{dh} [h - h_0] \right|_{h = h_0} \quad \text{a.s. } X.
\]
Following Ai and Chen (2003) and Chen and Pouzo (2009), we introduce two pseudo-metrics $|| \cdot ||$ and $|| \cdot ||_0$ on $A_{os}$ as: for any $\alpha_1, \alpha_2 \in A_{os}$,

\begin{align}
||\alpha_1 - \alpha_2||^2 &\equiv E \left[ \left( \frac{dm(X, \alpha_0)}{d\alpha} \right) (\alpha_1 - \alpha_2) \right]' \Sigma(X)^{-1} \left( \frac{dm(X, \alpha_0)}{d\alpha} \right) (\alpha_1 - \alpha_2) \right] ; \\
||\alpha_1 - \alpha_2||_0^2 &\equiv E \left[ \left( \frac{dm(X, \alpha_0)}{d\alpha} \right) (\alpha_1 - \alpha_2) \right]' \Sigma_0(X)^{-1} \left( \frac{dm(X, \alpha_0)}{d\alpha} \right) (\alpha_1 - \alpha_2) \right].
\end{align}

It is clear that, under Assumption 3.1(iv), these two pseudo-metrics are equivalent, i.e., $|| \cdot || \asymp || \cdot ||_0$ on $A_{os}$. This is why Assumption 3.1(iv) is imposed throughout the paper.

Let $A_{osn} = A_{os} \cap A_{k(n)}$. Let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence of positive real values such that $\delta_n = o(1)$ and $\delta_n \leq \delta_{m,n}$.

**Assumption 3.4** (i) There exists a convex $|| \cdot ||_s$-neighborhood of $\alpha_0$, $A_{os}$, such that $m(\cdot, \alpha)$ is continuously pathwise differentiable with respect to $\alpha \in A_{os}$, and there is a finite constant $C > 0$ such that $||\alpha - \alpha_0|| \leq C||\alpha - \alpha_0||_s$ for all $\alpha \in A_{os}$; (ii) $Q(\alpha) \asymp ||\alpha - \alpha_0||^2$ for all $\alpha \in A_{os}$; (iii) $\hat{Q}_n(\alpha) \geq cQ(\alpha) - O_{P_{Z \to \infty}}(\delta_n^2)$ uniformly over $A_{osn}$, and $\max\{\delta_n^2, Q(\Pi_n \alpha_0), \lambda_n, o(n^{-1})\} = \delta_n^2$; (iv) $\lambda_n \times \sup_{\alpha, \alpha' \in A_{os}} |\text{Pen}(h) - \text{Pen}(h')| = o(n^{-1})$ or $\lambda_n = o(n^{-1})$.

Assumption 3.4(ii) is about the local curvature of the population criterion $Q(\alpha)$ at $\alpha_0$. It can be weakened to Assumption 4.1(ii) in Chen and Pouzo (2012a). When $\hat{Q}_n(\alpha)$ is computed using the series LS estimator (2.5), Lemma C.2 of Chen and Pouzo (2012a) shows that $\hat{Q}_n(\alpha) \asymp Q(\alpha) - O_{P_{Z \to \infty}}(\delta_n^2)$ uniformly over $A_{osn}$ and hence Assumption 3.4(iii) is satisfied.
Recall the definition of the sieve measure of local ill-posedness

\[ \tau_n \equiv \sup_{\alpha \in \mathcal{A}_{osn} : ||\alpha - \Pi_n \alpha_0|| \neq 0} ||\alpha - \Pi_n \alpha_0||. \]

The problem of estimating \( \alpha_0 \) under \( ||\cdot||_s \) is locally ill-posed in rate if and only if \( \limsup_{n \to \infty} \tau_n = \infty \). We say the problem is mildly ill-posed if \( \tau_n = O([k(n)]^a) \), and severely ill-posed if \( \tau_n = O(\exp\{a^2k(n)\}) \) for some finite \( a > 0 \). The following general rate result is a minor modification of Theorem 4.1 and Remark 4.1(i) of Chen and Pouzo (2012a), and hence we omit its proof.

**Lemma 3.2** Let \( \hat{\alpha}_n \) be the PSMD estimator defined in (2.2), and Assumptions 3.1, 3.2(ii)(iii), 3.3 and 3.4(i)(ii)(iii) hold. Then:

\[ ||\hat{\alpha}_n - \alpha_0|| = O_{P_{Z\infty}}(\delta_n) \quad \text{and} \quad ||\hat{\alpha}_n - \alpha_0||_s = O_{P_{Z\infty}}(||\alpha_0 - \Pi_n \alpha_0||_s + \tau_n \delta_n). \]

The above convergence rate result is applicable to any nonparametric estimator \( \hat{m}(X, \alpha) \) of \( m(X, \alpha) \) as soon as one could compute \( \delta_n^2 \), the rate at which \( \hat{Q}_n(\alpha) \) goes to \( Q(\alpha) \). See Chen and Pouzo (2012a) and Chen and Pouzo (2009) for low level sufficient conditions in terms of the series LS estimator (2.5) of \( m(X, \alpha) \).

Let \( \{\delta_{s,n} : n \geq 1\} \) be a sequence of real positive numbers such that \( \delta_{s,n} = ||h_0 - \Pi_n h_0|| + \tau_n \delta_n = o(1) \). Lemma 3.2 implies that \( \hat{\alpha}_n \in \mathcal{N}_{osn} \subseteq \mathcal{N}_{os} \) wpa1-\( P_{Z\infty} \), where

\[ \mathcal{N}_{os} \equiv \{\alpha \in \mathcal{A} : ||\alpha - \alpha_0|| \leq M_n \delta_n, ||\alpha - \alpha_0||_s \leq M_n \delta_{s,n}, \lambda_n Pen(h) \leq \lambda_n M_0\}, \]

\[ \mathcal{N}_{osn} \equiv \mathcal{N}_{os} \cap \mathcal{A}_{k(n)}, \quad \text{with} \quad M_n \equiv \min\{\log(\log(n + 1)), \log((\delta_{s,n}^{-1} + 1))\}. \]

We can regard \( \mathcal{N}_{os} \) as the effective parameter space and \( \mathcal{N}_{osn} \) as its sieve space in the rest of the paper. Assumption 3.4(iv) is not needed for establishing a convergence rate in Lemma 3.2 but, it will be imposed in the rest of the paper so that we can ignore penalty effect in the first order local asymptotic analysis.
3.2. (Sieve) Riesz representation and (sieve) variance

We first introduce a representation of the functional of interest $\phi()$ at $\alpha_0$ that is crucial for all the subsequent local asymptotic theories. Let $\phi : \mathbb{R}^d \times \mathcal{H} \to \mathbb{R}$ be continuous in $||\cdot||_s$. We assume that

$$\frac{d\phi(\alpha_0)}{d\alpha}[\cdot] : (\mathbb{R}^d \times \mathcal{H}, ||\cdot||_s) \to \mathbb{R}$$

is a $||\cdot||_s$-bounded linear functional (i.e., $\left|\left|\left| \frac{d\phi(\alpha_0)}{d\alpha}[v] \right|\right| \leq c||v||_s$ uniformly over $v \in \mathbb{R}^d \times \mathcal{H}$ for a finite positive constant $c$), which could be computed as a pathwise (directional) derivative of the functional $\phi(\cdot)$ at $\alpha_0$ in the direction of $v = \alpha - \alpha_0 \in \mathbb{R}^d \times \mathcal{H}$:

$$\frac{d\phi(\alpha_0)}{d\alpha}[v] = \left. \frac{\partial \phi(\alpha_0 + \tau v)}{\partial \tau} \right|_{\tau=0}.$$ 

Let $V$ be a linear span of $A_{os} - \{\alpha_0\}$, which is endowed with both $||\cdot||_s$ and $||\cdot||$ (in equation (3.1)) norms, and $||v|| \leq C||v||_s$ for all $v \in V$ (under Assumption 3.4(i)). Let $\overline{V} \equiv \text{clsp}(A_{os} - \{\alpha_0\})$, where $\text{clsp}(\cdot)$ is the closure of the linear span under $||\cdot||$. For any $v_1, v_2 \in \overline{V}$, we define an inner product induced by the metric $||\cdot||$:

$$\langle v_1, v_2 \rangle = E \left[ \left( \frac{dm(X, \alpha_0)}{d\alpha}[v_1] \right)^\prime \Sigma(X)^{-1} \left( \frac{dm(X, \alpha_0)}{d\alpha}[v_2] \right) \right],$$

and for any $v \in \overline{V}$ we call $v = 0$ if and only if $||v|| = 0$ (i.e., functions in $\overline{V}$ are defined in an equivalent class sense according to the metric $||\cdot||$). It is clear that $(\overline{V}, ||\cdot||)$ is an infinite dimensional Hilbert space (under Assumptions 3.1(i)(iii)(iv) and 3.4(i)(ii)).

If the linear functional $\frac{d\phi(\alpha_0)}{d\alpha}[\cdot]$ is bounded on $(V, ||\cdot||)$, i.e.

$$\sup_{v \in V, v \neq 0} \frac{\left|\left| \frac{d\phi(\alpha_0)}{d\alpha}[v] \right|\right|}{||v||} < \infty,$$

then there is a unique extension of $\frac{d\phi(\alpha_0)}{d\alpha}[\cdot]$ from $(V, ||\cdot||)$ to $(\overline{V}, ||\cdot||)$,
and a unique Riesz representer $v^* \in \mathcal{V}$ of $\frac{d\phi(\alpha_0)}{d\alpha}[]$ on $(\mathcal{V}, ||\cdot||)$ such that

$$\frac{d\phi(\alpha_0)}{d\alpha}[v] = \langle v^*, v \rangle \text{ for all } v \in \mathcal{V} \text{ and}$$

$$||v^*|| \equiv \sup_{v \in \mathcal{V}, v \neq 0} \frac{\frac{d\phi(\alpha_0)}{d\alpha}[v]}{||v||} = \sup_{v \in \mathcal{V}, v \neq 0} \frac{d\phi(\alpha_0)}{d\alpha}[v] < \infty.$$

If $\frac{d\phi(\alpha_0)}{d\alpha}[]$ is unbounded on $(\mathcal{V}, ||\cdot||)$, i.e.

$$\sup_{v \in \mathcal{V}, v \neq 0} \frac{d\phi(\alpha_0)}{d\alpha}[v] = \infty,$$

then there is no unique extension of the mapping $\frac{d\phi(\alpha_0)}{d\alpha}[]$ from $(\mathcal{V}, ||\cdot||)$ to $(\mathcal{V}, ||\cdot||)$, and nor existing any Riesz representer of $\frac{d\phi(\alpha_0)}{d\alpha}[]$ on $(\mathcal{V}, ||\cdot||)$.

Since $||v|| \leq C||v||_s$ for all $v \in \mathcal{V}$, it is clear that a $||\cdot||_s$-bounded linear functional $\frac{d\phi(\alpha_0)}{d\alpha}[]$ could be either bounded or unbounded on $(\mathcal{V}, ||\cdot||)$.

**Sieve Riesz representation.** Let $\alpha_{0,n} \in \mathbb{R}^d \times \mathcal{H}_k(n)$ be such that

$$||\alpha_{0,n} - \alpha_0|| \equiv \min_{\alpha \in \mathbb{R}^d \times \mathcal{H}_k(n)} ||\alpha - \alpha_0||.$$

Let $\mathcal{V}_{k(n)} = \text{clsp}(\mathcal{A}_{osn} - \{\alpha_{0,n}\})$, where $\text{clsp}(\cdot)$ denotes the closed linear span under $||\cdot||$. Then $\mathcal{V}_{k(n)}$ is a finite dimensional Hilbert space under $||\cdot||$. Moreover, $\mathcal{V}_{k(n)}$ is dense in $\mathcal{V}$ under $||\cdot||$. To simplify the presentation, we assume that $\dim(\mathcal{V}_{k(n)}) = \dim(\mathcal{A}_{k(n)}) = k(n)$, all of which grow to infinity with $n$. By definition we have $\langle v_n, \alpha_{0,n} - \alpha_0 \rangle = 0$ for all $v_n \in \mathcal{V}_{k(n)}$.

Note that $\mathcal{V}_{k(n)}$ is a finite dimensional Hilbert space. As any linear functional on a finite dimensional Hilbert space is bounded, we can invoke the Riesz representation theorem to deduce that there is a $v^*_n \in \mathcal{V}_{k(n)}$ such that

$$\langle v_n, v^*_n \rangle = 0 \text{ for all } v_n \in \mathcal{V}_{k(n)}.$$
\[
\frac{d\phi(\alpha)}{d\alpha}[v] = \langle v_n^*, v \rangle, \quad \forall v \in \mathcal{V}_{k(n)}, \quad \text{and} \quad \|v_n^*\| \equiv \sup_{v \in \overline{\mathcal{V}}_{k(n)}; \|v\| \neq 0} \frac{|d\phi(\alpha)| [v]}{\|v\|} < \infty.
\]

We call \(v_n^*\) the sieve Riesz representer of the functional \(\frac{d\phi(\alpha)}{d\alpha}[-]\) on \(\overline{\mathcal{V}}_{k(n)}\). By definition, for any non-zero linear functional \(\frac{d\phi(\alpha)}{d\alpha}[-]\), we have:

\[
0 < \|v_n^*\|^2 = E \left[ \left( \frac{dm(X, \alpha_0)}{d\alpha} [v_n^*] \right)^\prime \Sigma(X)^{-1} \left( \frac{dm(X, \alpha_0)}{d\alpha} [v_n^*] \right) \right]
\]

is non-decreasing in \(k(n)\).

We emphasize that the sieve Riesz representer \(v_n^*\) of a linear functional \(\frac{d\phi(\alpha)}{d\alpha}[-]\) on \(\overline{\mathcal{V}}_{k(n)}\) always exists regardless of whether \(\frac{d\phi(\alpha)}{d\alpha}[-]\) is bounded on the infinite dimensional space \((\mathcal{V}, \| \cdot \|)\) or not. Moreover, \(v_n^* \in \overline{\mathcal{V}}_{k(n)}\) and its norm \(\|v_n^*\|\) can be computed in closed form (see Subsection 4.1.1).

The next Lemma allows us to verify whether or not \(\frac{d\phi(\alpha)}{d\alpha}[-]\) is bounded on \((\mathcal{V}, \| \cdot \|)\) by checking whether or not \(\lim_{k(n) \to \infty} \|v_n^*\| < \infty\).

**Lemma 3.3** Let \(\{\mathcal{V}_k\}_{k=1}^\infty\) be an increasing sequence of finite dimensional Hilbert spaces that is dense in \((\mathcal{V}, \| \cdot \|)\), and \(v_n^* \in \overline{\mathcal{V}}_{k(n)}\) be defined in (3.6). (1) If \(\frac{d\phi(\alpha)}{d\alpha}[-]\) is bounded on \((\mathcal{V}, \| \cdot \|)\), then (3.4) holds, \(v_n^* = \arg\min_{v \in \mathcal{V}_{k(n)}} \|v^* - v\|\) and \(\|v^* - v_n^*\| \to 0, \lim_{k(n) \to \infty} \|v_n^*\| = \|v^*\| < \infty\); (2) Let \(\frac{d\phi(\alpha)}{d\alpha}[-]\) be bounded on \((\mathcal{V}, \| \cdot \|)\) and \(\{\overline{\mathcal{V}}_k\}_{k=1}^\infty\) be dense in \((\mathcal{V}, \| \cdot \|)\). If \(\frac{d\phi(\alpha)}{d\alpha}[-]\) is unbounded on \((\mathcal{V}, \| \cdot \|)\) then \(\lim_{k(n) \to \infty} \|v_n^*\| = \infty\).

**Sieve score and sieve variance.** For each sieve dimension \(k(n)\), we call

\[
S_{n,i}^* = \left( \frac{dm(X, \alpha_0)}{d\alpha} [v_n^*] \right)^\prime \Sigma(X)^{-1} \rho(Z_i, \alpha_0)
\]

the sieve score associated with the \(i\)-th observation, and \(\|v_n^*\|^2 \equiv \text{Var}(S_{n,i}^*)\) as the sieve variance. Recall that \(\Sigma_0(X) \equiv \text{Var}(\rho(Z; \alpha_0)|X)\) a.s.-\(X\). Then

\[
\|v_n^*\|^2 = E[S_{n,i}^* S_{n,i}^{-}]
\]

\[
= E \left[ \left( \frac{dm(X, \alpha_0)}{d\alpha} [v_n^*] \right)^\prime \Sigma(X)^{-1} \Sigma_0(X) \Sigma(X)^{-1} \left( \frac{dm(X, \alpha_0)}{d\alpha} [v_n^*] \right) \right].
\]
(See Subsection 4.1.1 for closed form expressions of \( \|v_n^*\|_{sd}^2 \)). Under Assumption 3.1(iv), we have \( \|v_n^*\|_{sd}^2 \gg \|v_n^*\|^2 \), and hence \( \lim_{k(n) \to \infty} \|v_n^*\|_{sd} < \infty \) (or = \( \infty \)) iff \( \lim_{k(n) \to \infty} \|v_n^*\| < \infty \) (or = \( \infty \)). Therefore, in this paper we call \( \phi() \) regular (or irregular) at \( \alpha_0 \) whenever \( \lim_{k(n) \to \infty} \|v_n^*\| < \infty \) (or = \( \infty \)), which, by Lemma 3.3, is also whenever \( \frac{d\phi(\alpha_0)}{d\alpha}[\cdot] \) is bounded (or unbounded) on \((V, \|\cdot\|)\). It is clear that our notion of a regular \( \phi(\cdot) \) at \( \alpha_0 \) is only necessary but not sufficient for the existence of root-\( n \) asymptotically normal regular estimators of \( \phi(\alpha_0) \). Moreover, if \( \phi(\cdot) \) is regular at \( \alpha_0 \) then we can define

\[
S_i^* = \left( \frac{dm(X_i, \alpha_0)}{d\alpha} [v^*] \right)' \Sigma(X_i)^{-1} \rho(Z_i, \alpha_0)
\]

as the score associated with the \( i \)-th observation, and \( \|v^*\|_{sd}^2 \equiv \text{Var}(S_i^*) \) as the asymptotic variance. By Lemma 3.3(1) for a regular functional we have: \( \|v^*\|_{sd}^2 \gg \|v^*\| < \infty \) and \( \text{Var}(S_i^* - S_{n,i}^*) \propto \|v^* - v_n^*\|^2 \to 0 \) as \( k(n) \to \infty \). See Appendix A for further discussions.

3.3. Two key local conditions

For all \( k(n) \), let

\[
u_n^* = \frac{v_n^*}{\|v_n^*\|_{sd}}
\]

be the “scaled sieve Riesz representer”. Since \( \|v_n^*\|_{sd}^2 \gg \|v_n^*\|^2 \) (under Assumption 3.1(iv)), we have: \( \|u_n^*\| \asymp 1 \) and \( \|u_n^*\|_s \leq c \tau_n \) for \( \tau_n \) defined in (3.3) and a finite constant \( c > 0 \).

Let \( \mathcal{T}_n \equiv \{ t \in \mathbb{R} : |t| \leq 4M_n^2 \delta_n \} \) with \( M_n \) and \( \delta_n \) given in the definition of \( \mathcal{N}_{osn} \).

**Assumption 3.5** (Local behavior of \( \phi \)) (i) \( \nu \mapsto \frac{d\phi(\alpha_0)}{d\alpha}[v] \) is a non-zero linear functional mapping from \( V \) to \( \mathbb{R} \); \( \{\nabla_k\}_{k=1}^{\infty} \) is an increasing
sequence of finite dimensional Hilbert spaces that is dense in \((V, \| \cdot \|)\); and \(\| v^*_n \| = o(1)\):

\[
(ii) \quad \sup_{(\alpha, t) \in \mathcal{N}_{osn} \times T_n} \frac{\sqrt{n} \left| \phi (\alpha + tu^*_n) - \phi (\alpha_0) - \frac{d\phi (\alpha_0)}{d\alpha} [\alpha + tu^*_n - \alpha_0] \right|}{\| v^*_n \|} = o(1);
\]

\[
(iii) \quad \sqrt{n} \left| \frac{d\phi (\alpha_0)}{d\alpha} [\alpha_0 - \alpha] \right| = o(1).
\]

Since \(\| v^*_n \|_{sd}^2 \asymp \| v^*_n \|^2\) (under Assumption 3.1(iv)), we could rewrite Assumption 3.5 using \(\| v^*_n \|_{sd}\) instead of \(\| v^*_n \|\). As it will become clear in Theorem 4.1 that \(\| v^*_n \|_{sd}^2 \) is the variance of \(\phi (\hat{\alpha}_n) - \phi (\alpha_0)\), Assumption 3.5(ii) puts a restriction on how fast the sieve dimension \(k(n)\) could grow with the sample size \(n\).

Assumption 3.5(ii) controls the nonlinearity bias of \(\phi (\cdot)\) (i.e., the linear approximation error of a possibly nonlinear functional \(\phi (\cdot)\)). It is automatically satisfied when \(\phi (\cdot)\) is a linear functional. For a nonlinear functional \(\phi (\cdot)\) (such as the quadratic functional), it can be verified using the smoothness of \(\phi (\cdot)\) and the convergence rates in both \(\| \cdot \|\) and \(\| \cdot \|_s\) metrics (the definition of \(\mathcal{N}_{osn}\)). See Section 6 for verification.

Assumption 3.5(iii) controls the linear bias part due to the finite dimensional sieve approximation of \(\alpha_{0,n}\) to \(\alpha_0\). It is a condition imposed on the growth rate of the sieve dimension \(k(n)\). When \(\phi (\cdot)\) is an irregular functional, we have \(\| v^*_n \| \not\asymp \| v^* \| < \infty\). Assumption 3.5(iii) requires that the sieve bias term, \(\left| \frac{d\phi (\alpha_0)}{d\alpha} [\alpha_{0,n} - \alpha_0] \right|\), is of a smaller order than that of the sieve standard deviation term, \(n^{-1/2} \| v^*_n \|_{sd}\). This is a standard condition imposed for the asymptotic normality of any plug-in nonparametric estimator of an irregular functional (such as a point evaluation functional of a nonparametric mean regression).

**Remark 3.1** When \(\phi (\cdot)\) is regular at \(\alpha_0\) (i.e., \(\| v^*_n \| \not\asymp \| v^* \| < \infty\)), since \(\langle v^*_n, \alpha_{0,n} - \alpha_0 \rangle = 0\) (by definition of \(\alpha_{0,n}\)) we have \(\left| \frac{d\phi (\alpha_0)}{d\alpha} [\alpha_{0,n} - \alpha_0] \right| \leq\)
∥v* − v∗n∥ × ∥α0,n − α0∥. And Assumption 3.5(iii) is satisfied if

\[ (3.10) \quad ∥v* − v∗n∥ × ∥α0,n − α0∥ = o(n^{-1/2}). \]

This is similar to assumption 4.2 in Ai and Chen (2003) and assumption 3.2(iii) in Chen and Pouzo (2009) for the root-n estimable Euclidean parameter θ0 of the model (1.1). As pointed out by Chen and Pouzo (2009), Condition (3.10) could be satisfied when \( \dim(\mathcal{A}_k(n)) \asymp k(n) \) is chosen to obtain optimal nonparametric convergence rate in \( ||·||_s \) norm. But this nice feature only applies to regular functionals.

The next assumption is about the local quadratic approximation (LQA) to the sample criterion difference along the scaled sieve Riesz representer direction \( u∗n = v∗n / ∥v∗n∥_{sd} \).

For any \( (\alpha, t) \in \mathcal{N}_{0sn} \times \mathcal{T}_n \), we let \( \tilde{\Lambda}_n(\alpha(t), \alpha) \equiv 0.5\{\hat{Q}_n(\alpha(t)) − \hat{Q}_n(\alpha)\} \) with \( \alpha(t) \equiv \alpha + tu∗n \). Denote

\[ (3.11) \quad Z_n \equiv n^{-1} \sum_{i=1}^{n} \left( \frac{dm(X_i; \alpha_0)}{d\alpha} [u∗]_i \right)' \Sigma(X_i)^{-1} ρ(Z_i, \alpha_0) = n^{-1} \sum_{i=1}^{n} \frac{S_{n,i}}{∥v∗n∥_{sd}}. \]

**Assumption 3.6 (LQA) (i)** \( \alpha(t) \in \mathcal{A}_k(n) \) for any \( (\alpha, t) \in \mathcal{N}_{0sn} \times \mathcal{T}_n \); and with \( r_n(t_n) = (\max\{t_n^2, n^{-1/2}, o(n^{-1})\})^{-1} \),

\[ \sup_{(\alpha, t_n) \in \mathcal{N}_{0sn} \times \mathcal{T}_n} r_n(t_n) \left| \tilde{\Lambda}_n(\alpha(t_n), \alpha) − t_n \{Z_n + \langle u∗n, \alpha − \alpha_0 \rangle\} − \frac{B_n t_n^2}{2} \right| = o_{P_2\infty}(1), \]

where, for each \( n \), \( B_n \) is a \( Z^n \) measurable positive random variable, and \( B_n = O_{P_2\infty}(1) \);

(ii) \( \sqrt{n}Z_n \Rightarrow N(0, 1) \).

Assumption 3.6(ii) is a standard one, and is implied by the following Lindeberg condition: For all \( \epsilon > 0 \),

\[ (3.12) \quad \limsup_{n \to \infty} E \left[ \left( \frac{S_{n,i}^*}{∥v∗n∥_{sd}} \right)^2 \left\{ \frac{S_{n,i}^*}{\epsilon \sqrt{n} ∥v∗n∥_{sd}} > 1 \right\} \right] = 0, \]
which, under Lemma 3.3(1) and Assumption 3.1(iv), is satisfied when the functional \( \phi(\cdot) \) is regular (\( \|v^*_n\|_{sd} \asymp \|v^*_n\| \to \|v^*\| < \infty \)). This is why Assumption 3.6(ii) is not imposed in Ai and Chen (2003) and Chen and Pouzo (2009) in their root-\( n \) asymptotically normal estimation of the regular functional \( \phi(\alpha) = \lambda' \theta \).

Assumption 3.6(i) implicitly imposes restrictions on the nonparametric estimator \( \hat{m}(x, \alpha) \) of \( m(x, \alpha) = E[\rho(Z, \alpha)|X = x] \) in a shrinking neighborhood of \( \alpha_0 \), so that the criterion difference could be well approximated by a quadratic form. It is trivially satisfied when \( \hat{m}(x, \alpha) \) is linear in \( \alpha \), such as the series LS estimator (2.5) when \( \rho(Z, \alpha) \) is linear in \( \alpha \). There are two potential difficulties in verifying this assumption for nonlinear conditional moment models with nonparametric endogeneity (such as the NPQIV model). First, due to the non-smooth residual function \( \rho(Z, \alpha) \), the estimator \( \hat{m}(x, \alpha) \) (and hence the sample criterion \( \hat{Q}_n(\alpha) \)) could be pointwise non-smooth with respect to \( \alpha \). Second, due to the slow convergence rates in the strong norm \( \| \cdot \| \) present in nonlinear nonparametric ill-posed inverse problems, it could be challenging to control the remainder of a quadratic approximation. When \( \hat{m}(x, \alpha) \) is the series LS estimator (2.5), Lemma 5.1 in Section 5 shows that Assumption 3.6(i) is satisfied by a set of relatively low level sufficient conditions (Assumptions A.4 - A.7 in Appendix A). See Section 6 for verification of these sufficient conditions for functionals of the NPQIV model.

4. ASYMPTOTIC PROPERTIES OF SIEVE WALD AND QLR STATISTICS

In this section, we first establish the asymptotic normality of the plug-in PSMD estimator \( \hat{\phi}(\hat{\alpha}_n) \) of \( \phi(\alpha_0) \) for the model (1.1), regardless of whether it is root-\( n \) estimable or not. We then provide a simple consistent variance estimator and hence the asymptotic standard normality of the corresponding sieve t statistic for a real-valued functional \( \phi : \mathbb{R}^{d_\theta} \times \mathcal{H} \to \)
We finally derive the asymptotic properties of SQLR tests for the hypothesis $\phi(\alpha_0) = \phi_0$. See Appendix A for the case of a vector-valued functional $\phi : \mathbb{R}^{d_\theta} \times \mathcal{H} \to \mathbb{R}^{d_\phi}$ (where $d_\phi$ could grow slowly with $n$).

4.1. Asymptotic normality of the plug-in PSMD estimator

The next result allows for a (possibly) nonlinear irregular functional $\phi()$ of the general model (1.1).

**Theorem 4.1** Let $\hat{\alpha}_n$ be the PSMD estimator (2.2) and Assumptions 3.1 - 3.4 hold. If Assumptions 3.5 and 3.6 hold, then:

$$\sqrt{n} \frac{\phi(\hat{\alpha}_n) - \phi(\alpha_0)}{||v_n^*||_{sd}} = -\sqrt{n}Z_n + o_{P_{\mathbb{Z}^\infty}}(1) \Rightarrow N(0,1).$$

When the functional $\phi(\cdot)$ is regular at $\alpha = \alpha_0$, we have $||v_n^*||_{sd} \asymp ||v_n^*|| = O(1)$ and $\phi(\hat{\alpha}_n)$ converges to $\phi(\alpha_0)$ at the parametric rate of $1/\sqrt{n}$. When the functional $\phi(\cdot)$ is irregular at $\alpha = \alpha_0$, we have $||v_n^*||_{sd} \asymp ||v_n^*|| \to \infty$; so the convergence rate of $\phi(\hat{\alpha}_n)$ becomes slower than $1/\sqrt{n}$.

For any regular functional of the semi/nonparametric model (1.1), Theorem 4.1 implies that

$$\sqrt{n}(\phi(\hat{\alpha}_n) - \phi(\alpha_0)) = -n^{-1/2}\sum_{i=1}^n S_{n,i}^* + o_{P_{\mathbb{Z}^\infty}}(1) \Rightarrow N(0,\sigma_{v^*}^2),$$

with

$$\sigma_{v^*}^2 = \lim_{n \to \infty} ||v_n^*||_{sd}^2 = ||v^*||_{sd}^2 = E\left[\left(\frac{dm(X,\alpha_0)}{d\alpha}[v^*]\right)\Sigma(X)^{-1}\Sigma_0(X)\Sigma(X)^{-1}\left(\frac{dm(X,\alpha_0)}{d\alpha}[v^*]\right)\right].$$

Thus, Theorem 4.1 is a natural extension of the asymptotic normality results of Ai and Chen (2003) and Chen and Pouzo (2009) for the specific regular functional $\phi(\alpha_0) = \lambda^\prime\theta_0$ of the model (1.1). See Remark A.1 in Appendix A for further discussions.
4.1.1. Closed form expressions of sieve Riesz representer and sieve variance

To apply Theorem 4.1, one needs to know the sieve Riesz representer $v_n^*$ defined in (3.6) and the sieve variance $\|v_n^*\|_{sd}^2$ given in (3.8). It turns out that both can be computed in closed form.

**Lemma 4.1** Let $\mathbf{V}_n = \mathbb{R}^d \times \{v_h(\cdot) = \psi^k(n)(\cdot) : \beta \in \mathbb{R}^{k(n)}\} = \{v(\cdot) = \psi^k(n)(\cdot) : \gamma \in \mathbb{R}^{d+k(n)}\}$ be dense in the infinite dimensional Hilbert space $(\mathbf{V}, \| \cdot \|)$ with the norm $\| \cdot \|$ defined in (3.1). Then: the sieve Riesz representer $v_n^* = (v^{\theta}_n, v^*_h(\cdot)) \in \mathbf{V}_n$ of $d\phi(\alpha_0) d\alpha$ has a closed form expression:

\[
\begin{align*}
& v_n^* = (v^{\theta}_n, \psi^k(n)(\cdot))' = \psi^k(n)(\cdot)\gamma_n^*, \text{ and } \\
& \gamma_n^* = D_n F_n
\end{align*}
\]

with $D_n = E \left( \left( \frac{dm(X,\alpha_0)}{d\alpha} \psi^k(n)(\cdot) \right)' \Sigma(X)^{-1} \left( \frac{dm(X,\alpha_0)}{d\alpha} \psi^k(n)(\cdot) \right) \right)$ and $F_n = \frac{d\phi(\alpha_0)}{d\alpha} \psi^k(n)(\cdot)$. Thus

\[
\begin{align*}
& \|v_n^*\|^2 = \gamma_n^* D_n \gamma_n^* = F_n' D_n^{-1} F_n.
\end{align*}
\]

The sieve variance (3.8) also has a closed form expression:

\[
\begin{align*}
& \|v_n^*\|_{sd}^2 = F_n' D_n^{-1} \overline{\delta}_n D_n^{-1} F_n,
\end{align*}
\]

\[
\begin{align*}
& \overline{\delta}_n = E \left( \left( \frac{dm(X,\alpha_0)}{d\alpha} \psi^k(n)(\cdot) \right)' \Sigma(X)^{-1} \rho(Z,\alpha_0) \rho(Z,\alpha_0)' \Sigma(X)^{-1} \left( \frac{dm(X,\alpha_0)}{d\alpha} \psi^k(n)(\cdot) \right) \right).
\end{align*}
\]

Let $A_n = \Theta \times \mathcal{H}_{k(n)}$ with $\mathcal{H}_{k(n)}$ given in (2.3). Then $\mathbf{V}_n = clsp \left(A_{k(n)} - \{\alpha_0, n\}\right)$ and one could let $\psi^{k(n)}(\cdot) = q^{k(n)}(\cdot)$ in Lemma 4.1 and (4.3) becomes the sieve variance expression given in (2.6).

Lemmas 3.3 and 4.1 imply that $\phi(\cdot)$ is regular (or irregular) at $\alpha = \alpha_0$ iff $\lim_{k(n) \to \infty} (F_n' D_n^{-1} F_n) < \infty$ (or $\infty$).
According to Lemma 4.1 we could use different finite dimensional linear sieve basis $\psi_k^{\mathcal{N}(n)}$ to compute sieve Riesz representer $v^*_n = (v'^{\mathcal{N}(n)}_{\theta,n}, v^*_{h,n}(\cdot))' \in \mathcal{V}_k^{\mathcal{N}(n)}$, $\|v^*_n\|^2$ and $\|v^*_n\|^2_{sd}$. Most typical choices include orthonormal bases and the original sieve basis $q_k^{\mathcal{N}(n)}$ (used to approximate unknown function $h_0$). It is typically easier to characterize the speed of $\|v^*_n\|^2 = F_n'D_nF_n$ as a function of $k(n)$ when an orthonormal basis is used, while there is a nice interpretation in terms of sieve variance estimation when the original sieve basis $q_k^{\mathcal{N}(n)}$ is used. See Sections 2.2, 4.2 and 6 for related discussions.

4.2. Consistent estimator of sieve variance of $\phi(\hat{\alpha}_n)$

In order to apply the asymptotic normality Theorem 4.1 we need an estimator of the sieve variance $\|v^*_n\|^2_{sd}$ defined in (3.8). We now provide one simple consistent estimator of the sieve variance when the residual function $\rho(\cdot)$ is pointwise smooth with respect to $\alpha_0$. See Appendix B for additional consistent variance estimators.

The theoretical sieve Riesz representer $v^*_n$ is unknown but can be estimated easily. Let $\|\cdot\|_{n,M}$ denote the empirical norm induced by the following empirical inner product

\[
(4.4) \quad (v_1, v_2)_{n,M} \equiv \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{d\alpha} [v_1] \right)' M_{n,i} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{d\alpha} [v_2] \right),
\]

for any $v_1, v_2 \in \mathcal{V}_{k(n)}$, where $M_{n,i}$ is some (almost surely) positive definite weighting matrix.

We define an empirical sieve Riesz representer $\hat{v}^*_n$ of the functional $\frac{d\phi(\hat{\alpha}_n)}{d\alpha}[\cdot]$ with respect to the empirical norm $\|\cdot\|_{n,\hat{\Sigma}^{-1}}$ as

\[
(4.5) \quad \frac{d\phi(\hat{\alpha}_n)}{d\alpha} [\hat{v}_n^*] = \sup_{v \in \mathcal{V}_{k(n)}} \frac{\|v\|^2_{n,\hat{\Sigma}^{-1}}}{\frac{d\phi(\hat{\alpha}_n)}{d\alpha}[v]^2} < \infty
\]

and

\[
(4.6) \quad \frac{d\phi(\hat{\alpha}_n)}{d\alpha}[v] = \langle \hat{v}_n^*, v \rangle_{n,\hat{\Sigma}^{-1}} \quad \text{for any} \quad v \in \mathcal{V}_{k(n)}.
\]
For \( \|v^*_n\|_{sd}^2 = \mathbb{E} \left( S^*_{n,i} S^*_{n,i}' \right) \) given in (3.8) we can define a simple plug-in sieve variance estimator:

\[
\|\hat{v}^*_n\|_{n,sd}^2 = \frac{1}{n} \sum_{i=1}^{n} \tilde{S}^*_n \tilde{S}^*_n
\]

(4.7)

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{da} [\hat{\omega}^*_n] \right) ' \hat{\Sigma}_i^{-1} (\hat{\rho}_i \hat{\rho}_i') \hat{\Sigma}_i^{-1} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{da} [\hat{\omega}^*_n] \right)
\]

with \( \hat{\rho}_i = \rho(\hat{Z}_i, \hat{\alpha}_n) \) and \( \hat{\Sigma}_i = \hat{\Sigma}(X_i) \).

Under the condition stated in Lemma 4.1 \( \hat{v}^*_n \) defined in (4.5-4.6) also has a closed form solution:

\[
\hat{v}^*_n = \overline{v}^{k(n)}(\cdot) \gamma^*_n, \quad \text{and} \quad \hat{\gamma}^*_n = \hat{D}_n \hat{\rho}_n,
\]

(4.8)

with \( \hat{D}_n = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{da} [\overline{v}^{k(n)}(\cdot)] \right) ' \hat{\Sigma}_i^{-1} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{da} [\overline{v}^{k(n)}(\cdot)] \right) \) and \( \hat{\rho}_n = \frac{d\phi(\hat{\alpha}_n)}{da} [\overline{v}^{k(n)}(\cdot)] \). Hence the sieve variance estimator given in (4.7) now becomes

\[
\|\hat{v}^*_n\|_{n,sd}^2 = \hat{V}_1 \equiv \hat{F}_n \hat{D}_n \hat{\rho}_n \hat{D}_n \hat{\rho}_n \hat{F}_n \quad \text{with}
\]

\[
\hat{\Sigma}_n = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{da} [\overline{v}^{k(n)}(\cdot)] \right) ' \hat{\Sigma}_i^{-1} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{da} [\overline{v}^{k(n)}(\cdot)] \right).
\]

(4.9)

In particular, with \( v^{k(n)} = q^{k(n)} \) the sieve variance estimator \( \|\hat{v}^*_n\|_{n,sd}^2 \) given in (4.9) becomes the one given in (2.10) in Subsection 2.2.

Let \( \langle v_1, v_2 \rangle_{M} \equiv \mathbb{E} \left[ \left( \frac{d\phi(\alpha)}{da} [v_1] \right)' M \left( \frac{d\phi(\alpha)}{da} [v_2] \right) \right] \). Then \( \langle v_1, v_2 \rangle_{\Sigma^{-1}} \equiv \langle v_1, v_2 \rangle \) for all \( v_1, v_2 \in \overline{v}_k(n) \). Denote \( \overline{v}_k(n) \equiv \{ v \in \overline{v}_k(n) : \|v\| = 1 \} \).

**Assumption 4.1**

1. \( \sup_{\alpha \in \mathcal{N}_a} \sup_{\mathbb{V}_k(n)} \left| \frac{d\phi(\alpha)}{da} [v] - \frac{d\phi(\alpha)}{da} [v] \right| = o(1) \);
2. for each \( k(n) \) and any \( \alpha \in \mathcal{N}_a \), \( v \in \mathbb{V}_k(n) \mapsto \frac{d\hat{m}(\cdot)}{da} [v] \in L^2(fX) \) is a linear functional measurable with respect to \( Z^n \); and
3. \( \sup_{v_1, v_2 \in \mathbb{V}_k(n)} \| \langle v_1, v_2 \rangle_{\Sigma^{-1}} - \langle v_1, v_2 \rangle_{\Sigma} \| = o_{P_{2 \infty}}(1) \);
4. \( \sup_{x \in X} ||\hat{\Sigma}(x) - \Sigma(x)||_e = o_{P_{2 \infty}}(1) \).
(iv) \( \sup_{x \in X} E \left[ \sup_{\alpha \in \mathcal{N}_{osn}} \| \rho(Z, \alpha) \rho(Z, \alpha)' - \rho(Z, \alpha_0) \rho(Z, \alpha_0)' \|_e | X = x \right] = o(1) \).

(v) \( \sup_{v \in \mathbf{V}^1} |\langle v, v \rangle_n M - \langle v, v \rangle_M| = o_P(Z) \) for \( |\langle v, v \rangle_n M - \langle v, v \rangle_M| = \frac{1}{\sqrt{n}} Z_n + o_P(Z) \Rightarrow N(0, 1) \).

Assumption 4.1(i) becomes vacuous if \( \phi \) is linear; otherwise it requires smoothness of the family \( \{ \frac{d \phi(\alpha)}{d \alpha}[v] : \alpha \in \mathcal{N}_{osn} \} \) uniformly in \( v \in \mathbf{V}^{1}_{k(n)} \).

Assumption 4.1(ii) implicitly assumes that the residual function \( \rho(z, \cdot) \) is "smooth" in \( \alpha \in \mathcal{N}_{osn} \) (see, e.g., 
Ai and Chen (2003)) or that \( \frac{d \hat{m}(X, \hat{\alpha}_n)}{d \alpha}[v] \) can be well approximated by numerical derivatives (see, e.g., 
Hong et al. (2010)). Assumption 4.1(iii) assumes the existence of consistent estimators for \( \Sigma \). In most applications, \( \Sigma(\cdot) \) is either completely known (such as the identity matrix) or \( \Sigma_0 \); while \( \Sigma_0(x) \) could be consistently estimated via kernel, series LS, local linear regression and other nonparametric procedures (see, e.g., 
Ai and Chen (2003) and Chen and Pouzo (2009)).

Theorem 4.2 Let Assumptions 3.1 - 3.4 hold. If Assumption 4.1 is satisfied, then:

1. \( \left| \frac{||\hat{v}_n^*||_{n,sd}}{||\tilde{v}_n||_{n,sd}} - 1 \right| = o_{P,Z}(1) \) for \( ||\tilde{v}_n||_{n,sd} \) given in (4.7).
2. If, in addition, Assumptions 3.5 and 3.6 hold, then:
   \( \hat{W}_n = \frac{\sqrt{n} \phi(\hat{\alpha}_n) - \phi(\alpha_0)}{||\tilde{v}_n||_{n,sd}} = -\sqrt{n} Z_n + o_{P,Z}(1) \Rightarrow N(0, 1) \).

Theorem 4.2(2) allows us to construct confidence sets for \( \phi(\alpha_0) \) based on a possibly non-optimally weighted plug-in PSMD estimator \( \phi(\hat{\alpha}_n) \). A potential drawback, is that it requires a consistent estimator for \( v \mapsto \frac{d m(\cdot, \alpha_0)}{d \alpha}[v] \), which may be hard to compute in practice when the residual function \( \rho(Z, \alpha) \) is not pointwise smooth in \( \alpha \in \mathcal{N}_{osn} \) such as in the NPQIV (2.22) example.

Remark 4.1 Let \( W_n = \left( \frac{\sqrt{n} \phi(\hat{\alpha}_n) - \phi(\alpha_0)}{||\tilde{v}_n||_{n,sd}} \right)^2 = \left( \frac{\hat{W}_n + \sqrt{n} \phi(\alpha_0) - \phi(\alpha_0)}{||\tilde{v}_n||_{n,sd}} \right)^2 \) be the Wald test statistic. Then Theorem 4.2 (with \( \frac{||v^*_n||_{n,sd}}{\sqrt{n}} \times \frac{||\tilde{v}_n||_{n,sd}}{\sqrt{n}} = o(1) \)) immediately implies the following results:
Under $H_0 : \phi(\alpha_0) = \phi_0$, $W_n = \left( \hat{W}_n \right)^2 \Rightarrow \chi^2_1$.

Under $H_1 : \phi(\alpha_0) \neq \phi_0$, $W_n = \left( O_P(1) + \sqrt{n} \|v_n^*\|_{sd}^{-1} [\phi(\alpha_0) - \phi_0] (1 + o_P(1)) \right)^2 \rightarrow \infty$ in probability.

See Theorem A.3 in Appendix A for asymptotic properties of $W_n$ under local alternatives.

4.3. Sieve QLR statistics

We now characterize the asymptotic behaviors of the possibly non-optimally weighted SQLR statistic $\hat{QLR}_n(\phi_0)$ defined in (2.13).

Let $A^R_{k(n)} \equiv \{ \alpha \in A_{k(n)} : \phi(\alpha) = \phi_0 \}$ be the restricted sieve space, and $\hat{\alpha}_n^R \in A^R_{k(n)}$ be a restricted approximate PSMD estimator, defined as

$$\hat{Q}_n(\hat{\alpha}_n^R) + \lambda_n Pen(\hat{h}_n^R) \leq \inf_{\alpha \in A^R_{k(n)}} \left\{ \hat{Q}_n(\alpha) + \lambda_n Pen(h) \right\} + o_{PZ}(n^{-1}).$$

Then:

$$\hat{QLR}_n(\phi_0) = n \left( \hat{Q}_n(\hat{\alpha}_n^R) - \hat{Q}_n(\hat{\alpha}_n) \right)$$

$$= n \left( \inf_{\alpha \in A^R_{k(n)}} \hat{Q}_n(\alpha) - \inf_{\alpha \in A_{k(n)}} \hat{Q}_n(\alpha) \right) + o_{PZ}(1).$$

Recall that $u_n^* \equiv v_n^*/\|v_n^*\|_{sd}$, and that $\hat{QLR}_n^0(\phi_0)$ denotes the optimally weighted (i.e., $\Sigma = \Sigma_0$) SQLR statistic in Subsection 2.2. We note that $\|u_n^*\| = 1$ for the optimally weighted case.

**Theorem 4.3** Let Assumptions 3.1 - 3.6 hold with $|B_n - \|u_n^*\|^2| = o_{P_{Z\infty}}(1)$. If $\hat{\alpha}_n^R \in N_{osn} wpa1-P_{Z\infty}$, then: (1) under the null $H_0 : \phi(\alpha_0) = \phi_0$,

$$\|u_n^*\|^2 \times \hat{QLR}_n(\phi_0) = \left( \sqrt{n} \hat{Z}_n \right)^2 + o_{P_{Z\infty}}(1) \Rightarrow \chi^2_1.$$
Further, let $\hat{\alpha}_n$ be the optimally weighted PSMD estimator (2.2) with $\Sigma = \Sigma_0$. Then: under $H_0 : \phi(\alpha_0) = \phi_0$,

$$\hat{QLR}_n^0(\phi_0) = (\sqrt{n}Z_n)^2 + o_{P_\infty}(1) \Rightarrow \chi^2_1.$$  

See Theorem A.1 in Appendix A for the asymptotic behavior under local alternatives.

Compared to Theorem 4.1 on the asymptotic normality of $\phi(\hat{\alpha}_n)$, Theorem 4.3 on the asymptotic null distribution of the SQLR statistic requires two extra conditions: $|B_n - ||u_n^*||^2| = o_{P_\infty}(1)$ and $\hat{\alpha}_n^R \in N_{osn} \text{ wpa1-}P_\infty$. Both conditions are also needed even for QLR statistics in parametric extremum estimation and testing problems. Lemma 5.1 in Section 5 provides a simple sufficient condition (Assumption B) for $|B_n - ||u_n^*||^2| = o_{P_\infty}(1)$.

Proposition B.1 in Appendix B establishes $\hat{\alpha}_n^R \in N_{osn} \text{ wpa1-}P_\infty$ under the null $H_0 : \phi(\alpha_0) = \phi_0$ and other conditions virtually the same as those for Lemma 3.2 (i.e., $\hat{\alpha}_n \in N_{osn} \text{ wpa1-}P_\infty$).

Theorem 4.3(2) recommends to construct an asymptotic $100(1 - \tau)\%$ confidence set for $\phi(\alpha)$ by inverting the optimally weighted SQLR statistic:

$$\left\{ r \in \mathbb{R} : \hat{QLR}_n^0(r) \leq c_{\chi^2_1(1 - \tau)} \right\}.$$  
This result extends that of Chen and Pouzo (2009) for a regular Euclidean functional $\phi(\alpha) = \lambda'\theta$ to possibly irregular nonlinear functionals.

Next, we consider the asymptotic behavior of $\hat{QLR}_n^0(\phi_0)$ under the fixed alternatives $H_1 : \phi(\alpha_0) \neq \phi_0$.

**Theorem 4.4** Let Assumptions 3.1, 3.2 and 3.3 hold. Suppose that $\sup_{h \in H} Pen(h) < \infty$ and $\phi$ is continuous in $|| \cdot ||_s$. Then: under $H_1 : \phi(\alpha_0) \neq \phi_0$, there is a constant $C > 0$ such that

$$\frac{\hat{QLR}_n^0(\phi_0)}{n} \geq C > 0 \text{ wpa1.}$$
5. INFERENCE BASED ON GENERALIZED RESIDUAL BOOTSTRAP

The inference procedures described in Subsections 4.2 and 4.3 are based on the asymptotic critical values. For many parametric models it is known that bootstrap based procedures could approximate finite sample distributions more accurately. In this section we establish the consistency of the bootstrap sieve Wald and SQLR statistics under virtually the same conditions as those imposed for the original-sample sieve Wald and SQLR statistics.

A bootstrap procedure is described by an array of “weights” \( \{ \omega_{i,n} \}_{i=1}^{n} \) for each \( n \), where each bootstrap sample is drawn independently of the original data \( \{ Z_i \}_{i=1}^{n} \). Different bootstrap procedures correspond to different choices of the weights \( \{ \omega_{i,n} \}_{i=1}^{n} \) but all satisfy \( \omega_{i,n} \geq 0 \) and \( E[\omega_{i,n}] = 1 \).

For the time being we assume that \( \lim_{n \to \infty} \text{Var}(\omega_{i,n}) = \sigma_\omega^2 \in (0, \infty) \) for all \( i \).

In this paper we focus on two types of bootstrap weights:

**Assumption Boot.1 (I.i.d Weights)** Let \( (\omega_i)_{i=1}^{n} \) be a sequence such that \( \omega_i \in \mathbb{R}_+, \omega_i \sim \text{iid} P_\omega, E[\omega] = 1, \text{Var}(\omega) = \sigma_\omega^2, \) and \( \int_0^\infty \sqrt{P(|\omega - 1| \geq t)} dt < \infty. \)

The condition \( \int_0^\infty \sqrt{P(|\omega - 1| \geq t)} dt < \infty \) is implied by \( E[|\omega - 1|^{2+\epsilon}] < \infty \) for some \( \epsilon > 0. \)

**Assumption Boot.2 (Multinomial Weights)** Let \( (\omega_{i,n})_{i=1}^{n} \) be a triangular array of random variables such that \( (\omega_{1,n}, \ldots, \omega_{n,n}) \sim \text{Multinomial}(n; n^{-1}, \ldots, n^{-1}) \).

We sometimes omit the \( n \) subscript from the weight series. Note that under Assumption **Boot.2** \( E[\omega_1] = 1, \text{Var}(\omega_1) = (1 - 1/n) \to 1 \equiv \sigma_\omega^2 \) and \( \text{Cov}(\omega_i, \omega_j) = -n^{-1} \) (for \( i \neq j \)). Finally, \( n^{-1} \max_{1 \leq i \leq n} (\omega_i - 1)^2 = o_P(1) \). We use these facts in the proofs.
Let \( V_i \equiv (Z_i, \omega_{i,n}) \) and \( \rho^B(V_i, \alpha) \equiv \omega_{i,n} \rho(Z_i, \alpha) \), be the bootstrap residual function. Let \( \hat{m}^B(x, \alpha) \) be a bootstrap version of \( \hat{m}(x, \alpha) \), that is, \( \hat{m}^B(x, \alpha) \) is computed in the same way as that of \( \hat{m}(x, \alpha) \) except that we use \( \rho^B(V_i, \alpha) \) instead of \( \rho(Z_i, \alpha) \). In particular, \( \hat{m}^B(x, \alpha) = \sum_{i=1}^{n} \omega_{i,n} \rho(Z_i, \alpha) A_n(X_i, x) \) for any linear estimator \( \hat{m}(x, \alpha) \) (2.4) of \( m(x, \alpha) \). For example, if \( \hat{m}(x, \alpha) \) is a series LS estimator (2.5), then \( \hat{m}^B(x, \alpha) \) is the bootstrap series LS estimator (2.16) defined in Subsection 2.2.

Let \( \hat{Q}_B^n(\alpha) \equiv \frac{1}{n} \sum_{i=1}^{n} \hat{m}^B(X_i, \alpha)' \hat{\Sigma}(X_i)^{-1} \hat{m}^B(X_i, \alpha) \) be a bootstrap version of \( \hat{Q}_n(\alpha) \), and \( \hat{\alpha}_n^B \) be the bootstrap PSMD estimator, i.e., \( \hat{\alpha}_n^B \) is an approximate minimizer of \( \{ \hat{Q}_n^B(\alpha) + \lambda_n PEn(h) \} \) on \( A_{k(n)} \). Denote \( \hat{\phi}_n \equiv \phi(\hat{\alpha}_n) \). Then

\[
\hat{QLR}_n^B(\hat{\phi}_n) = n \left( \inf_{\{ A_{k(n)} : \phi(\alpha) = \hat{\phi}_n \}} \hat{Q}_n^B(\alpha) - \hat{Q}_n^B(\hat{\alpha}_n^B) \right)
\]

is the (generalized residual) bootstrap SQLR test statistic. And \( W_{1,n}^B \equiv \left( \sqrt{n} \frac{\hat{\phi}(\hat{\alpha}^B) - \hat{\phi}_n}{\sigma_{\hat{\phi}(\hat{\alpha}^B)|Z_n}} \right)^2 \) is one simple bootstrap Wald test statistic (see Subsection 5.2 for another simple bootstrap Wald statistic).

**Additional notation.** To be more precise, we introduce some definitions associated with the new random variables \( V_i \equiv (Z_i, \omega_{i,n}) \) and the enlarged probability spaces. Let \( \Omega = \{ \omega_{i,n} : i = 1, \ldots, n; n = 1, \ldots \} \) be the space of weights, defined as a triangle array with elements in \( \mathbb{R} \), the corresponding \( \sigma \)-algebra and probability are \( (\mathcal{B}_\Omega, P_\Omega) \). Let \( \mathcal{V}^\infty \equiv Z^\infty \times \Omega \), \( \mathcal{B}^\infty \equiv \mathcal{B}_Z^\infty \times \mathcal{B}_\Omega \) be the \( \sigma \)-algebra, and \( P_{\mathcal{V}^\infty} \) be the joint probability over \( \mathcal{V}^\infty \). Finally, for each \( n \), let \( \mathcal{V}^n \) be the \( \sigma \)-algebra generated by \( \mathcal{V}^n \equiv Z^n \times (\omega_{1,n}, \ldots, \omega_{n,n}) \), where each \( \omega_{i,n} \) acts as a “weight” of \( Z_i \). Let \( A_n \) be a random variable that is measurable with respect to \( \mathcal{B}^n \), and \( \mathcal{L}_{\mathcal{V}^\infty | Z^n}(A_n|Z^n) \) (or \( P_{\mathcal{V}^\infty | Z^n} (A_n \leq \cdot | Z^n) \)) be the conditional law.
(or conditional distribution) of $A_n$ given $Z^n$. Let $B_n$ be a random variable measurable with respect to $\mathcal{B}_{\mathcal{Z}}$, and $\mathcal{L}(B_n)$ (or $P_{Z_n}$ ($B_n \leq \cdot$)) be the law (or distribution) of $B_n$. For two real valued random variables, $A_n$ (measurable with respect to $\mathcal{B}_n$) and $B$ (measurable with respect to some $\sigma$-algebra $\mathcal{B}_B$), we say $|\mathcal{L}_{V_n} \mid Z^n (A_n | Z^n) - \mathcal{L}(B)| = o_{P_{Z_n}} (1)$ if for any $\delta > 0$, there exists a $N(\delta)$ such that

$$P_{Z_n} \left( \sup_{f \in BL_1} |E[f(A_n)]| \right) \leq \delta \geq 1 - \delta \quad \text{for all } n \geq N(\delta),$$

(i.e., $\sup_{f \in BL_1} |E[f(A_n)]| \leq \delta (1)$), where $BL_1$ denotes the class of uniformly bounded Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f(z) - f(z')| \leq |z - z'|$. See chapter 1.12 of Van der Vaart and Wellner (1996) (henceforth, VdV-W) for more details.

We say $\Delta_n$ is of order $o_{P_{V_n} | Z^n} (1)$ in $P_{Z_n}$ probability, and denote it as

$$\Delta_n = o_{P_{V_n} | Z^n} (1) \quad \text{wpa} 1 \quad P_{Z_n},$$

if for any $\epsilon > 0$, $P_{Z_n} \left( P_{V_n} | Z^n (|\Delta_n| > \epsilon | Z^n) > \epsilon \right) \to 0$ as $n \to \infty$.

We say $\Delta_n$ is of order $O_{P_{V_n} | Z^n} (1)$ in $P_{Z_n}$ probability, and denote it as

$$\Delta_n = O_{P_{V_n} | Z^n} (1) \quad \text{wpa} 1 \quad (P_{Z_n}),$$

if for any $\epsilon > 0$ there exists a $M \in (0, \infty)$, such that $P_{Z_n} \left( P_{V_n} | Z^n (|\Delta_n| > M | Z^n) > \epsilon \right) \to 0$ as $n \to \infty$.

### 5.1. Bootstrap local quadratic approximation (LQA$^B$)

Lemma A.1 in Appendix A shows that the bootstrap PSMD estimator $\hat{\alpha}_n^B \in \mathcal{N}_{osn}$ wpa1 under Assumptions A.1 and 3.1-3.4. This allows us to introduce a condition that is a bootstrap version of the LQA Assumption 3.6. For any $\alpha \in \mathcal{N}_{osn}$, we let $\hat{\alpha}_n^B(\alpha(t_n), \alpha) \equiv 0.5 \{ \hat{Q}_n^B(\alpha(t_n)) - \hat{Q}_n^B(\alpha) \}$ with $\alpha(t_n) \equiv \alpha + t_n u_n^*$ for $t_n \in T_n$. For any sequence of non-negative weights $(b_i)_i$, let

$$Z_n^\alpha \equiv n^{-1} \sum_{i=1}^n \frac{b_i}{\left( \frac{d\alpha}{d\alpha} \right)} \cdot \Sigma(X_i)^{-1} \rho(Z_i, \alpha_0) = n^{-1} \sum_{i=1}^n \frac{b_i}{\left\| u_n^* \right\|_{sd}}.$$
Assumption Boot.3 (LQA$^B$) (i) $\alpha(t) \in A_{k(n)}$ for any $(\alpha, t) \in N_{osn} \times T_n$, and with $r_n(t_n) = (\max\{t_n^2, t_n n^{-1/2}, o(n^{-1})\})^{-1}$,

$$
\sup_{(\alpha,t_n) \in N_{osn} \times T_n} r_n(t_n) \left| \hat{\Lambda}^B_n (\alpha(t_n), \alpha) - t_n \left\{ \frac{Z_{\omega, n}^n + \langle u^*_n, \alpha - \alpha_0 \rangle}{2} \right\} - B^\omega_n t_n^2 \right| = o_{P_{V_{\infty} | Z_{\infty}}}(1) \ \text{wpa1}(P_{Z_{\infty}})
$$

where $B^\omega_n$ is a $V^n$ measurable positive random variable such that $B^\omega_n = O_{P_{V_{\infty} | Z_{\infty}}}(1) \ \text{wpa1}(P_{Z_{\infty}})$;

(ii) $\left| \mathcal{L}_{V^n | Z^n} \left( \sqrt{n} \frac{Z_{\omega, n}^{n-1}}{\sigma_{\omega}} | Z^n \right) - \mathcal{L}(Z) \right| = o_{P_{Z_{\infty}}}(1),$

where $Z$ is a standard normal random variable.

Assumption Boot.3(i) implicitly imposes restrictions on the bootstrap estimator $\hat{m}^B(x, \alpha)$ of the conditional mean function $m(x, \alpha)$. Below we provide low level sufficient conditions for Assumption Boot.3(i) when $\hat{m}^B(x, \alpha)$ is a bootstrap series LS estimator.

Let $g(X, u^*_n) = \{ \frac{dm(X, \alpha_0)}{d\alpha} [u^*_n] \}' \Sigma(X)^{-1}$. Then $E [g(X_i, u^*_n) \Sigma(X_i) g(X_i, u^*_n)' ] = ||u^*_n||^2$.

Assumption B For $\Gamma(\cdot) \in \{ \Sigma(\cdot), \Sigma_0(\cdot) \}$,

$$
n^{-1} \sum_{i=1}^{n} g(X_i, u^*_n) \Gamma(X_i) g(X_i, u^*_n)' - E \left[ g(X_i, u^*_n) \Gamma(X_i) g(X_i, u^*_n)' \right] = o_{P_{Z_{\infty}}}(1).
$$

Lemma 5.1 Let Assumptions $3.1 - 3.4$ and $A.4 - A.7$ hold.

(1) Let $\hat{m}$ be the series LS estimator (2.5). Then Assumption 3.6(i) is satisfied. Further, if Assumption B holds then $B_n - ||u^*_n|| = o_{P_{Z_{\infty}}}(1)$.

(2) Let $\hat{m}^B(\cdot, \alpha)$ be the bootstrap series LS estimator (2.16), Assumption $A.4$ and either Assumption Boot.1 or Boot.2 hold. Then Assumption Boot.3(i) holds with $B^\omega_n = B_n$. Further, if Assumption B holds then $B^\omega_n - ||u^*_n|| = o_{P_{V_{\infty} | Z_{\infty}}}(1) \ \text{wpa1}(P_{Z_{\infty}})$. 


Lemma 5.1 indicates that the low level Assumptions A.4 - A.7 are sufficient for both the original-sample LQA Assumption 3.6(i) and the bootstrap LQA Assumption Boot.3(i).

Assumption Boot.3(ii) can be easily verified by applying some central limit theorems. For example, if the weights are independent (Assumption Boot.1), we can use Lindeberg-Feller CLT; if the weights are multinomial (Assumption Boot.2) we can apply Hayek CLT (see Van der Vaart and Wellner (1996) p. 458). The next lemma provides some simple sufficient conditions for Assumption Boot.3(ii).

**Lemma 5.2** Let either Assumption Boot.1 or Assumption Boot.2 hold. If there is a positive real sequence \( (b_n) \) such that \( b_n = o(\sqrt{n}) \) and

\[
\limsup_{n \to \infty} E \left[ \left( g(X, u_n^*) \rho(Z, \alpha_0) \right)^2 \right] < 1
\]

then Assumptions Boot.3(ii) and 3.6(ii) hold.

5.2. Bootstrap sieve Student t statistic

Lemma A.1 shows that \( \hat{\alpha}_n^B \in \mathcal{N}_{osn} \) wpa1 under virtually the same conditions as those for the original-sample estimator \( \hat{\alpha}_n \in \mathcal{N}_{osn} \) wpa1. This would easily lead to the consistency of the simplest bootstrap sieve t statistic \( \hat{W}_{1,n}^B \equiv \sqrt{n} \frac{\phi(\hat{\alpha}_n^B) - \phi(\hat{\alpha}_n)}{\sigma_{\hat{\alpha}_n}} \).

We now establish the consistency of another bootstrap sieve t statistic \( \hat{W}_{2,n}^B \equiv \sqrt{n} \frac{\phi(\hat{\alpha}_n^B) - \phi(\hat{\alpha}_n)}{||\hat{v}_n^*||_{B, sd}} \), where \( ||\hat{v}_n^*||_{B, sd} \) is a bootstrap sieve variance estimator:

\[
||\hat{v}_n^*||_{B, sd}^2 \equiv \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{d\alpha} \right)^T \hat{\Sigma}_i^{-1} \varphi(V_i, \hat{\alpha}_n) \varphi(V_i, \hat{\alpha}_n)^T \hat{\Sigma}_i^{-1} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{d\alpha} \right)
\]

with \( \varphi(V_i, \alpha) \equiv (\omega_i,n - 1) \rho(Z_i, \alpha) \equiv \rho^B(V_i, \alpha) - \rho(Z_i, \alpha) \) for any \( \alpha \).
We note that \(|\hat{v}\|^2_{B,sd}\) is an analog to \(|\hat{v}\|^2_{n,sd}\) defined in (4.7) but using the bootstrapped generalized residual \(\hat{\rho}(V_i, \hat{\alpha}_n)\) instead of the original sample fitted residual \(\rho(Z_i, \hat{\alpha}_n)\). It also has a closed form expression:

\[
|\hat{v}\|^2_{B,sd} = \hat{\mathcal{V}}' \hat{D} - \hat{\mathcal{M}} \hat{D} \hat{\mathcal{V}}
\]

where \(\hat{\mathcal{M}} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d\bar{m}(X_i, \hat{\alpha}_n)}{d\alpha} \left[ \psi_{\hat{\alpha}_n}(\cdot) \right]' \right) \omega_{i,n} (\omega_{i,n} - 1)^2 \hat{M}_i \left( \frac{d\bar{m}(X_i, \hat{\alpha}_n)}{d\alpha} \left[ \psi_{\hat{\alpha}_n}(\cdot) \right]' \right)^2 \hat{M}_i \).

This assumption can be verified given Assumptions Boot.1 or Boot.2. The following result is a bootstrap version of Theorem 4.2(1).

**Theorem 5.1** Let Assumptions 3.1 - 3.4, 4.1 and Boot.4 hold. Then:

\[
\left| \frac{|\hat{v}\|^2_{B,sd} - 1}{\sigma_{\hat{v}}|v\|^2_{n,sd}} \right| = o_{P_{\infty|Z|\infty}}(1) \text{ wpa1}(P_{Z^\infty})
\]

Recall that \(\hat{W}_n = \sqrt{n} \frac{\phi(\hat{\alpha}_n) - \phi(\alpha)}{\sigma_{\hat{V}}|V||n,sd|}\), whose probability distribution \(P_{Z^\infty}\left(\hat{W}_n \leq \cdot \right)\) converges to the standard normal cdf \(\Phi(\cdot)\). The next result is about the consistency of the bootstrap sieve t statistic \(\hat{W}_{2,n}\).

**Theorem 5.2** Let \(\hat{\alpha}_n\) be the PSMD estimator (2.2) and \(\hat{\alpha}_B^n\) the bootstrap PSMD estimator. Let Assumptions 3.1 - 3.4 and A.1 hold. Let Assumptions 3.5, 3.6 and Boot.3 hold.

(1) Let Assumptions 4.1 and Boot.4 hold. Then:

\[
\sup_{t \in \mathbb{R}} \left| P_{V|Z|\infty}\left(\hat{W}^B_{2,n} \leq t \mid Z\right) - P_{Z^\infty}\left(\hat{W}_n \leq t \right) \right| = o_{P_{V|Z|\infty}}(1) \text{ wpa1}(P_{Z^\infty}).
\]
(2) If \(\phi()\) is regular at \(\alpha_0\), without imposing Assumptions 4.1 and Boot.4, we have:
\[
\sup_{t \in \mathbb{R}} \left| P_{\psi_\infty | Z_\infty} \left( \frac{\sqrt{n} (\hat{\phi}_n) - \phi(\hat{\alpha}_n)}{\sigma_\omega} \leq t \mid Z^n \right) - P_{Z_\infty} \left( \sqrt{n} (\phi(\hat{\alpha}_n) - \phi(\alpha_0)) \leq t \right) \right| = o_{P_{\psi_\infty | Z_\infty}}(1) \text{ wpa1}(P_{Z_\infty}).
\]

For a regular functional, Theorem 5.2(2) provides one way to construct its confidence sets without the need to compute any variance estimator. This extends the result in Chen and Pouzo (2009) for a regular Euclidean parameter \(\lambda^T \theta\) to a general regular functional \(\phi(\alpha)\). Unfortunately for an irregular functional, we need to compute a consistent bootstrap sieve variance estimator \(||\hat{v}_n^*|||_{B,sd}^2\) to apply Theorem 5.2(1). Luckily \(||\hat{v}_n^*|||_{B,sd}^2\) is easy to compute when the residual function \(\rho(Z_i, \alpha)\) is pointwise smooth in \(\alpha_0\). Moreover, since \(E \left( ||\hat{v}_n^*||^2_{B,sd} \mid Z^n \right) = \sigma_\omega^2 ||\hat{v}_n^*||^2_{n,sd}\) we suspect that the bootstrap sieve t statistic \(\hat{W}_B^2,n\) might have second order refinement property by choices of bootstrap weights \{\omega_{i,n}\}. This will be a subject of future research.

The bootstrap sieve t statistic \(\hat{W}_B^2,n\) requires to compute the original sample PSMD estimator \(\hat{\alpha}_n\) and the bootstrap PSMD estimator \(\hat{\alpha}_{B,n}\). In the online Appendix D we present a sieve score test and its bootstrap version, which only use the original sample restricted PSMD estimator \(\hat{\alpha}_{R,n}\) and do not use \(\hat{\alpha}_{B,n}\), and hence are computationally simple.

**Remark 5.1** Theorems 4.2(2) and 5.2(1) imply that the bootstrap Wald test statistic \(W_{2,n}^B \equiv \left( \hat{W}_{2,n}^B \right)^2\) always has the same limiting distribution \(\chi^2_1\) (conditional on the data) under the null and the alternatives. Let \(\hat{c}_{2,n}(\alpha)\) be the \(a - \text{th}\) quantile of the distribution of \(W_{2,n}^B\) (conditional on the data \(\{Z_i\}_{i=1}^n\)). Let \(W_n \equiv \left( \sqrt{n} \frac{\phi(\hat{\alpha}_n) - \phi(\alpha_0)}{||\hat{v}_n^*||^2_{n,sd}} \right)^2\) be the original sample Wald test statistic. Then Remark 4.1 and Theorem 5.2(1) immediately imply that for any \(\tau \in (0, 1)\),
\[
\text{under } H_0 : \phi(\alpha_0) = \phi_0, \lim_{n \to \infty} \Pr(W_n \geq \hat{c}_{2,n}(1 - \tau)) = \tau;
\]
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under \( H_1 : \phi(\alpha_0) \neq \phi_0 \), \( \lim_{n \to \infty} \Pr (\mathcal{W}_n \geq \hat{c}_{2,n}(1 - \tau)) = 1 \).

See Theorem A.4 in Appendix A for properties under local alternatives.

See online supplemental Appendix B for consistency of \( \mathcal{W}^B_{1,n} \equiv (\sqrt{n} \phi(\hat{\alpha}_n) - \hat{\phi}_n)^2 \) and other bootstrap sieve Wald (t) statistics based on different sieve variance estimators.

5.3. Bootstrap SQLR statistic

If \( \Sigma \neq \Sigma_0 \), the SQLR statistic \( \widehat{QLR}_n(\phi_0) = n \left( \tilde{Q}_n(\hat{\alpha}_{\Sigma}^R,B) - \tilde{Q}_n(\hat{\alpha}_{\Sigma}^B) \right) \) is no longer asymptotically chi-square even under the null; Theorem 4.3(1), however, implies that the SQLR statistic converges weakly to a tight limit under the null. In this subsection we show that the asymptotic null distribution of the SQLR can be consistently approximated by that of the (generalized residual) bootstrap SQLR statistic \( \widehat{QLR}^B_n(\hat{\phi}_n) \). Recall that

\[
\widehat{QLR}^B_n(\hat{\phi}_n) = n \left( \tilde{Q}_n(\hat{\alpha}_{\Sigma}^{R,B}) - \tilde{Q}_n(\hat{\alpha}_{\Sigma}^B) \right) + o_{P_{V:\Xi} \to \infty} \left( \frac{1}{n} \right) \text{wpa1}(P_{Z^\infty})
\]

where \( \hat{\phi}_n \equiv \phi(\hat{\alpha}_n) \), and \( \hat{\alpha}_{\Sigma}^{R,B} \) is the restricted bootstrap PSMD estimator, defined as

\[
(5.3) \tilde{Q}_n(\hat{\alpha}_{\Sigma}^{R,B}) + \lambda_n \text{Pen}(\hat{h}_{\Sigma}^{R,B}) \leq \inf_{\alpha \in \mathcal{A}_{k(\alpha)}: \phi(\alpha) = \hat{\phi}_n} \left\{ \tilde{Q}_n(\alpha) + \lambda_n \text{Pen}(h) \right\}
\]

(5.4)

Lemma A.1 in Appendix A implies that \( \hat{\alpha}_{\Sigma}^{R,B} \), \( \hat{\alpha}_{\Sigma}^B \in \mathcal{N}_{\text{osn}} \) wpa1 under both the null \( H_0 : \phi(\alpha_0) = \phi_0 \) and the alternatives \( H_1 : \phi(\alpha_0) \neq \phi_0 \). This indicates that the bootstrap SQLR statistic \( \widehat{QLR}^B_n(\hat{\phi}_n) \) is always properly centered and should be stochastically bounded under both the null and the alternatives, as shown in the next theorem. Let \( P_{Z^\infty} \left( \widehat{QLR}_n(\phi_0) \leq \cdot | H_0 \right) \) denote the probability distribution of \( \widehat{QLR}_n(\phi_0) \) under the null \( H_0 : \phi(\alpha_0) = \phi_0 \), which would converge to the cdf of \( \chi^2_1 \) when \( \widehat{QLR}_n(\phi_0) = \widehat{QLR}^0_n(\phi_0) \) (the optimally weighted SQLR).
Theorem 5.3 Let Assumptions 3.1 - 3.4 and A.1 hold. Let Assumptions 3.5, 3.6 and Boot.3 hold with \( |B^\omega_n - ||u^*_n||^2| = o_{P_{V^\infty|Z^\infty}}(1) \) wp1(\( P_{Z^\infty} \)). Then:

\[
(1) \quad \frac{QLR^B_n(\hat{\phi}_n)}{\sigma^2_\omega} = \left( \sqrt{n} \frac{Z^{\omega-1}_n}{\sigma_\omega ||u^*_n||} \right)^2 + o_{P_{V^\infty|Z^\infty}}(1) = O_{P_{V^\infty|Z^\infty}}(1) \text{ wp1}(P_{Z^\infty});
\]

and

\[
(2) \quad \sup_{t \in \mathbb{R}} \left| P_{P_{V^\infty|Z^\infty}} \left( \frac{QLR^B_n(\hat{\phi}_n)}{\sigma^2_\omega} \leq t \mid Z^n \right) - P_{Z^\infty} \left( \frac{QLR_n(\phi_0)}{\sigma^2_\omega} \leq t \mid H_0 \right) \right| = o_{P_{V^\infty|Z^\infty}}(1) \text{ wp1}(P_{Z^\infty}).
\]

Theorem 5.3 allows us to construct valid confidence sets (CS) for \( \phi(\alpha_0) \) based on inverting possibly non-optimally weighted SQLR statistic without the need to compute a variance estimator. We recommend this procedure when it is difficult to compute any consistent variance estimator for \( \hat{\phi}(\hat{\alpha}) \), such as in the cases when the residual function \( \rho(Z; \alpha) \) is pointwise non-smooth in \( \alpha_0 \). See, e.g., Andrews and Buchinsky (2000) for a thorough discussion about how to construct CS via bootstrap.

Remark 5.2 Let \( \hat{c}_n(a) \) be the \( a \) th quantile of the distribution of \( \frac{QLR_n(\hat{\phi}_n)}{\sigma^2_\omega} \) (conditional on the data \( \{Z_i\}_{i=1}^n \)). Then Theorems 4.3, 4.4 and 5.3 immediately imply that for any \( \tau \in (0, 1) \),

under \( H_0 : \phi(\alpha_0) = \phi_0 \), \( \lim_{n \to \infty} \Pr \left( \frac{QLR_n(\phi_0)}{\sigma^2_\omega} \geq \hat{c}_n(1 - \tau) \right) = \tau; \)

under \( H_1 : \phi(\alpha_0) \neq \phi_0 \), \( \lim_{n \to \infty} \Pr \left( \frac{QLR_n(\phi_0)}{\sigma^2_\omega} \geq \hat{c}_n(1 - \tau) \right) = 1. \)

See Theorem A.2 in Appendix A for properties under local alternatives.

6. VERIFICATION OF ASSUMPTIONS 3.5 AND 3.6

In this section, we illustrate the verification of the two key regularity conditions, Assumption 3.5 and Assumption 3.6(i), via some functionals \( \phi(h) \) of the (nonlinear) nonparametric IV regressions:

\[
(6.1) \quad E[\rho(Y_1; h_0(Y_2))|X] = 0 \quad a.s. - X,
\]
where the scalar valued residual function $\rho()$ could be nonlinear and pointwise non-smooth in $h$. This model includes the NPIV and NPQIV as special cases. To be concrete, we consider a PSMD estimator $\hat{h} \in \mathcal{H}_{k(n)}$ of $h_0$ with $\hat{\Sigma} = \Sigma = 1$, and $\hat{m}(\cdot, h)$ being the series LS estimator (2.5) of $m(\cdot, h) = E[\rho(Y_1; h(Y_2))|X = \cdot]$ with $J_n = ck(n)$ for a finite constant $c \geq 1$. We assume that $h_0 \in \mathcal{H} = \Lambda_c^\varsigma([-1, 1])$ with smoothness $\varsigma > 1/2$. This Hölder ball condition and several other conditions assumed in this subsection are for illustration only, and can be replaced by weaker sufficient conditions.

It is well known that the convergence rates of $\hat{h}$ to $h_0$ in both $\| \cdot \|$ and $\| \cdot \|_s$ metrics have already been established in Chen and Pouzo (2012a), and hence will not be repeated here.

We use $\mathcal{H}_{os}$ and $\mathcal{H}_{osn}$ for $\mathcal{A}_{os}$ and $\mathcal{A}_{osn}$ defined in Subsection 3.1 (since there is no $\theta$ here). Denote $T \equiv \frac{dm(h)}{dt} : \mathcal{H}_{os} \subset L^2(f_{Y_2}) \rightarrow L^2(f_X)$, i.e., for any $h \in \mathcal{H}_{os} \subset L^2(f_{Y_2})$,

$$Th \equiv \frac{dE[\rho(Y_1; h_0(Y_2) + \tau h(Y_2))|X = \cdot]}{d\tau}igg|_{\tau = 0}.$$  

Let $T^*$ be the adjoint of $T$. Then for all $h \in \mathcal{H}_{os}$, we have $\|h\|^2 \equiv \|Th\|_{L^2(f_X)}^2 = \|(T^*T)^{1/2}h\|_{L^2(f_{Y_2})}^2$. Under mild conditions as stated in Chen and Pouzo (2012a), $T$ and $T^*$ are compact. Then $T$ has a singular value decomposition $\{\mu_k; \psi_k, \phi_0\}_{k=1}^\infty$, where $\{\mu_k > 0\}_{k=1}^\infty$ is the sequence of singular values in non-increasing order ($\mu_k \geq \mu_{k+1} \geq \ldots$) with $\text{lim inf}_{k \rightarrow \infty} \mu_k = 0$, $\{\psi_k \in L^2(f_{Y_2})\}_{k=1}^\infty$ and $\{\phi_0 \in L^2(f_X)\}_{k=1}^\infty$ are sequences of eigenfunctions of the operators $(T^*T)^{1/2}$ and $(TT^*)^{1/2}$:

$$T\psi_k = \mu_k \phi_0, \quad (T^*T)^{1/2}\psi_k = \mu_k \psi_k \quad \text{and} \quad (TT^*)^{1/2}\phi_0 = \mu_k \phi_0 \quad \text{for all } k.$$  

Since $\{q_k\}_{k=1}^\infty$ is a Riesz basis of $(\mathcal{H}, \| \cdot \|_s)$ we could also have $\mathcal{H}_{k(n)} = \text{clsp}\{\psi_1, \ldots, \psi_{k(n)}\}$. The sieve measure of local ill-posedness now becomes
\[ \tau_n = \mu_{k(n)}^{-1} \] (see, e.g., Blundell et al. (2007) and Chen and Pouzo (2012a), and hence \( \| u_{n,s} \| \leq c \mu_{k(n)}^{-1} \) for a finite constant \( c > 0 \). Also, \( \Pi_n h_0 \equiv \arg \min_{h \in \mathcal{H}_{k(n)}} \| h - h_0 \|_s = \sum_{k=1}^{k(n)} \langle h_0, \psi_k \rangle_s \psi_k \) is the LS projection of \( h_0 \) onto the sieve space \( \mathcal{H}_n \) under the strong norm \( \| \cdot \|_s = \| \cdot \|_{L^2(f_Y^2)} \). Recall that \( h_{0,n} \equiv \arg \min_{h \in \mathcal{H}_{k(n)}} \| h - h_0 \|_s = \Pi_n h_0 \).

The next remark specializes Theorem 4.1 to a general functional \( \phi(h) \) of the model (6.1).

**Remark 6.1** Let \( \hat{m} \) be the series LS estimator (2.5) for the model (6.1) with \( \hat{\Sigma} = \Sigma = 1 \), and Assumptions 3.1(i)(ii), 3.2(ii)(iii), and 3.4 hold with \( \delta_n = O \left( \sqrt{\frac{k(n)}{n}} \right) = o(n^{-1/4}) \) and \( \delta_{s,n} = O \left( \{ k(n) \}^{-\zeta} + \mu_{k(n)}^{-1} \sqrt{\frac{k(n)}{n}} \right) = o(1) \). Let Assumption 3.3, equation (3.12) and Assumptions A.4 - A.7 hold. Then:

\[
\frac{\sqrt{n} \phi(\hat{m}_n) - \phi(h_0)}{\| v_{n,s} \|_{sd}} \Rightarrow N(0,1),
\]

with \( \| v_{n,s} \|_{sd}^2 = \left( \frac{d \phi(h_0)}{dh} [q^{k(n)}(\cdot)]' \right)' D_n^{-1} \mathcal{U}_n D_n^{-1} \left( \frac{d \phi(h_0)}{dh} [q^{k(n)}(\cdot)]' \right) \), and \( D_n = E \left[ (T[q^{k(n)}(\cdot)]')' (T[q^{k(n)}(\cdot)]') \right] \) and \( \mathcal{U}_n = E \left[ (T[q^{k(n)}(\cdot)]')' \rho(Z, h_0)^2 (T[q^{k(n)}(\cdot)]') \right] \).

**Remark 6.1** includes the NPIV and NPQIV examples in Subsection 2.2 as special cases. In particular, the sieve variance expression (6.3) reproduces the one for the NPIV model (2.18) with \( T[q^{k(n)}(\cdot)]' = E[q^{k(n)}(Y_2)'|X] \), and the one for the NPQIV model (2.22) with \( T[q^{k(n)}(\cdot)]' = E[f_{U|Y_2,X}(0)q^{k(n)}(Y_2)'|X] \).

By the result in Chen and Pouzo (2012a), the sieve dimension \( k_n^* \) satisfying \( \{ k_n^* \}^{-\zeta} \times \mu_{k(n)}^{-1} \times \sqrt{\frac{k_n^*}{n}} \) leads to the nonparametric optimal convergence rate of \( \| \hat{m} - h_0 \|_s \equiv O_{p_{2\alpha}}(\delta_{s,n}^*) = o(1) \) in strong norm, where
\( \delta_{s,n} \asymp \{k_n^*\}^{-\varsigma}. \) In particular, \( k_n^* \asymp n^{\frac{1}{2(\varsigma + a) + 1}} \) and \( \delta_{s,n}^* = n^{-\frac{\varsigma}{2(\varsigma + a) + 1}} \) for the mildly ill-posed case \( \mu_k \asymp k^{-a} \) for a finite \( a > 0 \); and \( \delta_{s,n}^* = \{\ln n\}^{-\varsigma} \) for the severely ill-posed case \( \mu_k \asymp \exp\{-0.5ak\} \) for a finite \( a > 0 \). However this paper aims at simple valid inferences on functional \( \phi(h_0) \). As will be illustrated in the next subsection, although the nonparametric optimal choice \( k_n^* \) is compatible with the sufficient conditions for the asymptotic normality of \( \sqrt{n}(\hat{\phi}(\hat{h}) - \phi(h_0)) \) for a regular linear functional \( \phi(h_0) \) (see Remark 3.1), it is typically ruled out by Assumption 3.5(iii) for irregular functionals.

6.1. Verification of Assumption 3.5

Let \( b_j \equiv \frac{d\phi(h_0)}{dh}[\psi_j(\cdot)] \) for all \( j \). By Lemma 4.1

\[ D_n = E \left[ \left(T[q^{k(n)}(\cdot)'] \right)' \left(T[q^{k(n)}(\cdot)'] \right) \right] = \text{Diag} \left\{ \mu_1^2, \ldots, \mu_{k(n)}^2 \right\} \]

and

\[ \|v_n^*\|^2 = \left( \frac{d\phi(h_0)}{dh}[q^{k(n)}(\cdot)] \right)' D_n^{-1} \left( \frac{d\phi(h_0)}{dh}[q^{k(n)}(\cdot)] \right) = \sum_{j=1}^{k(n)} \mu_j^{-2} b_j^2. \]  

(6.4)

By Lemma 3.3, \( \phi(h) \) of the model (6.1) is regular (at \( h = h_0 \)) iff \( \sum_{j=1}^{\infty} \mu_j^{-2} b_j^2 < \infty \), and is irregular (at \( h = h_0 \)) iff \( \sum_{j=1}^{\infty} \mu_j^{-2} b_j^2 = \infty \).

For the same functional \( \phi(h) \) of a model (6.5) without endogeneity:

\[ E[\rho(Y_1; h_0(Y_2))|Y_2] = 0 \quad a.s. - Y_2, \]

we have \( D_n \asymp I_{k(n)} \) and \( \|v_n^*\|^2 \asymp \sum_{j=1}^{k(n)} b_j^2 \). Thus, \( \phi(h) \) of the model (6.5) is regular (or irregular) iff \( \sum_{j=1}^{\infty} b_j^2 < \infty \) (or \( = \infty \)).

Since \( \mu_{k(n)} \rightarrow 0 \) as \( k(n) \rightarrow \infty \), if a functional \( \phi(h) \) is irregular for the model (6.5) without endogeneity, then it is irregular for the model (6.1). But, even if a functional \( \phi(h) \) is regular for the model (6.5) without endogeneity, it could still be irregular for the model (6.1) with endogeneity.
6.1.1. Linear functionals of the model (6.1)

For a linear functional $\phi(h)$ of the model (6.1), given relation (6.2), Assumption 3.5 is satisfied provided that the sieve dimension $k(n)$ satisfies (6.6):

\[
\frac{\|v_n^*\|}{\sqrt{n}} = o(1) \quad \text{and} \quad \sqrt{n} \frac{\left| \frac{d\phi(h_0)}{dh}[\Pi_nh_0 - h_0] \right|}{\|v_n^*\|} = o(1).
\]

When $\phi(h)$ of the model (6.1) is regular, Remark 3.1 implies that (6.6) is satisfied provided

\[
\sum_{j=1}^{\infty} \mu_j^{-2}b_j^2 < \infty \quad \text{and} \quad n \times \sum_{j=k(n)+1}^{\infty} \mu_j^{-2}b_j^2 \times \|\Pi_nh_0 - h_0\|^2 = o(1).
\]

We shall illustrate below that both these sufficient conditions allow for severely ill-posed problems.

**Example 1 (evaluation functional).** For $\phi(h) = h(\bar{y}_2)$, we have:

\[
\|v_n^*\|^2 = \sum_{j=1}^{k(n)} \mu_j^{-2}[\psi_j(\bar{y}_2)]^2. \quad \text{Let } H_{k(n)} \text{ be the spline or the CDV wavelet sieve as described in Chen and Christensen (2013), say. Then}
\]

\[
\left| \frac{d\phi(h_0)}{dh}[\Pi_nh_0 - h_0] \right| = \|(\Pi_nh_0)(\bar{y}_2) - h_0(\bar{y}_2)\| \leq \|\Pi_nh_0 - h_0\|_{\infty} \leq \text{const.} \{k(n)\}^{-\varsigma}.
\]

To provide concrete sufficient condition for (6.6), we assume $\|v_n^*\|^2 \asymp E\left(\sum_{j=1}^{k(n)} \mu_j^{-2}[\psi_j(Y_2)]^2\right) = \sum_{k=1}^{k(n)} \mu_k^{-2}$. Since $\lim_{k(n) \to \infty} \|v_n^*\|^2 = \infty$, the evaluation functional is irregular. Condition (6.6) is satisfied provided that

\[
\frac{\|v_n^*\|^2}{n} = \frac{\sum_{k=1}^{k(n)} \mu_k^{-2}}{n} = o(1) \quad \text{and} \quad \frac{1}{n} \left\{ k(n) \right\}^{-2\varsigma} \frac{1}{n} \sum_{k=1}^{k(n)} \mu_k^{-2} = o(1).
\]

Condition (6.8) allows for both mildly and severely ill-posed cases.
(a) Mildly ill-posed: \( \mu_k \propto k^{-a} \) for a finite \( a > 0 \). Then \( \|v_n^*\|^2 \propto \{k(n)\}^{2a+1} \). Condition (6.8) is satisfied by a wide range of sieve dimensions, such as \( k(n) \propto n^{2(\xi + a+1)/\alpha} \) or \( n^{2(\xi + a+1)/(\alpha + 1)} \) for any finite \( \alpha > 0 \), or \( k(n) \propto n^\epsilon \) for any \( \epsilon \in (2(\xi + a+1)/\alpha, 2a+1) \). Note that any \( k(n) \) satisfying Condition (6.8) also ensures \( \delta_{s,n} = o(1) \). However, it does require \( k(n)/k_n^* \to \infty \), where \( k_n^* \propto n^{2(\xi + a+1)/(\alpha + 1)} \) is the choice for the nonparametric optimal convergence rate in strong norm.

(b) Severely ill-posed: \( \mu_k \propto \exp\{-0.5ak\} \) for a finite \( a > 0 \). Then \( \|v_n^*\|^2 \propto \exp\{ak(n)\} \). Condition (6.8) is satisfied with \( k(n) \propto a^{-1} [\ln n - \alpha \ln(\ln n)] \) for \( 0 < \alpha < 2\xi \). In addition we need \( \alpha > 1 \) (and hence \( \xi > 1/2 \)) to ensure \( \delta_{s,n} = O(\{k(n)\}^{-\epsilon} + k_{k(n)}^{-1} \sqrt{k(n)/n}) = o(1) \).

Example 2 (weighted derivative functional). For \( \phi(h) = \int w(y)v h(y)dy \), where \( w(y) \) is a weight satisfying the integration by part formula: \( \phi(h) = \int w(y)\nabla h(y)dy = -\int h(y)\nabla w(y)dy \), we have: \( \|v_n^*\|^2 = \sum_{j=1}^{k(n)} \mu_j^{-2} b_j^2 \) with \( b_j = \int \psi_j(y)\nabla w(y)dy \) for all \( j \), and

\[
\frac{d\phi(h_0)}{dh} \left[ \Pi_nh_0 - h_0 \right] = \left[ \int \left[ \Pi_nh_0(y) - h_0(y) \right]\nabla w(y)dy \right] \leq C \times \|\Pi_nh_0 - h_0\|_{L^2(V_\gamma)} \leq const.\{k(n)\}^{-\epsilon}
\]

provided that \( E \left( \left[ \frac{\nabla w(y_{\gamma 2})}{f_{V_\gamma}(Y_{\gamma 2})} \right]^2 \right) = \sum_{j=1}^{\infty} b_j^2 = C < \infty \). That is, the weighted derivative is assumed to be regular for the model (6.5) without endogeneity.

(i) When the weighted derivative is regular (i.e., \( \sum_{j=1}^{\infty} \mu_j^{-2} b_j^2 < \infty \)) for the model (6.1), Condition (6.7) is satisfied provided that \( n \times \sum_{j=k(n)+1}^{\infty} \mu_j^{-2} b_j^2 \times \delta_{s,n}^2 = o(1) \), which is the condition imposed in Ai and Chen (2007) for their root-n estimation of an average derivative of NPIV example, and is shown to allow for severely ill-posed inverse case in Ai and Chen (2007).

(ii) When the weighted derivative is irregular (i.e., \( \sum_{j=1}^{\infty} \mu_j^{-2} b_j^2 = \infty \)
for the model \( (6.1) \), Condition \( (6.6) \) is satisfied provided that
\[
\frac{||v^*_n||^2}{n} = \frac{\sum_{j=1}^{k(n)} \mu_j^{-2}b_j^2}{n} = o(1) \quad \text{and} \quad \frac{1}{n} ||v^*_n||^2 = \frac{1}{n} \frac{\sum_{j=1}^{k(n)} \mu_j^{-2}b_j^2}{n} = o(1).
\]
Condition \( (6.9) \) allows for both mildly and severely ill-posed cases. To provide concrete sufficient conditions for \( (6.9) \) we assume \( b_j^2 \approx (j \ln(j))^{-1} \) in the following calculations.

(a) Mildly ill-posed: \( \mu_k \approx k^{-a} \) for a finite \( a > 0 \). Then \( ||v^*_n||^2 \in [c \frac{k(n)^{a_0}}{\ln(k(n))}, c' k(n)^{2a}] \) for some \( 0 < c \leq c' < \infty \). Condition \( (6.9) \) and \( \delta_{s,n} = o(1) \) are jointly satisfied by a wide range of sieve dimensions, such as \( k(n) \approx n^{\frac{1}{2(\varsigma + a)}}(\ln n)^{\omega} \) for any finite \( \omega > \frac{1}{2(\varsigma + a)} \), or \( k(n) \approx n^\epsilon \) for any \( \epsilon \in (\frac{1}{2(\varsigma + a)}, \frac{1}{2a + 1}) \) and \( \varsigma > 1/2 \).

(b) Severely ill-posed: \( \mu_k \approx \exp\{-0.5ak\} \) for \( a > 0 \). Then \( ||v^*_n||^2 \in [c \exp\{ak(n)\}, c' \exp\{ak(n)\}] \) for some \( 0 < c \leq c' < \infty \). Condition \( (6.9) \) and \( \delta_{s,n} = o(1) \) are jointly satisfied by \( k(n) \approx a^{-1}[\ln(n) - \omega \ln(\ln(n))] \) for \( \omega \in (1, 2\varsigma - 1) \) and \( \varsigma > 1 \).

6.1.2. Nonlinear functionals

For a nonlinear functional \( \phi(h) \) of the model \( (6.1) \), Assumption 3.5 is satisfied provided that the sieve dimension \( k(n) \) satisfies \( (6.6) \) (or \( (6.7) \) if \( \phi(h) \) is regular) and Assumption \( 3.5\text{(ii)} \), which is implied by the following condition:

**Assumption 3.5(ii)**: there are finite non-negative constants \( C \geq 0, \omega_1, \omega_2 \geq 0 \) such that for all \( (\alpha, t) \in N_{osn} \times T_n \),
\[
\left| \phi(\alpha + tu^*_n) - \phi(\alpha_0) - \frac{d\phi(\alpha_0)}{d\alpha} [\alpha + tu^*_n - \alpha_0] \right| \leq C \times (||\alpha - \alpha_0 + tu^*_n||^{\omega_1} \times ||\alpha - \alpha_0 + tu^*_n||^{\omega_2},
\]
and
\[
C \times \sqrt{n} \times (\delta_n(1 + M_n^2))^{\omega_1} \times (\delta_{s,n} + M_n^2 \delta_n ||u^*_n||_s)^{\omega_2} ||v^*_n|| = o(1).
\]
Assumption 3.5(ii) or (ii)’ controls the nonlinearity bias of φ(·) (i.e., the linear approximation error of a nonlinear functional φ(·)). It typically rules out nonlinear regular functionals of severely illposed inverse problems, but allows for nonlinear irregular functionals of severely illposed inverse problems.

Example 3 (weighted quadratic functional). For \( \phi(h) = \frac{1}{2} \int w(y) |h(y)|^2 \, dy \), we have

\[
|v_n^*|^2 = \sum_{j=1}^{k(n)} \mu_j^{-2} b_j^2
\]

with \( b_j = \int h_0(y) w(y) \psi_j(y) \, dy \) for all \( j \), and

\[
\left| \frac{d\phi(h_0)}{dh} - \Pi_n h_0 - h_0 \right| = \left| \int w(y) h_0(y) [\Pi_n h_0(y) - h_0(y)] \, dy \right| \leq \text{const.} \times ||\Pi_n h_0 - h_0||_{L^2(f_{Y_2})}
\]

provided that \( \sup_y \frac{w(y)}{f_{Y_2}(y)} < \infty \). This and \( E \left( [h_0(Y_2)]^2 \right) < \infty \) imply that \( \sum_{j=1}^\infty b_j^2 < \infty \). That is, the weighted quadratic functional is regular for the model (6.5) without endogeneity. Also,

\[
\left| \phi(h) - \phi(h_0) - \frac{d\phi(h_0)}{dh} [h - h_0] \right| = \frac{1}{2} \int w(y) |h(y) - h_0(y)|^2 \, dy \leq \text{const.} \times ||h - h_0||^2_{L^2(f_{Y_2})}.
\]

(i) When the weighted quadratic functional is regular (i.e., \( \sum_{j=1}^\infty \mu_j^{-2} b_j^2 < \infty \)) for the model (6.1), Condition (6.7) is satisfied provided that \( n \times \sum_{j=k(n)+1}^\infty \mu_j^{-2} b_j^2 \times \delta_n^2 = o(1) \), which allows for severely ill-posed cases. But Assumption 3.5(ii)’ requires that \( \sqrt{n} \times \delta^2_{s,n} = \sqrt{n} \times \left( \{ k(n) \}^{-\epsilon} + \mu_{k(n)} \right) \sqrt{\frac{k(n)}{n}} = o(1) \), which clearly rules out severely ill-posed inverse case where \( \mu_k \approx \exp\{-0.5ak\} \) for some finite \( a > 0 \).

(ii) When the weighted quadratic functional is irregular (i.e., \( \sum_{j=1}^\infty \mu_j^{-2} b_j^2 = \infty \)) for the model (6.1), Condition (6.6) is satisfied provided that Condition (6.9) holds with \( b_j = \int h_0(y) w(y) \psi_j(y) \, dy \) for Example 3. Assumption 3.5(ii)’ is satisfied provided that

\[
\sqrt{n} \frac{\delta_{s,n}^2}{||v_n^*||} = \frac{\sqrt{n} \times \left( \{ k(n) \}^{-\epsilon} + \mu_{k(n)} \right) \sqrt{k(n)} \sqrt{n}}{||v_n^*||} \leq n^{-1/2} \frac{\mu_{k(n)}^{-2} k(n)}{\sqrt{\sum_{j=1}^\infty \mu_j^{-2} b_j^2}} = o(1).
\]
Any $k(n)$ satisfying Conditions (6.9) and (6.10) automatically satisfies $\delta_{s,n} = o(1)$. In addition, both conditions allow for mildly and severely ill-posed cases. To provide concrete sufficient conditions we assume $b_j^2 \asymp (j \ln(j))^{-1}$ in the following calculations.

(a) Mildly ill-posed: $\mu_k \asymp k^{-a}$ for a finite $a > 0$. Then $||v_n^*||^2 \in [c\exp\{ak(n)\}, c'\exp\{ak(n)\}]$ for some $0 < c \leq c' < \infty$. Conditions (6.9) and (6.10) are satisfied by a wide range of sieve dimensions, such as $k(n) \asymp n^{\frac{\varsigma}{2}a}$ for any finite $\varsigma > 2$. Here, we present simple (albeit somewhat strong) sufficient conditions for Assumptions A.6 and A.7 for irregular functionals of the NPIV and NPQIV examples.

6.2. Verification of Assumption 3.6(i)

By Lemma 5.1(1), to verify Assumption 3.6(i), it suffices to verify Assumptions A.4 - A.7 in Appendix A. Note that Assumptions A.4 and A.5 do not depend on sieve Riesz representer at all, and have already been verified in Chen and Pouzo (2009), Ai and Chen (2007) and others for (penalized) SMD estimators for the model (6.1). Assumptions A.6 and A.7 do depend on the scaled sieve Riesz representer $u_n^* \equiv v_n^* / ||v_n^*||_{sd}$. Both these assumptions are also verified in Ai and Chen (2003), Chen and Pouzo (2009), Ai and Chen (2007) for examples of regular functionals of the model (6.1). Here, we present simple (albeit somewhat strong) sufficient conditions for Assumptions A.6 and A.7 for irregular functionals of the NPIV and NPQIV examples.

**Condition 6.1** (i) $\{E[h(Y_2)|\cdot] : h \in \mathcal{H}\} \subseteq \Lambda_\gamma^1(\mathcal{X})$, with $\gamma > 0.5$; (ii) $\sup_{x,y_2} \frac{\mathcal{f}_{y_2|x}}{\mathcal{f}_{y_2|y_2}} \leq Const. < \infty$. 

Proposition 6.1 Let all conditions for Remark 6.1 hold. Under Condition 6.1 Assumptions A.6 and A.7 hold for the NPIV model (2.18).

Proposition 6.1 allows for irregular functionals of the NPIV model with severely ill-posed case.

Condition 6.2 (i) \( \{ E[F_{Y_1|Y_2,X}(h(Y_2),Y_2,\cdot)|\cdot] : h \in \mathcal{H} \} \subseteq \Lambda_c^\gamma(X) \), with \( \gamma > 0.5 \); (ii) \( \sup_{y_1,y_2,x} |\frac{\partial F_{Y_1|Y_2,X}(y_1,y_2,x)}{\partial y_1}| \leq C < \infty \).

Condition 6.3 \( n(\log \log n)^4 \delta_s^n = o(1) \)

Proposition 6.2 Let all conditions for Remark 6.1 hold. Under conditions 6.1(ii) and 6.2-6.3 Assumptions A.6 and A.7 hold for the NPQIV model (2.22).

It is clear that Condition 6.3 rules out severely ill-posed case, and hence Proposition 6.2 only allows for irregular functionals of the NPQIV model with mildly ill-posed case.

7. SIMULATION STUDIES AND AN EMPIRICAL ILLUSTRATION

This section first presents simulation studies for SQLR and sieve t tests of linear and nonlinear hypotheses for the NPQIV and NPIV models respectively. It then provides an empirical illustration of the optimally weighted SQLR inferences for a NPQIV Engel curve. In this section, we use the series LS estimator (2.5) of \( m(x,h) \) with \( p'^n(x) \) as its basis, and \( q^{k(n)} \) as the basis approximating the unknown structure function \( h_0 \). We use \( p'^r = P - \text{Spline}(r,k) \) to denote \( r \)th degree polynomial spline with \( k \) (quantile) equally spaced knots, hence \( J = (r+1)+k \) is the total number of sieve terms. We use \( p'^r = \text{Pol}(J) \) to denote power series up to \( (J-1) \)th degree. See, e.g., Chen (2007) for definitions of these and other sieve bases.
7.1. Simulation Studies

We run Monte Carlo (MC) studies to assess the finite sample performance of SQLR and sieve $t$ tests of linear and nonlinear hypotheses in two models: the NPQIV (2.22) and the NPIV (2.18).

For all cases, our design is based on the MC design of Newey and Powell (2003) and Santos (2012) for a NPIV model, which we adapt to cover both NPIV and NPQIV models. Specifically, we generate i.i.d. draws of $(Y^*_2, X, U^*)$ from

$$
\begin{bmatrix}
Y^*_2 \\
X^* \\
U^*
\end{bmatrix}
\sim
N
\left(0, \begin{bmatrix}
1 & 0.8 & 0.5 \\
0.8 & 1 & 0 \\
0.5 & 0 & 1
\end{bmatrix}\right),
$$

and $Y_2 = 2(\Phi(Y^*_2/3) - 0.5)$ and $X = 2(\Phi(X^*/3) - 0.5)$. The true function $h_0$ is given by $h_0(\cdot) = 2\sin(\pi \cdot)$. We consider 5,000 MC repetitions and $n = 750$ for each of the cases studied below. We use $Pen(h) = ||h||^2_{L_2} + ||\nabla h||^2_{L_2}$ in all the simulations, and have used a very small $\lambda_n = 10^{-5}$ in most cases (except for the cases where we study the sensitivity to the choice of $\lambda_n$).

**Summary of sensitivity checks**: For NPQIV and NPIV models, for both SQLR and sieve $t$ tests of linear and nonlinear hypotheses, as long as $J_n > k(n) + 1$ with not too large $k(n)$, the MC sizes of the tests are good and insensitive to the choices of basis $q_{k(n)}$ and $p_j$ or the very small penalty $\lambda_n$. This is consistent with previous MC findings in Blundell et al. (2007) and Chen and Pouzo (2012a) for PSMD estimation of NPIV and NPQIV respectively.

**NPQIV model: SQLR test for an irregular linear functional.**

We consider the NPQIV model $Y_1 = h_0(Y_2) + U = 2\sin(\pi Y_2) + U$ with $U = 2(\Phi(U^*) - \gamma)$. This last transformation is done to ensure that $E[1\{U \leq 0\}|X] = \gamma$. To save space we only present the case with $\gamma = 0.5$. The parameter of interest is $\phi(h_0) = h_0(0)$, hence $\phi$ is an irregular linear functional. We study the finite sample properties of the SQLR and
bootstrap-SQLR tests. The SQLR-based confidence intervals are specially well-suited for models like NPQIV where the generalized residual function is non-smooth yet the optimal weighting matrix is easy to compute.

**Size.** Table 7.1 reports the simulated size of the SQLR test of $H_0: \phi(h_0) = 0$ as a function of the nominal size (NS), for different choices of $q^{k(n)}$ and $p^{J_n}$, and different values of the tuning parameters ($\lambda_n, k(n), J_n$).

<table>
<thead>
<tr>
<th>$q^{k(n)}$</th>
<th>$p^{J_n}$</th>
<th>$\lambda_n$</th>
<th>10%</th>
<th>5%</th>
<th>1%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pol(4)</td>
<td>Pol(7)</td>
<td>$(1 \times 10^{-3})$</td>
<td>0.099</td>
<td>0.055</td>
<td>0.008</td>
</tr>
<tr>
<td>Pol(4)</td>
<td>Pol(7)</td>
<td>$(2 \times 10^{-4})$</td>
<td>0.096</td>
<td>0.048</td>
<td>0.008</td>
</tr>
<tr>
<td>Pol(4)</td>
<td>Pol(7)</td>
<td>$(4 \times 10^{-5})$</td>
<td>0.107</td>
<td>0.053</td>
<td>0.010</td>
</tr>
<tr>
<td>Pol(6)</td>
<td>Pol(7)</td>
<td>$(1 \times 10^{-3})$</td>
<td>0.133</td>
<td>0.068</td>
<td>0.011</td>
</tr>
<tr>
<td>Pol(6)</td>
<td>Pol(7)</td>
<td>$(2 \times 10^{-4})$</td>
<td>0.091</td>
<td>0.036</td>
<td>0.006</td>
</tr>
<tr>
<td>Pol(6)</td>
<td>Pol(7)</td>
<td>$(4 \times 10^{-5})$</td>
<td>0.105</td>
<td>0.052</td>
<td>0.008</td>
</tr>
<tr>
<td>Pol(6)</td>
<td>Pol(9)</td>
<td>$(1 \times 10^{-5})$</td>
<td>0.107</td>
<td>0.055</td>
<td>0.012</td>
</tr>
<tr>
<td>Pol(6)</td>
<td>Pol(15)</td>
<td>$(1 \times 10^{-5})$</td>
<td>0.109</td>
<td>0.058</td>
<td>0.014</td>
</tr>
<tr>
<td>Pol(6)</td>
<td>Pol(21)</td>
<td>$(1 \times 10^{-5})$</td>
<td>0.112</td>
<td>0.058</td>
<td>0.013</td>
</tr>
<tr>
<td>P-Spline(3,2)</td>
<td>Pol(9)</td>
<td>$(1 \times 10^{-5})$</td>
<td>0.103</td>
<td>0.049</td>
<td>0.010</td>
</tr>
<tr>
<td>P-Spline(3,2)</td>
<td>Pol(10)</td>
<td>$(1 \times 10^{-5})$</td>
<td>0.104</td>
<td>0.051</td>
<td>0.010</td>
</tr>
<tr>
<td>P-Spline(3,2)</td>
<td>Pol(15)</td>
<td>$(1 \times 10^{-5})$</td>
<td>0.105</td>
<td>0.049</td>
<td>0.009</td>
</tr>
<tr>
<td>P-Spline(3,2)</td>
<td>Pol(21)</td>
<td>$(1 \times 10^{-5})$</td>
<td>0.105</td>
<td>0.052</td>
<td>0.009</td>
</tr>
<tr>
<td>P-Spline(3,2)</td>
<td>P-Spline(5,3)</td>
<td>$(1 \times 10^{-5})$</td>
<td>0.098</td>
<td>0.049</td>
<td>0.008</td>
</tr>
<tr>
<td>P-Spline(3,2)</td>
<td>P-Spline(5,9)</td>
<td>$(1 \times 10^{-5})$</td>
<td>0.103</td>
<td>0.050</td>
<td>0.009</td>
</tr>
<tr>
<td>P-Spline(3,2)</td>
<td>P-Spline(5,18)</td>
<td>$(1 \times 10^{-5})$</td>
<td>0.106</td>
<td>0.051</td>
<td>0.009</td>
</tr>
</tbody>
</table>

Table 7.1 shows that for small value of $k(n)$, say in $(k(n), J_n) = (4, 7)$ (i.e., rows 1-3), the SQLR test performs well and is fairly insensitive to different choices of $\lambda_n$. For a fixed relatively small $J_n = 7$, rows 1-6 indicate that as $k(n)$ increases, the results become a bit more sensitive to the choice of $\lambda_n$. For a fixed very small penalty $\lambda_n = 10^{-5}$, rows 7-16 show that the results are fairly insensitive to different choices of $J_n$ and basis for $p^{J_n}$ and $q^{k(n)}$ as long as $J_n > k(n) + 1$. 

Local power. Figure 7.3 shows the rejection probabilities at 5% (lower panel) and 1% (upper panel) level of the null hypothesis as a function of \( r \) where \( r : \phi(h_0) = r \) for the SQLR (solid red line) and the bootstrap SQLR (dashed blue line) with multinomial weights. To save space we only report the local power results corresponding to the case of P-Spline(3,2) for \( q^k(n) \), Pol(10) for \( p_f \) and \( \lambda_n = 10^{-5} \) in Table 7.1. We employ 500 bootstrap evaluations per MC replication, and lower the number of MC repetitions to 1,000 to ease the computational burden. We note that since our functional \( \phi(h) = h(0) \) is estimated at a slower than root-\( n \) rate, the deviations considered for \( r \) which are in the range of \([0, 8/\sqrt{n}]\) are indeed “small”. We can see from the figure that the bootstrap SQLR performance is similar to its non-bootstrapped counterpart. We expect that the performance will improve if we increase number of bootstrap runs. (We also run simulation studies corresponding to the case of Pol(4) for \( q^k(n) \), Pol(7) for \( p_f \) and \( \lambda_n = 2 \times 10^{-4} \) in Table 7.1, and the local power patterns are similar to the ones reported here.)

NPIV model: sieve variance estimators for an irregular linear functional. We now consider the NPIV model: \( Y_1 = h_0(Y_2) + 0.76U = 2 \sin(\pi Y_2) + 0.76U \), with \( U = U^* \) so the identifying condition of NPIV holds: \( E[U|X] = 0 \). The parameter of interest is \( \phi(h_0) = h_0(0) \), and the null hypothesis is \( H_0: \phi(h_0) = 0 \). We focus on the finite sample performance of the sieve variance estimators for irregular linear functionals. We compute two sieve variance estimators:

\[
\hat{V}_1 = q^{k(n)}(0)' \hat{D}_n^{-1} \hat{\Omega}_n \hat{D}_n^{-1} q^{k(n)}(0) \quad \text{and} \quad \hat{V}_2 = q^{k(n)}(0)' \hat{D}_n^{-1} \hat{\Omega}_n \hat{D}_n^{-1} q^{k(n)}(0),
\]

where \( \hat{D}_n = n^{-1} \left( \hat{C}_n(P'P)^{-1} \hat{C}_n' \right) \), \( \hat{C}_n \equiv \sum_{i=1}^n q^{k(n)}(Y_{2i}) p_f(X_i)' \), \( \hat{\Omega}_n \) is given in equation (2.21), and \( \hat{\Omega}_n = \frac{1}{n} \hat{C}_n(P'P)^{-1} \left( \sum_{i=1}^n p_f(X_i) \tilde{\Sigma}_0(X_i)p_f(X_i)' \right) (P'P)^{-1} \hat{C}_n' \) with \( \tilde{U}_j = Y_{1j} - \tilde{h}(Y_{2j}) \) and \( \tilde{\Sigma}_0(x) = \left( \sum_{j=1}^n \tilde{U}_j p_f(X_j)' \right) (P'P)^{-1} p_f(X_j) \).
Figure 7.1.— Rejection probabilities at 1% (upper panel) and 5% level (lower panel) of the null hypothesis as a function of $r = \phi(h_0)$ for the SQLR (solid red line) and for the bootstrap SQLR (dashed blue line) for NPQIV.

(See Theorem B.1 in Appendix B for the definition and consistency of $\hat{V}_2$ as another sieve variance estimator for any plug-in PSMD $\phi(\hat{\alpha})$.)

Table 7.2 reports the results for different choices of bases for $q^{k(n)}$ and $p^{J_n}$, and for different values of $k(n)$ and $J_n$; in all cases we use a very small $\lambda_n = 10^{-5}$. This table shows $Med_{MC} \left[ \frac{\hat{V}_j}{||v^*_n||_{sd}} - 1 \right]$ for $j = 1, 2$, where $||v^*_n||_{sd}$ is computed using the MC variance of $\sqrt{n}\hat{h}_n(0)$ and $Med_{MC}[\cdot]$ is the MC median. It also shows the nominal size and MC rejection frequencies of the two sieve t tests $\hat{t}_j = \sqrt{n}\hat{h}_n(0) - 0 / \sqrt{\hat{V}_j}$ for $j = 1, 2$.

We note that the two sieve variance estimators have almost identical performance and the associated sieve t tests have good rejection probabilities. These results are fairly robust to different choices of basis for $q^{k(n)}$ and $p^{J_n}$ and different values of $k(n)$ and $J_n$ as long as $J_n > k(n) + 1$. Fig-
<table>
<thead>
<tr>
<th>$q^{k(n)}$</th>
<th>$p^{l(n)}$</th>
<th>$\hat{V}_1$</th>
<th>$\hat{V}_2$</th>
<th>$\hat{V}_1$</th>
<th>$\hat{V}_2$</th>
<th>$\hat{V}_1$</th>
<th>$\hat{V}_2$</th>
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<td>Pol(4)</td>
<td></td>
<td>0.0946</td>
<td>0.0937</td>
<td>0.0512</td>
<td>0.0514</td>
<td>0.0980</td>
<td>0.0974</td>
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<td>Pol(10)</td>
<td>0.0922</td>
<td>0.0920</td>
<td>0.0536</td>
<td>0.0532</td>
<td>0.0992</td>
<td>0.0990</td>
</tr>
<tr>
<td></td>
<td>Pol(12)</td>
<td>0.0918</td>
<td>0.0917</td>
<td>0.0538</td>
<td>0.0532</td>
<td>0.1002</td>
<td>0.0998</td>
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<tr>
<td></td>
<td>Pol(16)</td>
<td>0.0911</td>
<td>0.0912</td>
<td>0.0540</td>
<td>0.0538</td>
<td>0.1000</td>
<td>0.0998</td>
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<td>P-Spline(3,2)</td>
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<td>0.051</td>
<td>0.0516</td>
<td>0.0984</td>
<td>0.0986</td>
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<td>0.0538</td>
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<tr>
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<td>Pol(12)</td>
<td>0.0938</td>
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<td>0.0572</td>
<td>0.0564</td>
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<td>0.0936</td>
<td>0.0582</td>
<td>0.0578</td>
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<td>0.0580</td>
<td>0.0574</td>
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</tr>
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<td>P-Spline(3,5)</td>
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<td>0.1116</td>
<td>0.0606</td>
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<td>0.0574</td>
<td>0.1122</td>
<td>0.1116</td>
</tr>
<tr>
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<td>0.0960</td>
<td>0.0572</td>
<td>0.0566</td>
<td>0.1100</td>
<td>0.1094</td>
</tr>
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<td>0.1082</td>
<td>0.1080</td>
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<tr>
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<td></td>
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</tr>
<tr>
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<td>P-Spline(5,12)</td>
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<td>0.1009</td>
<td>0.0586</td>
<td>0.0578</td>
<td>0.1100</td>
<td>0.1092</td>
</tr>
<tr>
<td></td>
<td>P-Spline(5,18)</td>
<td>0.1007</td>
<td>0.1009</td>
<td>0.0580</td>
<td>0.0572</td>
<td>0.1110</td>
<td>0.1096</td>
</tr>
</tbody>
</table>

Figure 7.ii (first row) shows the QQ-Plot for the sieve t tests $\hat{t}_j = \sqrt{\frac{\hat{h}_n(0)-0}{\hat{V}_j}}$ under the null for $j = 1, 2$ for the case Pol(4)-Pol(16) in the table; the right panel in the first row corresponds to $\hat{t}_1$ and the left panel in the first row to $\hat{t}_2$. Both sieve t tests are almost identical to each other and to the standard normal.

**NPIV model:** sieve variance estimators for an irregular *nonlinear functional*. This case is identical to the previous one for the NPIV model.
model, except that the functional of interest is \( \phi(h_0) = \exp\{h_0(0)\} \), and the null hypothesis is \( H_0 : \phi(h_0) = 1 \). This choice of \( \phi \) allows us to evaluate the finite sample performance of sieve t statistics for a nonlinear functional.

Table 7.3 shows \( \text{Med}_{MC} \) and rejection probabilities for this nonlinear case. By comparing the results with those in Table 7.2 we note that the results are very similar in both cases. Figure 7.ii (second row) shows the QQ-Plot for the two sieve t tests for the non-linear case; the right panel in the second row corresponds to \( \hat{t}_1 \) whereas the left panel in the second row corresponds to \( \hat{t}_2 \). These results suggest that our sieve t tests perform equally well for both functionals.

Finally we wish to point out that we have tried other bases such as Hermite polynomials and cosine series and even larger \( J_n \) in these two NPIV MC studies, the results are all similar to the ones reported here and hence are not presented due to the lack of space.
TABLE 7.3
Relative performance of \( \hat{V}_1 \) and \( \hat{V}_2 \): \( \text{Med}_{MC} \left[ \left\| \frac{V_j}{\text{Med}_d} - 1 \right\| \right] \), and Nominal size and MC rejection frequencies for t tests \( t_j \) for \( j = 1, 2 \) for a nonlinear functional of NPIV.

<table>
<thead>
<tr>
<th>( q^{(n)} )</th>
<th>( p^{(n)} )</th>
<th>( \text{Med}_{MC} )</th>
<th>5%</th>
<th>10%</th>
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<td>Pol(4)</td>
<td>Pol(6)</td>
<td>0.0990 0.0985 0.0528 0.0530 0.0982 0.0988</td>
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<tr>
<td>Pol(10)</td>
<td>Pol(6)</td>
<td>0.0971 0.0958 0.0524 0.0522 0.1014 0.1012</td>
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<td>Pol(12)</td>
<td>Pol(6)</td>
<td>0.0967 0.0959 0.0526 0.0526 0.1020 0.1018</td>
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<tr>
<td>Pol(16)</td>
<td>Pol(6)</td>
<td>0.0961 0.0958 0.0524 0.0528 0.1018 0.1014</td>
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<tr>
<td>P-Spline(3,2)</td>
<td>Pol(6)</td>
<td>0.0976 0.0983 0.0534 0.0530 0.0978 0.0976</td>
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</tr>
<tr>
<td>P-Spline(3,5)</td>
<td>Pol(6)</td>
<td>0.0982 0.0969 0.0534 0.0542 0.0990 0.0992</td>
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<tr>
<td>P-Spline(3,11)</td>
<td>Pol(6)</td>
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<tr>
<td>Pol(4)</td>
<td>Pol(12)</td>
<td>0.1011 0.1009 0.0580 0.0568 0.1120 0.1122</td>
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<td>Pol(12)</td>
<td>0.1014 0.1005 0.0588 0.0574 0.1128 0.1126</td>
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<tr>
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<td>Pol(12)</td>
<td>0.1014 0.1007 0.0582 0.0568 0.1130 0.1122</td>
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<td>0.1015 0.1006 0.0580 0.0568 0.1138 0.1128</td>
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<td></td>
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<td>P-Spline(3,2)</td>
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<td>P-Spline(3,2)</td>
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<tr>
<td>P-Spline(5,18)</td>
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<td>0.1069 0.1079 0.0594 0.0586 0.1122 0.1120</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7.2. An Empirical Application

We compute SQLR based confidence bands for nonparametric quantile IV Engel curves using the British FES data set from [Blundell et al. (2007)]:

\[
E[1\{Y_{1,i} \leq h_0(Y_{2,i})\} \mid X_i] = 0.5,
\]

where \( Y_{1,i} \) is the budget share of the \( i \)-th household on a particular non-durable goods, say food-in consumption; \( Y_{2,i} \) is the log-total expenditure of the household, which is endogenous, and hence we use \( X_i \), the gross
earnings of the head of the household, to instrument it. We work with the “no kids” sub-sample of the data set, which consists of \( n = 628 \) observations. Blundell et al. (2007) estimated NPIV Engel curves using this data set. But, as explained by Koenker (2005) and others, quantile Engel curves are more informative.

We estimate \( h_0(\cdot) \) for food-in quantile Engel curve via the optimally weighted PSMD procedure with \( \hat{\Sigma} = \Sigma_0 = 0.25 \), using a polynomial spline (P-spline) sieve \( H_k(n) \) with \( k(n) = 4 \), \( \text{Pen}(h) = ||h||^2_L + ||\nabla h||^2_L \) with \( \lambda_n = 0.0005 \), and a Hermite polynomial LS basis \( p^J_n(X) \) with \( J_n = 6 \). We also considered other bases such as P-splines as \( p^J_n(X) \) and results remained essentially the same. See Chen and Pouzo (2009) for PSMD estimates of NPQIV Engel curves for other non-durable goods.

We use the fact that the optimally weighted SQR of testing \( \phi(h) = h(y_2) \) (for any fixed \( y_2 \)) is asymptotically \( \chi^2_1 \) to construct pointwise confidence bands. That is, for each \( y_2 \) in the sample we construct a grid, \( (r_i)_{i=1}^{30} \). For each \( i = 1, ..., 30 \), we compute the value of the SQR test statistic under \( h(y_2) = r_i \) for \( (r_i)_{i=1}^{30} \). We then, take the smallest interval that included all points \( r_i \) that yield a corresponding value of the SQR test below the 95% percentile of \( \chi^2_1 \).12 Figure 7.iii presents the results, where the solid blue line is the point estimate and the red dashed lines are the 95% pointwise confidence bands. We can see that the confidence bands get wider towards the extremes of the sample, but are tighter in the middle.

To test whether the quantile IV Engel curve for food-in is linear or not,

\footnote{The grid \( (r_i)_{i=1}^{30} \) was constructed to have \( r_{15} = \hat{h}_n(y_2) \), for all \( i \leq 15 \), \( r_{i+1} \leq r_i \leq r_{15} \) decreasing in steps of length 0.002 (approx) and for all \( i \geq 15 \), \( r_{i+1} \geq r_i \geq r_{15} \) increasing in steps of length 0.008 (approx); finally, the extremes, \( r_1 \) and \( r_{30} \), were chosen so the SQR test at those points was above the 95% percentile of \( \chi^2_1 \). We tried different lengths and step sizes and the results remain qualitatively unchanged. For some observations, which only account for less than 4% of the sample, the confidence interval was degenerate at a point; this result was due to numerical approximation issues, and these observations were excluded from the reported results.}
one can test whether $\phi(h_0) \equiv \int |\nabla^2 h(y_2)|^2 w(y_2)dy_2 = 0$ using our SQL-R test. Let $w(\cdot) = (\sigma_{y_2})^{-1} \exp \left( -\frac{1}{2}(\sigma_{y_2}^{-1}(\cdot - \mu_{y_2}))^2 \right) 1\{t_{0.01} \leq \cdot \leq t_{0.99}\}$ where $\mu_{y_2}$, $\sigma_{y_2}$, $t_{0.01}$ and $t_{0.99}$ are the sample mean, standard deviation and the 1% and 99% quantiles of $Y_2$. The value of the SQLR is (approx.) 38 and the p-value is smaller than 0.0001, and hence we reject the null hypothesis of linearity.\footnote{We use the standard Riemann sum with 1000 terms to compute the integral. We also considered other choices of $w$ such that $w(\cdot) = 1\{t_{0.25} \leq \cdot \leq t_{0.75}\}$ and $w(\cdot) = 1\{t_{0.01} \leq \cdot \leq t_{0.99}\}$. Although the numerical value of the SQLR test changes, all produce p-values below 0.0001.}

8. CONCLUSION

In this paper, we provide unified asymptotic theories for PSMD based inferences on possibly irregular parameters $\phi(\alpha_0)$ of the general semi/nonparametric conditional moment restrictions $E[\rho(Y,X;\alpha_0)|X] = 0$. Under regularity
conditions that allow for any consistent nonparametric estimator of the conditional mean function \( m(X, \alpha) \equiv E[\rho(Y, X; \alpha)|X] \), we establish the asymptotic normality of the plug-in PSMD estimator \( \phi(\hat{\alpha}_n) \) of \( \phi(\alpha_0) \), as well as the asymptotically tight distribution of a possibly non-optimally weighted SQR statistic under the null hypothesis of \( \phi(\alpha_0) = \phi_0 \). As a simple yet useful by-product, we immediately obtain that an optimally weighted SQR statistic is asymptotically chi-square distributed under the null hypothesis. For (pointwise) smooth residuals \( \rho(Z; \alpha) \) (in \( \alpha \)), we propose several simple consistent sieve variance estimators for \( \phi(\hat{\alpha}_n) \) (in the text and in online Appendix [B]), and establish the asymptotic chi-square distribution of sieve Wald statistics. We also establish local power properties of SQR and sieve Wald tests in Appendix [A]. Under conditions that are virtually the same as those for the limiting distributions of the original-sample sieve Wald and SQR statistics, we establish the consistency of the generalized residual bootstrap sieve Wald and SQR statistics. All these results are valid regardless of whether \( \phi(\alpha_0) \) is regular or not. While SQR and bootstrap SQR are useful for models with (pointwise) non-smooth \( \rho(Z; \alpha) \), sieve Wald statistic is computationally attractive for models with smooth \( \rho(Z; \alpha) \). Monte Carlo studies and an empirical illustration of a nonparametric quantile IV regression demonstrate the good finite sample performance of our inference procedures.

This paper assumes that the semi/nonparametric conditional moment restrictions \( E[\rho(Y, X; \alpha_0)|X] = 0 \) uniquely identifies the unknown true parameter value \( \alpha_0 \equiv (\theta_0, h_0) \), and conduct inferences that are robust to whether a possibly nonlinear functional of \( \alpha_0 \) is root-\( n \) estimable or not. Recently, for the NPIV model \( E[Y_1 - h_0(Y_2)|X] = 0 \) without assuming point identification of \( h_0 \), Santos (2011) proposed a root-\( n \) asymptotically normal estimation of a regular linear functional of \( h_0 \) and Santos (2012) considered Bierens’ type test of the NPIV. Chen et al. (2011) is extending
the SQLR inference procedure to allow for partial identification of the general model $E[\rho(Y, X; \alpha_0)|X] = 0$.

REFERENCES


APPENDIX A: ADDITIONAL RESULTS AND SUFFICIENT CONDITIONS

Appendix A consists of several subsections. Subsection A.1 presents additional results on (sieve) Riesz representation of the functional of interest. Subsection A.2 derives the convergence rates of the bootstrap PSMD estimator. Subsection A.3 presents asymptotic properties under local alternatives of the SQLR and the sieve Wald tests, and of their bootstrap versions. Subsection A.4 provides some inference results for functionals of increasing dimension. Subsection A.5 provides some low level sufficient conditions for the high level LQA Assumption 3.6(i) and the bootstrap LQA Assumption Boot.3(i) with series LS estimated conditional mean functions of increasing dimension. Subsection A.6 provides some inference results for functionals of increasing dimension. Subsection A.7 provides some inference results for the root-n rate of convergence of the bootstrap PSMD estimator. Subsection A.8 provides some inference results for the root-n rate of convergence of the bootstrap PSMD estimator.

A.1. Additional discussion on (sieve) Riesz representation

The discussion in Subsection 3.2 on Riesz representation seems to depend on the weighting matrix $\Sigma$, but, under Assumption 3.1(iv), we have $\|v\| \asymp \|v\|_0$, (i.e., the norm $\|\cdot\|$ (using $\Sigma$) is equivalent to the norm $\|\cdot\|_0$ (using $\Sigma_0$) defined in (3.2)), and the space $\mathbf{V}$ (or $\mathbf{V}'$) under $\|\cdot\|$ is equivalent to that under $\|\cdot\|_0$. Therefore, under Assumption 3.1(iv), $\frac{d\phi(\alpha)}{d\alpha}[\cdot]$ is bounded on $\left(\mathbf{V}, \|\cdot\|\right)$ iff $\frac{d\phi(\alpha)}{d\alpha}[\cdot]$ is bounded on $\left(\mathbf{V}, \|\cdot\|_0\right)$, i.e.,

$$\sup_{v \in \mathbf{V}, v \neq 0} \frac{\left|\frac{d\phi(\alpha)}{d\alpha}[v]\right|}{\|v\|_0} < \infty,$$

and in this case we call $\phi(\cdot)$ regular at $\alpha_0$. Likewise, $\frac{d\phi(\alpha)}{d\alpha}[\cdot]$ is unbounded on $\left(\mathbf{V}, \|\cdot\|\right)$ iff $\frac{d\phi(\alpha)}{d\alpha}[\cdot]$ is unbounded on $\left(\mathbf{V}, \|\cdot\|_0\right)$ i.e., sup$_{v \in \mathbf{V}, v \neq 0} \left\{\frac{\left|\frac{d\phi(\alpha)}{d\alpha}[v]\right|}{\|v\|_0}\right\} = \infty$, and in this case we call $\phi(\cdot)$ irregular at $\alpha_0$.

For the specific Euclidean functional $\phi(\alpha) = X'\theta$ of the semi/nonparametric model (1.1), following the proof in appendix E of Ai and Chen (2003), it is easy to see the equivalence between sup$_{v \in \mathbf{V}, v \neq 0} \left\{\frac{\left|\frac{d\phi(\alpha)}{d\alpha}[v]\right|}{\|v\|_0}\right\} < \infty$ (or $= \infty$) and the semiparametric efficiency bound of $\theta_0$ being non-singular (or singular). Previously Kahn and Tamarc (2010) and Graham and Powell (2012) call $\theta_0$ irregular when the semiparametric efficiency bound of $\theta_0$ is singular. This motivates us to call a general functional $\phi(\cdot)$ irregular at $\alpha_0$ whenever $\frac{d\phi(\alpha)}{d\alpha}[v]$ is unbounded on $\left(\mathbf{V}, \|\cdot\|_0\right)$.

It is clear that $\phi(\cdot)$ being regular at $\alpha_0$ (i.e., $\frac{d\phi(\alpha)}{d\alpha}[v]$ being bounded on $\left(\mathbf{V}, \|\cdot\|_0\right)$) is a necessary condition for the root-n rate of convergence of $\phi(\hat{\alpha}_n) - \phi(\alpha_0)$. Unfortunately for a complicated semi/nonparametric model (1.1), it is difficult to compute sup$_{v \in \mathbf{V}, v \neq 0} \left\{\frac{\left|\frac{d\phi(\alpha)}{d\alpha}[v]\right|}{\|v\|_0}\right\}$ explicitly; and hence difficult to verify its root-n estimability. But, we could always compute $\|v_n^*\| = \sup_{v \in \mathbf{V}(n), v \neq 0} \left\{\frac{\left|\frac{d\phi(\alpha)}{d\alpha}[v]\right|}{\|v\|_0}\right\}$ in closed form (by Lemma 4.1) and then check whether $\lim_{k(n) \to \infty} \|v_n^*\|^2 < \infty$ or not (by Lemma 5.3).

Remark A.1 For a semi/nonparametric model (1.1) with $\alpha_0 = (\theta_0', h_0)$, it is conve-
nient to rewrite \( D_n \) and its inverse in Lemma 4.7 as

\[
D_n = \begin{pmatrix}
I_{11} & I_{12} & I_{13} \\
I_{21} & I_{22} & I_{23} \\
I_{31} & I_{32} & I_{33}
\end{pmatrix}
\quad \text{and} \quad
D_n^{-1} = \begin{pmatrix}
-I_{11} & -I_{12} & -I_{13} \\
-I_{21} & -I_{22} & -I_{23} \\
-I_{31} & -I_{32} & -I_{33}
\end{pmatrix},
\]

with

\[
I_{11} = E \left[ \left( \frac{\partial m(X,\alpha_0)}{\partial \theta} \right) \Sigma(X)^{-1} \frac{\partial m(X,\alpha_0)}{\partial \theta}^\prime \right],
\]

\[
I_{12} = E \left[ \left( \frac{\partial m(X,\alpha_0)}{\partial \theta} \right) \Sigma(X)^{-1} \frac{\partial m(X,\alpha_0)}{\partial v_\theta} \right],
\]

\[
I_{13} = E \left[ \left( \frac{\partial m(X,\alpha_0)}{\partial \theta} \right) \Sigma(X)^{-1} \frac{\partial m(X,\alpha_0)}{\partial \beta_\theta} \right],
\]

\[
I_{21} = E \left[ \left( \frac{\partial m(X,\alpha_0)}{\partial v_\theta} \right) \Sigma(X)^{-1} \frac{\partial m(X,\alpha_0)}{\partial \theta} \right],
\]

\[
I_{22} = E \left[ \left( \frac{\partial m(X,\alpha_0)}{\partial v_\theta} \right) \Sigma(X)^{-1} \frac{\partial m(X,\alpha_0)}{\partial v_\theta} \right],
\]

\[
I_{23} = E \left[ \left( \frac{\partial m(X,\alpha_0)}{\partial v_\theta} \right) \Sigma(X)^{-1} \frac{\partial m(X,\alpha_0)}{\partial \beta_\theta} \right],
\]

\[
I_{31} = E \left[ \left( \frac{\partial m(X,\alpha_0)}{\partial \beta_\theta} \right) \Sigma(X)^{-1} \frac{\partial m(X,\alpha_0)}{\partial \theta} \right],
\]

\[
I_{32} = E \left[ \left( \frac{\partial m(X,\alpha_0)}{\partial \beta_\theta} \right) \Sigma(X)^{-1} \frac{\partial m(X,\alpha_0)}{\partial v_\theta} \right],
\]

\[
I_{33} = E \left[ \left( \frac{\partial m(X,\alpha_0)}{\partial \beta_\theta} \right) \Sigma(X)^{-1} \frac{\partial m(X,\alpha_0)}{\partial \beta_\theta} \right].
\]

Assumption A.1 (Bootstrap sample criterion) is analogous to Assumption 3.4(iii) for the original sample. Again, when \( \hat{m}^B(x,\alpha) \) is the bootstrap series LS estimator to \( m(x,\alpha) \), under virtually the same sufficient conditions as those in [Chen and Pouzo (2012a) and Chen and Pouzo... page 71]
Let Assumption A.1(i)(ii) and conditions for Lemma 3.1 hold. Then:

1. $||\hat{\alpha}_R - \alpha_0||_s = \frac{1}{\sqrt{n}}\sqrt{\frac{\log n}{m}} \rightarrow 0$ with $\mathbb{P}(\hat{\alpha}_R \in N_{\alpha_0}) \rightarrow 1$.

2. In addition, let Assumption 3.4(i)(ii)(iii) and Assumption A.1(iii) hold, then:

   $||\hat{\alpha}_R - \alpha_0||_s = \frac{1}{\sqrt{n}}\sqrt{\frac{\log n}{m}} \rightarrow 0$ with $\mathbb{P}(\hat{\alpha}_R \in N_{\alpha_0}) \rightarrow 1$.

3. The above results remain true when $\hat{\alpha}_R$ is replaced by $\hat{\alpha}_R^B$.

Lemma A.1(2) and (3) show that $\hat{\alpha}_R \in N_{\alpha_0}$ regardless of whether the null $H_0 : \phi(\alpha_0) = \phi_0$ is true or not.

A.3. Asymptotic behaviors under local alternatives

In this subsection we consider the behavior of SQLR, sieve Wald and their bootstrap versions under local alternatives. That is, we consider local alternatives along the curve

$\{\alpha_n \in N_{\alpha_0} : n \in \{1, 2, \ldots\}\}$

where

$$\alpha_n = \alpha_0 + d_n \Delta_n,$$

with

$$\frac{d\phi(\alpha_0)}{d\alpha}[\Delta_n] = \kappa \times (1 + o(1)) \neq 0$$

for any $(d_n, \Delta_n) \in \mathbb{R}_+ \times \nabla_{(\kappa)}(\alpha_0)$ such that $d_n||\Delta_n||_s \leq M_0 \delta_n$, $d_n||\Delta_n||_s \leq M_n \delta_n$, for all $n$. The restriction on the rates under both norms is to ensure that the required assumptions for studying the asymptotic behavior under these alternatives (Assumption 3.5 in particular) hold. This choice of local alternatives is to simplify the presentation and could be relaxed somewhat.

Since we are now interested in the behavior of the test statistics under local alternatives, we need to be more explicit about the underlying probability, in a.s. or in probability statements. Henceforth, we use $P_{n,Z_\infty}$ to denote the probability measure over sequences $Z_\infty$ induced by the model at $\alpha_n$ (we leave $P_{Z_\infty}$ to denote the one associated to $\alpha_0$).

A.3.1. SQLR and SQLR$^B$ under local alternatives

In this subsection we consider the behavior of the SQLR and the bootstrap SQLR, under local alternatives along the curve $\{\alpha_n \in N_{\alpha_0} : n \in \{1, 2, \ldots\}\}$ defined in (A.2).

Theorem A.1 Let conditions for Lemma 3.2 and Proposition 3.7 and Assumption 3.6 (with $|B_n - ||u_n^*||^2| = o_{P_{n,Z_\infty}}(1)$) hold under the local alternatives $\alpha_n$ defined in (A.2). Let Assumption 3.5 hold and $\hat{\alpha}_R \in N_{\alpha_0}$. Then, under the local alternatives $\alpha_n$,

---

The verification is amounts to follow the proof of Lemma C.2 of Chen and Pouzo (2012a) except that the original-sample series LS estimator $\hat{m}(x, \alpha)$ is replaced by its bootstrap version $\hat{m}^B(x, \alpha)$.
In addition, let conditions for Theorem A.1 hold. Then: for any Sieve Wald and bootstrap sieve Wald tests under local alternatives

\[
\hat{\chi}^2_n(\phi) \Rightarrow \chi^2_1(n^2);
\]

(2) if \( n^{1/2}||v_n^*||_d \to 1 \), then \( \lim_{n \to \infty} \left( ||v_n^*||^2 \times \hat{QLR}_n(\phi_0) \right) = 1 \) in probability.

The statement that assumptions hold under the local alternatives \( \alpha_n \) really means that the assumptions hold when the true DGP model is indexed by \( \alpha_n \) (as opposed to \( \alpha_0 \)). For instance, this change impacts on Assumption 3.6 by changing the “centering” of the expansion to \( \alpha_n \) and also changing “in probability” statements to hold under \( P_{n,Z} = 1 \) as opposed to \( P_{Z} = 1 \).

If we had a likelihood function instead of our criterion function, we could adapt Le Cam’s 3rd Lemma to show that Assumption 3.6 under local alternatives holds directly. Since our criterion function is not a likelihood we cannot proceed in this manner, and we directly assume it. Also, if we only consider contiguous alternatives, i.e., curves \( \{\alpha_n\} \) that yield probability measures \( P_{n,Z} \) that are contiguous to \( P_Z = 1 \), then any statement in a.s. or wpa under \( P_Z = 1 \) holds automatically under \( P_{n,Z} = 1 \).

The next proposition presents the relative efficiency under local alternatives of tests based on the non- and optimally weighted SQLR statistics. We show —aligned with the literature for regular cases— that optimally weighted SQLR statistic is more efficient than the non-optimally weighted one.

**Proposition A.1** Let all conditions for Theorem A.1 hold. Then, under the local alternatives \( \alpha_n \) defined in (A.2) with \( d_n = n^{-1/2}||v_n^*||_d \), we have: for any \( t \),

\[
\lim_{n \to \infty} P_{n,Z = 1} (||v_n^*||^2 \times \hat{QLR}_n(\phi_0) \geq t) \leq \liminf_{n \to \infty} P_{n,Z = 1} (\hat{QLR}_n(\phi_0) \geq t).
\]

The next theorem shows the consistency of our bootstrap SQLR statistic under the local alternatives \( \alpha_n \) in (A.2). This result completes that in Remark 5.2.

**Theorem A.2** Let conditions for Theorem 5.2 hold under local alternatives \( \alpha_n \) defined in (A.2). Then: (1)

\[
\frac{\hat{QLR}^B_n(\hat{\alpha}_n)}{\sigma^2_\alpha} = \left( \frac{\sqrt{n}Z^{-1}(\alpha_n)}{\sigma_n||u_n^*||} \right)^2 + O_{P_{n,Z}}(1) = O_{P_{n,Z}}(1) \text{ wpa}(P_{n,Z} = 1); \quad \text{and}
\]

\[
\sup_{t \in \mathbb{R}} P_{n,Z = 1} \left( \frac{\hat{QLR}^B_n(\hat{\alpha}_n)}{\sigma^2_\alpha} \leq t \mid Z^n \right) - P_{Z = 1} \left( \frac{\hat{QLR}_n(\phi_0)}{\sigma^2_\alpha} \leq t \mid H_0 \right) = O_{P_{n,Z}}(1) \text{ wpa}(P_{n,Z} = 1).
\]

(2) In addition, let conditions for Theorem A.1 hold. Then: for any \( \tau \in (0,1) \),

\[
\tau < \lim_{n \to \infty} P_{n,Z = 1} (\hat{QLR}_n(\phi_0) \geq \hat{c}_n(1 - \tau)) < 1 \text{ under } d_n = n^{-1/2}||v_n^*||_d \rightarrow \infty;
\]

\[
\lim_{n \to \infty} P_{n,Z = 1} (\hat{QLR}_n(\phi_0) \geq \hat{c}_n(1 - \tau)) = 1 \text{ under } n^{1/2}||v_n^*||_d^{-1} d_n \rightarrow \infty,
\]

where \( \hat{c}_n(\alpha) \) is the \( \alpha - \text{th} \) quantile of the distribution of \( \frac{\hat{QLR}^B_n(\hat{\alpha}_n)}{\sigma^2_\alpha} \) (conditional on data \( \{Z_i\}^n_{i=1} \)).

A.3.2. Sieve Wald and bootstrap sieve Wald tests under local alternatives

The next result establishes the asymptotic behavior of the sieve Wald test statistic \( \hat{W}_n = \left( \sqrt{n} \frac{\phi(\hat{\alpha}_n) - \phi_0}{||v_n^*||_d} \right)^2 \) under the local alternative along the curve \( \alpha_n \) defined in (A.2).
Theorem A.3 Let \( \hat{\alpha}_n \) be the PSMD estimator (2.3), conditions for Lemma A.2 and Theorem A.2 and Assumption 3.6 hold under the local alternatives \( \alpha_n \) defined in (A.2). Let Assumption 3.3 hold. Then, under the local alternatives \( \alpha_n \),

(1) if \( d_n = n^{-1/2} ||v_n^*||_{sd} \), then \( W_n \Rightarrow \chi_n^2(\kappa^2) \);

(2) if \( n^{1/2} ||v_n^*||_{sd} d_n \rightarrow \infty \), then \( \lim_{n \rightarrow \infty} W_n = \infty \) in probability.

Remark A.2 By the same proof as that of Proposition A.1, one can establish the asymptotically relative efficiency results for the sieve Wald test statistic.

The next theorem shows the consistency of our bootstrap sieve Wald test statistic under the local alternatives \( \alpha_n \) in (A.2). This result completes that in Remark 5.1.

Theorem A.4 Let all conditions for Theorem 3.3(1) hold under local alternatives \( \alpha_n \) defined in (A.2). Then: (1) for \( j = 1, 2, \)

\[
\sup_{t \in \mathbb{R}} P_{n \rightarrow \infty} |Z_j^\ast - \hat{W}_{j,n}^B| = o_P(n \rightarrow \infty) \text{ (1) wpa}(P_n, Z_j^\ast).
\]

(2) In addition, let conditions for Theorem A.3 hold. Then: for any \( \tau \in (0, 1) \),

\[ P_{n \rightarrow \infty} \left( W_n \geq \tilde{c}_{j,n}(1 - \tau) \right) = \tau + \Pr \left( \chi_1^2(\kappa^2) \geq \tilde{c}_{j,n}(1 - \tau) \right) + o_P(n \rightarrow \infty) \text{ (1) wpa}(P_n, Z_j^\ast)
\]

and \( \tau < \lim_{n \rightarrow \infty} P_{n \rightarrow \infty} \left( W_n \geq \tilde{c}_{j,n}(1 - \tau) \right) < 1, \)

(2a) If \( d_n = n^{-1/2} ||v_n^*||_{sd} \) then:

\[
W_n \rightarrow \tilde{c}_{j,n}(1 - \tau)
\]

and \( \tau < \lim_{n \rightarrow \infty} P_{n \rightarrow \infty} \left( W_n \geq \tilde{c}_{j,n}(1 - \tau) \right) < 1, \)

(2b) If \( \sqrt{n} ||v_n^*||_{sd} \rightarrow \infty \) then: \( \lim_{n \rightarrow \infty} P_{n \rightarrow \infty} \left( W_n \geq \tilde{c}_{j,n}(1 - \tau) \right) = 1. \)

where \( \tilde{c}_{j,n}(a) \) be the \( a - \text{th} \) quantile of the distribution of \( W_j^B, n \equiv \left( \hat{W}_{j,n}^B \right)^2 \) (conditional on the data \( \{Z_i \}_{i=1}^n \)).

A.4. Local asymptotic theory under increasing dimension of \( \hat{\phi} \)

In this section we extend some inference results to the case of vector-valued functional \( \hat{\phi} \) (i.e., \( d_{\hat{\phi}} \equiv d(n) > 1 \)), and in fact \( d(n) \) could grow with \( n \).

We first introduce some notation. Let \( v^\ast_{j,n} \) be the sieve Riesz respresentor corresponding to \( \phi_j \) for \( j = 1, \ldots, d(n) \) and let \( \mathbf{v}_{n} = (v_{1,n}, \ldots, v_{d(n),n})^\top \). For each \( x \), we use \( \frac{dm(x, \alpha_0)}{d\alpha}[v_{n,j}^\ast] \) to denote a \( d_{\alpha} \times d(n) \)-matrix with \( \frac{dm(x, \alpha_0)}{d\alpha}[v_{n,j}^\ast] \) as its \( j \)-th column for \( j = 1, \ldots, d(n) \). Finally, let

\[
\Omega_{sd,n} \equiv E \left[ \left( \frac{dm(X, \alpha_0)}{d\alpha} [v_{n,j}^\ast] \right)^\top \Sigma^{-1}(X) \Sigma_0(X) \Sigma^{-1}(X) \left( \frac{dm(X, \alpha_0)}{d\alpha} [v_{n,j}^\ast] \right) \right] \in \mathbb{R}^{d(n) \times d(n)}
\]

and

\[
\Omega_n \equiv (\mathbf{v}_{n}^\ast, \mathbf{v}_{n}^\ast) \equiv E \left[ \left( \frac{dm(X, \alpha_0)}{d\alpha} [v_{n,j}^\ast] \right)^\top \Sigma^{-1}(X) \left( \frac{dm(X, \alpha_0)}{d\alpha} [v_{n,j}^\ast] \right) \right] \in \mathbb{R}^{d(n) \times d(n)}.
\]

Observe that for \( d(n) = 1, \Omega_{sd,n} = ||v_{n,j}^\ast||_{sd}^2 \) and \( \Omega_n = ||v_{n,j}^\ast||_{sd}^2 \). Also, for the case \( \Sigma = \Sigma_0 \), we would have

\[
\Omega_n = \Omega_{sd,n} = \Omega_{0,n} \equiv E \left[ \left( \frac{dm(X, \alpha_0)}{d\alpha} [v_{n,j}^\ast] \right)^\top \Sigma_0^{-1}(X) \left( \frac{dm(X, \alpha_0)}{d\alpha} [v_{n,j}^\ast] \right) \right].
\]
Let
\[ T_M^n \equiv \{ t \in \mathbb{R}^d(n) : \|t\|_e \leq M_n n^{-1/2} \sqrt{d(n)} \} \quad \text{and} \quad \alpha(t) \equiv \alpha + \mathbf{v}_n^*(\Omega_{sd,n})^{-1/2} t. \]

Let \((c_n)_n\) be a real-valued positive sequence that converges to zero as \(n \to \infty\). The following assumption is analogous to Assumption 3.5 but for vector-valued \(\phi\). Under Assumption 3.1(iv), we could use \(\Omega_n\) instead of \(\Omega_{sd,n}\) in Assumption A.2(ii)(iii) below.

**Assumption A.2** (i) for each \(j = 1, \ldots, d(n)\), \(\frac{d\phi_j(\alpha_n)}{d\alpha}\) satisfies Assumption 3.3(i); and for each \(v \neq 0\), \(\frac{d\phi(v)}{d\alpha}\) is linearly independent;

\(\sup_{(\alpha,t) \in \mathcal{N}_{s\alpha,n} \times T_M^n} \left\| (\Omega_{sd,n})^{-1/2} \left\{ \phi(\alpha(t)) - \phi(\alpha_0) - \frac{d\phi(\alpha)}{d\alpha}[\alpha(t) - \alpha_0] \right\} \right\|_e = O(c_n) \);

\(\sup_{(\alpha,t) \in \mathcal{N}_{s\alpha,n} \times T_M^n} \left\| (\Omega_{sd,n})^{-1/2} \frac{d\phi(\alpha)}{d\alpha} [\alpha_0, n - \alpha_0] \right\|_e = O(c_n); \) (iv) \(c_n = o(n^{-1/2})\).

For any \(v \in \mathbf{v}_k(n)\), we use \(\langle v_n^\prime, v \rangle\) to denote a \(d(n) \times 1\) vector with components \(\langle v_n^\prime_j, v \rangle\) for \(j = 1, \ldots, d(n)\). Then \(\frac{d\phi(v)}{d\alpha}\) is linearly independent.

**Assumption A.3** (i) For all \(n\), for all \((\alpha,t) \in \mathcal{N}_{s\alpha,n} \times T_M^n\) with \(\alpha(t) \in \mathcal{A}_k(n)\),

\[ \sup_{(\alpha,t) \in \mathcal{N}_{s\alpha,n} \times T_M^n} r_n(t_n) \left| \tilde{A}_n(\alpha(t_n), \alpha) - t_n^\prime (\Omega_{sd,n})^{-1/2} \{ Z_n + \langle v_n^\prime, \alpha - \alpha_0 \rangle \} - t_n^\prime \mathbb{B}_n \right| = O_{P_{\infty}} (1) \]

where \(r_n(t_n) = \left\{ \max_b \{ \| t_n^\prime b_n \|_e, \| t_n \|_{\alpha_n, s_n} \} \right\}^{-1}\) and \(\mathbb{B}_n\) is such that, for each \(n\), \(\mathbb{B}_n\) is a \(Z^n\) measurable positive definite matrix in \(\mathbb{R}^{d(n) \times d(n)}\) and \(\mathbb{B}_n = O_{P_{\infty}} (1); \) (ii) \(s_n d(n) = o(1), b_n \sqrt{d(n)} = o(1), \sqrt{d(n)} \times a_n = o(1)\).

In the rest of this section as well as in its proofs, since there is no risk of confusion, we use \(O_P\) and \(O_{P_{\infty}}\) to denote \(o_{P_{\infty}}\) and \(O_{P_{\infty}}\) respectively.

The next theorem extends Theorem 4.1 to the case of vector-valued functionals \(\phi\) (of increasing dimension). Let \(\mu_{\alpha,n} = \mathbb{E} \left[ \left\| (\Omega_{sd,n})^{-1/2} \left( \frac{d\phi(\alpha_n)}{d\alpha} \right)^\prime \right\|_e \right]\).

**Theorem A.5** Let Conditions for Lemma 3.3 Assumptions A.2 and A.3 hold. Then:

(1) \(n(\phi(\tilde{\alpha}_n) - \phi(\alpha_0)) \Omega_{sd,n}^{-1/2} \phi(\alpha_0) = n Z_n^\prime \Omega_{sd,n}^{-1/2} Z_n + O_{\mathbb{P}} \left( \sqrt{d(n)} \right); \)

(2) for a fixed \(d(n) = d\), if \(\sqrt{d(n)} Z_n \Rightarrow N(0, I_d)\) then

\[ n(\phi(\tilde{\alpha}_n) - \phi(\alpha_0))^\prime \Omega_{sd,n}^{-1} (\phi(\tilde{\alpha}_n) - \phi(\alpha_0)) \Rightarrow \chi^2_d; \]
(3) if \( d(n) \to \infty \), \( d(n) = o(\sqrt{n} \mu_{x,n}^{-1}) \), then:
\[
\frac{n(\phi(\hat{a}_n) - \phi(a_0))^\prime \Omega_{sd,n}^{-1}(\phi(\hat{a}_n) - \phi(a_0)) - d(n)}{\sqrt{2d(n)}} \Rightarrow N(0,1).
\]

Theorem [A.5(3)] essentially states that the asymptotic distribution of \( n(\phi(\hat{a}_n) - \phi(a_0))^\prime \Omega_{sd,n}^{-1}(\phi(\hat{a}_n) - \phi(a_0)) \) is close to \( \chi^2_{d(n)} \). Moreover, as \( N(d(n), 2d(n)) \) is close to \( \chi^2_{d(n)} \) for large \( d(n) \), we can simulate from either distribution. However, as \( d(n) \) grows slowly (depends on the rate of \( \mu_{x,n} \)) \[15\] it might be more convenient to use \( \chi^2_{d(n)} \) in finite samples.

Let
\[
D_n \equiv \Omega_{sd,n}^{-1/2} \Omega_n^{-1} \Omega_{sd,n}^{1/2}
\]
which, under Assumption [A.1(iv)], is bounded in the sense that \( D_n \approx I_{d(n)} \) (see Lemma [C.1] in Appendix [C]). It is obvious that if \( \Sigma = \Sigma_0 \) then \( D_n = I_{d(n)} \). Note that \( D_n \) becomes \( ||u_n||^{-2} \) for a scalar-valued function \( \phi \).

The next result extends Theorem [A.3] for the SCLR statistic to the case of vector-valued functions \( \phi \) (of increasing dimension). Recall that \( \hat{Q}L_R(n, \phi_0) \) is the SCLR statistic \( \hat{Q}L_R(n, \phi_0) \) when \( \Sigma = \Sigma_0 \).

**Theorem A.6**  Let Conditions for Lemma [A.2] and Proposition [R.2] (in Appendix [B]) hold. Let Assumptions [A.2] and [A.3] hold with \( \max_{2 \leq |\ell| \leq 1} \{|t| : t \in \{B_n - B_n^{-1}\}t\} = O_P(b_n) \). Then: under the null hypothesis of \( \phi(a_0) = \phi_0 \),

1. \( \hat{Q}L_R(n, \phi_0) = (\sqrt{n} \Omega_{sd,n}^{-1/2} Z_n)^\prime D_n (\sqrt{n} \Omega_{sd,n}^{-1/2} Z_n) + o_P(\sqrt{d(n)}) \);
2. if \( \Sigma = \Sigma_0 \), then \( \hat{Q}L_R(n, \phi_0) = n Z_n^\prime \Omega_n^{-1} Z_n + o_P(\sqrt{d(n)}) \); for a fixed \( d(n) = d \), if \( \sqrt{n} \Omega_{sd,n}^{-1/2} Z_n \Rightarrow N(0, I_d) \) then \( \hat{Q}L_R(n, \phi_0) \Rightarrow \chi^2_d \);
3. if \( \Sigma = \Sigma_0 \) and \( d(n) \to \infty \), \( d(n) = o(\sqrt{n} \mu_{x,n}^{-1}) \), then:
\[
\frac{\hat{Q}L_R(n, \phi_0) - d(n)}{\sqrt{2d(n)}} \Rightarrow N(0,1).
\]

Theorem [A.6(2)] is a multivariate version of Theorem [A.3(2)]. Theorem [A.6(3)] shows that the optimally weighted SCLR preserves the Wilks phenomenon that is previously shown for the likelihood ratio statistic for semiparametric likelihood models. Again, as \( d(n) \) grows slowly with \( n \), Theorem [A.6(3)] essentially states that the asymptotic null distribution of \( \hat{Q}L_R(n, \phi_0) \) is close to \( \chi^2_{d(n)} \).

Given Theorems [A.5] and [A.6] and their proofs, it is obvious that we can repeat the results on the consistency of the bootstrap SCLR and sieve Wald as well as the local power properties of SCLR and sieve Wald tests to vector-valued \( \phi \) (of increasing dimension). We do not state these results here due to the length of the paper. We suspect that one could slightly improve Assumptions [A.2] and [A.3] and the coupling condition \( d(n) = o(\sqrt{n} \mu_{x,n}^{-1}) \) so that the dimension \( d(n) \) might grow faster with \( n \), but this will be a subject of future research.

\[15\] The condition \( d(n) = o(\sqrt{n} \mu_{x,n}^{-1}) \) is used for a coupling argument regarding \( \Omega_{sd,n}^{-1/2} \sqrt{n} Z_n \) and a multivariate Gaussian \( N(0, I_{d(n)}) \). See, e.g., Section 10.4 of Pollard (2001).
A.5. Sufficient conditions for LQA(i) and LQA^B(i) with series LS estimator \( \hat{m} \)

**Assumption A.4**

(i) \( X \) is a compact connected subset of \( \mathbb{R}^d \) with Lipschitz continuous boundary, and \( f_X \) is bounded and bounded away from zero over \( X \); (ii) The smallest and largest eigenvalues of \( \mathbb{E}[p^n(X)p^{n'}(X)'] \) are bounded and bounded away from zero for all \( J_n \); (iii) \( \sup_{x \in X} |p_j(x)| \leq \text{const.} < \infty \) for all \( j = 1, \ldots, J_n \) and \( J_n \log(J_n) = o(n) \) for \( p^n(X) \) a polynomial spline or trigonometric polynomial sieve; (iv) There is \( p^n(X) \) such that \( \sup_{x \in X} |g(x) - p^n(x)\pi| = O(b_m, J_n) = o(1) \) uniformly in \( g \in \{m(\cdot, \alpha) : \alpha \in A_{b_0}(\mathbb{R}) \}\).

Thanks to lemma 5.2 in Chen and Christensen (2013), Assumption A.4(iii) now allows \( J_n \log(J_n) = o(n) \) for \( p^n(X) \) being a (tensor-product) trigonometric polynomial in addition to a polynomial spline sieve. Let \( O_{on} \equiv \{\rho(\cdot, \alpha) - \rho(\cdot, \alpha_0) : \alpha \in N_{on} \}\). Denote \( 1 \leq \sqrt{C_n} = \int_0^1 \sqrt{1 + \log(N_{\Omega}(w(M_n, \delta, n)))} dw < \infty. \)

**Assumption A.5**

(i) There is a sequence \( \{\hat{\rho}_n(Z)\}_n \) of measurable functions such that \( \sup_{A_{b_0}(\mathbb{R})} |\rho(Z, \alpha)| = \hat{\rho}_n(Z) \) a.s.-\( Z \) and \( E[|\hat{\rho}_n(Z)|^2] \leq \text{const.} < \infty; \) (ii) there exist some \( \kappa \in (0, 1] \) and \( K : X \to \mathbb{R} \) measurable with \( E[|K(X)|^2] \leq \text{const.} < \infty; \) such that \( \forall \delta > 0, \)

\[
E \left[ \sup_{\alpha \in N_{on} : ||\alpha - \alpha'|| \leq \delta} \|\hat{\rho}(Z, \alpha) - \rho(Z, \alpha')\|_e^2 \mid X = x \right] \leq K(x)^2 \delta^{2\kappa}; \forall \alpha' \in N_{on} \cup \{\alpha_0\} \text{ and all } n,
\]

and \( \max \{\{M_0 \delta_{n}^2, (M_n \delta_{n}, n)^{2\kappa}\} = (M_n \delta_{n})^{2\kappa}\} \max \{\{M_0 \delta_{n}, n)^{2\kappa}\} = o(1); \) (iv) \( \sup_{X} ||\Sigma(x) - \Sigma(x)|| \times (M_n \delta_{n}) = o_P \left(n^{-1/2}\right); \) \( \hat{\delta} \times \frac{\sqrt{\hat{\delta}}}{2} = \max \left\{ \sqrt{\frac{\hat{\delta}}{2}}, b_m, J_n \right\} = o(n^{-1/4}). \)

Let \( \hat{m}(X, \alpha) \equiv \left( \sum_{i=1}^n m(X, \alpha)p^n(X_i) \right)' (P'P)^{-1} p^n(X) \) be the LS projection of \( m(X, \alpha) \) onto \( p^n(X) \), and let \( g(X, u_n) \equiv \left\{ \frac{dm(X, \alpha)}{d\alpha} | u_n^* \right\} \Sigma(X)^{-1} \) and \( g(X, u_n) \) be its LS projection onto \( p^n(X) \).

**Assumption A.6**

(i) \( E_{P \sim \Xi} \left[ \left\| \frac{dm(X, \alpha)}{d\alpha} | u_n^* \right\|_e^2 - \frac{dm(X, \alpha)}{d\alpha} | u_n^* \right\|_e^2 \right] (M_n \delta_{n})^2 = o(n^{-1}); \)

(ii) \( E_{P \sim \Xi} \left[ \left\| \Sigma(X, u_n^*) - g(X, u_n^*) \right\|_e^2 \right] (M_n \delta_{n})^2 = o(n^{-1}); \)

(iii) \( \sup_{N_{on}} n^{-1} \sum_{i=1}^n \left\{ \left\| m(X, \alpha) \right\|_e^2 - E[\left\| m(X, \alpha) \right\|_e^2] \right\} = o_P(n^{-1/2}); \)

(iv) \( \sup_{N_{on}} n^{-1} \sum_{i=1}^n \left\{ g(X, u_n^*) | m(X, \alpha) - E[g(X, u_n^*) m(X, \alpha)] \right\} = o_P(n^{-1/2}). \)

**Assumption A.7**

(i) \( m(X, \alpha) \) is twice continuously pathwise differentiable in \( \alpha \in N_{on}, \) a.s.-\( X; \)

(ii) \( E \left[ \sup_{\alpha \in N_{on}} \left\| \frac{dm(X, \alpha)}{d\alpha} | u_n^* \right\|_e^2 - \frac{dm(X, \alpha)}{d\alpha} | u_n^* \right\|_e^2 \right] (M_n \delta_{n})^2 = o(n^{-1}); \)
(iii) \( E \left[ \sup_{\alpha \in \mathcal{N}_{osn}} \left| \frac{d^2 m(X, \alpha)}{d \alpha^2} [u_n^*, u_n^*] \right|^2 \right] \times (M_n \delta_n)^2 = o(1) \); (iv) Uniformly over \( \alpha_1 \in \mathcal{N}_{os} \) and \( \alpha_2 \in \mathcal{N}_{osn} \),

\[
E \left[ g(X, u_n^*) \left( \frac{dm(X, \alpha_1)}{d \alpha} [\alpha_2 - \alpha_0] - \frac{dm(X, \alpha_0)}{d \alpha} [\alpha_2 - \alpha_0] \right) \right] = o(n^{-1/2}).
\]

Assumptions \( A.4 \) and \( A.5 \) are comparable to those imposed in Chen and Pouzo (2009), except that we use the scaled sieve Riesz representer \( u \). These assumptions ensure that the sample criterion function \( Q_n \) is well approximated by a “smooth” version of it. Assumptions \( A.6 \) and \( A.7 \) are similar to those imposed in Ai and Chen (2003), Ai and Chen (2007) and Chen and Pouzo (2009), except that we use the scaled sieve Riesz representer \( u_n^* \). This is because we allow for possibly irregular functionals (i.e., possibly \( \|v_n^*\| \to \infty \)), while the above mentioned papers only consider regular functionals (i.e., possibly \( \|v_n^*\| \to \|v^*\| < \infty \)). We refer readers to these papers for detailed discussions and verifications of these assumptions in examples of the general model (1.1).

A.6. Lemmas for series LS estimator \( \hat{m}(x, \alpha) \) and its bootstrap version

The next lemma (Lemma A.2) extends Lemma C.3 of Chen and Pouzo (2012a) and Lemma A.1 of Chen and Pouzo (2009) to the bootstrap version. Denote

\[
\ell_n(x, \alpha) \equiv \hat{m}(x, \alpha) + m(x, \alpha_0) \quad \text{and} \quad \ell_n^B(x, \alpha) \equiv \hat{m}(x, \alpha) + m^B(x, \alpha_0).
\]

Lemma A.2 Let \( \hat{m}^B(\cdot, \alpha) \) be the bootstrap series LS estimator (2.16). Let Assumptions \( 3.1, 3.4 \), \( 4.1\)(ii), \( A.4 \), \( A.5\)(i)(ii), and \( \text{Boot.1 or Boot.2} \) hold. Then:

1. For all \( \delta > 0 \), there is a \( M(\delta) > 0 \) such that for all \( M \geq M(\delta) \),

\[
P_{\infty < \infty} \left( \sup_{\alpha \in \mathcal{N}_{osn}} \frac{\tau_n}{n} \sum_{i=1}^{n} \| \hat{m}^B(X_i, \alpha) - \ell_n^B(X_i, \alpha) \|_c^2 \geq M | Z_n^* \| \geq \delta \right) < \delta
\]

eventually, with \( \tau_n^{-1} \equiv (\delta_n)^2 (M_n \delta_{zn})^{2c} \).

2. For all \( \delta > 0 \), there is a \( M(\delta) > 0 \) such that for all \( M \geq M(\delta) \),

\[
P_{\infty < \infty} \left( \sup_{\alpha \in \mathcal{N}_{osn}} \frac{\tau_n}{n} \sum_{i=1}^{n} \| \ell_n^B(X_i, \alpha) \|_c^2 \geq M | Z_n^* \| \geq \delta \right) < \delta
\]

eventually, with

\[
(\tau_n^{-1}) = \max \left\{ \frac{J_n}{n}, \ell_{m,J_n}^2, (M_n \delta_n)^2 \right\} = \text{const.} \times (M_n \delta_n)^2.
\]

3. Let Assumption \( A.3 \)(iii) hold. For all \( \delta > 0 \), there is \( N(\delta) \) such that, for all \( n \geq N(\delta) \),

\[
P_{\infty < \infty} \left( \sup_{\mathcal{N}_{osn}} \frac{s_n}{n} \sum_{i=1}^{n} \| \hat{m}^B(X_i, \alpha) \|_c^2 - \sum_{i=1}^{n} \| \ell_n^B(X_i, \alpha) \|_c^2 \geq \delta | Z_n^* \| \geq \delta \right) < \delta
\]
with
\[s_n^{-1} \leq (\delta_n)^2 (M_n \delta_{s,n}) \sqrt{C_n} \max \left\{ (M_n \delta_{s,n})^2 \sqrt{C_n}, M_n \right\} \] \(L_n = o(n^{-1})\),

where \(\{L_n\}_{n=1}^\infty\) is a slowly divergent sequence of positive real numbers (such a choice of \(L_n\) exists under assumption \(A.5(iii)\)).

Recall that
\[Z_{\omega} = \frac{1}{n} \sum_{i=1}^{n} \omega_{i,n} \left( \frac{dm(X_i, \alpha)}{d\alpha} \right) [u_{i,n}^*] \Sigma(X_i)^{-1} \rho(Z_i, \alpha_0) = \frac{1}{n} \sum_{i=1}^{n} g(X_i, u_{i,n}^*) \omega_{i,n} \rho(Z_i, \alpha_0).\]

**Lemma A.3** Let all of the conditions for Lemma A.2(2) hold. If Assumptions A.5(iv), A.6 and A.7(i)(ii)(iv) hold, then: for all \(\delta > 0\), there is a \(N(\delta)\) such that for all \(n \geq N(\delta)\),

\[P_Z \sup_{N_{osn}} \sqrt{n} \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \left( \frac{d \tilde{m}(X_i, \alpha)}{d\alpha} \right) [u_{i,n}^*] \right) \right] \Sigma(X_i)^{-1} \rho(Z_i, \alpha_0) \geq \delta \rightarrow Z_n^* \geq \delta < \delta.\]

**Lemma A.4** Let all of the conditions for Lemma A.2(2) hold. If Assumption A.7(i)(iii) holds, then: for all \(\delta > 0\), there is a \(N(\delta)\) such that for all \(n \geq N(\delta)\),

\[P_Z \sup_{N_{osn}} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d \tilde{m}(X_i, \alpha)}{d\alpha} \right) [u_{i,n}^*] \Sigma(X_i)^{-1} \rho(Z_i, \alpha_0) \right) \geq \delta \rightarrow Z_n^* \geq \delta < \delta.\]

**Lemma A.5** Let Assumptions 3.1(iv), 3.4(i), 4.1(iii), A.4, A.6(i), A.7(ii) hold. Then:
(1) For all \(\delta > 0\) there is a \(M(\delta) > 0\), such that for all \(M \geq M(\delta)\),

\[P_Z \sup_{N_{osn}} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d \tilde{m}(X_i, \alpha)}{d\alpha} \right) [u_{i,n}^*] \right) \Sigma^{-1}(X_i) \left( \frac{d \tilde{m}(X_i, \alpha)}{d\alpha} \right) [u_{i,n}^*] \geq M \) \(< \delta.\)

(2) If in addition, Assumption B holds, then: For all \(\delta > 0\), there is a \(N(\delta)\) such that for all \(n \geq N(\delta)\),

\[P_Z \sup_{N_{osn}} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d \tilde{m}(X_i, \alpha)}{d\alpha} \right) [u_{i,n}^*] \right) \Sigma^{-1}(X_i) \left( \frac{d \tilde{m}(X_i, \alpha)}{d\alpha} \right) [u_{i,n}^*] - ||u_{i,n}^*||^2 \geq \delta \rightarrow Z_n^* \geq \delta.\]
Supplement to “Sieve Wald and QLR Inference on Semi/Nonparametric Conditional Moment Models” by X. Chen and D. Pouzo

This supplemental material consists of Appendices B, C and D to the paper “Sieve Wald and QLR Inference on Semi/Nonparametric Conditional Moment Models” by X. Chen and D. Pouzo.

APPENDIX B: ADDITIONAL RESULTS AND PROOFS OF THE RESULTS IN THE MAIN TEXT

In Appendix B, we provide the proofs of all the lemmas, theorems and propositions stated in the main text. Additional results on consistent sieve variance estimators and bootstrap sieve t statistics are also presented.

B.1. Proofs for Section 3 on basic conditions

Proof of Lemma 3.3. For Result (1). Observe that \( \frac{d\phi(\alpha_0)}{d\alpha} \) is bounded on \( (V, \| \cdot \|) \); and in this case equation (3.4) holds. By definitions of \( v_n^* \) and \( v^* \), we have: \( \frac{d\phi(\alpha_0)}{d\alpha} [v] = \langle v^*_n, v \rangle \) and \( \frac{d\phi(\alpha_0)}{d\alpha} [v] = \langle v^*, v \rangle \) for all \( v \in V_{k(n)} \). Thus

\[
\langle v^* - v_n^*, v \rangle = 0 \quad \text{for all } v \in V_{k(n)} \quad \text{and} \quad \| v^* \|^2 = \| v^* - v_n^* \|^2 + \| v_n^* \|^2.
\]

Since \( V_{k(n)} \) is a finite dimensional Hilbert space we have \( v_n^* = \arg \min_{v \in V_{k(n)}} \| v^* - v \| \). Since \( V_{k(n)} \) is dense in \((V, \| \cdot \|)\) we have \( \| v^* - v_n^* \| \to 0 \) and \( \| v_n^* \| \to \| v^* \| < \infty \) as \( k(n) \to \infty \).

For Result (2). We show this part by contradiction. That is, assume that \( \lim_{k(n) \to \infty} \| v_n^* \| = C^* < \infty \). Since \( \frac{d\phi(\alpha_0)}{d\alpha} \) is unbounded under \( \| \cdot \| \) in \( V \), we have: for any \( M > 0 \), there exists a \( v_M \in V \) such that \( \| v_M^* \| = \inf_{v \in V_{k(n)}} \| v^* - v \| \) implies \( \frac{d\phi(\alpha_0)}{d\alpha} [v_M] > M \| v_M \| \).

Since \( v_M \in V \), and \( \{ V_k \}_k \) is dense (under \( \| \cdot \|_s \) in \( V \), there exists a sequence \( (v_{n,M})_n \) such that \( v_{n,M} \in V_{k(n)} \) and \( \lim_{n \to \infty} \| v_{n,M} - v_M \|_s = 0 \). This result and the fact that \( \| \cdot \| \leq C \| \cdot \|_s \) for some finite \( C > 0 \), imply that \( \lim_{n \to \infty} \| v_{n,M} \| = \| v_M \| \). Also, since \( \frac{d\phi(\alpha_0)}{d\alpha} [v] \) is continuous or bounded on \((V, \| \cdot \|_s)\), we have:

\[
\lim_{n \to \infty} \left| \frac{d\phi(\alpha_0)}{d\alpha} [v_{n,M} - v_M] \right| = 0.
\]

Hence, there exists a \( N(M) \) such that

\[
\left| \frac{d\phi(\alpha_0)}{d\alpha} [v_{n,M}] \right| \geq M \| v_{n,M} \|
\]

for all \( n \geq N(M) \). Since \( v_{n,M} \in V_{k(n)} \), the previous inequality implies that

\[
\| v_n^* \| = \sup_{v \in V_{k(n)}, \| v \| \neq 0} \left| \frac{d\phi(\alpha_0)}{d\alpha} [v] \right| \geq M
\]
for all \( n \geq N(M) \). Since \( M \) is arbitrary we have \( \lim_{n \to \infty} \| v_n^* \| = \infty \). A contradiction. 

Q.E.D.

B.2. Proofs for Section 4 on sieve t (Wald) and SCLR

**Lemma B.1** Let \( \hat{\alpha}_n \) be the PSMD estimator (2.2) and conditions for Lemma 3.2 hold. Let Assumptions 3.5(i) and 3.6(i) hold. Then:

\[
\sqrt{n}(u_n^*, \hat{\alpha}_n - \alpha_0) = -\sqrt{n}Z_n + o_{P_\infty}(1).
\]

**Proof of Lemma B.1** We note that \( n^{-1} \sum_{i=1}^n \| \hat{m}(X_i, \alpha) \|_2^2 = \tilde{Q}_n(\alpha) \). By Assumption 3.6(i), we have: for any \( \epsilon \in T_n \),

\[
\begin{align*}
 &\quad n^{-1} \sum_{i=1}^n \| \hat{m}(X_i, \hat{\alpha}_n + \epsilon_n u_n^*) \|_2^2 - n^{-1} \sum_{i=1}^n \| \hat{m}(X_i, \hat{\alpha}_n) \|_2^2 \\
&= 2\epsilon_n \{ Z_n + (u_n^*, \hat{\alpha}_n - \alpha_0) \} + \epsilon_n^2 B_n + o_{P_\infty}(r_n^{-1}),
\end{align*}
\]

where \( r_n^{-1} = \max \{ \epsilon_n, \epsilon_n n^{-1/2}, s_n^{-1} \} \) with \( s_n^{-1} = o(n^{-1}) \), and

\[
Z_n = n^{-1} \sum_{i=1}^n \left( \frac{dm(X_i, \alpha_0)}{d\alpha} [u_n^*] \right) \Sigma(X_i)^{-1} \rho(Z_i, \alpha_0).
\]

By adding

\[
E_n(\hat{\alpha}_n, \epsilon_n) \equiv o(n^{-1}) + \lambda_n \left( P_{en} \left( \hat{h}_n + \epsilon_n \frac{v_{n,\hat{\alpha}_n}^*}{\|v_{n,\hat{\alpha}_n}^*\|_2} \right) - P_{en} \left( \tilde{h}_n \right) \right)
\]

to both sides of equation (B.1), we have, by the definition of the approximate minimizer \( \hat{\alpha}_n \) and the fact \( \hat{\alpha}_n + \epsilon_n u_n^* \in A_{k(n)} \) that, for all \( \epsilon_n \in T_n \)

\[
2\epsilon_n \{ Z_n + (u_n^*, \hat{\alpha}_n - \alpha_0) \} + \epsilon_n^2 B_n + E_n(\hat{\alpha}_n, \epsilon_n) + o_{P_\infty}(r_n^{-1}) \geq 0.
\]

Or, equivalently, for any \( \delta > 0 \) and some \( N(\delta) \)

\[
(B.2)
\]

\[
P_{Z_\infty} \left( \forall \epsilon_n : \hat{\alpha}_n + \epsilon_n u_{n,\hat{\alpha}_n}^* \in N_{\infty, n}, 2\epsilon_n \{ Z_n + (u_n^*, \hat{\alpha}_n - \alpha_0) \} + \epsilon_n^2 B_n + E_n(\hat{\alpha}_n, \epsilon_n) \geq -\delta r_n^{-1} \right) \geq 1 - \delta
\]

for all \( n \geq N(\delta) \). In particular, this holds for \( \epsilon_n \equiv \pm \{ s_{n,\hat{\alpha}_n} + o(n^{-1/2}) \} \equiv \pm o(n^{-1/2}) \) since \( s_{n,\hat{\alpha}_n} = o(n^{-1/2}) \). Under this choice of \( \epsilon_n, r_n^{-1} = \max \{ s_{n,\hat{\alpha}_n}^{-1}, s_{n,\hat{\alpha}_n}^{-1} n^{-1/2} \} \). Moreover Assumptions 3.2(i)(ii) and 3.4(iv) imply that \( E(\hat{\alpha}_n, \epsilon_\alpha) = o_{P_\infty}(n^{-1}) \). Thus \(
\sqrt{\epsilon_n} E(\hat{\alpha}_n, \epsilon_\alpha) = o_{P_\infty}(\sqrt{\epsilon_n} n^{-1}) = o_{P_\infty}(1). \) Thus, from equation (B.2), it follows,

\[
P_{Z_\infty} \left( A_n, A_{n,\delta} \geq \sqrt{\epsilon_n} \{ Z_n + (u_n^*, \hat{\alpha}_n - \alpha_0) \} \geq B_n, \delta \right) \geq 1 - \delta \]

eventually, where

\[
A_n, \delta \equiv -0.5\sqrt{\epsilon_n} B_n - \delta \sqrt{\epsilon_n} r_n^{-1} + 0.5\delta
\]
and
\[ B_{n,\delta} \equiv -0.5\sqrt{n}\epsilon_n B_n - 0.5\sqrt{n}\delta\epsilon_n^{-1} r_n^{-1} - 0.5\delta \]

(here the 0.5\delta follows from the previous algebra regarding \( \sqrt{n}\epsilon_n^{-1}E(\hat{\alpha}_n, \epsilon_n) \)). Note that \( \sqrt{n}\epsilon_n = o(1) \), \( B_n = O_{P_2}\approx (1) \), and \( \sqrt{n}\epsilon_n^{-1} r_n^{-1} = \pm \max\{s_n^{-1/2} \sqrt{n}, 1\} \asymp \pm 1 \). Thus
\[ P_2\approx \{ 2\delta \geq \sqrt{n}\{Z_n + \langle u^*_\epsilon, \hat{\alpha}_n - \alpha_0 \rangle \} \geq -2\delta \geq 1 - \delta, \text{ eventually.} \]

Hence we have established \( \sqrt{n}(u^*_n, \hat{\alpha}_n - \alpha_0) = -\sqrt{n}Z_n + o_{P_2}\approx (1) \). Q.E.D.

**Proof of Theorem 4.1** By Lemma [B.1] and Assumption 3.6(ii), we immediately obtain: \( \sqrt{n}(u^*_n, \hat{\alpha}_n - \alpha_0) \Rightarrow N(0, 1) \). Hence, in order to show the result, it suffices to prove that
\[ \sqrt{n} \frac{\phi(\hat{\alpha}_n) - \phi(\alpha_0)}{\|v^*_n\|_{sd}} = \sqrt{n} \left( u^*_n, \hat{\alpha}_n - \alpha_0 \right) + o_{P_2}\approx (1). \]

By Riesz representation Theorem and the orthogonality property of \( \alpha \), it follows
\[ \frac{d\phi(\alpha_0)}{d\alpha}[\hat{\alpha}_n - \alpha_0, n] = \langle v^*_n, \hat{\alpha}_n - \alpha_0, n \rangle = \langle v^*_n, \hat{\alpha}_n - \alpha_0 \rangle. \]

By Assumptions 3.1(iv) and 3.5(i) we have \( ||v^*_n||_{sd} \asymp ||v^*_n|| \). This and Assumption 3.5(ii)(iii) imply
\[ \sqrt{n} \frac{\phi(\hat{\alpha}_n) - \phi(\alpha_0)}{\|v^*_n\|_{sd}} = \sqrt{n} \left( u^*_n, \hat{\alpha}_n - \alpha_0 \right) + o_{P_2}\approx (1) \]
\[ = \sqrt{n} ||v^*_n||_{sd}^{-1} d\phi(\alpha_0) \hat{\alpha}_n - \alpha_0, n \]  
\[ = \sqrt{n} ||v^*_n||_{sd}^{-1} d\phi(\alpha_0) \hat{\alpha}_n - \alpha_0, n + o_{P_2}\approx (1) \]
\[ = \sqrt{n} ||v^*_n||_{sd}^{-1} \left( u^*_n, \hat{\alpha}_n - \alpha_0 \right) + o_{P_2}\approx (1). \]

Thus
\[ \sqrt{n} \frac{\phi(\hat{\alpha}_n) - \phi(\alpha_0)}{\|v^*_n\|_{sd}} = \sqrt{n} \left( u^*_n, \hat{\alpha}_n - \alpha_0 \right) + o_{P_2}\approx (1), \]

and the claimed result now follows from Lemma [B.1] and Assumption 3.6(ii). Q.E.D.

**Proof of Lemma 4.1** By the definitions of \( \nabla_{k(n)} \) and the sieve Riesz representer \( v^*_n \in \nabla_{k(n)} \) of \( \frac{d\phi(\alpha_0)}{d\alpha} \) given in 3.6, we know that \( v^*_n = (v^*_n, v^*_n, \cdot) = (v^*_n, v^*_n, \psi_{k(n)}(\cdot)\beta^n) \in \nabla_{k(n)} \) solves the following optimization problem:
\[ \frac{d\phi(\alpha_0)}{d\alpha}[v^*_n] = \|v^*_n\|^2 = \sup_{v = (v_n, v_n) \in \nabla_{k(n)}, v \neq 0} \frac{\left| \frac{d\phi(\alpha_0)}{d\alpha} v_n + \frac{d\phi(\alpha_0)}{d\alpha} [v_n, \cdot] \right|^2}{\Sigma(X)^{-1} \left( \frac{d\phi(\alpha_0)}{d\alpha} \right) v_n} \]
\[ \sup_{\gamma = (v_n, \beta^n) \in \mathbb{R}^{k(n)}, \gamma \neq 0} \frac{\gamma^T (I_\Sigma\gamma)^\frac{1}{2}}{\gamma^T D_n \gamma}, \]

(B.3)
where $D_n = E \left[ \left( \frac{d m(X, \alpha_0)}{d \alpha} \left[ \psi^{(n)}(\cdot) \right] \right)' \Sigma(X)^{-1} \left( \frac{d m(X, \alpha_0)}{d \alpha} \left[ \psi^{(n)}(\cdot) \right] \right) \right]$ is a $(d_0 + k(n)) \times (d_0 + k(n))$ positive definite matrix such that

$$\gamma' D_n \gamma \equiv E \left[ \left( \frac{d m(X, \alpha_0)}{d \alpha} \left[ \psi^{(n)}(\cdot) \right] \right)' \Sigma(X)^{-1} \left( \frac{d m(X, \alpha_0)}{d \alpha} \left[ \psi^{(n)}(\cdot) \right] \right) \right]$$

for all $v = (v', \psi^{(n)}(\cdot) \beta)^' \in \mathbf{v}(k(n))$.

The sieve Riesz representation (4.6) becomes: for all $v = (v', \psi^{(n)}(\cdot) \beta)^' \in \mathbf{v}(k(n))$,

$$\frac{d \phi(\alpha)}{d \alpha}[v] = F_n \gamma = \langle v, v \rangle = \gamma_n^* D_n \gamma \quad \text{for all } \gamma = (\psi, \beta)^' \in \mathbb{R}^{d_0 + k(n)}.$$

It is obvious that the optimal solution of $\gamma$ in (B.3) or in (B.4) has a closed-form expression:

$$\gamma_n^* = (v_n^*, \beta_n^*)^' = D_n^* F_n.$$

The sieve Riesz representer is then given by

$$v_n^* = (v_n^*, \psi_{n,\cdot}(\cdot))^' = (v_n^*, \psi^{(n)}(\cdot) \beta_n^*)^' \in \mathbf{v}(k(n)).$$

Consequently, $\|v_n^*\|^2 = \gamma_n^* D_n \gamma_n^* = F_n^* D_n F_n \gamma_n^*$. Q.E.D.

**Another consistent variance estimator.** For $\|v_n^*\|^2_{\text{sd}} = E \left[ \langle S_n, S_n^* \rangle \right]$ given in (3.8) and (4.3), by Lemma 4.1, it has an alternative closed form expression:

$$\|v_n^*\|^2_{\text{sd}} = F_n^* D_n \Omega_n D_n F_n,$$

$$\Omega_n \equiv E \left[ \left( \frac{d m(X, \alpha_0)}{d \alpha} \left[ \psi^{(n)}(\cdot) \right] \right)' \Sigma(X)^{-1} \Omega_0(X) \Sigma(X)^{-1} \left( \frac{d m(X, \alpha_0)}{d \alpha} \left[ \psi^{(n)}(\cdot) \right] \right) \right] = \Omega_n.$$

Therefore, in addition to the sieve variance estimator $\|v_n^*\|^2_{\text{sd}}$ given in (4.7), we can define another simple plug-in sieve variance estimator:

$$\|\hat{v}_n\|^2_{\text{sd}} = \|\hat{v}_n\|^2_{n,\Sigma^{-1} \hat{\Sigma}_0 \Sigma^{-1}} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d m(X_i, \hat{\alpha}_n)}{d \alpha} \left[ \psi^{(n)}(\cdot) \right] \right)' \hat{\Sigma}_i^{-1} \hat{\Sigma}_0 \hat{\Sigma}_i^{-1} \left( \frac{d m(X_i, \hat{\alpha}_n)}{d \alpha} \left[ \psi^{(n)}(\cdot) \right] \right).$$

with $\hat{\Sigma}_0 = \hat{\Sigma}_0(X_i)$ where $\hat{\Sigma}_0(x)$ is a consistent estimator of $\Sigma_0(x)$, e.g. $\hat{E}_n[\rho(Z, \hat{\alpha}_n) \rho(Z, \hat{\alpha}_n)' | X = x]$, where $\hat{E}_n[\cdot | X = x]$ is some consistent estimator of a conditional mean function of $X$, such as a series, kernel or local polynomial based estimator.

The sieve variance estimator given in (B.5) can also be expressed as

$$\|\hat{v}_n\|^2_{\text{sd}} = \hat{\Omega}_n = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d m(X_i, \hat{\alpha}_n)}{d \alpha} \left[ \psi^{(n)}(\cdot) \right] \right)' \hat{\Sigma}_i^{-1} \hat{\Sigma}_0 \hat{\Sigma}_i^{-1} \left( \frac{d m(X_i, \hat{\alpha}_n)}{d \alpha} \left[ \psi^{(n)}(\cdot) \right] \right).$$

**Assumption B.1** (i) $\sup_{v \in \mathbf{v}(k(n))} | \langle v, v \rangle_{\Sigma^{-1} \Omega^{-1} \Sigma^{-1}} - \langle v, v \rangle_{\Sigma^{-1} \Omega^{-1} \Sigma^{-1}} | = o_p(1)$; and
(ii) \( \sup_{\sigma \in N_{\alpha,m}} \sup_{x \in X} \| \hat{E}_n(\rho(z, \alpha)\rho(z, \alpha)'|X = x) - E[\rho(z, \alpha)\rho(z, \alpha)'|X = x]\|_c = o_{P_{\infty}}(1). \)

**Theorem B.1** Let Assumption 4.4(i)-(iv), Assumption B.1 and assumptions for Lemma 3.2 hold. Then: Results (1) and (2) of Theorem 4.2 hold with \( \| \hat{\sigma}_n^* \|_{n,sd}^2 \) given in B.9.

Monte Carlo studies indicate that both sieve variance estimators perform well and similarly in finite samples.

**Proof of Theorems 4.2 and B.1** In the proof we use simplified notation \( o_{P_{\infty}}(1) = o_P(1) \). Also, Result (2) trivially follows from Result (1) and Theorem 4.1. So we only show Result (1). For Result (1), by the triangle inequality, we have that:

\[
\left| \frac{\| \hat{\sigma}_n^* \|_{n,sd} - \| \sigma_n^* \|_{n,sd}}{\| \sigma_n^* \|_{n,sd}} \right| \leq \left( \frac{\| \hat{\sigma}_n^* - \sigma_n^* \|_{n,sd}}{\| \sigma_n^* \|_{n,sd}} \right) + \left( \frac{\| \sigma_n^* \|_{n,sd} - \| \sigma_n^* \|_{n,sd}}{\| \sigma_n^* \|_{n,sd}} \right).
\]

This and the fact \( \frac{\| \hat{\sigma}_n^* - \sigma_n^* \|_{n,sd}}{\| \sigma_n^* \|_{n,sd}} \geq \frac{\| \hat{\sigma}_n^* - \sigma_n^* \|_{n,sd}}{\| \sigma_n^* \|_{n,sd}} \) (under Assumption 3.1(iv)) imply that Result (1) follows from:

\[
\| \hat{\sigma}_n^* - \sigma_n^* \|_{n,sd} = o_P(1),
\]

and

\[
\left| \frac{\| \hat{\sigma}_n^* \|_{n,sd} - \| \sigma_n^* \|_{n,sd}}{\| \sigma_n^* \|_{n,sd}} \right| = o_P(1).
\]

We will establish results (B.7) and (B.8) in Step 1 and Step 2 below.

**Step 1.** Observe that result (B.7) is about the consistency of the empirical sieve Riesz representer \( \hat{\sigma}_n^* \) in the \( \| \cdot \| \) norm, which is the same whether we use \( \hat{\rho}_n \hat{\rho}_n' \) or \( \hat{\Sigma}_0^i \) to compute the sieve variance estimators (4.7) or (4.5). By the Riesz representation theorem, we have for all \( v \in \nabla_{k(n)} \),

\[
\frac{d\phi(\hat{\sigma}_n)}{d\alpha}(v) = \langle \hat{\sigma}_n^*, v \rangle_{n,\Sigma} - \langle \sigma_n^*, v \rangle_{n,\Sigma}.
\]

Hence, by Assumption 4.1(i), we have:

\[
op(1) = \sup_{v \in \nabla_{k(n)}} \left| \langle \hat{\sigma}_n^*, v \rangle_{n,\Sigma} - \langle \sigma_n^*, v \rangle_{n,\Sigma} \right| \]

\[
= \sup_{v \in \nabla_{k(n)}} \left| \langle \hat{\sigma}_n^*, v \rangle_{n,\Sigma} - \langle \sigma_n^*, v \rangle_{n,\Sigma} \right| \| \sigma_n^* \|_{n,\Sigma} + \langle \sigma_n^*, v \rangle_{n,\Sigma} \]

\[
\leq \sup_{v \in \nabla_{k(n)}} \left| \langle \hat{\sigma}_n^* - \sigma_n^*, v \rangle_{n,\Sigma} \right| \| \sigma_n^* \|_{n,\Sigma} + \sup_{v \in \nabla_{k(n)}} \| \sigma_n^* \|_{n,\Sigma} \times \| \sigma_n^* \|_{n,\Sigma},
\]

where \( \Sigma \equiv v/\| v \| \) and \( \hat{\Sigma}_n^i \equiv \hat{\sigma}_n^* / \| \hat{\sigma}_n^* \|_{n,\Sigma} \). First note that

\[
\left| \langle \hat{\sigma}_n^*, \Sigma \rangle_{n,\Sigma} - \langle \hat{\sigma}_n^*, \Sigma \rangle_{n,\Sigma} \right| \leq \left| \langle \hat{\sigma}_n^*, \Sigma \rangle_{n,\Sigma} - \langle \hat{\sigma}_n^*, \Sigma \rangle_{n,\Sigma} \right| + \left| \langle \sigma_n^*, \Sigma \rangle_{n,\Sigma} - \langle \sigma_n^*, \Sigma \rangle_{n,\Sigma} \right| \equiv |T_{1n}(\Sigma)| + |T_{2n}(\Sigma)|.
\]
By Assumption 4.1(ii), we have: $$\sup_{\omega \in \nabla_{k(n)}||\omega||=1} |T_{2n}(\omega)| = o_P(1).$$ Note that

$$T_{1n}(\omega) = n^{-1} \sum_{i=1}^{n} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{da} [\hat{\epsilon}_n] \right)^{'} \left\{ \Sigma^{-1}(X_i) - \Sigma^{-1}(X_i) \right\} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{da} [\omega] \right).$$

By the triangle inequality, Assumptions 3.1(iv) and 4.1(ii)(iii), we obtain

$$|T_{1n}(\omega)| \leq \sup_{x \in X} ||\Sigma^{-1}(x) - \Sigma^{-1}(x)|| \leq n^{-1} \sum_{i=1}^{n} \left| \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{da} [\hat{\epsilon}_n] \right|^2 \leq \sup_{x \in X} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{da} [\omega] \right)^2 = \sigma_P(1) \times \sigma_P(1) = o_P(1).$$

Hence

$$0 < \sup_{\omega \in \nabla_{k(n)}, \omega \neq 0} \left| \frac{\hat{\epsilon}_n - \epsilon_n^*}{|\hat{\epsilon}_n^*|} \right| = o_P(1 + ||\hat{\epsilon}_n||).$$

In particular, for $$\omega = \hat{\epsilon}_n - \epsilon_n^*$$, this implies

$$\frac{||\hat{\epsilon}_n^* - \epsilon_n^*||}{||\hat{\epsilon}_n^*||} = o_P(1 + ||\hat{\epsilon}_n^*||).$$

Note that $$||\epsilon_n^*|| \geq \text{const.} > 0$$ and $$||\hat{\epsilon}_n^*|| \leq ||\hat{\epsilon}_n - \epsilon_n^*|| + 1$$, and thus, the previous equation implies

$$\frac{||\hat{\epsilon}_n^* - \epsilon_n^*||}{||\hat{\epsilon}_n^*||} (1 - o_P(1)) = o_P(1) \quad \text{and} \quad \frac{||\hat{\epsilon}_n^*||}{||\epsilon_n^*||} = O_P(1).$$

Step 2. We now show that result (B.8) holds for the sieve variance estimators $$||\hat{\sigma}_{n,sd}||^2$$ defined in (B.7) and (B.5). By Assumption 3.1(iv), we have:

$$\frac{||\hat{\sigma}_{n,sd}||^2}{||\epsilon_n^*||^2} = \frac{||\hat{\sigma}_{n,sd}||^2}{||\epsilon_n^*||^2} \times \frac{||\epsilon_n^*||^2}{||\epsilon_n^*||^2} = \left( \frac{||\epsilon_n^*||^2}{||\epsilon_n^*||^2} + 1 \right) \times \frac{||\epsilon_n^*||^2}{||\epsilon_n^*||^2} - 1 \times \frac{||\epsilon_n^*||^2}{||\epsilon_n^*||^2} = \left| \frac{||\epsilon_n^*||^2}{||\epsilon_n^*||^2} \right| \left| \frac{||\epsilon_n^*||^2}{||\epsilon_n^*||^2} \right| - 1 \times \frac{||\epsilon_n^*||^2}{||\epsilon_n^*||^2} = \left| \frac{||\epsilon_n^*||^2}{||\epsilon_n^*||^2} \right| \left| \frac{||\epsilon_n^*||^2}{||\epsilon_n^*||^2} \right| - 1 \times \frac{||\epsilon_n^*||^2}{||\epsilon_n^*||^2} = O_P(1),$$

where $$\hat{\sigma}_{n,sd}^2 = \frac{||\epsilon_n^*||^2}{||\epsilon_n^*||^2} = O_P(1)$$ (by Step 1), and $$||\epsilon_n^*||^2 = O_P(1)$$ (by Assumption 3.1(iv) and i.i.d. data). Thus, it suffices to show that

$$||\hat{\sigma}_{n,sd}||^2 = o_P(1).$$

Step 2a for the estimator $$||\hat{\sigma}_{n,sd}||^2$$ defined in (B.7). We now establish the result (B.9) when the sieve variance estimator is defined in (4.7).

Let $$\tilde{M}(Z; \alpha) = \Sigma^{-1} \rho(Z, \alpha) \rho(Z, \alpha) \Sigma^{-1}$$ and $$M(z, \alpha_0) \equiv \Sigma^{-1}(z) \rho(z, \alpha_0) \rho(z, \alpha_0) \Sigma^{-1}(z)$$ and $$M(z, \alpha_0) = M(z, \alpha_0)$$. Also let $$\tilde{T}_i\epsilon_n = \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{da}[\epsilon_n], \tilde{T}_i\epsilon_n = \frac{d\hat{m}(X_i, \hat{\alpha}_n)}{da}[\epsilon_n],$$ and $$\Sigma(x, \alpha) \equiv \mathbb{E} [\rho(Z, \alpha) \rho(Z, \alpha)^{'} | x].$$
It turns out that \(|\|\hat{\Sigma}_n^*\|_{n, sd}^2 - \|\hat{\Sigma}_n\|_{sd}^2|\) can be bounded above by
\[
\sup_{v_n \in \mathbf{v}_n} \left| n^{-1} \sum_{i=1}^{n} \hat{T}_i[v_n]\hat{M}(Z_i, \hat{\alpha}_n)\hat{T}_i[v_n] - n^{-1} \sum_{i=1}^{n} \hat{T}_i[v_n]\hat{M}\hat{T}_i[v_n] \right|
\]
\[
+ \sup_{v_n \in \mathbf{v}_n} \left| n^{-1} \sum_{i=1}^{n} \hat{T}_i[v_n]\hat{M}\hat{T}_i[v_n] - E[T_i[v_n]\hat{M}\hat{T}_i[v_n]] \right|
\]
\[
+ \sup_{v_n \in \mathbf{v}_n} \left| E[T_i[v_n]\hat{M}\hat{T}_i[v_n]] - E[T_i[v_n]\Sigma^{-1}(X_i)\Sigma(X_i, \alpha_0)\Sigma^{-1}(X_i)\hat{T}_i[v_n]] \right|
\]
\[\equiv A_{1n} + A_{2n} + A_{3n}.\]

Note that \(A_{3n} = 0\) by the fact that \(E[M_i|X_i] = \Sigma^{-1}(X_i)\Sigma(X_i, \alpha_0)\Sigma^{-1}(X_i),\) and that \(A_{2n} = o_P(1)\) by Assumption 4.1(v). Thus it remains to show that \(A_{1n} = o_P(1).\) We note that
\[
A_{1n} \leq \sup_{z} \sup_{\alpha \in \mathcal{N}_{o,n}} \|\hat{M}(z, \alpha) - M(z, \alpha_0)\|_e \sup_{v_n \in \mathbf{v}_n} \left| n^{-1} \sum_{i=1}^{n} \hat{T}_i[v_n]\hat{M}\hat{T}_i[v_n] \right|
\]
\[\leq \text{Const.} \times \sup_{z} \sup_{\alpha \in \mathcal{N}_{o,n}} \|\hat{M}(z, \alpha) - M(z, \alpha_0)\|_e \sup_{v_n \in \mathbf{v}_n} \left| n^{-1} \sum_{i=1}^{n} \hat{T}_i[v_n]\hat{M}(Z_i, \alpha_0)\hat{T}_i[v_n] \right|
\]
where the first inequality follows from the fact that for matrices \(A\) and \(B, |A'B| \leq ||A||_e^2 ||B||_e\) and Assumption 3.1(iv). Observe that by Assumptions 4.1(iii)(iv) and 3.1(iv),
\[
\sup_{z} \sup_{\alpha \in \mathcal{N}_{o,n}} \|\hat{M}(z, \alpha) - M(z, \alpha_0)\|_e \leq \sup_{z} \sup_{\alpha \in \mathcal{N}_{o,n}} \|\hat{\Sigma}^{-1}(x)(\rho(z, \alpha)\rho(z, \alpha)' - \rho(z, \alpha_0)\rho(z, \alpha_0)'\hat{\Sigma}^{-1}(x)\|_e
\]
\[+ \sup_{z} \|\hat{\Sigma}^{-1}(x)\rho(z, \alpha_0)\rho(z, \alpha_0)'\hat{\Sigma}^{-1}(x) - \Sigma^{-1}(x)\rho(z, \alpha_0)\rho(z, \alpha_0)'\Sigma^{-1}(x)\|_e.
\]
The first term in the RHS is \(o_P(1)\) by Assumptions 4.1(iii)(iv) and 3.1(iv); the second term in the RHS is also of order \(o_P(1)\) by Assumptions 4.1(iii) and 3.1(iv) and the fact that \(\rho(Z, \alpha_0)\rho(Z, \alpha_0)' = O_P(1).\) By Assumption 4.1(v), \(\sup_{v_n \in \mathbf{v}_n} \left| n^{-1} \sum_{i=1}^{n} \hat{T}_i[v_n]\hat{M}(Z_i, \alpha_0)\hat{T}_i[v_n] \right| = O_P(1).\) Hence \(A_{1n} = o_P(1)\) and result \((B.9)\) holds.

**Step 2b for the Estimator \(|\|\hat{\Sigma}_n^*\|_{n, sd}^2|\)** defined in \((B.5).\) Since we already provide a detailed proof for result \((B.9)\) in Step 2a for the case of \((4.7),\) here we present a more succinct proof for the case of \((B.5).\)

By the triangle inequality,
\[
\|\|\hat{\Sigma}_n^*\|_{n, sd}^2 - \|\hat{\Sigma}_n\|_{sd}^2\| \leq \|\|\hat{\Sigma}_n^*\|_{n, sd}^2 - \|\hat{\Sigma}_n\|_{n, \Sigma^{-1}\Sigma_0\Sigma^{-1}}\| + \|\|\hat{\Sigma}_n\|_{n, \Sigma^{-1}\Sigma_0\Sigma^{-1}} - \|\hat{\Sigma}_n^*\|_{n, sd}^2\| \equiv R_{1n} + R_{2n}.
\]

By Assumptions 3.1(iv), 4.1(iii)(iv) and \((B.1)\) we have:
\[
\sup_{x \in X} \|\hat{\Sigma}^{-1}(x)\hat{\Sigma}_0(x)\hat{\Sigma}^{-1}(x) - \Sigma^{-1}(x)\Sigma_0(x)\Sigma^{-1}(x)\|_e = o_P(1),
\]
where \(\hat{\Sigma}_0(x) = \hat{E}_n[\rho(Z, \alpha)\rho(Z, \alpha)'|x]\). Therefore, by Assumptions 3.1(iv) and 4.1(ii)
and similar algebra to the one used to bound $T_{1n} (\pi)$, we have:

$$R_{1n} \leq o_P (1) \times n^{-1} \sum_{i=1}^{n} \left| \frac{d \hat{m}_n(X_i, \hat{\alpha}_n)}{d \alpha} [\hat{\alpha}_n]^{2} \right| = o_P (1) \times O_P (1) = o_P (1).$$

Also by Assumption [B.1] $R_{2n} = o_P (1)$. Thus result (B.9) holds. Q.E.D.

Before we prove Theorem 4.3 we introduce some notation that will simplify the presentation of the proofs. For any $\phi \in \mathbb{R}$ let $A(\phi) \equiv \{ \alpha \in \mathcal{A} : \phi (\alpha) = \phi \}$, and $A_{k(n)}(\phi) \equiv A(\phi) \cap A_{k(n)}$. In particular, let $\mathcal{A}^0 \equiv A(\phi (\alpha_0))$ and $A_{k(n)}^0 \equiv A_{k(n)}(\phi (\alpha_0))$.

Also, we need to show that for any deviation of $\alpha$ of the type $\alpha + tu_n$, there exists a $t$ such that $\hat{\phi}(\alpha + tu_n)$ is “close” to $\hat{\phi}(\alpha_0)$. Formally,

**Lemma B.2** Let Assumption 3.3 hold. (1) For any $n \in \{1, 2, \ldots \}$, any $r \in \{ r : |r| \leq 2M_n ||v_n^*|| \delta_n \}$, and any $\alpha \in \mathcal{N}_{osn}$, there exists a $t \in \mathcal{T}_n$ such that $\hat{\phi}(\alpha + tu_n) - \hat{\phi}(\alpha_0) = r$ and $\alpha + tu_n \in A_{k(n)}$. (2) For any $r \in \{ r : |r| \leq ||v_n^*|| \tau_n \}$ and any $\alpha \in \mathcal{A}_{k(n)} : ||\alpha - \alpha_0|| \leq \tau_n$ with some positive sequence $(\tau_n)_n$ such that $\tau_n = O(\delta_n)$, the $t$ in part (1) also satisfies $|t| \leq \max \{ C \tau_n, o(n^{-1/2}) \}$ for some constant $C > 0$.

**Proof of Lemma B.2** For Part (1), we first show that there exists a $t \in \mathcal{T}_n$ such that $\hat{\phi}(\alpha + tu_n) - \hat{\phi}(\alpha_0) = r$. By Assumption 3.5 there exists a $(F_n)_n$ such that $F_n \geq 0$ and $F_n = o(n^{-1/2} ||v_n^*||)$ and, for any $\alpha \in \mathcal{N}_{osn}$ and $t \in \mathcal{T}_n$,

$$\left| \hat{\phi}(\alpha + tu_n) - \hat{\phi}(\alpha_0) - \langle v_n^*, \alpha - \alpha_0 \rangle - t \frac{||v_n^*||^2}{||v_n^*||} \right| \leq F_n. \tag{B.10}$$

(note that by assumption 3.5 $F_n$ does not depend on $\alpha$ nor $t$).

For any $r \in \{ r : |r| \leq 2M_n ||v_n^*|| \delta_n \}$, we define $(t_l)_{l=1,2}$ as

$$t_l ||u_n^*||^2 = \langle u_n^*, \alpha - \alpha_0 \rangle + a_{l,n} F_n ||v_n^*||^{-1} \leq r ||v_n^*||^{-1}.$$

where $a_l = (-1)^{l+1}$. Note that, by assumption 3.5 (the second part), $||u_n^*||^{-2} \leq c^{-2}$, and thus

$$|t_l| \leq c^{-2} (||u_n^*|| \times ||\alpha - \alpha_0|| + 2F_n \times ||v_n^*||^{-1} + |r| \times ||v_n^*||^{-1}).$$

Without loss of generality, we can re-normalize $M_n$ so that $c^{-2}C < M_n$ and $C \geq 1$. Hence,

$$|t_l| \leq c^{-2} (||u_n^*|| \times ||\alpha - \alpha_0|| + 2F_n \times ||v_n^*||^{-1} + |r| \times ||v_n^*||^{-1})$$

$$= c^{-2} (||u_n^*|| \times ||\alpha - \alpha_0|| + 2F_n \times ||v_n^*||^{-1} + |r| \times ||v_n^*||^{-1} ||u_n^*||)$$

$$\leq c^{-2} (||u_n^*|| \times ||\alpha - \alpha_0|| + 2F_n \times ||v_n^*||^{-1} + |r| \times ||v_n^*||^{-1} ||u_n^*||) \leq 4M_n \delta_n,$$

where the third inequality follows from Assumption 3.5 (the second part), and the last inequality follows from the facts that $\alpha \in \mathcal{N}_{osn}$, $c^{-2}C2F_n \times ||v_n^*||^{-1} = o(n^{-1/2}) \leq M_n \delta_n$, $\alpha \in \{ r \leq 2M_n ||v_n^*|| \delta_n \}$. Thus, $t_l$ is a valid choice in the sense that $t_l \in \mathcal{T}_n$ for $l = 1, 2$. 

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Thus, this result and equation (B.10) imply
\[
\phi(\alpha + t_1 u_n^*) - \phi(\alpha_0) \leq (v_n^*, \alpha - \alpha_0) + t_1 \frac{\|v_n^*\|^2}{\|v_n^*\|_{sd}} + F_n
\]
\[
= \|v_n^*\|_{sd} \left( (u_n^*, \alpha - \alpha_0) + t_1 \|u_n^*\|^2 + F_n \right)
\]
\[
= r - F_n < r.
\]
Hence, \(\phi(\alpha + t_1 u_n^*) - \phi(\alpha_0) < r\). Similarly,
\[
\phi(\alpha + t_2 u_n^*) - \phi(\alpha_0) \geq (v_n^*, \alpha - \alpha_0) + t_2 \frac{\|v_n^*\|^2}{\|v_n^*\|_{sd}} - F_n
\]
\[
= \|v_n^*\|_{sd} \left( (u_n^*, \alpha - \alpha_0) + t_2 \|u_n^*\|^2 - F_n \right)
\]
\[
= r + F_n > r
\]
and thus \(\phi(\alpha + t_2 u_n^*) - \phi(\alpha_0) > r\). Since \(t \mapsto \phi(\alpha + tu_n^*)\) is continuous, there exists a \(t \in [t_1, t_2]\) such that \(\phi(\alpha + tu_n^*) - \phi(\alpha_0) = r\). Clearly, \(t \in \mathcal{T}_n\).

The fact that \(\alpha(t) \equiv \alpha + tu_n^* \in \mathcal{A}_k(n)\) for \(\alpha \in \mathcal{N}_{asn}\) and \(t \in \mathcal{T}_n\) follows from the fact that the sieve space \(\mathcal{A}_k(n)\) is assumed to be convex with non-empty interior. Part (2) can be proved in the same way as that for Part (1). Q.E.D.

**Proof of Theorem 4.3** Result (2) directly follows from Result (1) with \(\Sigma = \Sigma_0\) and \(\|u_n^*\| = 1\). The proof of Result (1) consists of several steps.

**Step 1.** For any \(t_n \in \mathcal{T}_n\) wpa1., by Assumption 3.6 and Lemma B.1, we have:
\[
0.5 \left( \tilde{Q}_n(\alpha_n(-t_n)) - \tilde{Q}_n(\alpha_n) \right) = -t_n \{Z_n^* + (u_n^*, \alpha_n - \alpha_0)\} + \frac{B_n}{2} t_n^2 + o_{P_{Z_{\infty}}} (r_n^{-1})
\]
(B.11)
\[
= \frac{B_n}{2} t_n^2 + o_{P_{Z_{\infty}}} (r_n^{-1}),
\]
where \(r_n^{-1} = \max \{t_n^2, t_n n^{-1/2}, s_n^{-1} \}\) and \(s_n^{-1} = o(n^{-1})\).

And under the null hypothesis, \(\hat{\alpha}_n^R \in \mathcal{N}_{asn} \cap \mathcal{A}_{k(n)}^0\) wpa1,
\[
0.5 \left( \tilde{Q}_n(\alpha_n^R(t_n)) - \tilde{Q}_n(\hat{\alpha}_n^R) \right) = t_n \{Z_n + (u_n^*, \alpha_n^R - \alpha_0)\} + \frac{B_n}{2} t_n^2 + o_{P_{Z_{\infty}}} (r_n^{-1})
\]
(B.12)
\[
= t_n \{Z_n + \frac{B_n}{2} t_n^2 + o_{P_{Z_{\infty}}} (r_n^{-1})\},
\]
where the last line follows from the fact that \(t_n (u_n^*, \alpha_n^R - \alpha_0) = o_{P_{Z_{\infty}}} (r_n^{-1})\). To show this, note that under the null hypothesis, \(\hat{\alpha}_n^R \in \mathcal{N}_{asn} \cap \mathcal{A}_{k(n)}^0\) wpa1. This and Assumption 3.5 ii) imply that
\[
\left| \phi(\alpha_n^R) - \phi(\alpha_0) \right| - \frac{d \phi(\alpha_0)}{d \alpha} \left| \hat{\alpha}_n^R - \alpha_0 \right| = o_{P_{Z_{\infty}}} (n^{-1/2} \|u_n^*\|).
\]
Thus
\[
P_{Z_{\infty}} \left( \frac{\sqrt{n}}{\|u_n^*\|} \left| \frac{d \phi(\alpha_0)}{d \alpha} \left| \hat{\alpha}_n^R - \alpha_0 \right| < \delta \right. \right) \geq 1 - \delta
\]
eventually. By similar calculations to those in the proof of Theorem 4.1, we have

$$P_{Z,\delta} \left( \frac{\sqrt{n}}{\delta} \left| \left( u_n^*, \hat{\alpha}_n - \alpha_0 \right) \right| < 1 - \delta \right) \geq 1 - \delta, \text{ eventually.}$$

Hence, \( (u_n^*, \hat{\alpha}_n - \alpha_0) = o_{P_{Z,\infty}}(n^{-1/2}) \), and thus \( t_n(u_n^*, \hat{\alpha}_n - \alpha_0) = o_{P_{Z,\infty}}(n^{-1/2}t_n) = o_{P_{Z,\infty}}(r_n^{-1}) \).

**Step 2.** We choose \( t_n = -Z_n B_n^{-1} \). Note that under assumption 3.6 \( t_n \in T_n \) wpal. By the definition of \( \hat{\alpha}_n \), we have, under the null hypothesis,

$$0.5 \left( \hat{Q}_n(\alpha_n^R) - \hat{Q}_n(\alpha_n) \right) \geq 0.5 \left( \hat{Q}_n(\alpha_n^R) - \hat{Q}_n(\alpha_n(t_n)) \right) - o_{P_{Z,\infty}}(n^{-1})
= 2B_n B_n^{-1} - o_{P_{Z,\infty}}(\max\{B_n^2 Z_n, B_n^{-1} Z_n^{-1} s_n^{-1}\}) - o_{P_{Z,\infty}}(n^{-1})
= 2B_n Z_n B_n^{-1} + o_{P_{Z,\infty}}(n^{-1}),$$

where the first inequality follows from the fact that, since \( t_n \in T_n \) and \( \hat{\alpha}_n R \in N_{\alpha_0} \) wpal, then \( \hat{\alpha}_n R(t_n) \in A_{k(n)} \) wpal; and the second line follows from equation 3.12 with \( t_n = -Z_n B_n^{-1} \).

**Step 3.** We choose \( t_n^* \in T_n \) wpal such that (a) \( \phi(\hat{\alpha}_n(t_n^*)) = \phi(\alpha_0) \), \( \hat{\alpha}_n (t_n^*) \in A_{k(n)} \); and (b) \( t_n^* = Z_n |\nu_n^*|/\|\nu_n^*\|_2^2 + o_{P_{Z,\infty}}(n^{-1/2}) = O_{P_{Z,\infty}}(n^{-1/2}) \).

Suppose such a \( t_n^* \) exists, then \( |r_n(t_n^*)|^{-1} = \max\{ (t_n^*)^2, t_n^* n^{-1/2}, o(n^{-1}) \} = O_{P_{Z,\infty}}(n^{-1}) \).

By the definition of \( \hat{\alpha}_n R \), we have, under the null hypothesis,

$$0.5 \left( \hat{Q}_n(\alpha_n^R) - \hat{Q}_n(\alpha_n) \right) \leq 0.5 \left( \hat{Q}_n(\alpha_n(t_n^*)) - \hat{Q}_n(\alpha_n) \right) + o_{P_{Z,\infty}}(n^{-1})
= t_n^* \left( \frac{Z_n}{\|\nu_n^*\|_2^2} + B_n \frac{t_n^*^2}{2} + o_{P_{Z,\infty}}(n^{-1}) \right)
= B_n \left( \frac{Z_n}{\|\nu_n^*\|_2^2} + o_{P_{Z,\infty}}(n^{-1}) \right)^2 + o_{P_{Z,\infty}}(n^{-1})
= 2B_n Z_n B_n^{-1} + o_{P_{Z,\infty}}(n^{-1}) = 2Z_n B_n^{-1} + o_{P_{Z,\infty}}(n^{-1}),$$

where the second line follows from Assumption 3.6(i) and the fact that \( t_n^* \) satisfying (b), \( |r_n(t_n^*)|^{-1} = O_{P_{Z,\infty}}(n^{-1}) \); the third line follows from equation 3.11 and the fact that \( t_n^* \) satisfying (b); and the last line follows from Assumptions 3.5(i) and 3.6(ii), \( B_n - ||\nu_n^*||_2^2 = O_{P_{Z,\infty}}(1) \) and \( u_n^* = v_n^*/\|v_n^*\|_{sd} \).

We now show that there is a \( t_n^* \in T_n \) wpal such that (a) and (b) hold. Denote \( r = \phi(\hat{\alpha}_n) - \phi(\alpha_0) \). Since \( \hat{\alpha}_n \in N_{\alpha_0} \) wpal and \( \phi(\hat{\alpha}_n) - \phi(\alpha_0) = O_{P_{Z,\infty}}(\|v_n^*\|/\sqrt{n}) \) (see the proof of Theorem 4.1), we have \(|r_1| \leq 2M_\alpha\|v_n^*\|\delta_n \). Thus, by Lemma 3.2, there is a \( t_n^* \in T_n \) wpal such that \( \hat{\alpha}_n(t_n^*) = \alpha_n + t_n^* u_n^* \in A_{k(n)} \) and \( \phi(\hat{\alpha}_n(t_n^*)) = \phi(\alpha_0) \), so (a) holds. Moreover, by Assumption 3.5(ii), such a choice of \( t_n^* \) also satisfies

$$\left| \phi(\hat{\alpha}_n(t_n^*)) - \phi(\alpha_0) - \frac{d\phi(\alpha_0)}{d\alpha_1} (\hat{\alpha}_n - \alpha_0 + t_n^* u_n^*) \right| = O_{P_{Z,\infty}}(\|v_n^*\|/\sqrt{n}).$$
By Assumption 3.5(i) and the definition of $u^*_n = v^*_n / \| v^*_n \|_{sd}$ we have: $\frac{d\phi(\alpha)}{d\alpha}[t^*_n u^*_n] = t^*_n \| v^*_n \|_{sd}$. Thus

$$P_Z \approx \left( \frac{\sqrt{n}}{\| v^*_n \|} \left| \frac{d\phi(\alpha)}{d\alpha}[\hat{\alpha}_n - \alpha_0] + t^*_n \| v^*_n \|_{sd} \right| < \delta \right) \geq 1 - \delta$$

eventually. By similar algebra to that in the proof of Theorem 4.1 it follows that the LHS of the equation above is majorized by

$$P_Z \approx \left( \frac{\sqrt{n}}{\| v^*_n \|} \left| \frac{d\phi(\alpha)}{d\alpha}[\hat{\alpha}_n - \alpha_0] + t^*_n \| v^*_n \|_{sd} \right| < \delta \right) + \delta$$

$$= P_Z \approx \left( \frac{\sqrt{n}}{\| v^*_n \|} - Z_n \| v^*_n \|_{sd} + t^*_n \| v^*_n \|_{sd}^2 < \delta \right) + \delta$$

where the second line follows from the proof of Lemma B.1. Since $\| v^*_n \|_{sd} \approx const.$ (by Assumption 3.5(i)), we obtain:

$$P_Z \approx \left( \frac{\sqrt{n}}{\| v^*_n \|} t^*_n - Z_n \| v^*_n \|_{sd} + t^*_n \| v^*_n \|_{sd}^2 < \delta \right) \geq 1 - \delta, \ \text{eventually.}$$

Since $\sqrt{n}Z_n = O_{P_{\infty}}(1)$ (Assumption 3.6(ii)), we have: $t^*_n = O_{P_{\infty}}(n^{-1/2})$, and in fact, $\sqrt{n}t^*_n = \sqrt{n}Z_n \| v^*_n \|_{sd} + o_{P_{\infty}}(1)$ and hence (b) holds. Q.E.D.

Let $A^R \equiv \{ \alpha \in A : \phi(\alpha) = \phi_0 \}$ be the restricted parameter space. Then $\alpha_0 \in A^R$ iff the null hypothesis $H_0 : \phi(\alpha) = \phi_0$ holds. Also, $A_{k(n)} \equiv \{ \alpha \in A_{k(n)} : \phi(\alpha) = \phi_0 \}$ is a sieve space for $A^R$. Let $\{\hat{\alpha}_{0,n} \in A_{k(n)}^R\}$ be a sequence such that $\| \hat{\alpha}_{0,n} - \alpha_0 \|_s \leq \inf_{\alpha \in A_{k(n)}} \| \alpha - \alpha_0 \|_s + o(n^{-1})$.

**Assumption B.2** (i) $|\text{Pen}(\hat{h}_{0,n}) - \text{Pen}(h_0)| = O(1)$ and $\text{Pen}(h_0) < \infty$; (ii) $\hat{Q}_n(\hat{\alpha}_{0,n}) \leq c_0 Q(\hat{\alpha}_{0,n}) + o_{P_{\infty}}(n^{-1})$.

This assumption on $\hat{\alpha}_{0,n} \in A_{k(n)}^R$ is the same as Assumptions 3.2(ii) and 3.3(i) imposed on $\Pi_n \omega_0 \in A_{k(n)}$, and can be verified in the same way provided that $\alpha_0 \in A^R$.

**Proposition B.1** Let $\hat{\alpha}_n^R \in A_{k(n)}^R$ be the restricted PMSE estimator of $4.10$ and $\alpha_0 \in A^R$. Let Assumptions 3.1, 3.3(iii), 3.3(iv), B.2 and $Q(\hat{\alpha}_{0,n}) + o(n^{-1}) = O(\lambda_n) = o(1)$ hold. Then:

1. $\text{Pen}(\hat{h}_{0,n}^R) = O_{P_{\infty}}(1)$ and $\| \hat{\alpha}_n^R - \alpha_0 \|_s = o_{P_{\infty}}(1)$;
2. Further, let $Q(\hat{\alpha}_{0,n}) = Q(\Pi_n \omega_0)$ and Assumptions 3.2(ii), 3.3(i), B.2 and Assumptions 3.2(iii)(ii)(ii) hold. Then: $\| \hat{\alpha}_n^R - \alpha_0 \|_s = O_{P_{\infty}}(\delta_n)$ and $\| \hat{\alpha}_n^R - \alpha_0 \|_s = O_{P_{\infty}}(\| \alpha_0 - \Pi_n \omega_0 \|_s + \tau_n \delta_n)$.

16Sufficient conditions for $\hat{\alpha}_{0,n} \in A_{k(n)}^R$ to solve $\inf_{\alpha \in A_{k(n)}} \| \alpha - \alpha_0 \|_s$ under the null include either (a) $A_{k(n)}$ is compact (in $\| \cdot \|_s$) and $\phi$ is continuous (in $\| \cdot \|_s$), or (b) $A_{k(n)}$ is convex and $\phi$ is linear.
Proof of Proposition B.1. The proof is very similar to those for theorem 3.2 and remark 4.1 in [Chen and Pouzo (2012a)] by recognizing that $A_{k(n)}^{R}$ is a sieve for $\alpha_0 \in A^{R}$.

For Result (1), we first want to show that $\hat{\alpha}_n \in A_{k(n)}^{R} \cap \{\text{Pen}(\hat{h}) \leq M\}$ for some $M > 0$ wpa1-$P_{2\infty}$. By definitions of $\hat{\alpha}_n^R$ and $\hat{\alpha}_0$, Assumption B.2(i)(ii) and the condition that $Q(\hat{\alpha}_0, n) + o(n^{-1}) = O(\lambda_n)$, we have:

$$\text{Pen}(\hat{h}_n^R) \leq \frac{\hat{Q}_n(\hat{\alpha}_0, n) + \text{Pen}(\hat{h}_0, n)}{\lambda_n} \leq Q(\hat{\alpha}_0, n) + o(n^{-1}) + O_P \leq \epsilon$$

Therefore, for any $\epsilon > 0$, $\Pr(\text{Pen}(\hat{h}_n^R) \geq M) < \epsilon$ for some $M$, eventually.

We now show that $\Pr(||\hat{\alpha}_n^R - \alpha_0|| \geq \epsilon) = o(1)$ for any $\epsilon > 0$. Let $A_{k(n)}^{R,M} \equiv A_{k(n)}^{R} \cap \{\text{Pen}(\hat{h}) \leq M\}$ and $A^{R,M} \equiv A^{R} \cap \{\text{Pen}(\hat{h}) \leq M\}$. These sets are compact under $|| \cdot ||$, by Assumption 3.2(ii) and the $|| \cdot ||_s$ continuity of $\phi$. Assumptions 3.1(i)(iv) and B.2(i) imply that $\alpha_0 \in A^{R,M}$ and $\hat{\alpha}_0 \in A_{k(n)}^{R}$. Under assumption 3.1(ii), $\text{cl}(\cup_k A_k) \supseteq A$ and thus $\text{cl}(\cup_k A_{k(n)}^{R,M}) \supseteq A^{R,M}$. Therefore $||\hat{\alpha}_0 - \alpha_0||_s = o(1)$ by the definition of $\hat{\alpha}_0$ and the fact that $A_{k(n)}^{R,M}$ is dense in $A^{R,M}$.

By standard calculations, it follows that, for any $\epsilon > 0$,

$$\Pr(||\hat{\alpha}_n^R - \alpha_0|| \geq \epsilon) \leq \Pr \left( \inf_{A_{k(n)}^{R,M}, ||\alpha - \alpha_0|| \geq \epsilon} \{Q(\alpha) + \lambda_n \text{Pen}(\hat{h})\} \leq Q(\hat{\alpha}_0, n) + \lambda_n \text{Pen}(\hat{h}_0, n) + o_P(n^{-1}) \right) + 0.5 \epsilon$$

Moreover (up to omitted constants)

$$\Pr \left( ||\hat{\alpha}_n^R - \alpha_0|| \geq \epsilon \right) \leq \Pr \left( \inf_{A_{k(n)}^{R,M}, ||\alpha - \alpha_0|| \geq \epsilon} \{Q(\alpha) + \lambda_n \text{Pen}(\hat{h})\} \leq Q(\hat{\alpha}_0, n) + \lambda_n \text{Pen}(\hat{h}_0, n) + O_P(\delta^2) + o_P(n^{-1}) \right) + \epsilon$$

where the first line follows by Assumptions 3.3(ii) and B.2(ii) and the second by $A_{k(n)}^{R,M} \subseteq A^{R,M}$. Since $A^{R,M}$ is compact under $|| \cdot ||_s$, $\alpha_0 \in A^{R,M}$ is unique and $Q$ is continuous (Assumption 3.3.2), then $\inf_{A^{R,M}, ||\alpha - \alpha_0|| \geq \epsilon} \{Q(\alpha) + \lambda_n \text{Pen}(\hat{h})\} \geq c(\epsilon) > 0$; however, the term $Q(\hat{\alpha}_0, n) + \lambda_n \text{Pen}(\hat{h}_0, n) + O_P(\delta^2) + o_P(n^{-1}) = o_P(1)$ and thus the desired result follows.

For Result (2), we now show that $||\hat{\alpha}_n^R - \alpha_0|| = O_{P_{2\infty}}(\kappa_\beta)$ where $\kappa_\beta \equiv \max \{\delta^2, ||\hat{\alpha}_0 - \alpha_0||^2, \lambda_n, o(n^{-1})\}$. Let $A_{\alpha_0}^{R} = \{\alpha \in A_{\alpha_0} : \phi(\alpha) = \phi(\alpha_0)\}$ and $A_{\alpha_0}^R = \{\alpha \in A_{\alpha_0} : \phi(\alpha) = \phi(\alpha_0)\}$. Result (1) implies that $\hat{\alpha}_n \in A_{\alpha_0}^{R}$ wpa1. To show Result (2), we em-
Thus Assumption 3.3(ii), we have:
\[
\Pr\left(\|\tilde{\alpha}^R_n - \alpha_0\| \geq K\kappa_n\right) \\
\leq \Pr\left(\inf_{\lambda_n Pen(h) \leq Q(\tilde{\alpha}_0, n) + \lambda_n Pen(h_0, n) + O_P(\delta_n^2) + o_P(n^{-1})} Q(\alpha) + \lambda_n Pen(h) \leq Q(\tilde{\alpha}_0, n) + \lambda_n Pen(h_0, n) + O_P(\delta_n^2) + o_P(n^{-1}) + \epsilon\right) \\
\leq \Pr\left(\inf_{\lambda_n \leq Q(\tilde{\alpha}_0, n) + \lambda_n Pen(h_0, n) + O_P(\delta_n^2) + o_P(n^{-1})} ||\alpha - \alpha_0||^2 \leq \text{Const.}\cdot \|\tilde{\alpha}_0, n - \alpha_0\|^2 + \lambda_n Pen(h_0, n) + O_P(\delta_n^2) + o_P(n^{-1}) + \epsilon\right) \\
\leq \Pr\left(K^2\kappa_n^2 \leq \text{Const.}\cdot ||\tilde{\alpha}_0, n - \alpha_0||^2 + O(\lambda_n) + O_P(\delta_n^2) + o_P(n^{-1}) + \epsilon\right),
\]
where the first inequality is due to Assumption B.2(ii) and the assumption that \(\hat{Q}_n(\alpha) \geq cQ(\alpha) - O_P(\delta_n^2)\) uniformly over \(A_{\alpha, n}\); the second inequality is due to Assumption 3.4.

By our choice of \(\kappa_n\), the first term in the RHS is zero for large \(K\). So the desired result follows. The fact that \(\kappa_n\) coincides with \(\delta_n\) follows from the fact that \(||\tilde{\alpha}_0, n - \alpha_0||^2 \approx Q(\tilde{\alpha}_0, n)\) by assumption in the Proposition.

Finally, the convergence rate under \(||\cdot||_{s}\) is obtain by applying the previous result and the definition of \(\tau_n\). \(Q.E.D.\)

**Proof of Theorem 4.4** Since \(\sup_{h \in H} Pen(h) < \infty\), the relevant parameter set is \(A^M \equiv \{\alpha \in A : Pen(h) \leq M\}\) with \(M = \sup_{h \in H} Pen(h)\), which is non-empty and compact (in \(||\cdot||_{s}\) under Assumptions 3.1(i)(ii) and 3.2(iii)). Let \(A^{R,M} = A^M \cap \{\alpha \in A : \phi(\alpha) = \tilde{\phi}_0\}\). Since \(\phi\) is continuous in \(||\cdot||_{s}\), \(A^{R,M}\) is also compact (in \(||\cdot||_{s}\)). Note that \(\alpha_0 \in A^{R,M}\) iff the null \(H_0 : \phi(\alpha_0) = \tilde{\phi}_0\) holds.

If \(A^{R,M}\) is empty, then there does not exist any \(\alpha \in A^M\) such that \(\phi(\alpha) = \tilde{\phi}_0\), and hence it holds trivially that \(\hat{Q}_n(\tilde{\phi}_0) \geq nC\) for some \(C > 0\) wpa1.

If \(A^{R,M}\) is non-empty, under Assumption 3.1(iii) we have: \(\min_{\alpha \in A^{R,M}} Q(\alpha)\) is achieved at some point within \(A^{R,M}\), say, \(\alpha \in A^{R,M}\). This and Assumption 3.1(i)(iv) imply that \(Q(\alpha) = \min_{\alpha \in A^{R,M}} Q(\alpha) > 0 = Q(\alpha_0)\) under the fixed alternatives \(H_1 : \phi(\alpha) \neq \tilde{\phi}_0\).

By definitions of \(\tilde{\alpha}_n\) and \(H_0, \alpha_0\) and Assumption 3.3(i), we have:
\[
\hat{Q}_n(\tilde{\alpha}_n) \leq \hat{Q}_n(H_0, \alpha_0) \leq c\alpha_0 Q(\Pi_0, \alpha_0) + o_P(\delta_n^2) (n^{-1}).
\]

Since \(M = \sup_{h \in H} Pen(h) < \infty\), we also have that \(\tilde{\alpha}_n^R \in A^{R,M}_n \subseteq A^M_n\) wpa1, so by Assumption 4.3(ii), we have:
\[
\hat{Q}_n(\tilde{\alpha}_n^R) \geq cQ(\tilde{\alpha}_n^R) - O_P(\delta_n^2) (\tilde{\delta}_m,n) \geq c\times \min_{\alpha \in A^{R,M}} Q(\alpha) - O_P(\delta_n^2) (\tilde{\delta}_m,n).
\]

Thus
\[
\hat{Q}_n(\tilde{\alpha}_n^R) - \hat{Q}_n(\tilde{\alpha}_n) \geq c\times \min_{\alpha \in A^{R,M}} Q(\alpha) - c_0 Q(\Pi_0, \alpha_0) + o_P(\delta_n^2) (n^{-1}) - O_P(\delta_n^2) (\tilde{\delta}_m,n) = cQ(\alpha) + o_P(\delta_n^2) (1).
\]

Thus under the fixed alternatives \(H_1 : \phi(\alpha) \neq \tilde{\phi}_0\),
\[
\frac{\hat{Q}_n(\alpha)}{n} \geq cQ(\alpha) > 0 \quad \text{wpa1}.
\]

A consistent variance estimator for optimally weighted PSMD estimator.

To stress the fact that we consider the optimally weighted PSMD procedure, we use
\( u_n^0 \) and \( \|v_n^0\|_0 \) to denote the corresponding \( v_n^* \) and \( \|v_n^*\| \) computed using the optimal weighting matrix \( \Sigma = \Sigma_0 \). That is,
\[
\|v_n^0\|_0^2 = E \left[ \left( \frac{dm(X, \alpha_0)}{d\alpha} \right)_{\alpha_0} \right] \Sigma_0(X)^{-1} \left( \frac{dm(X, \alpha_0)}{d\alpha} \right)_{\alpha_0} \|v_n^0\|_0^2.
\]

We call the corresponding sieve score, \( S_{n,i}^0 = \left( \frac{dm(X, \alpha_0)}{d\alpha} \right)_{\alpha_0} \Sigma_0(X_i)^{-1} \rho(Z_i, \alpha_0) \), the optimal sieve score. Note that \( \|v_n^0\|_0^2 = Var(S_{n,i}^0) = \|v_n^0\|_0^2 \). By Theorem 4.1, \( \|v_n^0\|_0^2 = \|v_n^0\|_0^2 \) is the variance of the optimally weighted PSMD estimator \( \phi(\alpha_n) \). We could compute a consistent estimator \( \|v_n^0\|_0^2 \) of the variance \( \|v_n^0\|_0^2 \) by looking at the “slope” of the optimally weighted criterion \( \hat{Q}_n(\alpha) \):
\[
(B.13) \quad \|v_n^0\|_0^2 \equiv \left( \hat{Q}_n(\tilde{\alpha}_n) - \hat{Q}_n(\alpha_n) \right)^{-1} \varepsilon_n^2,
\]
where \( \tilde{\alpha}_n \) is an approximate minimizer of \( \hat{Q}_n(\alpha) \) over \( \{\alpha \in A_{k(n)} : \phi(\alpha) = \phi(\tilde{\alpha}_n) - \varepsilon_n \} \).

**Theorem B.2** Let \( \tilde{\alpha}_n \) be the optimally weighted PSMD estimator \( 2.2 \) with \( \Sigma = \Sigma_0 \), and conditions for Lemma 3.2, Assumptions 3.5 and 3.6 hold with \( \|v_n^0\|_0^2 = \|v_n^0\|_0^2 \). Let \( n^{-1/2} \leq \frac{\varepsilon_n}{\|v_n^0\|_0} \leq C \delta_n \) for finite constants \( c, C > 0 \). Then: \( \tilde{\alpha}_n \in \mathcal{N}_{\alpha = 0} \) wpa1 \(-P_{\alpha = 0}^\infty\), and
\[
\|\tilde{\alpha}_n\|_0^2 = 1 + o_{P_{\alpha = 0}^\infty}(1).
\]

When \( \tilde{\alpha}_n \) is the optimally weighted PSMD estimator of \( \alpha_0 \), Theorem B.2 suggests \( \|v_n^0\|_0^2 \) defined in (B.13) as an alternative consistent variance estimator for \( \phi(\tilde{\alpha}_n) \). Compared to Theorems 4.2 and B.1, this alternative variance estimator \( \|v_n^0\|_0^2 \) allows for a non-smooth residual function \( \rho(Z, \alpha) \) (such as the one in NPQIV), but is only valid for an optimally weighted PSMD estimator.

**Proof of Theorem B.2** Recall that for the optimally weighted criterion case \( u_n^* = v_n^0/\|v_n^0\|_0 \), and hence \( \|u_n^*\| = 1 \), \( B_n = 1 + o_{P_{\alpha = 0}^\infty}(1) \). To simplify notation, in this proof we use \( \langle \cdot, \cdot \rangle, \| \cdot \| \) and \( \hat{Q}(\cdot) \) for the ones corresponding to the optimal weighting matrix \( \Sigma = \Sigma_0 \).

We first show that \( \tilde{\alpha}_n \in \mathcal{N}_{\alpha = 0} \) wpa1. Recall that \( \tilde{\alpha}_n \) is defined as an approximate optimally weighted PSMD estimator constrained to \( \{\alpha \in A_{k(n)} : \phi(\alpha) = \phi(\tilde{\alpha}_n) - \varepsilon_n \} \). In the following since there is no risk of confusion, we use \( \rho \) instead of \( \rho_{\alpha = 0} \).

Let \( r = \phi(\tilde{\alpha}_n) - \phi(\tilde{\alpha}_0) - \varepsilon_n \). Since \( \varepsilon_n \leq C\|v_n^0\|_0 \delta_n \) (by assumption), and \( \tilde{\alpha}_n \in \mathcal{N}_{\alpha = 0} \) wpa1, \( \phi(\tilde{\alpha}_n) - \phi(\tilde{\alpha}_0) = O_P(\|\tilde{\alpha}_n^0\|_0/\sqrt{n}) \) (by Theorem 4.1), we have \( \|r\| \leq C\|v_n^0\|_0 O(\delta_n + n^{-1/2}) \leq C\|v_n^0\|_0 \delta_n \) for some \( C > 0 \). Also note that \( \|\tilde{\alpha}_n - \alpha_0\| \leq C \delta_n \) wpa1. Thus, by Lemma B.2, there exists a \( t_n^* \in T_n \) such that \( \phi(\tilde{\alpha}_n(t_n^*)) = \phi(\tilde{\alpha}_n) - \varepsilon_n \) and \( \tilde{\alpha}_n(t_n^*) = \alpha_n + t_n^* u_n^* \in A_{k(n)} \) and \( t_n^* = O(\delta_n) \). Henceforth, let \( \tilde{\alpha}_n \equiv \tilde{\alpha}_n(t_n^*) \). Observe that
\[
\|\tilde{\alpha}_n - \alpha_0\| \leq \delta_n + t_n^* = O(\delta_n),
\]
using the definition of sieve measure of local ill-posedness $\tau_n$ under our assumptions $B_\alpha$, $\tilde{A}$, and $\sigma_n \in T$ with $t_n = O(\delta_n)$; and (b) $\tilde{\alpha}_n \in \{\alpha \in A_k(n) : \phi(\alpha) = \phi(\tilde{\alpha}_n) - \varepsilon_n\}$.

We now establish the consistency of $\tilde{\alpha}_n$ using the properties of $\tilde{\alpha}_n$. We observe that, for any $\epsilon > 0$,

$$\Pr(||\tilde{\alpha}_n - \alpha_0||_s \geq \epsilon) \leq \Pr \left( \inf_{g_n:||a - \alpha_0||_s \geq \epsilon} \tilde{Q}_n(a) \leq \tilde{Q}_n(\tilde{\alpha}_n) + o(n^{-1}) + \lambda_n \text{Pen}(\tilde{h}_n) \right)$$

where $B_n \equiv \{\alpha \in A_k(n) : \phi(\alpha) = \phi(\tilde{\alpha}_n) - \varepsilon_n\}$ and the inequality is valid because $\tilde{\alpha}_n \in B_n$ by (a) and (b). Under (a) and Lemma 3.2 $\lambda_n \text{Pen}(\tilde{h}_n) = O_P(\lambda_n) = O(n^{-1})$.

By (a), under assumption 3.6(i)

$$\tilde{Q}_n(\tilde{\alpha}_n) = \tilde{Q}_n(\tilde{\alpha}_n) + t_n^*(Z_n + \langle u_n^*, \tilde{\alpha}_n - \alpha_0 \rangle) + 0.5(t_n^*)^2 + o_P \left( t_n^* n^{-1/2} + (t_n^*)^2 + o(n^{-1}) \right).$$

By Lemma B.1, $Z_n + \langle u_n^*, \tilde{\alpha}_n - \alpha_0 \rangle = O(n^{-1/2})$ and thus, given that $t_n = O(\delta_n)$, the previous display implies that

$$\tilde{Q}_n(\tilde{\alpha}_n) \leq \tilde{Q}_n(\tilde{\alpha}_n) + o_P(n^{-1/2} + \delta_n + o(n^{-1})) \leq O_P(\delta_n^2).$$

Therefore,

$$\Pr(||\tilde{\alpha}_n - \alpha_0||_s \geq \epsilon) \leq \Pr \left( \inf_{g_n:||a - \alpha_0||_s \geq \epsilon} \tilde{Q}_n(a) \leq \tilde{Q}_n(\tilde{\alpha}_n) + O(\lambda_n + \delta_n^2) \right).$$

Since $\tilde{Q}_n(\tilde{\alpha}_n) \leq \tilde{Q}_n(\Pi_n, \alpha_0) + O(\lambda_n)$ by definition of $\tilde{\alpha}_n$ and from the fact that $B_n \subseteq A_k(n)$, it follows that

$$\Pr(||\tilde{\alpha}_n - \alpha_0||_s \geq \epsilon) \leq \Pr \left( \inf_{g_n:||a - \alpha_0||_s \geq \epsilon} \tilde{Q}_n(a) \leq \tilde{Q}_n(\Pi_n, \alpha_0) + O(\lambda_n + \delta_n^2) \right).$$

The rest of the consistency proof follows from identical steps to the standard one; see [Chen and Pouzo (2009)].

In order to show the rate, by similar arguments to the previous ones

$$\tilde{Q}_n(\tilde{\alpha}_n) \leq \tilde{Q}_n(\Pi_n, \alpha_0) + O(\lambda_n + \delta_n^2),$$

under our assumptions $\tilde{Q}_n(\tilde{\alpha}_n)$$ \geq c||\tilde{\alpha}_n - \alpha_0||^2 - O_P(\delta_n^2)$ and $\tilde{Q}_n(\Pi_n, \alpha_0) \leq c_0 Q(\Pi_n, \alpha_0) + o_P(n^{-1})$, so the desired rate under $||.||_s$ follows. The rate under $||.||_s$ immediately follows using the definition of sieve measure of local ill-posedness $\tau_n$. Thus $\tilde{\alpha}_n \in N_{\alpha_0}$ wpa1.

**We now show that** $||\tilde{\alpha}_n - \alpha_0||_0 = 1 + o_{P_2}(1)$. This part of proof consists of several steps that are similar to those in the proof of Theorem 4.3 and hence we omit some details. We first provide an asymptotic expansion for $n(\tilde{Q}_n(\alpha_n) - \tilde{Q}_n(\tilde{\alpha}_n))$ using Assumption 3.6(i) (with $B_n = 1 + o_{P_2}(1)$), and then show that this is enough to establish the desired result.

In the following we let $t_n = \varepsilon_n/||u_n^0||_0$. By the assumption on $\varepsilon_n$ we have: $cn^{-1/2} \leq \varepsilon_n \leq \frac{1}{2}cn^{-1/2}$
\( t_n \leq C\delta_n \). Therefore, \( t_n \in T_n, t_n = o_{P_{\infty}}(1) \) and \( o_{P_{\infty}} \left( \frac{1}{n} n^{-1/2} \right) = o_{P_{\infty}}(1) \).

**Step 1:** First, we note that \( \tilde{\alpha}_n \in N_{o_{\infty}} \), wpa1, that \( -t_n \in T_n \) and \( \tilde{\alpha}_n(-t_n) \in A_{k(n)} \).

So we can apply Assumption 3.6(i) with \( \alpha = \tilde{\alpha}_n \) and \( -t_n \) as the direction, and obtain:

\[
\frac{(\tilde{\beta}_n(\tilde{\alpha}_n(-t_n)) - \tilde{\beta}_n(\tilde{\alpha}_n))}{t_n^2} = -\frac{2}{t_n} \{ Z_n + \langle u_n^*, \tilde{\alpha}_n - \alpha_0 \rangle \} + 1 + o_P \left( \max \left\{ 1, \frac{1}{t_n^2}, \frac{o(n^{-1})}{t_n^2} \right\} \right)
\]

(B.14)

where the last equality follows from the fact that \( \langle u_n^*, \tilde{\alpha}_n - \alpha_0 \rangle + Z_n = o_{P_{\infty}}(n^{-1/2}) \) (by Lemma B.1), and that \( o_{P_{\infty}} \left( \frac{1}{n} n^{-1/2} \right) = o_{P_{\infty}}(1) \) (by our choice of \( t_n \)).

**Step 2:** Since \( \tilde{\alpha}_n \in N_{o_{\infty}} \), wpa1, \( t_n \in T_n \) and \( \tilde{\alpha}_n(t_n) \in A_{k(n)} \), we can apply Assumption 3.6(i) with \( \alpha = \tilde{\alpha}_n \) and \( t_n \) as the direction, and obtain:

\[
\frac{(\tilde{\beta}_n(\tilde{\alpha}_n(t_n)) - \tilde{\beta}_n(\tilde{\alpha}_n))}{t_n^2} = \frac{2}{t_n} \{ Z_n + \langle u_n^*, \tilde{\alpha}_n - \alpha_0 \rangle \} + 1 + o_P \left( \max \left\{ 1, \frac{1}{t_n^2}, \frac{o(n^{-1})}{t_n^2} \right\} \right)
\]

(B.15)

where the last line follows from the definition of the restricted estimator \( \tilde{\alpha}_n \). This is because \( \phi(\tilde{\alpha}_n) = \phi(\tilde{\alpha}_n) - \varepsilon_n \), by Assumptions 3.5(i)(ii),

\[
\left| -\varepsilon_n - \frac{d\phi(\alpha_0)}{d\alpha} [\tilde{\alpha}_n - \alpha_0] \right| = o_{P_{\infty}}(\|v_n^0\|/\sqrt{n}).
\]

Hence \( \langle u_n^0, \tilde{\alpha}_n - \alpha_0 \rangle = \langle v_n^0, \tilde{\alpha}_n - \alpha_0 \rangle - \varepsilon_n + o_{P_{\infty}}(\|v_n^0\|/\sqrt{n}) \). This implies that \( Z_n + \langle u_n^*, \tilde{\alpha}_n - \alpha_0 \rangle = -\varepsilon_n + o_{P_{\infty}}(n^{-1/2}) = -t_n + o_{P_{\infty}}(n^{-1/2}) \).

**Step 3:** It is easy to see that, from equation B.15 and by the definition of \( \tilde{\alpha}_n \),

\[
\frac{(\tilde{\beta}_n(\tilde{\alpha}_n) - \tilde{\beta}_n(\tilde{\alpha}_n))}{t_n^2} \geq \frac{(\tilde{\beta}_n(\tilde{\alpha}_n) - \tilde{\beta}_n(\tilde{\alpha}_n))}{t_n^2} - o_{P_{\infty}}(1) = 1 + o_{P_{\infty}}(1).
\]

Also, from equation B.14, Assumption 3.6(i) and by the definition of \( \tilde{\alpha}_n \),

\[
\frac{(\tilde{\beta}_n(\tilde{\alpha}_n) - \tilde{\beta}_n(\tilde{\alpha}_n))}{t_n^2} \leq \frac{(\tilde{\beta}_n(\tilde{\alpha}_n(t_n^*)) - \tilde{\beta}_n(\tilde{\alpha}_n))}{t_n^2} + o_{P_{\infty}}(1)
\]

\[
= \frac{2t_n^* \{ Z_n + \langle u_n^*, \tilde{\alpha}_n - \alpha_0 \rangle \} + \langle t_n^* \rangle^2}{t_n^2} + o_P \left( \max \left\{ \langle t_n^* \rangle^2, t_n^* n^{-1/2}, o(n^{-1}) \right\} \right) + o_P(1)
\]

\[
= \frac{2}{t_n} \{ Z_n + \langle u_n^*, \tilde{\alpha}_n - \alpha_0 \rangle \} + 1 + o_{P_{\infty}}(1)
\]

\[
= 1 + o_{P_{\infty}}(1),
\]

provided that there is a \( t_n^* \in T_n \) such that (3a) \( \phi(\tilde{\alpha}_n(t_n^*)) = \phi(\tilde{\alpha}_n) - \varepsilon_n \) and (3b) \( t_n^*/t_n = -1 + o_{P_{\infty}}(1) \). In Step 5 we verify that such a \( t_n^* \) exists.

By putting these inequalities together, it follows

\[
\frac{||v_n^0||}{\varepsilon_n^2} \frac{(\tilde{\beta}_n(\tilde{\alpha}_n) - \tilde{\beta}_n(\tilde{\alpha}_n))}{t_n^2} = \frac{(\tilde{\beta}_n(\tilde{\alpha}_n) - \tilde{\beta}_n(\tilde{\alpha}_n))}{t_n^2} = 1 + o_{P_{\infty}}(1).
\]
STEP 4: By equation (3.16) we have:

\[ \frac{||v_n^0||^2}{||v_n^0||_0^2} = 1 + o_{P_\infty}(1), \quad \text{with} \quad \frac{||v_n^0||^2}{||v_n^0||_0^2} \equiv \left( \frac{\hat{Q}_n(\hat{a}_n) - \hat{Q}_n(\bar{a}_n)}{v_n^0} \right)^{-1}, \]

which implies that 0.5 \leq \frac{||v_n^0||^2}{||v_n^0||_0^2} \leq 1.5 with probability \( P_{Z_\infty} \) approaching one. By continuous mapping theorem, we obtain:

\[ \frac{||v_n^0||^2}{||v_n^0||_0^2} = 1 + o_{P_\infty}(1). \]

STEP 5: We now show that there is a \( t_n^* \in T_n \) such that (3a) and (3b) in Step 3 hold. Denote \( r \equiv \phi(\hat{a}_n) - \phi(\bar{a}_0) - \varepsilon_n \). Since \( \varepsilon_n \leq C||v_n^0||_0d_n \), and \( \hat{a}_n \in \mathcal{N}_{\infty} \) wpal, \( \phi(\hat{a}_n) - \phi(\bar{a}_0) = O_p(||v_n^0||_0/\sqrt{n}) \) (by Theorem 4.1), we have \( |r| \leq ||v_n^0||_0d_n(M_n + C) \leq 2M_n||v_n^0||_0d_n \) (since \( C < M_n \) eventually). Thus, by Lemma B.2 there exists a \( t_n^* \in T_n \) such that \( \phi(\hat{a}_n(t_n^*)) = \phi(\hat{a}_n) - \varepsilon_n \) and \( \hat{a}_n(t_n^*) = \hat{a}_n + t_n^*u_n \in \mathcal{A}_{k(a)} \), and hence (3a) holds. Moreover, by Assumption 3.5(i)(ii), such a choice of \( t_n^* \) also satisfies

\[ \phi(\hat{a}_n(t_n^*)) - \phi(\hat{a}_n) - \frac{d\phi(\bar{a}_0)}{da}[t_n^*u_n] \approx o_{P_{\infty}}(||v_n^0||_0n^{-1/2}). \]

Since \( u_n^* = v_n^0/||v_n^0||_0 \) for optimally weighted criterion case, we have: \( \frac{d\phi(\bar{a}_0)}{da}[u_n^*] = ||v_n^0||_0. \) Thus

\[ \varepsilon_n - t_n^* ||v_n^0||_0 = o_{P_{\infty}}(||v_n^0||_0n^{-1/2}). \]

Since \( t_n \equiv \varepsilon_n/||v_n^0||_0 \), we obtain: \( |t_n - t_n^*| = o_{P_{\infty}}(n^{-1/2}) \), and hence

\[ \left| t_n^*/t_n + 1 \right| = o_{P_{\infty}}(n^{-1/2}/t_n) = o_{P_{\infty}}(1) \]

due to the fact that \( c_n^{-1/2} \leq t_n \leq C\delta_n \). Thus (3b) holds. Q.E.D.

B.3. Proofs for Section 5 on bootstrap inference

Throughout the Appendices, we sometimes use the simplified term “wpal” in the bootstrap world while its precise meaning is given in Section 5.

Recall that \( Z_n^0 \equiv \frac{1}{n} \sum_{i=1}^{n} \omega_i g(X_i, u_n^*) \rho(Z_i, \bar{a}_0) \) with \( g(X_i, u_n^*) \equiv \left( \frac{d\omega(X_i, u_n^*)}{du_n^*} \right)' \Sigma(X_i)^{-1} \).

Lemma B.3 Let \( \hat{a}_n^B \) be the bootstrap PSMD estimator and conditions for Lemma 3.2 and Lemma 4.1 hold. Let Assumption Boot.3(i) hold. Then: (1) for all \( \delta > 0 \), there exists a \( N(\delta) \) such that for all \( n \geq N(\delta) \),

\[ P_{\infty} \left( \sqrt{n} \left| \left( u_n^*, \hat{a}_n^B - \bar{a}_0 \right) + Z_n^0 \right| \geq \delta \right) \zeta(n) < \frac{1}{\delta}. \]
(2) If, in addition, assumptions of Lemma B.1 hold, then
\[ \sqrt{n}(u_n^*, \hat{\alpha}_n^B - \hat{\alpha}_n) = -\sqrt{n}Z_{n,\delta}^{-1} + o_{P_\infty|Z_n}(1) \text{ wpa1}(P_\infty) \]

Proof of Lemma B.3 The proof is very similar to that of Lemma B.1, so we only present the main steps.

For Result (1). Under Assumption Boot.3(i) and using the fact that \( \hat{\alpha}_n^B \) is an approximate minimizer of \( \hat{Q}_n^B(\alpha) + \lambda_n \text{Pen}(h) \) on \( A_{k(n)} \) and \( \hat{\alpha}_n^B \in N_{\alpha,n} \), it follows (see the proof of Lemma B.1 for details), for sufficiently large \( n \),
\[ P_\infty \left( P_{V_\infty|Z_n} \left( 2\epsilon_n \{Z_n^\alpha + \langle u_n^*, \hat{\alpha}_n^B - \alpha \rangle \} + \epsilon_n^2 B_n^w + E_n(\hat{\alpha}_n^B, \epsilon_n) \geq -\delta r_n^{-1}|Z_n| \right) \geq 1 - \delta \right) > 1 - \delta, \]
where \( r_n \) and \( E_n \) are defined as in the proof of Lemma B.1 and \( \epsilon_n = \pm \{\delta_n^{-1/2} + o(n^{-1/2})\} \). Dividing by \( 2\epsilon_n \) and multiplying by \( \sqrt{n} \), it follows that
\[ P_\infty \left( P_{V_\infty|Z_n} \left( A_n^\alpha, \delta \geq \sqrt{n}\{Z_n^\alpha + \langle u_n^*, \hat{\alpha}_n^B - \alpha \rangle \} \geq B_n^w|Z_n| \right) \geq 1 - \delta \right) > 1 - \delta \]
eventually, where
\[ A_n^\alpha, \delta \equiv -0.5\sqrt{n}\epsilon_n B_n^w - \delta \sqrt{n}\epsilon_n^{-1} r_n^{-1} + 0.5\delta \]
\[ B_n^w, \delta \equiv -0.5\sqrt{n}\epsilon_n B_n^w - \delta \sqrt{n}\epsilon_n^{-1} r_n^{-1} - 0.5\delta. \]

Since \( \sqrt{n}\epsilon_n = o(1) \) and \( B_n^w = O_{P_\infty|Z_n}(1) \text{ wpa1}(P_\infty) \) and \( |\sqrt{n}\epsilon_n^{-1} r_n^{-1}| \approx 1 \), it follows, for sufficiently large \( n \),
\[ P_\infty \left( P_{V_\infty|Z_n} \left( 2\delta \geq \sqrt{n}\{Z_n^\alpha + \langle u_n^*, \hat{\alpha}_n^B - \alpha \rangle \} \geq -2\delta|Z_n| \right) \geq 1 - \delta \right) > 1 - \delta. \]
Or equivalently, for sufficiently large \( n \),
\[ P_\infty \left( P_{V_\infty|Z_n} \left( \{\sqrt{n}\{Z_n^\alpha + \langle u_n^*, \hat{\alpha}_n^B - \alpha \rangle \} \right) \geq 2\delta|Z_n| \right) < \delta \geq 1 - \delta. \]

Result (2) directly follows from Result (1) and Lemma B.1 Q.E.D.

Proof of Theorem 5.1 We note that Assumption Boot.4 implies that \( |n^{-1} \sum_{i=1}^n \hat{T}_i[v_n]^\alpha \hat{M}_i v_n| - \sigma_n^2 - \sum_{i=1}^n \hat{T}_i[v_n] M_i v_n| = o_{P_\infty|Z_n}(1) \), uniformly over \( v_n \in V_{k(n)} \) with \( \hat{M}_i = \hat{M}(Z_i, \hat{\alpha}_n) \) and \( \hat{T}_i[v_n] \equiv \frac{\partial h(X_i, \hat{\alpha}_n)}{\partial \alpha_n}[v_n] \). The rest of the proof follows directly from that of Theorem 4.1 for the sieve variance defined in (4.7) case. Q.E.D.

Proof of Theorem 5.2 By Lemma B.3 and steps analogous to those used to show Theorem 4.1 it follows
\[ \sqrt{n} \left( \frac{\phi(\hat{\alpha}_n^B) - \phi(\hat{\alpha}_n)}{\sigma_n} \right)_{sd}^{\alpha} = -\sqrt{n}Z^{-1} \left( \frac{\phi(\hat{\alpha}_n^B) - \phi(\hat{\alpha}_n)}{\sigma_n} \right)_{sd}^{\alpha} + o_{P_\infty}(1) \text{ wpa1}(P_\infty). \]

For Result (1), we note that the result for \( \hat{W}_1 \) follows directly from Theorem 5.1 and the proof of the Result (1) for \( \hat{W}_1 \) follows from Theorem 5.1.
In fact, for both $j = 1, 2$, Theorem 4.2 (1), equation (B.17) and Theorem 5.1 imply that
\begin{equation}
\hat{W}_{j,n}^B = -\sqrt{n} \frac{Z_{\alpha}^w}{\sigma_{\alpha}} + o_P\left(1\right) \text{ wpa1}\left(P_{Z^\infty}\right);
\end{equation}
Equation (B.18) and Assumptions 3.6(ii) and Boot.3(ii) imply that:
\[|L_{\|Z\|=\infty}(\hat{W}_{j,n}^B) - L\left(\hat{W}_{n}\right)| = o_P\left(1\right).\]
Result (1) now follows from the following two equations:
\begin{equation}
\sup_{t \in \mathbb{R}} |P_{\|Z\|=\infty}(\hat{W}_{j,n}^B \leq t) - \Phi(t)| = o_P\left(1\right) \text{ wpa1}\left(P_{Z^\infty}\right),
\end{equation}
and
\begin{equation}
\sup_{t \in \mathbb{R}} |P_{Z^\infty}(\hat{W}_n \leq t) - \Phi(t)| = o_P\left(1\right),
\end{equation}
where $\Phi()$ is the cdf of a standard normal. Equation (B.20) follows directly from Theorem 4.2 (2) and Polya’s theorem (see e.g., Bickel and Millar (1992)).

**Result (2)** follows from equation (B.17) and the fact that $||v^*_n||_{sd} \rightarrow ||v^*_n||_{sd} \in (0, \infty)$ for regular functionals. Q.E.D.

**Proof of Theorem 5.3** For Result (1), denote
\[F_n = \frac{n}{\text{inf}_{\alpha \in A_{k(n)}(\phi_n)} \hat{Q}_n^B(\alpha) - \hat{Q}_n^B(\hat{\phi}_n)} = \frac{\text{QLR}_n^B(\hat{\phi}_n)}{\sigma^2} = \frac{n}{\hat{Q}_n^B(\hat{\phi}_n) - \hat{Q}_n^B(\hat{\phi}_n)} + o_P\left(1\right) \text{ wpa1}\left(P_{Z^\infty}\right)\]
where $A_{k(n)}(\phi_n) \equiv \{\alpha \in A_{k(n)}: \phi(\alpha) = \phi(\hat{\phi}_n)\}$. Since $o_P\left(1\right) \text{ wpa1}\left(P_{Z^\infty}\right)$ will not affect the asymptotic results we omit it from the rest of the proof to ease the notational burden. We want to show that for all $\delta > 0$, there exists a $N(\delta)$ such that
\[P_{Z^\infty} \left( P_{\|Z\|=\infty} \left( F_n - \left( \sqrt{n} \frac{Z_{\alpha}^w}{\sigma_{\alpha} \sqrt{n}} \right) \right)^2 \right) \geq \delta | Z^\infty \right) \geq 1 - \delta\]
for all $n \geq N(\delta)$. We divide the proof in several steps.

**Step 1.** By assumption $|B_{n}\| - \|u^*_n\|^2| = o_P\left(1\right) \text{ wpa1}\left(P_{Z^\infty}\right)$ and $\|u^*_n\| \in (c, C)$, we have:
\[\frac{\|u^*_n\|^2}{B_{n}} = 1 = o_P\left(1\right) \text{ wpa1}\left(P_{Z^\infty}\right).\]
Therefore, it suffices to show that
\begin{equation}
P_{Z^\infty} \left( P_{\|Z\|=\infty} \left( F_n - \left( \sqrt{n} \frac{Z_{\alpha}^w}{\sigma_{\alpha} \sqrt{n}} B_{n} \right) \right)^2 \right) \geq \delta | Z^\infty \right) < \delta \right) \geq 1 - \delta
\end{equation}
eventually.
STEP 2. By Assumption [Boo.3(i)], for all $\delta > 0$, there is a $M > 0$ such that

$$P_{2|z} = (P_{V \infty|z} \left( \sqrt{n} |Z_{n}^{-1}/B_n| \geq M \mid Z_{n}^{n} \right) < \delta \right) \geq 1 - \delta$$

eventually. Thus $t_n = -Z_{n}^{-1}/B_n \in T_n$ wpa1. By the definition of $\hat{\alpha}_{n}$, and the fact that $\hat{\alpha}_{n}^{R,B} \in N_{\alpha_{n}}$ wpa1 (by Lemma [A.1(3)]),

$$F_n \geq \frac{\sqrt{n}}{\sqrt{\sigma_{Z}}} \left| \phi(\hat{\alpha}_{n}^{R,B} - \hat{\alpha}_{n}) - \frac{d\phi(\alpha)}{d\alpha} \right| \geq \frac{(Z_{n}^{-1}-1)^{2}}{2B_{n}} + o_{P_{V \infty|z} \left( n^{-1} \right) \text{ wpa1}(P_{2\infty}).$$

By specializing Assumption [Boo.3(i)] to $\alpha = \hat{\alpha}_{n}^{R,B}$ and $t_n = -Z_{n}^{-1}/B_n$, it follows

$$(B.22) \quad 0.5(\sqrt{n}(\hat{\alpha}_{n}^{R,B} - \hat{\alpha}_{n})) = -\frac{Z_{n}^{-1}}{B_n}(\hat{\alpha}_{n}^{R,B} - \hat{\alpha}_{n}) \geq \frac{(Z_{n}^{-1}-1)^{2}}{2B_{n}} + o_{P_{V \infty|z} \left( n^{-1} \right) \text{ wpa1}(P_{2\infty}).$$

By Assumption [Boo.3(i)(ii)], and the fact that $\hat{\alpha}_{n}^{R,B} \in N_{\alpha_{n}}$ wpa1,

$$P_{2\infty} \left( P_{V \infty|z} \left( \frac{\sqrt{n}}{\|u_{n}\|} \left| \phi(\hat{\alpha}_{n}^{R,B} - \hat{\alpha}_{n}) - \frac{d\phi(\alpha)}{d\alpha} \right| \geq \frac{(Z_{n}^{-1}-1)^{2}}{2B_{n}} \right) \right) \geq \frac{1}{1 - \delta}$$

eventually. Also by definition $\frac{d\phi(\alpha)}{d\alpha} = (u_{n}^{*}, \hat{\alpha}_{n}^{R,B} - \hat{\alpha}_{n}).$ This and Assumption [Boo.3(i)] imply that

$$(B.23) \quad \sqrt{n}(u_{n}^{*}, \hat{\alpha}_{n}^{R,B} - \hat{\alpha}_{n}) = o_{P_{V \infty|z} \left( n^{-1} \right) \text{ wpa1}(P_{2\infty}).$$

Equation [B.23] and $\sqrt{n}(u_{n}^{*}, \hat{\alpha}_{n} - \alpha) = -\sqrt{n}Z_{n} + o_{P_{2\infty} \left( n^{-1} \right) \text{ wpa1}(P_{2\infty}).}$ (Lemma [B.1]) imply that

$$\sqrt{n}(u_{n}^{*}, \hat{\alpha}_{n}^{R,B} - \alpha) = -\sqrt{n}Z_{n} + o_{P_{V \infty|z} \left( n^{-1} \right) \text{ wpa1}(P_{2\infty}).$$

Thus we can infer from equation (B.22) that

$$(B.24) \quad 0.5(\sqrt{n}(\hat{\alpha}_{n}^{R,B} - \hat{\alpha}_{n})) = -\frac{(Z_{n}^{-1}-1)^{2}}{2B_{n}} + o_{P_{V \infty|z} \left( n^{-1} \right) \text{ wpa1}(P_{2\infty}).$$

Since $nr_{n}^{-1} = O(1)$, multiplying both sides by $-2n\sigma_{Z}^{-2}$, we obtain:

$$F_n \geq \left( \frac{\sqrt{n}}{\|u_{n}\|} \right)^{2} - o_{P_{V \infty|z} \left( n^{-1} \right) \text{ wpa1}(P_{2\infty}).$$

STEP 3. In order to show

$$(B.25) \quad F_n \leq \left( \frac{\sqrt{n}}{\|u_{n}\|} \right)^{2} + o_{P_{V \infty|z} \left( n^{-1} \right) \text{ wpa1}(P_{2\infty}),}$$

we can repeat the same calculations as in Step 2, provided there exists a $t_{n}^{*} \in T_n$ wpa1 such that (a) $\phi(\hat{\alpha}_{n}^{R,B}(t_{n}^{*})) = \phi(\hat{\alpha}_{n})$ with $\hat{\alpha}_{n}^{B}(t_{n}^{*}) \in A_{k(\alpha)}$, and (b) $t_{n}^{*} = Z_{n}^{-1}/\|u_{n}\|^2 + o_{P_{V \infty|z} \left( n^{-1} \right) \text{ wpa1}(P_{2\infty}).}$
Because, by (a) and the definition of $\hat{\alpha}_n^{R,B}$,
\[
n\frac{n}{\sigma^2} (\hat{Q}^B_n(\hat{\alpha}_n^{R,B}) - \hat{Q}^B_n(\hat{\alpha}_n^{B})) \leq n\frac{\hat{Q}^B_n(\hat{\alpha}_n^{B}(t_n^*)) - \hat{Q}^B_n(\hat{\alpha}_n^{B})}{\sigma^2} + o_{P_{\infty|Z^n}} (1) \text{ wpal}(P_{Z^n}).
\]
By specializing Assumption \textbf{Boot.3(i)} to $\alpha = \hat{\alpha}_n^{B} \in \mathcal{N}_\infty$ wpal (by Lemma \textbf{A.1}(2)), and $t_n^*$ as the direction, it follows
\[
0.5\hat{Q}^B_n(\hat{\alpha}_n^{B}(t_n^*)) = t_n^* \{Z_n^n + \langle u_n^*, \hat{\alpha}_n^{B} - \alpha_0 \rangle \} + \frac{B_n^w}{2}(t_n^*)^2 + o_{P_{\infty|Z^n}} (r_n^{-1}) \text{ wpal}(P_{Z^n})
\]
\[
= \frac{B_n^w}{2} \left( \frac{Z_n^n - 1}{||u_n^*||^2} + o_{P_{\infty|Z^n}} (n^{-1/2}) \right)^2 + o_{P_{\infty|Z^n}} (r_n^{-1}) \text{ wpal}(P_{Z^n})
\]
\[
= \frac{1}{2} \left( \frac{Z_n^n - 1}{||u_n^*||^2} \right)^2 + o_{P_{\infty|Z^n}} (r_n^{-1}) \text{ wpal}(P_{Z^n}),
\]
where the second equality is due to Lemma \textbf{B.3}(2) and (b), the third equality is due to the assumption $||B_n^w - ||u_n^*||^2|| = o_{P_{\infty|Z^n}} (1) \text{ wpal}(P_{Z^n})$ and $||u_n^*|| \in (c, C)$. Thus equation \textbf{(B.25)} holds.

**STEP 4.** We now show that there exists a $t_n^*$ such that (a) and (b) hold in Step 3.

Let $r \equiv \phi(\hat{\alpha}_n) - \phi(\alpha_0)$. Since $\hat{\alpha}_n^{B} \in \mathcal{N}_\infty$ wpal, and $\phi(\hat{\alpha}_n) - \phi(\alpha_0) = O_{P_{\infty}} (||v_n^*||/\sqrt{n})$, by Lemma \textbf{B.2} there is a $t_n^* \in T_n$ wpal satisfying (a) with $\hat{\alpha}_n^{B}(t_n^*) = \hat{\alpha}_n^{B} + t_n^* u_n^* \in \mathcal{A}_n^{(n)}$ and $\phi(\hat{\alpha}_n^{B}(t_n^*)) = \phi(\alpha_0) = r$. Moreover, by assumption \textbf{3.5(i)(ii)}, such a choice of $t_n^*$ also satisfies
\[
\left| \phi(\hat{\alpha}_n^{B}(t_n^*)) - \phi(\hat{\alpha}_n) - \frac{d\phi(\alpha_0)}{d\alpha} |\hat{\alpha}_n^{B} - \hat{\alpha}_n + t_n^* u_n^*| \right| = o_{P_{\infty|Z^n}} (||v_n^*||/\sqrt{n}) \text{ wpal}(P_{Z^n}).
\]
Thus, for sufficiently large $n$,
\[
P_{Z^n} \geq \left( P_{P_{\infty|Z^n}} \left( \frac{\sqrt{n}}{||v_n^*||} \left| \frac{d\phi(\alpha_0)}{d\alpha} \right| |\hat{\alpha}_n^{B} - \hat{\alpha}_n + t_n^* u_n^*| \geq \delta | Z^n \right) < \delta \right) \geq 1 - \delta.
\]
By Assumption \textbf{B.3(i)} and Lemma \textbf{B.3}(2), it follows that the LHS of the above equation is majorized by
\[
P_{Z^n} \geq \left( P_{P_{\infty|Z^n}} \left( \frac{\sqrt{n}}{||v_n^*||} \left| \frac{d\phi(\alpha_0)}{d\alpha} \right| |\hat{\alpha}_n^{B} - \hat{\alpha}_n| + t_n^* ||v_n^*||^2 \geq 2\delta | Z^n \right) < \delta \right) + \delta
\]
\[
= P_{Z^n} \left( P_{P_{\infty|Z^n}} \left( \frac{\sqrt{n}}{||v_n^*||} \left| -Z_n^n -1||v_n^*||_{sd} + t_n^* ||v_n^*||^2 \geq 2\delta | Z^n \right) < \delta \right) + \delta.
\]
Therefore,
\[
\sqrt{m} t_n^* = \sqrt{n} Z_n^n \rightarrow 1/||u_n^*||^2 + o_{P_{\infty|Z^n}} (1) \text{ wpal}(P_{Z^n}).
\]
Since $\sqrt{n} Z_n^{-1} = O_{P_{\infty|Z^n}} (1)$ with probability $P_{Z^n}$ approaching one (assumption \textbf{Boot.3(ii)}) and $||u_n^*||^2 = O(1)$, we have $t_n^* = O_{P_{\infty|Z^n}} (n^{-1/2})$ with probability $P_{Z^n}$ approaching one. Thus (b) holds.
Before we prove Result (2), we wish to establish the following equation (B.26):

\[
\text{\textbf{(B.26)}} \quad \mathcal{L}_{\mathcal{V} \sim \mathcal{Z}} \left( \frac{QLR_n^B (\phi_0) | Z^n}{\sigma_Z} \right) - \mathcal{L} \left( QLR_n (\phi_0) | H_0 \right) = o_{\mathcal{P}_{\mathcal{Z}}} (1),
\]

where \( \mathcal{L} (QLR_n (\phi_0) | H_0) \) denotes the law of \( QLR_n (\phi_0) \) under the null \( H_0 : \phi (\alpha) = \phi_0 \), which will be simply denoted as \( \mathcal{L} \left( QLR_n (\phi_0) \right) \) in the rest of the proof. By Result (1), it suffices to show that for any \( \delta > 0 \), there exists a \( N(\delta) \) such that

\[
P_{\mathcal{Z}} \left( \sup_{f \in \mathcal{B}_1} \left| E \left[ f \left( \frac{\sqrt{n} Z_n^{\alpha-1}}{\sigma_\alpha u_\alpha |u_\alpha|^\alpha} \right)^2 \right] | Z^n \right| - E \left[ f (QLR_n (\phi_0)) \right] \right) \leq \delta \geq 1 - \delta
\]

for all \( n \geq N(\delta) \). Let \( Z \) denote a standard normal random variable (i.e., \( Z \sim N(0,1) \)). If the following equation (B.27) holds, which will be shown at the end of the proof of equation (B.26),

\[
\text{\textbf{(B.27)}} \quad T_n \equiv \sup_{f \in \mathcal{B}_1} \left| E \left[ f \left( \frac{Z}{||u_\alpha||} \right)^2 \right] - E \left[ f (QLR_n (\phi_0)) \right] \right| = o(1),
\]

then, it suffices to show that

\[
\text{\textbf{(B.28)}} \quad P_{\mathcal{Z}} \left( \sup_{f \in \mathcal{B}_1} \left| E \left[ f \left( \frac{\sqrt{n} Z_n^{\alpha-1}}{\sigma_\alpha u_\alpha |u_\alpha|^\alpha} \right)^2 \right] | Z^n \right| - E \left[ f \left( \frac{Z}{||u_\alpha||} \right)^2 \right] \leq \delta \right) \geq 1 - \delta
\]

for all \( n \geq N(\delta) \).

Suppose we could show that

\[
\text{\textbf{(B.29)}} \quad \sup_{f \in \mathcal{B}_1} \left| E \left[ f \left( \frac{\sqrt{n} Z_n^{\alpha-1}}{\sigma_\alpha u_\alpha |u_\alpha|^\alpha} \right) | Z^n \right] - E \left[ f \left( \frac{Z}{||u_\alpha||} \right) \right] \right| \to 0, \quad \text{wpa1} (P_{\mathcal{Z}}),
\]

or equivalently,

\[
P_{\mathcal{Z}} \left( | \mathcal{L}_{\mathcal{V} \sim \mathcal{Z}} \left( \sqrt{n} \frac{Z_n^{\alpha-1}}{\sigma_\alpha |u_\alpha|^\alpha} | Z^n \right) - \mathcal{L} (Z | ||u_\alpha||^{-1}) \right| \leq \delta \geq 1 - \delta, \quad \text{eventually.}
\]

Then, by the continuous mapping theorem (see Kosorok (2008) Theorem 10.8 and the discussion in section 10.1.4), we have:

\[
P_{\mathcal{Z}} \left( \left| \mathcal{L}_{\mathcal{V} \sim \mathcal{Z}} \left( \frac{\sqrt{n} Z_n^{\alpha-1}}{\sigma_\alpha |u_\alpha|^\alpha} \right)^2 | Z^n \right| - \mathcal{L} \left( (Z | ||u_\alpha||^{-1})^2 \right) \right| \leq \delta \geq 1 - \delta, \quad \text{eventually,}
\]

and hence equation (B.28) follows.

It remains to show equation (B.29). By Assumption (Boot.3 ii), and the fact that if a sequence converges in probability, for all subsequence, there exists a subsubsequence that converges almost surely, it follows for all subsequence \( (n_k)_k \), there exists a
The first term vanishes because
\[
\mathcal{L}_{V_\infty}^{Z \infty} \left( \frac{2^{n(k(j))}}{2 \sigma^2} | Z^{n(k(j))} \right) - \mathcal{L}(Z) \to 0, \text{ a.s. } - \mathcal{P}_{Z \infty}.
\]

Since \( |u_{n_k(j)}^*| \in (c, C) \), then there exists a further subsequence (which we still denote as \( n_k(j) \)), such that \( \lim_{j \to \infty} |u_{n_k(j)}^*| = d_\infty \in [c, C] \). Also, since \( \sqrt{n} \frac{\bar{Z}^{n-1}}{\tilde{Z}^{n-1}} \) is a real valued sequence, by Helly’s theorem, convergence in distribution also holds for \( (n_k(j)) \). Therefore, by Slutsky theorem,
\[
\mathcal{L}_{V_\infty}^{Z \infty} \left( \frac{2^{n_k(j)}}{2 \sigma \omega \| u_{n_k(j)}^* \|} | Z^{n_k(j)} \right) - \mathcal{L}(Zd_\infty) \to 0, \text{ a.s. } - \mathcal{P}_{Z \infty}.
\]

Since \( \lim_{j \to \infty} |u_{n_k(j)}^*| = d_\infty \in [c, C] \) and \( Z \) is bounded in probability, this readily implies
\[
\mathcal{L}_{V_\infty}^{Z \infty} \left( \frac{2^{n_k(j)}}{2 \sigma \omega \| u_{n_k(j)}^* \|} | Z^{n_k(j)} \right) - \mathcal{L}(Z|u_{n_k(j)}^*|^{-1}) \to 0, \text{ a.s. } - \mathcal{P}_{Z \infty}.
\]

Therefore, it follows that
\[
\sup_{f \in \mathcal{BL}_1} \left| E \left[ f \left( \frac{Z^{n_k(j)}}{2 \sigma \omega \| u_{n_k(j)}^* \|} \right) | Z^{n_k(j)} \right] - E \left[ f \left( Z|u_{n_k(j)}^*|^{-1} \right) \right] \right| \to 0, \text{ a.s. } - \mathcal{P}_{Z \infty}.
\]

Since the argument started with an arbitrary subsequence \( n_k \), equation (B.29) holds.

To conclude the proof of equation (B.26), we now show that equation (B.27) in fact holds (i.e., \( T_n = o(1) \)). Again, it suffices to show that for any sub-sequence, there exists a sub-sub-sequence such that \( T_n(j) = o(1) \). For any sub-sequence, since \( \| u_n^* \| \) is a bounded sequence (under Assumption 3.1 iv)), there exists a further sub-sub-sequence (which we denote as \( (n(j)) \)) such that \( \lim_{j \to \infty} |u_{n(j)}^*| = d_\infty \in [c, C] \) for finite \( c, C > 0 \). Observe that

\[
T_n(j) \leq \sup_{f \in \mathcal{BL}_1} \left| E \left[ f \left( \frac{Z}{d_\infty \| u_{n(j)}^* \|} \right) \right] - E \left[ f \left( \frac{Z}{d_\infty} \right) \right] \right|
\]

\[
+ \sup_{f \in \mathcal{BL}_1} \left| E \left[ f \left( \frac{Z}{d_\infty} \right) \right] - E \left[ f \left( \frac{\| u_{n(j)}^* \|}{d_\infty} \right) \right] \right|
\]

\[
+ \sup_{f \in \mathcal{BL}_1} \left| E \left[ f \left( Q \mathcal{L}R_n(j)(\phi_0) \right) \right] - E \left[ f \left( \frac{\| u_{n(j)}^* \|}{d_\infty} \right) \right] \right|
\]

The first term vanishes because \( Z \) is bounded in probability and \( \lim_{j \to \infty} |u_{n(j)}^*| = d_\infty > 0 \); the third term follows by the same reason (by Theorem 4.3 and Assumption 3.6 ii)), \( Q \mathcal{L}R_n(j)(\phi_0) \) is bounded in probability).

Finally, for any \( f \in \mathcal{BL}_1 \), let \( f(d_\infty^{-1}) \equiv f \circ d_\infty^{-2} \). Since \( f \circ d_\infty^{-2} \) is bounded and \( |f \circ d_\infty^{-2}(t) - f \circ d_\infty^{-2}(s)| \leq d_\infty^{-2} |t - s| \leq c^{-2} |t - s| \), we have \( \{ f \circ d_\infty^{-2} : f \in \mathcal{BL}_1 \} \subseteq \mathcal{BL}_{c^{-2}} \). Therefore, the second term in the previous display is majorized by
Denote \(Z\) whose distribution is continuous. Therefore, by Polya’s theorem, the second line follows because by assumption, \(A\) as \(A\) (which we still denote as \((\text{Theorem } 4.3\text{ (i.e., }|||\text{)}\)).

Finally for Result (2), we want to show that

\[
\sup_{t \in \mathbb{R}} \left| P_{V^\infty |Z^\infty} \left( \frac{Q^R_n(B)}{\sigma_n^2} \leq t \mid Z^n \right) - P_{Z^\infty} \left( \frac{Q^R_n(\phi_0)}{\sigma_n^2} \leq t \mid H_0 \right) \right| = o_{P_{Z^\infty}}(1).
\]

Let \(f_t(\cdot) \equiv 1\{\cdot \leq t\}\) for \(t \in \mathbb{R}\). Under this notation, the previous display can be cast as

\[
A_n \equiv \sup_{t \in \mathbb{R}} \left| E_{P_{V^\infty |Z^\infty}} \left( f_t \left( \frac{Q^R_n(B)}{\sigma_n^2} \right) \mid Z^n \right) - E_{P_{Z^\infty}} \left( f_t \left( \frac{Q^R_n(\phi_0)}{\sigma_n^2} \right) \right) \right| = o_{P_{Z^\infty}}(1).
\]

Denote \(Z^2 \sim \chi^2_1\) and

\[
A_{1,n} \equiv \sup_{t \in \mathbb{R}} \left| E_{P_{V^\infty |Z^\infty}} \left( f_t \left( \frac{||u^*_n||^2 \times Q^R_n(B)}{\sigma_n^2} \right) \mid Z^n \right) - E \left[ f_t \left( Z^2 \right) \right] \right|
\]

\[
A_{2,n} \equiv \sup_{t \in \mathbb{R}} \left| E_{P_{Z^\infty}} \left( f_t \left( ||u^*_n||^2 \times Q^R_n(\phi_0) \right) \right) - E \left[ f_t \left( Z^2 \right) \right] \right|
\]

Notice that

\[
A_n = \sup_{t \in \mathbb{R}} \left| E_{P_{V^\infty |Z^\infty}} \left( f_{t||u^*_n||^2} \left( \frac{||u^*_n||^2 Q^R_n(B)}{\sigma_n^2} \right) \mid Z^n \right) - E_{P_{Z^\infty}} \left( f_{t||u^*_n||^2} \left( ||u^*_n||^2 Q^R_n(\phi_0) \right) \right) \right|
\]

\[
\leq \sup_{t \in \mathbb{R}} \sup_{d \in [c, C]} \left| E_{P_{V^\infty |Z^\infty}} \left( f_{td^2} \left( ||u^*_n||^2 Q^R_n(\phi_0) \right) \mid Z^n \right) - E_{P_{Z^\infty}} \left( f_{td^2} \left( ||u^*_n||^2 Q^R_n(\phi_0) \right) \right) \right|
\]

\[
\leq A_{1,n} + A_{2,n}
\]

where the first line follows from the property that \(f_t(\cdot) = f_{\lambda \times \cdot}\) for any \(\lambda \in \mathbb{R}_+\); the second line follows because by assumption, \(||u^*_n||^2 \in [c^2, C^2]\); the third line follows simply because \(\{1\{\cdot \leq t\lambda\} : t \in \mathbb{R} \text{ and } \lambda \in \mathbb{R}_+\} \subseteq \{1\{\cdot \leq t\} : t \in \mathbb{R}\}\). Finally, the last line is due to the triangle inequality and the definitions of \(A_{1,n}\) and \(A_{2,n}\).

By Theorem 4.3 under the null, \(||u^*_n||^2 \times Q^R_n(\phi_0)\) converges weakly to \(Z^2 \sim \chi^2_2\), whose distribution is continuous. Therefore, by Polya’s theorem, \(A_{2,n} = o(1)\). Similarly,

\[
A_{1,n} = \sup_{t \in \mathbb{R}} \left| P_{V^\infty |Z^\infty} \left( ||u^*_n||^2 \times Q^R_n(B) \leq t' \mid Z^n \right) - P \left( Z^2 \leq t' \right) \right| = o_{P_{Z^\infty}}(1)
\]
by equation (B.26) and by the same arguments in Lemma 10.11 in [Kosorok (2008)], Q.E.D.

We first recall some notation introduced in the main text. Let \( T_n \equiv \{ t \in \mathbb{R} : |t| \leq 4M_2^2 \delta_n \} \). For \( t_n \in T_n \), \( \alpha(t_n) \equiv \alpha + t_n u_n^* \) where \( u_n^* = v_n^* / \| v_n^* \|_a \) and \( v_n^* = (v_{n,n}^* \ldots v_{n,n}^* )' \).

To simplify presentation we use \( r_n = r_n(t_n) \equiv (\max\{ t_n^2, t_n n^{-1/2}, o(n^{-1}) \})^{−1} \).

**Proof of Lemma 5.1** For Result (1), if \( \omega \equiv 1 \), then Assumption [Boot.3(i)] simplifies to

\[
P_Z \approx \left( P_{V^2 \otimes Z} \left( \sup_{(\alpha,t_n) \in \mathcal{N}_{a,n} \otimes T_n} r_n \left| \hat{\Lambda}_n(\alpha(t_n), \alpha) - t_n \{Z_n + (u_n^*, \alpha - \alpha_0) \} - \frac{B_n}{2} t_n^2 \right| \geq \delta \mid Z^n \right) \leq \delta \right) \geq 1 - \delta;
\]

iff

\[
P_Z \approx \left( \sup_{(\alpha,t_n) \in \mathcal{N}_{a,n} \otimes T_n} r_n \left| \hat{\Lambda}_n(\alpha(t_n), \alpha) - t_n \{Z_n + (u_n^*, \alpha - \alpha_0) \} - \frac{B_n}{2} t_n^2 \right| \leq \delta \right) \geq 1 - \delta,
\]

where \( \hat{\Lambda}_n(\alpha(t_n), \alpha) \equiv 0.5(\hat{Q}_n(\alpha(t_n)) - \hat{Q}_n(\alpha)) \) and \( B_n \) is a \( Z^n \) measurable random variable with \( B_n = Q_{P_{Z^2}}(1) \). Therefore, if we could verify Assumption [Boot.3(i)] in Result (2), we also verify Assumption (3.3(i)).

For Result (2), we divide its proof in several steps.

**Step 1:** We first introduce some notation. Let

\[
P_n(Z^n) \equiv P_{V^2 \otimes Z} \left( \sup_{(\alpha,t_n) \in \mathcal{N}_{a,n} \otimes T_n} r_n \left| \hat{\Lambda}_n^B(\alpha(t_n), \alpha) - t_n \{Z_n + (u_n^*, \alpha - \alpha_0) \} - \frac{B_n}{2} t_n^2 \right| \geq \delta \mid Z^n \right) \geq 1 - \delta.
\]

Recall that \( \ell_n^B(x, \alpha) \equiv \tilde{m}(x, \alpha) + \hat{m}^B(x, \alpha_0) \). Let

\[
\hat{\ell}_n^B(\alpha(t_n), \alpha) \equiv \frac{1}{2n} \sum_{i=1}^n \{ \ell_n^B(X_i, \alpha(t_n))' \hat{\Sigma}(X_i)^{-1} \ell_n^B(X_i, \alpha(t_n))' - \ell_n^B(X_i, \alpha)' \hat{\Sigma}(X_i)^{-1} \ell_n^B(X_i, \alpha) \}.
\]

We need to show that \( P_Z \approx (P_n(Z^n) < \delta) \cap S_n \} + P_Z \approx (S_n^C) \leq \delta \), eventually, for some event \( S_n \) that is measurable with respect to \( Z^n \), and some \( P_n'(Z^n) \geq P_n(Z^n) \) a.s., here \( S_n^C \) denotes the complement of \( S_n \). In the following we take \( S_n \) and \( P_n'(Z^n) \) as

\[
S_n \equiv \left\{ Z^n : P_{V^2 \otimes Z} \left( \sup_{(\alpha,t_n) \in \mathcal{N}_{a,n} \otimes T_n} r_n \left| \hat{\Lambda}_n^B(\alpha(t_n), \alpha) - \hat{\ell}_n^B(\alpha(t_n), \alpha) \right| \geq 0.5\delta \mid Z^n \right) < 0.5\delta \right\},
\]

and

\[
P_n'(Z^n) \equiv P_{V^2 \otimes Z} \left( \sup_{(\alpha,t_n) \in \mathcal{N}_{a,n} \otimes T_n} r_n \left| \hat{\Lambda}_n^B(\alpha(t_n), \alpha) - \hat{\ell}_n^B(\alpha(t_n), \alpha) \right| \geq 0.5\delta \mid Z^n \right) + P_{V^2 \otimes Z} \left( \sup_{(\alpha,t_n) \in \mathcal{N}_{a,n} \otimes T_n} r_n \left| \hat{\Lambda}_n^B(\alpha(t_n), \alpha) - \hat{\ell}_n^B(\alpha(t_n), \alpha) \right| \geq 0.5\delta \mid Z^n \right) \geq 1 - \delta.
\]
It follows that we “only” need to show that
\[ P_{Z^\infty} (S_n^c) \leq 0.5\delta \quad \text{and} \quad P_{Z^\infty} (\{P_n' (Z^n) > \delta\} \cap S_n) \leq 0.5\delta, \text{ eventually.} \]

Since \( P_{Z^\infty} (S_n^c) \) can be expressed as
\[
P_{Z^\infty} \left( P_{V^\infty|Z^\infty} \left( \sup_{(\alpha, t_n) \in N_{\alpha,n} \times T_n} r_n \left| \hat{\Lambda}_n^B (\alpha(t_n), \alpha) - \hat{\Lambda}_n^B (\alpha(t_n), \alpha) \right| \geq 0.5\delta \right) \right),
\]
which, by Lemma A.2(3), is in fact less than 0.5. We only need to verify
\[ P_{Z^\infty} (\{P_n' (Z^n) > \delta\} \cap S_n) \leq 0.5\delta, \text{ eventually.} \]

It is easy to see that
\[
P_{Z^\infty} (\{P_n' (Z^n) > \delta\} \cap S_n) \leq P_{Z^\infty} \left( P_{V^\infty|Z^\infty} \left( \sup_{(\alpha, t_n) \in N_{\alpha,n} \times T_n} r_n \left| \hat{\Lambda}_n^B (\alpha(t_n), \alpha) - t_n \{Z_{n^s}^\omega + (u_n^s, \alpha - \alpha_0)\} - \frac{B_n}{2} t_n^2 \right| \geq 0.5\delta \mid Z^n \right) \right).
\]
Hence, in order to prove the desired result, it suffices to show that
\[
(B.31) \quad P_{Z^\infty} \left( P_{V^\infty|Z^\infty} \left( \sup_{(\alpha, t_n) \in N_{\alpha,n} \times T_n} r_n \left| \hat{\Lambda}_n^B (\alpha(t_n), \alpha) - t_n \{Z_{n^s}^\omega + (u_n^s, \alpha - \alpha_0)\} - \frac{B_n}{2} t_n^2 \right| \geq 0.5\delta \mid Z^n \right) \right) < \delta
\]

eventually.

**Step 2:** For any \( \alpha \in N_{\alpha,n} \) and \( t_n, \alpha(t_n) = \alpha + t_n u_n^s, \) under Assumption A.7(i), we can apply the mean value theorem (wrt \( t_n \)) and obtain
\[
\hat{\Lambda}_n^B (\alpha(t_n), \alpha) = \frac{t_n}{n} \sum_{i=1}^{n} \left( \frac{d\bar{m}(X_i, \alpha)}{\alpha} [u_n^s] \right)' \hat{\Sigma}(X_i)^{-1} \hat{\epsilon}_n^B (X_i, \alpha)
\]
\[
+ \frac{t_n^2}{2n} \int_0^1 \sum_{i=1}^{n} \left( \frac{d^2\bar{m}(X_i, \alpha(s))}{\alpha^2} [u_n^s] \right)' \hat{\Sigma}(X_i)^{-1} \left( \frac{d\bar{m}(X_i, \alpha(s))}{\alpha} [u_n^s] \right) ds
\]
\[
+ \frac{t_n^2}{2n} \int_0^1 \sum_{i=1}^{n} \left( \frac{d^2\bar{m}(X_i, \alpha(s))}{\alpha^2} [u_n^s, u_n^s] \right)' \hat{\Sigma}(X_i)^{-1} \hat{\epsilon}_n^B (Z_i, \alpha(s)) ds
\]
\[
\equiv t_n T_{1n}^B (\alpha) + \frac{t_n^2}{2} (T_{2n} (\alpha) + T_{3n}^B (\alpha)),
\]
where \( \alpha(s) \equiv \alpha + s t_n u_n^s \in N_{\alpha,n}. \)

From these calculations and the fact that \( P_{V^\infty|Z^\infty} (a_n + b_n \geq d | Z^n) \leq P_{V^\infty|Z^\infty} (a_n \geq 0.5d | Z^n) + P_{V^\infty|Z^\infty} (b_n \geq 0.5d | Z^n) \) a.s. for any two measurable random variables.
Lemma B.4 Let \((X_{i,n})_{n=1}^\infty\) be a triangular array of real valued random variables such that (a) \(X_{1,n}, \ldots, X_{n,n}\) are independent and \(X_{i,n} \sim P_{i,n}\) for all \(n\), (b) \(E[X_{i,n}] = 0\) for all \(i\) and \(n\), and (c) there is a sequence of non-negative real numbers \((b_n)_n\) such that \(b_n = o(\sqrt{n})\) and

\[
\limsup_{n \to \infty} n^{-1} \sum_{i=1}^{n} E[|X_{i,n}|1\{|X_{i,n}| \geq b_n\}] = 0.
\]
Then: for all $\epsilon > 0$, there is a $N(\epsilon)$ such that
\[
\Pr \left( \left| \sum_{i=1}^{n} X_{i,n} \right| \geq \epsilon \right) < \epsilon \quad \text{for all } n \geq N(\epsilon).
\]

**Proof of Lemma B.4** We obtain the result by modifying the proofs of [Billingsley 1995] theorem 22.1 and of [Feller 1970] (p. 248). For any $\epsilon > 0$, let
\[
X_{i,n} = X_{i,n,1} \{ |X_{i,n}| \leq b_n \} + X_{i,n,1} \{ |X_{i,n}| > b_n \} \equiv X_{i,n}^B + X_{i,n}^U.
\]
Thus,
\[
\Pr \left( \left| \sum_{i=1}^{n} X_{i,n} \right| \geq \epsilon \right) \leq \Pr \left( \left| \sum_{i=1}^{n} X_{i,n}^B \right| \geq 0.5\epsilon \right) + \Pr \left( \left| \sum_{i=1}^{n} X_{i,n}^U \right| \geq 0.5\epsilon \right)
\]
\[
\equiv T_{1,\epsilon} + T_{2,\epsilon}.
\]
By conditions (b) and (c), it is easy to see that, for large enough $n$,
\[
T_{1,\epsilon} \leq \Pr \left( \left| \sum_{i=1}^{n} \{X_{i,n}^B - E[X_{i,n}^B]\} \right| \geq 0.25\epsilon \right) + 1 \{ \sum_{i=1}^{n} E[X_{i,n}^B] \geq 0.25\epsilon \}
\]
\[
= \Pr \left( \sum_{i=1}^{n} \{X_{i,n}^B - E[X_{i,n}^B]\} \geq 0.25\epsilon \right) \leq 2 \exp \left( -\text{const.} \frac{\epsilon^2 n}{b_n^2} \right),
\]
for some finite constant $\text{const} > 0$, where the last inequality is due to Hoeffding inequality (cf. [Van der Vaart and Wellner 1996] Appendix A.6). Thus, there is a $N(\epsilon)$ such that for all $n \geq N(\epsilon)$, $T_{1,\epsilon} < 0.5\epsilon$.

For $T_{2,\epsilon}$, by Markov inequality and then by condition (c), we have:
\[
T_{2,\epsilon} \leq (\epsilon/2)^{-1} n^{-1} \sum_{i=1}^{n} \int_{|x| > b_n} |x| P_{1,n}(dx)
\]
\[
= (\epsilon/2)^{-1} n^{-1} \sum_{i=1}^{n} \int |x| 1\{|x| > b_n\} P_{1,n}(dx) < 0.5\epsilon
\]
eventually. Q.E.D.

**Proof of Lemma 5.2** We divide the proof into several steps.

**Step 1.** We first show that the event
\[
S_n \equiv \left\{ Z^n : \left| \sum_{i=1}^{n} (g(X_i, u_{n,i})^2 - E[g(X, u_{n,i})] \Sigma_0(X) g(X, u_{n,i})] \right| \leq \delta \right\}
\]
occurs $\mathbb{P}_Z = 1$. For this we apply Lemma B.4. Using the notation in the lemma, we let $X_{i,n} = (g(X_i, u_{n,i})^2 - E[g(X, u_{n,i})] \Sigma_0(X) g(X, u_{n,i})]$, and thus conditions (a) and (b) of Lemma B.4 immediately follow (note that $E[g(X, u_{n,i})\Sigma_0(X) g(X, u_{n,i})] = 1$). In order to check condition (c), note first that for any generic random variable $X$ with
mean $\mu < \infty$, it follows

$$E[|X - \mu| \{ |X - \mu| \geq b_n \}] \leq E[|X| \{ |X| \geq b_n - |\mu| \}] + |\mu| \Pr\{ |X| \geq b_n - |\mu| \}.$$  

Since $b_n$ is taken to diverge, we can “redefine” $b_n$ as $b_n - |\mu|$. Moreover,

$$\Pr\{ |X| \geq b_n - |\mu| \} \leq E[\max\{|X|, 1\} \{ |X| \geq b_n - |\mu| \}].$$

Again, since $b_n$ is taken to diverge the only relevant case is $|X| \geq 1$. Therefore, it suffices to study $E[|X| \{ |X| \geq b_n \}]$ in order to bound $E[|X - \mu| \{ |X - \mu| \geq b_n \}]$.

Thus, applied to our case, it is sufficient to verify that

$$\limsup_{n \to \infty} n^{-1} \sum_{i=1}^{n} E\left[ \left( g(X_i, \omega_n) \rho(Z_i, \alpha_0) \right)^2 \right] \leq 0,$$

which holds under our assumption equation (5.1).

**Step 2.** Let $\sqrt{n} \frac{Z_n^{\omega \omega^{-1}}}{\sigma_\omega} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \zeta_i s_{i,n}$ where $s_{i,n} \equiv g(X_i, \omega_n) \rho(Z_i, \alpha_0)$, and either $\{ \zeta_i \}_{i=1}^{n}$ is IID with $\zeta_i = (\omega_i - 1) \sigma_\omega^{-1}$ (under assumption Boot.1) or $\{ \zeta_i \}_{i=1}^{n}$ is multinomial with $\zeta_i = (\omega_i, n - 1)$ (under assumption Boot.2). In the following we let $P_\Omega$ denote the conditional distribution of $\{ \zeta_i \}_{i=1}^{n}$ given the data $Z^n$, which is also the unconditional distribution of $\{ \zeta_i \}_{i=1}^{n}$ since $\{ \zeta_i \}_{i=1}^{n}$ is independent of $Z^n$. We want to establish that

$$\sup_{f \in BL_1} E \left[ f \left( \sqrt{n} \frac{Z_n^{\omega \omega^{-1}}}{\sigma_\omega} \right) | Z^n \right] - E[f(Z)] = o_{P_{Z \sim \infty}}(1),$$

where $Z \sim N(0, 1)$. Which is equivalent to show that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \zeta_i s_{i,n} \Rightarrow Z, \ wpa1(P_{Z \sim \infty}).$$

Which, by [Billingsley (1995) (Theorem 20.5, p. 268), in turn suffices to show that any sub-sequence, contains a further sub-sequence, $(n_k)_k$, such that

$$\sup_{f \in BL_1} E \left[ f \left( \sqrt{n} \frac{Z_n^{\omega \omega^{-1}}}{\sigma_\omega} \right) | Z^n \right] - E[f(Z)] = o_{P_{Z \sim \infty}}(1),$$

where $Z \sim N(0, 1)$. Which is equivalent to show that

$$\frac{1}{\sqrt{n_k}} \sum_{i=1}^{n_k} \zeta_i s_{i,n_k} \Rightarrow Z, \ a.s. - (P_{Z \sim \infty}).$$

**Step 3** below establishes (B.32) under assumption Boot.1, while **Step 4** below establishes (B.32) under assumption Boot.1.

**Step 3.** (under assumption Boot.1) Since the event $S_n$ occurs $wpa1(P_{Z \sim \infty})$ (Step 1), it follows that any sub-sequence, contains a further sub-sequence such that

$$n_k^{-1} \sum_{i=1}^{n_k} (s_{i,n_k})^2 \to 1, \ a.s. - (P_{Z \sim \infty}).$$

Moreover, $\max_{i \leq n_k} |s_{i,n_k}| / \sqrt{n_k} = o(1), \ a.s. - (P_{Z \sim \infty})$. This follows since, for any $\epsilon > 0$,

$$P_{Z \sim \infty} \left( \max_{i \leq n_k} |s_{i,n_k}| \geq \epsilon \sqrt{n} \right) \leq \sum_{i=1}^{n_k} \int_{|s| \geq \epsilon \sqrt{n}} P_{i,n}(ds) \leq \epsilon^{-2} n^{1/2} \sum_{i=1}^{n_k} \int_{|s| \geq \epsilon \sqrt{n}} s^2 P_{i,n}(ds)$$

$$= \epsilon^{-2} n^{1/2} \sum_{i=1}^{n_k} E[|s_{i,n_k}|^2 \{ |s_{i,n_k}| \geq \epsilon \sqrt{n} \}].$$
We note that $1\{|s_{i,n}| \geq \sqrt{n}\} \leq 1\{|s_{i,n}| \geq b_n\}$ (provided that $|s_{i,n}| \geq 1$, but if it is not, then the proof is trivial). Hence by equation [5.1] and the fact that $s_{i,n}$ are row-wise iid, the RHS is of order $o(1)$. Going to a sub-sequence establishes the result.

Under assumption Boot.1 $\zeta_i = (\omega_i - 1)\sigma_{\omega}^{-1}$ is IID with mean zero, variance 1, hence conditional on the event $S_{\omega}$, for any $\epsilon > 0$,

$$n_k^{-1} \sum_{i=1}^{n_k} E_{\Omega} \left[ (\zeta_i s_{i,n_k})^2 \mathbb{1}\{|\zeta_i s_{i,n_k}| > \epsilon \sqrt{n_k}\} \right]$$

\leq \left( n_k^{-1} \sum_{i=1}^{n_k} |s_{i,n_k}|^2 \right) \times E_{\Omega} \left[ \zeta_i^2 \times 1 \left\{ |\zeta_i| \times \max_{1 \leq i \leq n} |s_{i,n_k}| > \epsilon \sqrt{n_k} \right\} \right]$$

\leq \left( n_k^{-1} \sum_{i=1}^{n_k} |s_{i,n_k}|^2 \right) \times E_{\Omega} \left[ \zeta_i^2 \times 1 \left\{ |\zeta_i| > \epsilon' \right\} \right] \rightarrow 0, \text{ a.s. - } (P_{Z\sim})

where the second inequality follows from the fact that $\max_{1 \leq i \leq n_k} |s_{i,n_k}|/\sqrt{n_k} < \epsilon', \text{ a.s. - } (P_{Z\sim})$ eventually. Since $\zeta_i$ are IID, by choosing the $\epsilon'$ (small relative to $\epsilon$), one can make the term $E_{\Omega}[\zeta_i^2 \cdot \{ |\zeta_i| > \epsilon' \}]$ arbitrarily small. The Lindeberg-Feller CLT then implies that $\frac{1}{\sqrt{n_k}} \sum_{i=1}^{n_k} \zeta_i s_{i,n_k} \Rightarrow Z$, a.s. - $P_{Z\sim}$ where $Z \sim N(0,1)$.

We have thus showed that any sub-sequence, contains a further sub-sequence such that the above equation holds; therefore

$$\sup_{f \in B_{L_1}} \left| E \left[ f \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \zeta_i s_{i,n} \right) \right] - E[ f(Z) ] \right| = o_{P_{Z\sim}}(1).$$

**STEP 4.** (under assumption Boot.2) We proceed as in Step 3 to establish equation [B.32]. The difference is that now $\zeta_i n_k^{-1}$ is not iid, but exchangeable with $\zeta_i = \zeta_{i,n} \equiv (\omega_i/n_k - 1)$. To overcome this, we follow lemma 3.6.15 (or really proposition A.5.3) in VdV-W for a given sub-sequence $(n_k)_k$. To simplify notation we let $n = n_k$ and $s_{i,n} = s_{i,n_k}$.

Under assumption Boot.2 we have: $n^{-1} \sum_{i=1}^n \zeta_i n_k = 0, n^{-1} \sum_{i=1}^n \zeta_i^2 n_k \rightarrow 1, n^{-1} \max_{1 \leq i \leq n} \zeta_i^2 n_k = o_{P_{\Omega}}(1)$ and $\max_{1 \leq i \leq n} E[|\zeta_i n_k|^2] \leq c < \infty$. Conditional on the event $S_{\omega}$, we also have $n^{-1} \sum_{i=1}^n s_{i,n} \rightarrow 0, n^{-1} \sum_{i=1}^n s_{i,n}^2 \rightarrow 1$ and $n^{-1} \max_{1 \leq i \leq n} s_{i,n}^2 = o(1)$ (this has already been established in Step 3), and finally we need:

$$\limsup_{n \to \infty} n^{-2} \sum_{i=1}^n \sum_{j=1}^n (s_{i,n} \zeta_{j,n})^2 \mathbb{1}\{|s_{i,n} \zeta_{j,n}| > \epsilon \sqrt{n}\} = 0, \text{ a.s. - } P_{Z\sim}.$$  

To show equation [B.33], we note that

$$\limsup_{n \to \infty} n^{-2} \sum_{i=1}^n \sum_{j=1}^n (s_{i,n} \zeta_{j,n})^2 \mathbb{1}\{|s_{i,n} \zeta_{j,n}| > \epsilon \sqrt{n}\}$$

\leq \limsup_{n \to \infty} \left( \frac{\sum_{i=1}^n \zeta_{i,n}^2}{n} \times \frac{\sum_{i=1}^n (s_{i,n})^2}{n} \mathbb{1}\{|s_{i,n} \times \max_{1 \leq j \leq n} |\zeta_{j,n}| > \epsilon \sqrt{n}\} \right).$$

Since $n^{-1} \sum_{i=1}^n \zeta_{i,n}^2 = 1 = o_{P_{\Omega}}(1)$, we have (with possibly going to a subsequence)

$$\limsup_{n \to \infty} n^{-2} \sum_{i=1}^n \zeta_{i,n}^2 = 1 \text{ a.s. - } P_{\Omega} \text{ a.s. - } P_{Z\sim}.$$ Hence, in order to show equation...
that is constructed as follows. First, we define
\[ \|\|B.34\|\| \]
Let \( L_n = \sqrt{\frac{2}{n}} \to \infty \) (such a choice exists since \( b_n = o(\sqrt{n}) \) in equation (5.1)). It then follows from the properties of \( \zeta_{j,n} \) that
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (s_{i,n})^2 1 \left\{ |s_{i,n}| \times \max_{1 \leq j \leq n} |\zeta_{j,n}| > \epsilon \sqrt{n} \right\} = 0 \text{ a.s.} - P_{2\infty}. \]

So, by lemma 3.6.15 (or proposition A.5.3) in Van der Vaart and Wellner (1996),
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (s_{i,n})^2 1 \left\{ |s_{i,n}| \times \max_{1 \leq j \leq n} |\zeta_{j,n}| > \epsilon \sqrt{n} \right\} \leq \frac{1}{n} \sum_{i=1}^{n} (s_{i,n})^2 1 \{ |s_{i,n}|^2 > 2b_n \} \to 0, \text{ a.s.} - P_{2\infty}. \]

Hence (by possibly going to subsequence) equation (B.33) follows from equation (5.1).

So, by lemma 3.6.15 (or proposition A.5.3) in Van der Vaart and Wellner (1996),
\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \zeta_{i,s_{i,n}} \Rightarrow Z, \text{ a.s.} - (P_{2\infty}). \]

The rest of the steps are analogous to those in Step 3 and will not be repeated here.

Q.E.D.

B.3.1. Alternative Bootstrap sieve t statistics

In this subsection we present additional bootstrap sieve \( t \) statistics. Recall that \( \widehat{W}_a \equiv \sqrt{n} \delta(\hat{\alpha}_n) - \hat{\phi}(\alpha_n) \) is the original sieve \( t \) statistic. The first one is \( \widehat{W}_{1,n}^B \equiv \sqrt{n} \delta(\hat{\alpha}_n) - \hat{\phi}(\alpha_n) \). In the definition of \( \widehat{W}_{2,n}^B \) one could also define \( ||\hat{\alpha}_n||_{B,sd} \) using \( \widehat{\Sigma}_0^B = E_{\alpha_n}[\delta(V, \hat{\alpha}_n) \delta(V, \alpha_n)] = E_{\alpha_n}[\delta(V, \hat{\alpha}_n) \delta(V, \hat{\alpha_n})'] \), which will be a bootstrap analog to \( ||\hat{\alpha}_n||_{B,sd} \) defined in equation (B.5).

Let \( \widehat{W}_{3,n}^B \equiv \sqrt{n} \delta(\hat{\alpha}_n) - \hat{\phi}(\alpha_n) \) where \( ||\hat{\alpha}_n||_{B,sd} \) is a bootstrap sieve variance estimator that is constructed as follows. First, we define
\[ ||\cdot||_{B,M} \equiv n^{-1} \sum_{i=1}^{n} \left( \frac{d\hat{m}^B(X_i, \hat{\alpha}_n)}{d\alpha} \right)' M_{n,i} \left( \frac{d\hat{m}^B(X_i, \hat{\alpha}_n)}{d\alpha} \right), \]
where \( M_{n,i} \) is some (almost surely) positive definite weighting matrix. Let \( \hat{\alpha}_n^B \) be a bootstrap empirical Riesz representer of the linear functional \( \frac{d\hat{m}^B}{d\alpha} : [-1,1] \) under \( ||\cdot||_{B,\hat{\Sigma}} \). We compute a bootstrap sieve variance estimator as:
\[ (B.34) \]
\[ ||\hat{\alpha}_n^B||_{B,sd}^2 \equiv n \sum_{i=1}^{n} \left( \frac{d\hat{m}^B(X_i, \hat{\alpha}_n^B)}{d\alpha} \right)' \hat{\Sigma}_i^{-1} \delta(V_i, \hat{\alpha}_n) \delta(V_i, \hat{\alpha}_n)' \hat{\Sigma}_i^{-1} \left( \frac{d\hat{m}^B(X_i, \hat{\alpha}_n^B)}{d\alpha} \right) \]
with \( g(V, \alpha) \equiv (\omega_i, n - 1) \rho(Z_i, \alpha) \equiv \rho^B(V, \alpha) - \rho(Z_i, \alpha) \) for any \( \alpha \). That is, \( \| \hat{\sigma}_n^B \|_{B, sd}^2 \) is a bootstrap analog to \( \| \hat{\sigma}_n^\omega \|_{B, sd}^2 \) defined in equation (4.7). One could also define \( \| \hat{\sigma}_n^B \|_{B, sd}^2 \) using \( \hat{E}_n[g(V, \hat{\alpha}_n^B)g(V, \hat{\alpha}_n^B)]|X = X_i \) instead of \( g(V, \hat{\alpha}_n)g(V, \hat{\alpha}_n) \), which will be a bootstrap analog to \( \| \hat{\sigma}_n^\omega \|_{B, sd}^2 \) defined in equation (B.5). In addition, one could also define \( \| \hat{\sigma}_n^B \|_{B, sd} \) using \( \hat{\alpha} \) instead of \( \hat{\alpha}_n \). In terms of the first order asymptotic approximation, this alternative definition yields the same asymptotic results. Due to space considerations, we omit these alternative bootstrap sieve variance estimators.

The bootstrap sieve variance estimator \( \| \hat{\sigma}_n^B \|_{B, sd}^2 \) also has a closed form expression:

\[
\hat{F}_n^B = \frac{d\hat{\sigma}_n^B}{d\alpha} [\hat{\Psi}^{k(n)}(\cdot)]', \quad \hat{D}_n = \frac{1}{n}\sum_{i=1}^{n} \left( \frac{d\hat{m}_B(X_i, \hat{\alpha}_n^B)}{d\alpha} \right)^2 \hat{\Sigma}_i^{-1} \left( \frac{d\hat{m}_B(X_i, \hat{\alpha}_n^B)}{d\alpha} \right) \hat{\Psi}^{k(n)}(\cdot)', \quad \hat{\Omega}_{\hat{\alpha}_n} = \frac{1}{n}\sum_{i=1}^{n} \left( \frac{d\hat{m}_B(X_i, \hat{\alpha}_n^B)}{d\alpha} \right)^2 \hat{\Sigma}_i^{-1} (\omega_i, n - 1)^2 \rho(Z_i, \hat{\alpha}_n^B) \rho(Z_i, \hat{\alpha}_n^B) \hat{\Sigma}_i^{-1} \left( \frac{d\hat{m}_B(X_i, \hat{\alpha}_n^B)}{d\alpha} \right) \hat{\Psi}^{k(n)}(\cdot)'.
\]

This expression is computed in the same way as \( \| \hat{\sigma}_n^\omega \|_{B, sd}^2 \), but using bootstrap analogs. Note that this bootstrap sieve variance only uses \( \hat{\alpha}_n \), and is easy to compute.

When specialized to the NPIV model (2.18) in subsection 2.2.1, the expression \( \| \hat{\sigma}_n^B \|_{B, sd}^2 \) simplifies further, with \( \hat{F}_n^B = \frac{d\hat{\sigma}_n^B}{d\alpha} [\hat{\Psi}^{k(n)}(\cdot)]', \quad \hat{D}_n = \hat{\Sigma}_i^{-1} \left( \frac{d\hat{m}_B(X_i, \hat{\alpha}_n^B)}{d\alpha} \right)^2 \hat{\Psi}^{k(n)}(\cdot)' \)

\[
\hat{\Omega}_{\hat{\alpha}_n} = \frac{1}{n}\sum_{i=1}^{n} \left( \frac{d\hat{m}_B(X_i, \hat{\alpha}_n^B)}{d\alpha} \right)^2 \hat{\Sigma}_i^{-1} (\omega_i, n - 1)^2 \rho(Z_i, \hat{\alpha}_n^B) \rho(Z_i, \hat{\alpha}_n^B) \hat{\Sigma}_i^{-1} \left( \frac{d\hat{m}_B(X_i, \hat{\alpha}_n^B)}{d\alpha} \right) \hat{\Psi}^{k(n)}(\cdot)'.
\]

This expression is analogous to that for a 2SLS t-bootstrap test; see Davidson and MacKinnon (2010). We leave it to further work to study whether this bootstrap sieve t statistic might have second order refinement by choice of some IID bootstrap weights.

Recall that \( M_i = (\omega_i, n - 1)^2 \hat{M}_i \) and \( \hat{M}_i = \hat{\Sigma}_i^{-1} \rho(Z_i, \hat{\alpha}_n) \rho(Z_i, \hat{\alpha}_n) \hat{\Sigma}_i^{-1} \).

**Assumption B.3**

(i) \( \sup_{v_1, v_2 \in \mathbf{V}_{k(n)}} |\langle v_1, v_2 \rangle_{B, \Sigma^{-1}} - \langle v_1, v_2 \rangle_{n, \Sigma^{-1}}| = o_{P_{V_{\infty}|Z_{\infty}}}(1) \) wp1(\( P_{2_{\infty}} \));

(ii) \( \sup_{v \in \mathbf{V}_{k(n)}} \| v \|_{B, M} - \| v \|_{n, M} \| = o_{P_{V_{\infty}|Z_{\infty}}}(1) \) wp1(\( P_{2_{\infty}} \));

(iii) \( \sup_{v \in \mathbf{V}_{k(n)}} n^{-1} \sum_{i=1}^{n} (\omega_i, n - 1)^2 \left\| \frac{d\hat{m}_B(X_i, \hat{\alpha}_n^B)}{d\alpha} \right\|^2 = O_{P_{V_{\infty}|Z_{\infty}}}(1) \) wp1(\( P_{2_{\infty}} \)).

Assumption B.3(i) (ii) is analogous to Assumption A.1(ii) (v). Assumption B.3(iii) is a mild one, for example, it is implied by Assumptions for Lemma A.1 and uniformly bounded bootstrap weights (i.e., \( |\omega_i, n| \leq C < \infty \) for all \( i \)).

The following result is a bootstrap version of Theorem 4.2.

**Theorem B.3**

Let Conditions for Theorem 4.2(1) and Lemma A.1, Assumption B.3 hold. Then:

\[
(1) \quad \frac{\| \hat{\sigma}_n^B \|_{B, sd}^2 - 1}{\sigma_v^2 \| v \|_{B, sd}} = o_{P_{V_{\infty}|Z_{\infty}}}(1) \text{ wp1}(P_{2_{\infty}}).
\]
4.2 step 2, it suffices to show that
\[ \text{(B.36)} \]

Proof of Theorem B.3. For Result (1), the proof is analogous to the one for Theorem 4.2(1). As in the proof of Theorem 4.2(1), it suffices to show that
\[ wpa \langle (B.35), it suffices to show \]

Regarding equation (B.36), following the same derivations as in the proof of Theorem 4.2(1) step 1, for equation
\[ \text{B.35}, it suffices to show that \]

Following the same derivations as in the proof of theorem 4.2(1) step 1, for equation
\[ \text{B.35}, it suffices to show that \]

by the triangle inequality,
\[ \sup_{v \in \nabla_{k(n)}} |(v, v)_{B, \hat{W}^B} - (v, v)_{n, \hat{M}}| \leq \sup_{v \in \nabla_{k(n)}} |(v, v)_{B, \hat{W}^B} - (v, v)_{B, \hat{M}}| \]

with \( \hat{W}^B = \hat{\Sigma}_{i-1} g(V_i, \hat{\alpha}_n) g(V_i, \hat{\alpha}_n) \hat{\Sigma}_{i-1} = (\omega_i - 1)^2 \hat{\Sigma}_{i-1} \rho(Z_i, \hat{\alpha}_n) \rho(Z_i, \hat{\alpha}_n) \Sigma_{i-1} \) and \( \hat{M}^B = (\omega_i - 1)^2 \hat{M}_i = \Sigma_i \rho(Z_i, \hat{\alpha}_n) \rho(Z_i, \hat{\alpha}_n) \Sigma_i^{-1} \).

It is easy to see that \( A_{1n}^B \) is bounded above by
\[ \sup_{x} ||\hat{\Sigma}_{i}^{-1}(x)\{\rho(z, \hat{\alpha}_n)\rho(z, \hat{\alpha}_n) - \rho(z, \hat{\alpha}_n)\rho(z, \hat{\alpha}_n)\} ||_{c} n^{-1} \sum_{i=1}^{n} (\omega_i - 1)^2 ||\hat{T}_i^B[v]||_c^2 \]

\[ \leq 2 \sup_{x, \alpha \in \nabla_{n}} ||\hat{\Sigma}_{i}^{-1}(x)\{\rho(z, \alpha')\rho(z, \alpha') - \rho(z, \alpha)\rho(z, \alpha)\} ||_{c} n^{-1} \sum_{i=1}^{n} (\omega_i - 1)^2 ||\hat{T}_i^B[v]||_c^2 \]

where \( \hat{T}_i^B[v] \equiv \hat{\alpha}_n^B(X_i, \hat{\alpha}_n^B)[v] \). The second line follows because \( \hat{\alpha}_n^B \in N_{\alpha n} \) wpa. The first term in the RHS is of order \( o_P(Z_n) \) by Assumption 4.1(iv). The second term is
where $E$ holds. Regarding Assumption A.6(iv). By Theorem 2.14.2 in VdV-W, (up to omitted 6.1(i) implies that \[ \{ \\]

Since $\Sigma = 1$, Assumption A.6 (ii) is the same as Assumption A.6 (i).

Proof of Proposition 6.1. We claim that, under our assumptions, there exists a $l_n, u_n \in \mathbb{R}$ such that: they are valid brackets of $A^*_n$ under condition 6.1, it follows that there exists a $\{l_{n,j}, u_{n,j}\}_{j=1}^{N(\delta)}$ such that: they are valid brackets of $F_n$, under $\| \cdot \|_{L^2(f_X)}$. To show this, observe that, for any $f_n \in F_n$, there exists a $\alpha \in \mathcal{N}_{osn}$, such that $f_n = g(\cdot, \alpha) - m(\cdot, \alpha_0)$, and under condition 6.1 it follows that there exists a $j \in \{1, ..., N(\delta)\}$ such that

\[
(l_{n,j}) = \{ g(\cdot, u_{n,j}) > 0 \} l_j(x) + \{ g(\cdot, u_{n,j}) < 0 \} u_j(x) g(x, u_{n,j}),
\]

hence, there exists a $\{l_{n,j}, u_{n,j}\}$ such that, for all $x,$

\[
(l_{n,j}) = \{ g(\cdot, u_{n,j}) > 0 \} l_j(x) + \{ g(\cdot, u_{n,j}) < 0 \} u_j(x) g(x, u_{n,j}),
\]

and

$$u_{n,j}(x) = \{ g(\cdot, u_{n,j}) > 0 \} u_j(x) + \{ g(\cdot, u_{n,j}) < 0 \} l_j(x) g(x, u_{n,j}).$$

B.4. Proofs for Section 6 on examples

Proof of Proposition 6.1. By our assumption over $\text{clsp}\{p_j : j = 1, ..., J\}, \frac{d_m(x,\alpha_n)}{d_o} [u_n^*] \in \text{clsp}\{p_j : j = 1, ..., J_n\}$ provided $k(n) \leq j_n$, and thus Assumption A.6(i) trivially holds. Since $\Sigma = 1$, Assumption A.6(ii) is the same as Assumption A.6(i).

We now show that Assumption A.6(iii) holds under condition 6.1. First, condition 6.1(i) implies that $\{(E[h(Y_2) - h_0(Y_2)]^2 : h \in \mathcal{H})\}$ is a P-Donsker class and, moreover,

$$E[(E[h(Y_2) - h_0(Y_2)]^2)^2] = 2c \cdot ||h - h_0||^2 \to 0$$

as $||h - h_0||_{L^2(f_Y)} \to 0$. So by Lemma 1 in Chen et al. (2003), Assumption A.6(iii) holds. Regarding Assumption A.6(iv). By Theorem 2.14.2 in VdV-W, (up to omitted constants)

$$E\left[ \sup_{f \in F_n} n^{-1/2} \sum_{i=1}^{n} f(X_i) - E[f(X_i)] \right] \leq \int_0^{||F_n||_{L^2(f_X)}} \sqrt{1 + \log N_u(u, F_n, || \cdot ||_{L^2(f_X)})} du$$

where $F_n \equiv \{ f : f = g(\cdot, u^*_n)(m(\cdot, \alpha) - m(\cdot, \alpha_0)) \text{, some } \alpha \in \mathcal{N}_{osn} \}$ and

$$F_n(x) \equiv \sup_{f \in F_n} |f(x)| = \sup_{\alpha \in \mathcal{N}_{osn}} |g(x, u^*_n)(m(x, \alpha) - m(x, \alpha_0))|.$$
such that \( t_{n,j} \leq f_n \leq u_{n,j} \). Also, observe that

\[
||a_{n,j} - u_{n,j}||_{L^2(f_X)} = \sqrt{E[(g(X, u_{n,j}^*) - l_j(X))^2]} \leq ||u_j - l_j||_{L^\infty} \leq \delta
\]

because \( E[(g(X, u_{n,j}^*))^2] = ||u_{n,j}^*||^2 = 1 \) and \( ||u_j - l_j||_{L^\infty} \leq \delta \) by construction.

Therefore,

\[
E \left[ \sup_{f \in F_n} n^{-1/2} \sum_{i=1}^n [f(X_i) - E[f(X_i)]] \right] \leq \int_0^{||F_n||_{L^2(f_X)}} \sqrt{1 + \log N(||u, \Lambda^2(X)|| \cdot ||L_{\infty})} du.
\]

Since by assumption \( \gamma > 0.5 \), it is well-known that \( \sqrt{1 + \log N(||u, \Lambda^2(X)|| \cdot ||L_{\infty})} \) is integrable, so in order to show that \( E \left[ \sup_{f \in F_n} n^{-1/2} \sum_{i=1}^n [f(X_i) - E[f(X_i)]] \right] = o(1) \), it suffices to show that \( ||F_n||_{L^2(f_X)} = o(1) \). In order to show this,

\[
||F_n||_{L^2(f_X)} \leq \sqrt{E[(g(X, u_{n,j}^*))^2(\sup_{N_{x,\alpha}} [m(X, \alpha) - m(X, \alpha_0)])^2]}
\]

\[
= \sqrt{E[(g(X, u_{n,j}^*))^2(\sup_{N_{x,\alpha}} [E[h(Y_2) - h_0(Y_2)]^2)]}
\]

\[
= \sqrt{E[(g(X, u_{n,j}^*))^2(\sup_{N_{x,\alpha}} \int (h(y_2) - h_0(y_2))^2 f_{Y_2|X}(y_2, X)dy_2)]}
\]

\[
= \sqrt{\sup_{x,y_2} \int f_{Y_2|X}(y_2, x) \sup_{N_{x,\alpha}} \|h - h_0\|_{L^2(f_{Y_2|X})} \sqrt{E[(g(X, u_{n,j}^*))^2]}}
\]

\[
\leq \text{Const.} \times M_0 \delta_{n} \rightarrow 0
\]

where the last expression follows from the fact that \( E[(g(X, u_{n,j}^*))^2] = ||u_{n,j}^*||^2 = 1 \) and condition (6.1ii), that states that

\[
\sup_{x,y_2} \frac{f_{Y_2|X}(y_2, x)}{f_{Y_2}(y_2)f_X(x)} \leq \text{Const.} < \infty.
\]

Hence, \( E \left[ \sup_{f \in F_n} n^{-1/2} \sum_{i=1}^n [f(X_i) - E[f(X_i)]] \right] = o(1) \) which implies assumption (A.7iv). Finally, Assumption (A.7) is automatically satisfied with the NPIV model. Q.E.D.

**Proof of Proposition 6.2.** Assumptions (A.6)i and (ii) hold by the same calculations as those in the proof of Proposition 6.1 (for the NPIV model). Also, under Condition (A.6i), \( \{E[F_{Y_2|X}(h(Y_2), Y_2)]: h \in \mathcal{H}\} \subseteq \Lambda^2(X') \) with \( \gamma > 0.5 \), Assumptions (A.6iii) and (iv) hold by similar calculations to those in the proof of Proposition 6.1.

Assumption (A.7i) is standard in the literature. Regarding Assumption (A.7ii), ob-
serve that for any $h \in \mathcal{N}_{\text{ass}}$,
\[
\frac{dm(x, h)}{dh}\left[u_0^*\right] - \frac{dm(x, h_0)}{dh}\left[u_0^*\right]
= \left| E \left[ \left\{ f_{Y_1|Y_2}(y(2), X=x) - f_{Y_2|X}(x) \right\} u_0^* \right| X = x \right| \n= \left| \int_0^1 f_{Y_1|Y_2}(y(2), y_2, x) \frac{d\gamma_{Y_2}(y_2)}{dy_1} f_{X|Y_2}(y_2, x) dy_2 \right|
= \left| \int \left( \int_0^1 f_{Y_1|Y_2}(y(2), y_2, x) \frac{d\gamma_{Y_2}(y_2)}{dy_1} (h(y_2) - h_0(y_2)) u_0^* (y_2) dy_2 \right) df_{Y_2|X}(y_2, x) \right|
= \left| \int \Gamma_1(y_2, x) \Gamma_2(y_2, x) (h(y_2) - h_0(y_2)) u_0^* (y_2) f_{Y_2}(y_2) dy_2 \right|
\leq ||\Gamma_1(\cdot, x)\Gamma_2(\cdot, x)||_{L^\infty} ||h - h_0||_{L^2(f_{Y_2})} ||u_0^*||_{L^2(f_{Y_2})}
\]
where $h_0(t) \equiv h_0 + t(h - h_0)$ and $\Gamma_1(y_2, x) \equiv \int_0^1 f_{Y_1|Y_2}(y(2), y_2, x) \frac{d\gamma_{Y_2}(y_2)}{dy_1} dy_2$, the last line follows from Cauchy-Swarz inequality.

Under Condition \ref{6.2}(ii), it follows that
\[
\sup_{y_1, y_2, x} \left| \frac{d\gamma_{Y_2}(y_2)}{dy_1} \right| \leq C < \infty
\]
and, under Condition \ref{6.1}(ii), it follows that
\[
\sup_{y_1, y_2, x} \left| \frac{f_{Y_2}(y_2, x)}{f_{Y_2}(y_2, f_{X}(x))} \right| \leq C < \infty.
\]
Then it is easy to see that $||\Gamma_j(\cdot, x)||_{L^\infty(f_{Y_2})} \leq C < \infty$ for both $j = 1, 2$. Thus
\[
\left| \frac{dm(x, h)}{dh}\left[u_0^*\right] - \frac{dm(x, h_0)}{dh}\left[u_0^*\right] \right| \leq C^2 \times ||h - h_0||_{L^2(f_{Y_2})} ||u_0^*||_{L^2(f_{Y_2})}
\]
and thus, Assumption \ref{A.7}(ii) is satisfied provided that $n \times M_{\text{ass}}^2 \delta^2 \sup_{h \in \mathcal{N}_{\text{ass}}} ||h - h_0||_{L^2(f_{Y_2})} ||u_0^*||_{L^2(f_{Y_2})} = o(1)$. Since $||u_0^*||_{L^2(f_{Y_2})} \leq \mu_{k(n)}^{-1}$ it suffices to show that
\[
nM_{\text{ass}}^2 \left( ||\Pi_n h_0 - h_0||_{L^2(f_{Y_2})} + \mu_{k(n)}^{-1} \delta_n \right) \mu_{k(n)}^{-2} = o(1).
\]
By assumption, $||\Pi_n h_0 - h_0||_{L^2(f_{Y_2})} \leq \text{Const} \times \mu_{k(n)}^{-1} \delta_n$ and $\delta_n = \text{Const} \times k(n)/n$; then it suffices to show that
\[
nM_{\text{ass}}^2 \delta_n = o(1),
\]
which holds by Condition \ref{6.3}.

Regarding Assumption \ref{A.7}(iii), observe that for any $h \in \mathcal{N}_{\text{ass}}$,
\[
\frac{d^2 m(x, h)}{dh^2}\left[u_0^*, u_0^*\right] = \int \frac{d\gamma_{Y_2}(y(2), y_2, x)}{dy_1} \left( f_{Y_2}(y_2) \right)^2 dy_2.
\]
Again by Conditions \ref{6.2}(ii) and \ref{6.1}(ii), it follows that $\left| \frac{d^2 m(x, h)}{dh^2}\left[u_0^*, u_0^*\right] \right| \leq C^2 \times$
\[ \|u^*_n\|^2_{L^2(Y_2^2)} \]. Since \( \|u^*_n\|_{L^2(Y_2)} \leq const \times \mu_{k(n)}^{-1} \), Assumption A.7(iii) holds because
\[ \mu_{k(n)}^{-2} \times (M_n \delta_n)^2 = o(1) \], or \( M_n^2 \delta_n^2 = o(1) \).

Finally, we verify Assumption A.7(iv). By our previous calculations
\[ \left| \frac{dm(x, h_1)}{dh}[h_2 - h_0] - \frac{dm(x, h_0)}{dh}[h_2 - h_0] \right| = \left| \int \left( \frac{dy_y}{Y_2} \left( f_{Y_2}(y_2) \right) \right) \frac{dy_y}{Y_2} [h_1(y_2) - h_0(y_2)](h_2(y_2) - h_0(y_2)) \right| \leq C^2 \times \int \left( ||h_1(y_2) - h_0(y_2)||L^2(Y_2) \right) \right| ||h_2(y_2) - h_0(y_2)||L^2(Y_2) \right| dy_y \]
\[ \leq C^2 \times \left( ||h_1 - h_0||L^2(Y_2) \right) \right| ||h_2 - h_0||L^2(Y_2) \right| \]
where the first inequality follows from Conditions 6.2(ii) and 6.1(ii), and the last one from Cauchy-Swarz inequality. This result and Cauchy-Swarz inequality together imply that
\[ \left| \left( \frac{dm(x, h_1)}{dh}[h_2 - h_0] \right) - \left( \frac{dm(x, h_0)}{dh}[h_2 - h_0] \right) \right| \leq C^2 \sqrt{E[(g(X, u^*_n))^2]} \leq const \times ||h_1 - h_0||L^2(Y_2) \right| ||h_2 - h_0||L^2(Y_2) \right| \]
where the last line follows from \( E[(g(X, u^*_n))^2] = ||u^*_n||^2 = 1 \). Thus, Assumption A.7(iv) follows if
\[ \delta^2_{s,n} = (||\Pi_n h_0 - h_0||L^2(Y_2) + \mu_{k(n)}^{-1} \delta_n)^2 = o(n^{-1/2}) \]
which holds by Condition 6.3. Q.E.D.
APPENDIX C: PROOFS OF THE RESULTS IN APPENDIX A

In Appendix [A], we provide the proofs of all the lemmas, theorems and propositions stated in Appendix [A].

C.1. Proofs for Section [A.2] on convergence rates of bootstrap PSMD estimators

Proof of Lemma [A.1]: For Result (1), we prove this result in two steps. First, we show that $\hat{\alpha}_n^B \in A_{k(n)}^{M_0(\alpha_0)}$ wpa1-$P_{V_{\infty}|Z}\approx$ for any $Z_{\infty}$ in a set that occurs with $P_{Z}\approx$ probability approaching one, where $A_{k(n)}^{M_0}$ is defined in the text. Second, we establish consistency, using the fact that we are in the $A_{k(n)}^{M_0}$ set.

Step 1. We show that for any $\delta > 0$, there exists a $N(\delta)$ such that

$$P_{Z_{\infty}}(P_{V_{\infty}|Z_{\infty}}(\hat{\alpha}_n^B \notin A_{k(n)}^{M_0}|Z_{\infty}^{n}) < \delta) \geq 1 - \delta, \forall n \geq N(\delta).$$

To show this, note that, by definition of $\hat{\alpha}_n^B$,

$$\lambda_n Pen(\hat{\alpha}_n^B) \leq \tilde{Q}_n(\hat{\alpha}_n) + \lambda_n Pen(\hat{h}_n) + o_{P_{V_{\infty}|Z_{\infty}}(\frac{1}{n})}, \ wpa1(P_{Z_{\infty}}).$$

By Assumption [A.1(i)] and the definition of $\hat{\alpha}_n \in A_{k(n)}$,

$$\lambda_n Pen(\hat{h}_n^B) \leq c^*_0 \left( \tilde{Q}_n(\hat{\alpha}_n) + \lambda_n Pen(\hat{h}_n) \right) + o_{P_{V_{\infty}|Z_{\infty}}(\frac{1}{n})}, \ wpa1(P_{Z_{\infty}}).$$

By Assumptions [3.2(ii)] and [3.3(i)],

$$\lambda_n Pen(\hat{h}_n^B) \leq c^*_0 \alpha Q(\Pi n \alpha_0) + \lambda_n Pen(h_n) + O_{P_{V_{\infty}|Z_{\infty}}(\lambda_n + o(\frac{1}{n}))}, \ wpa1(P_{Z_{\infty}}).$$

By the fact that $Q(\Pi n \alpha_0) + o(\frac{1}{n}) = O(\lambda_n)$, the desired result follows.

Step 2. We want to show that for any $\delta > 0$, there exists a $N(\delta)$ such that

$$P_{Z_{\infty}}(P_{V_{\infty}|Z_{\infty}}(||\hat{\alpha}_n^B - \alpha_0||_{s} \geq \delta |Z_{\infty}^{n}) < \delta) \geq 1 - \delta, \forall n \geq N(\delta),$$

which is equivalent to show that $P_{Z_{\infty}}(P_{V_{\infty}|Z_{\infty}}(||\hat{\alpha}_n^B - \alpha_0||_{s} \geq \delta |Z_{\infty}^{n}) > \delta) \leq \delta$ eventually. Note that

$$P_{Z_{\infty}}(P_{V_{\infty}|Z_{\infty}}(||\hat{\alpha}_n^B - \alpha_0||_{s} \geq \delta |Z_{\infty}^{n}) > \delta) \leq P_{Z_{\infty}}(P_{V_{\infty}|Z_{\infty}}(||\hat{\alpha}_n^B - \alpha_0||_{s} \geq \delta) \cap \{\hat{\alpha}_n^B \in A_{k(n)}^{M_0}|Z_{\infty}^{n} > 0.5\delta\} + P_{Z_{\infty}}(P_{V_{\infty}|Z_{\infty}}(\hat{\alpha}_n^B \notin A_{k(n)}^{M_0}|Z_{\infty}^{n}) > 0.5\delta).$$

By step 1, the second summand in the RHS is negligible. Thus, it suffices to show that

$$P_{Z_{\infty}}(P_{V_{\infty}|Z_{\infty}}(\hat{\alpha}_n^B \in A_{k(n)}^{M_0}, ||\hat{\alpha}_n^B - \alpha_0||_{s} \geq \delta |Z_{\infty}^{n}) < \delta) \geq 1 - \delta, \forall n \geq N(\delta).$$
(henceforth, we omit $\hat{\alpha}_n^B \in \mathcal{A}_{k(n)}^{M_0}$). Note that, conditioning on $Z^n$, by Assumption A.1(i)(ii), the definition of $\hat{\alpha}_n \in \mathcal{A}_{k(n)}^{M_0}$, Assumption 3.2(i)(ii) and $\max\{\lambda_n, o(\frac{1}{n})\} = O(\lambda_n)$, we have:

$$P_{V \mid Z^n} \left( \|\hat{\alpha}_n^B - \alpha_0\|_s \geq \delta \right) \leq P_{V \mid Z^n} \left( \inf_{(A_{k(n)}^{M_0}) \cdot \|a - \alpha_0\|_s \geq \delta} \left\{ \hat{Q}_n^B(\alpha) + \lambda_n Pen(h) \right\} \leq \hat{Q}_n^B(\hat{\alpha}_n) + \lambda_n Pen(\hat{h}_n) + o(\frac{1}{n}) \right) \leq P_{V \mid Z^n} \left( \inf_{(A_{k(n)}^{M_0}) \cdot \|a - \alpha_0\|_s \geq \delta} \left\{ c^* \hat{Q}_n(\alpha) + \lambda_n Pen(h) \right\} \leq c^*_0 \left[ \hat{Q}_n(\hat{\alpha}_n) + \lambda_n Pen(\hat{h}_n) \right] + O(\lambda_n) + \tilde{\delta}_{m,n}^2 |Z^n| \right) \leq P_{V \mid Z^n} \left( \inf_{(A_{k(n)}^{M_0}) \cdot \|a - \alpha_0\|_s \geq \delta} c^* \hat{Q}_n(\alpha) \leq c^*_0 \left[ \hat{Q}_n(\Pi_n \alpha_0) + \lambda_n Pen(\Pi_n h_0) \right] + O(\lambda_n) + \tilde{\delta}_{m,n}^2 |Z^n| \right).

Thus, wpa1($P_{Z^n}$),

$$P_{V \mid Z^n} \left( \|\hat{\alpha}_n^B - \alpha_0\|_s \geq \delta \right) \leq P_{V \mid Z^n} \left( \inf_{(A_{k(n)}^{M_0}) \cdot \|a - \alpha_0\|_s \geq \delta} c^* \hat{Q}_n(\alpha) \leq c^*_0 \left[ \hat{Q}_n(\Pi_n \alpha_0) + M(\lambda_n + \tilde{\delta}_{m,n}^2) \right] \right),$$

which can be bounded above by

$$P_{V \mid Z^n} \left( \inf_{(A_{k(n)}^{M_0}) \cdot \|a - \alpha_0\|_s \geq \delta} c^* Q(\alpha) \leq c^*_0 c_0 Q(\Pi_n \alpha_0) + M(\lambda_n + (\tilde{\delta}_{m,n} + \tilde{\delta}_{m,n}^2)^2) \right) + P_{V \mid Z^n} \sup_{(A_{k(n)}^{M_0}) \cdot \|a - \alpha_0\|_s \geq \delta} \hat{Q}_n(\alpha) - cQ(\alpha) < -M\tilde{\delta}_{m,n}^2 |Z^n| + P_{V \mid Z^n} \left( \hat{Q}_n(\Pi_n \alpha_0) - c_0 Q(\Pi_n \alpha_0) > -o(\frac{1}{n}) |Z^n| \right).$$

Therefore, for sufficiently large $n$,

$$P_Z \left( P_{V \mid Z^n} \left( \|\hat{\alpha}_n^B - \alpha_0\|_s \geq \delta \right) \right) \leq 0.25 \delta$$

$$+ P_Z \left( \inf_{(A_{k(n)}^{M_0}) \cdot \|a - \alpha_0\|_s \geq \delta} c^* Q(\alpha) \leq c^*_0 c_0 Q(\Pi_n \alpha_0) + M(\lambda_n + (\tilde{\delta}_{m,n} + \tilde{\delta}_{m,n}^2)^2) \right)$$

$$+ P_Z \sup_{(A_{k(n)}^{M_0}) \cdot \|a - \alpha_0\|_s \geq \delta} \hat{Q}_n(\alpha) - cQ(\alpha) < -M\tilde{\delta}_{m,n}^2$$

$$+ P_Z \left( \hat{Q}_n(\Pi_n \alpha_0) - c_0 Q(\Pi_n \alpha_0) > -o(\frac{1}{n}) \right).$$

By Assumption 3.3 the third and fourth terms in the RHS are less than 0.5$\delta$. The second term in the RHS is not random. By Assumptions 3.1(ii) and 3.2(iii), $\mathcal{A}_{k(n)}^{M_0}$ is compact, and so is $\mathcal{A}_{k(n)}^{M_0} \equiv \{ \alpha = (\theta', h) \in \mathcal{A} : \lambda_n Pen(h) \leq \lambda_n M_0 \}$. This fact, and Assumption 3.1(iii) imply that $\inf_{(A_{k(n)}^{M_0}) \cdot \|a - \alpha_0\|_s \geq \delta} c^* Q(\alpha) \geq Q(\alpha(\delta))$ some
\( \alpha(\delta) \in \mathcal{A}^{M_0} \cap \{||\alpha - \alpha_0||_s \geq \delta\} \). By Assumption 3.4(i), \( Q(\alpha) > 0 \), so eventually, since \( c_0 c_0 Q(\Pi_n o_0) + M(\lambda_n + (\delta_{m,n} + \delta^*_{m,n})^2) = o(1) \),

\[
P_Z \approx \left( \inf_{(\mathcal{A}^{M_0}_{k(n)}, ||\alpha - \alpha_0||_s \geq \delta)} c^* c Q(\alpha) \leq c_0^* c_0 Q(\Pi_n o_0) + M(\lambda_n + (\delta_{m,n} + \delta^*_{m,n})^2) \right) = 0.
\]

For Result (2), we want to show that for any \( \delta > 0 \), there exists a \( M(\delta) \) such that

\[
P_Z \approx P_{\gamma_{\Pi} z} \left( \delta_n^{-1} ||\alpha - \alpha_0|| \geq M' | Z^n \right) < \delta \]

eventually. By Assumptions 3.4(iii) and A.1(iii), following the similar algebra as before, we have: for \( M' \) large enough,

\[
P_{\gamma_{\Pi} z} \left( \delta_n^{-1} ||\alpha - \alpha_0|| \geq M' | Z^n \right) \leq \inf_{(\mathcal{A}_0, ||\alpha - \alpha_0|| \geq M')} c^* c Q(\alpha) \leq M(\lambda_n + \delta_{m,n}^2) | Z^n | + \delta.
\]

By Assumption 3.4(ii) and \( \delta_n = \sqrt{\max \{\lambda_n, \delta_{m,n}^2\} \}, \) we have:

\[
P_{\gamma_{\Pi} z} \left( \delta_n^{-1} ||\alpha - \alpha_0|| \geq M' | Z^n \right) \leq 1(\max (c^* c, M' \delta_n)) \leq M(\lambda_n + \delta_{m,n}^2) \}
\]

which is eventually naught, because \( M' \) can be chosen to be large. The rate under \( || \cdot ||_s \) immediately follows from this result and the definition of the sieve measure of local ill-posedness \( \tau_n \).

For Result (3), we note that both \( \alpha_n^R, \tilde{\alpha}_n \in \{\alpha \in \mathcal{A}_{k(n)}: \phi(\alpha) = \phi(\tilde{\alpha}_n)\} \), and hence all the above proofs go through with \( \alpha_n^R, \tilde{\alpha}_n \) replacing \( \alpha_n^R \). In particular, let \( \mathcal{A}^{M_0}_{k(n)}(\tilde{\alpha}_n) \equiv \{\alpha \in \mathcal{A}^{M_0}_{k(n)} : \phi(\alpha) = \phi(\tilde{\alpha}_n)\} \subseteq \mathcal{A}^{M_0}_{k(n)} \). Then: for any \( \delta > 0 \),

\[
P_{\gamma_{\Pi} z} \left( \delta_n^{-1} ||\tilde{\alpha}_n - \alpha_0|| \geq \delta | Z^n \right) \leq \inf_{(\mathcal{A}^{M_0}_{k(n)}(\tilde{\alpha}_n), ||\tilde{\alpha}_n - \alpha_0|| \geq \delta)} \left\{ c^* Q_n(\alpha) + \lambda_n Pen(h) \right\} \leq Q_n(\tilde{\alpha}_n) + \lambda_n Pen(h) + o(1/\Pi)|Z^n |
\]

where \( A_n \equiv c_0^* \left[ Q_n(\tilde{\alpha}_n) + \lambda_n Pen(h) \right] + O(\lambda_n) + \delta_{m,n}^2 | Z^n | \). The rest follows from the proof of Results (1) and (2). Q.E.D.
C.2. Proofs for Section 4.3 on behaviors under local alternatives

Proof of Theorem A.1 The proof is analogous to that of Theorem 4.3 hence we only present the main steps. Let \( \alpha_n = \alpha_0 + d_n \Delta_n \) with \( \frac{d\phi(\alpha_0)}{d\alpha} |\Delta_n| = \langle v_n^*, \Delta_n \rangle = \kappa_n = \kappa \times (1 + o(1)) \neq 0. \)

**Step 1.** By assumption 3.6(i) under the local alternatives, for any \( t_n \in T_n \),

\[
0 \leq 0.5 \left( \hat{Q}_n(\alpha_n(t_n)) - \hat{Q}_n(\alpha_0) \right) = t_n \left( Z_n(\alpha_n(t_n)) + \langle u_n^*, \hat{\alpha}_n - \alpha_n \rangle + \frac{B_n}{2} t_n^2 + o_P(n^{-2}) \right)
\]

where \( [r_n(t_n)^{-1}] = \max\{t_n^2, t_n n^{-1/2}, s_n^{-1}\} \) and \( s_n^{-1} = o(n^{-1}) \). The LHS is always positive (up to possibly a negligible terms given by the penalty function, see the proof of Theorem 4.3) by definition of \( \hat{\alpha}_n \). Hence, by choosing \( t_n = \pm\{s_n^{-1/2} + o(n^{-1/2})\} \), it follows that \( \langle Z_n(\alpha_n) + \langle u_n^*, \hat{\alpha}_n - \alpha_n \rangle \rangle = o_P(n^{-1/2}) \). Since \( \langle u_n^*, \alpha_n - \alpha_0 \rangle = \frac{d_n \kappa_n}{\|v_n^*\|_{\|d\|}} \) by the definition of local alternatives \( \alpha_n \), we obtain equation C.2:

\[
\left\{ Z_n(\alpha_n(t_n)) + \langle u_n^*, \hat{\alpha}_n - \alpha_0 \rangle - \frac{d_n \kappa_n}{\|v_n^*\|_{\|d\|}} \right\} = Z_n(\alpha_n) + \langle u_n^*, \hat{\alpha}_n - \alpha_n \rangle = o_P(n^{-1/2}),
\]

where \( Z_n(\alpha_n) \) is defined as that of \( Z_n \) but using \( \rho(z, \alpha_n) \) instead of \( \rho(z, \alpha_0) \) (since \( m(X, \alpha_n) = 0 \) a.s.- \( X \) under the local alternative).

Next, by Assumption 3.6(ii) under the local alternative, we have: for any \( t_n \in T_n \),

\[
0.5 \left( \hat{Q}_n(\alpha_n^R(t_n)) - \hat{Q}_n(\alpha_n^R) \right) = t_n \left( Z_n(\alpha_n) + \langle u_n^*, \hat{\alpha}_n^R - \alpha_n \rangle \right) + \frac{B_n}{2} t_n^2 + o_P(\infty) \left( [r_n(t_n)^{-1}] \right).
\]

By Assumption 3.3(ii)

\[
\sup_{\alpha \in N_{\alpha_n}} |\phi(\alpha) - \phi(\alpha_0) - \frac{d\phi(\alpha_0)}{d\alpha} |\alpha - \alpha_0| = o(n^{-1/2} |\|v_n^*\||),
\]

and assumption \( \alpha_n^R \in N_{\alpha_n} \) w.p.1-\( P_{\alpha_n^R} \), and the fact that \( \phi(\alpha_n^R) - \phi(\alpha_0) = 0 \), following the same calculations as those in Step 1 of the proof of Theorem 4.3 we have:

\[
\langle u_n^*, \hat{\alpha}_n^R - \alpha_0 \rangle = o_P(n^{-1/2}).
\]

Since \( \alpha_n = \alpha_0 + d_n \Delta_n \in N_{\alpha_n} \) with \( \frac{d\phi(\alpha_0)}{d\alpha} |\Delta_n| = \langle v_n^*, \Delta_n \rangle = \kappa_n \), we have:

\[
\langle u_n^*, \hat{\alpha}_n^R - \alpha_0 \rangle = \langle u_n^*, \hat{\alpha}_n^R - \alpha_0 \rangle - \frac{d_n \kappa_n}{\|v_n^*\|_{\|d\|}} + o_P(\infty) (n^{-1/2}) = - \frac{d_n \kappa_n}{\|v_n^*\|_{\|d\|}} + o_P(n^{-1/2}).
\]

Therefore, by choosing \( t_n = -(Z_n(\alpha_n) - \frac{d_n \kappa_n}{\|v_n^*\|_{\|d\|}}) B_n^{-1} \) in (C.3) with \( [r_n(t_n)^{-1}] = \)

\[
\]
max\{t_n^2, t_n n^{-1/2}, o(n^{-1})\} (which is a valid choice), we obtain:
\[
0.5 \left( \hat{Q}_n(\alpha_n) - \hat{Q}_n(\tilde{\alpha}_n^R) \right) \leq 0.5 \left( \hat{Q}_n(\alpha_n) - \hat{Q}_n(\tilde{\alpha}_n^R) \right) + op_{n, \infty}(n^{-1})
\]
\[
= -\frac{1}{2} \left( \mathbb{E}_n(\alpha_n) - \frac{d_{\alpha_n}}{||\alpha_n||_{sd}} \right)^2 + op_{n, \infty}\left( [r_n(t_n)]^{-1} \right).
\]
By our assumption and the fact that \(|u_n^*| \geq c > 0\) for all \(n\), it follows that \(B_n \geq c > 0\) eventually, so
\[
0.5 \left( \hat{Q}_n(\alpha_n) - \hat{Q}_n(\tilde{\alpha}_n^R) \right) \leq \frac{1}{2} \left( \mathbb{E}_n(\alpha_n) - \frac{d_{\alpha_n}}{||\alpha_n||_{sd}} \right)^2 \times (1 + op_{n, \infty}(1)).
\]

**STEP 2.** On the other hand, suppose there exists a \(t_n^*\), such that (a) \(\phi(\alpha_n(t_n^*)) = \phi(\alpha_0), \tilde{\alpha}_n(t_n^*) \in A_{k(n)}\), and (b) \(t_n^* = (\mathbb{E}_n(\alpha_n) - \frac{d_{\alpha_n}}{||\alpha_n||_{sd}}) (||u_n^*||)^{-2} + op_{n, \infty}(n^{-1/2})\).
Substituting this into equation (C.1) with \([r_n(t_n^*)]^{-1} = \max\{t_n^*2, t_n^* n^{-1/2}, o(n^{-1})\}\), we obtain:
\[
0.5 \left( \hat{Q}_n(\alpha_n) - \hat{Q}_n(\tilde{\alpha}_n^R) \right) \geq 0.5 \left( \hat{Q}_n(\alpha_n) - \hat{Q}_n(\alpha_n) \right) - op_{n, \infty}(n^{-1})
\]
\[
= -\frac{B_n}{2} \left( t_n^* \right)^2 + op_{n, \infty}\left( [r_n(t_n^*)]^{-1} \right)
\]
\[
= -\frac{B_n}{2} \left( \mathbb{E}_n(\alpha_n) - \frac{d_{\alpha_n}}{||\alpha_n||_{sd}} \right)^2 (||u_n^*||)^{-4} + op_{n, \infty}\left( [r_n(t_n^*)]^{-1} \right)
\]
\[
= \frac{1}{2} \left( \mathbb{E}_n(\alpha_n) - \frac{d_{\alpha_n}}{||\alpha_n||_{sd}} \right)^2 \times (1 + op_{n, \infty}(1))
\]
where the second line follows from equation (C.2). Finally, we observe that point (a) follows from Lemma [B.2] with \(r = 0\). Point (b) follows by analogous calculations to those in Step 3 of the proof of Theorem 4.3 except that now with \(\tilde{\alpha}(t_n^*) = \alpha_n + t_n^* u_n^*\),
\[
\phi(\tilde{\alpha}(t_n^*)) - \phi(\alpha_0) = \frac{d\phi(\alpha_0)}{d\alpha_0}|_{\tilde{\alpha}(\alpha_0) = \alpha_n} + t_n^* ||u_n^*||^2 + ||u_n^*||^2 + op_{n, \infty}(n^{-1/2}||u_n^*||)
\]
\[
= -\mathbb{E}_n(\alpha_n)||u_n^*||^2 + \frac{d_{\alpha_n} \kappa_n}{||\alpha_n||_{sd}} ||u_n^*||^2
\]
\[
+ \left( \mathbb{E}_n(\alpha_n) - \frac{d_{\alpha_n} \kappa_n}{||\alpha_n||_{sd}} ||u_n^*||^2 \right) ||u_n^*||^2 + op_{n, \infty}(n^{-1/2}||u_n^*||)
\]
\[
= \mathbb{E}_n(\alpha_n)||u_n^*||^2 + op_{n, \infty}(n^{-1/2}||u_n^*||)
\]
where the second line follows from equation (C.2) and some straightforward algebra.

**STEP 3.** Finally, the above calculations and \(\kappa_n = \kappa (1 + o(1))\) imply that
\[
||u_n^*||^2 \times \hat{Q}_n(\tilde{\alpha}_n^R) = \left( \mathbb{E}_n(\alpha_n) - \frac{d_{\alpha_n} \kappa (1 + o(1))}{||\alpha_n||_{sd}} \right)^2 \times (1 + op_{n, \infty}(1)).
\]

For **Result (1)**, equation (C.4) with \(d_n = n^{-1/2}||u_n^*||_{sd}\) implies that
\[
||u_n^*||^2 \times \hat{Q}_n(\tilde{\alpha}_n^R) = \left( \sqrt{\mathbb{E}_n(\alpha_n) - \kappa (1 + o(1))} \right)^2 \times (1 + op_{n, \infty}(1)) = \chi_n^2(\kappa^2),
\]
which is due to $\sqrt{n}\mathcal{Z}_n(\alpha_n) \Rightarrow N(0, 1)$ under the local alternatives.

For **Result (2)**, equation (C.4) with $\sqrt{n} \frac{d_{\alpha}}{\|v^n\|_d} \to \infty$ implies that

$$||u_n^*||^2 \times \overline{QLR}_n(\phi_0) = \left( \sqrt{n}\mathcal{Z}_n(\alpha_n) - \sqrt{n} \frac{d_{\alpha}}{\|v^n\|_d} (1 + o(1)) \right)^2 \times (1 + o_{P_n,Z\to\infty}(1))$$

and thus

$$\overline{QLR}_n(\phi_0) = \left( \sqrt{n}\mathcal{Z}_n(\alpha_n) - \kappa \frac{||v^n||_d}{\|v^n\|_0} \right)^2 + o_{P_n,Z\to\infty}(1),$$

where the second line is due to $\sqrt{n}\mathcal{Z}_n(\alpha_n) \Rightarrow N(0, 1)$ under the local alternatives. Since $\sqrt{n} \frac{d_{\alpha}}{\|v^n\|_d} \to \infty$ (or $-\infty$) if $\kappa > 0$ (or $\kappa < 0$), we have that $\lim_{n \to \infty} \left( ||u_n^*||^2 \times \overline{QLR}_n(\phi_0) \right) = \infty$ in probability (under the alternative). Q.E.D.

**Proof of Proposition A.1** Recall that $\overline{QLR}_n^0(\phi_0)$ denotes the optimally-weighted SCLR. By inspection of the proof of Theorem A.1 it is easy to see that

$$||u_n^*||^2 \times \overline{QLR}_n(\phi_0) = \left( \sqrt{n}\mathcal{Z}_n(\alpha_n) - \kappa \right)^2 + o_{P_n,Z\to\infty}(1)$$

and

$$\overline{QLR}_n^0(\phi_0) = \left( \sqrt{n}\mathcal{Z}_n(\alpha_n) - \kappa \frac{||v^n||_d}{\|v^n\|_0} \right)^2 + o_{P_n,Z\to\infty}(1)$$

for local alternatives of the form described in equation (A.2) with $d_n = n^{-1/2}||v^n||_d$. Hence, the distribution of $||u_n^*||^2 \times \overline{QLR}_n(\phi_0)$ is, asymptotically close to $\chi^2(\kappa^2)$ and the distribution of $\overline{QLR}_n^0(\phi_0)$ is, asymptotically close to $\chi^2 \left( \frac{||u_n^*||^2}{\|v^n\|_0} \kappa^2 \right)$.

Let $A_n(Z) \equiv \left( \frac{d_{X,0}}{d\alpha} \right)\{v^n\}^T (\Sigma_0^{-1} \rho) \{v^n\}$ and $A_n(Z) \equiv \left( \frac{d_{X,0}}{d\alpha} \right)\{v^n\}^T (\Sigma^{-1} \rho) \{v^n\}$ where $v^n$ is the Riesz respecenent under $\| \cdot \|_0$. Since $\left( \frac{v^n}{\|v^n\|_0} \right) \left( \frac{v^n}{\|v^n\|_0} \right)^T \left( \frac{d_{X,0}}{d\alpha} \right)\{v^n\}^T \left( \Sigma_0^{-1} \rho \right) \{v^n\} = \left( \frac{v^n}{\|v^n\|_0} \right) \left( \frac{v^n}{\|v^n\|_0} \right)^T \left( \frac{d_{X,0}}{d\alpha} \right)\{v^n\}^T \left( \Sigma_0^{-1} \rho \right) \{v^n\}$ and $E \left[ (A_n(Z))^T (A_n(Z)) \right] = \left( \frac{v^n}{\|v^n\|_0} \right) \left( \frac{v^n}{\|v^n\|_0} \right)^T = \left( \frac{v^n}{\|v^n\|_0} \right) \left( \frac{v^n}{\|v^n\|_0} \right)^T \left( \frac{d_{X,0}}{d\alpha} \right)\{v^n\}^T \left( \Sigma_0^{-1} \rho \right) \{v^n\}$, we have:

$$E \left[ (A_n(Z)) (A_n(Z))^T \right] = \left( \frac{v^n}{\|v^n\|_0} \right) \left( \frac{v^n}{\|v^n\|_0} \right)^T$$

$$E \left[ (A_n(Z)) (A_n(Z))^T \right] = \left( \frac{v^n}{\|v^n\|_0} \right) \left( \frac{v^n}{\|v^n\|_0} \right)^T$$

Since the LHS is non-negative, the previous equation implies that $||v^n||^2_2 - \left( \frac{v^n}{\|v^n\|_0} \right) \left( \frac{v^n}{\|v^n\|_0} \right)^T \geq 0$. By definition of $v_n^*$ and $v_0^*$, it follows that

$$\langle v_n^*, v_0^* \rangle = \frac{d\phi(\alpha_0)}{d\alpha} \{v_0\} = ||v^n||^2_0,$$

and thus $||v_n^*||_d^2 \geq ||v_0^*||_d^2$ for all $n$.

Observe that for a noncentral chi-square, $\chi^2_r(r)$, $Pr(\chi^2_r(r) \leq t)$ is decreasing in the noncentrality parameter $r$ for each $t$; thus $Pr(\chi^2_r(r_1) > t) > Pr(\chi^2_r(r_2) > t)$ for $r_1 > r_2$. 

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Therefore, the previous results imply that, for any \( t \),

\[
\lim_{n \to \infty} P_{n,Z} \left( \|u_n^*\|^2 \times QLR_n(\phi_0) \geq t \right) = \Pr(\chi_1^2(\kappa^2) \geq t) \\
\leq \liminf_{n \to \infty} \Pr \left( \chi_1^2 \left( \frac{\|v_n^0\|^2_2 \kappa^2}{\|v_n^0\|^2_0} \right) \geq t \right) \\
= \liminf_{n \to \infty} P_{n,Z} \left( QLR_n(\phi_0) \geq t \right).
\]

Q.E.D.

**Proof of Theorem A.2** The proof of Result (1) is similar to that of Theorem 5.3 so we only present a sketch here. By assumptions 3.6(i) and Boot.3(i) under local alternative, it follows that

\[
0.5 \left( \hat{Q}_n^B(\alpha_n) - \frac{Z_{n-1}(\alpha_n)}{B_n^2} \right) = \frac{Z_{n-1}(\alpha_n)}{B_n^2} \left( \frac{Z_{n-1}(\alpha_n)}{B_n^2} \right)^2 + \frac{1}{2B_n^2} + o_{P_{V\to Z\to}}(n^{-1}), \ wpa1(P_{n,Z}^∞),
\]

where \( r_n^{-1} = \max \left( -\frac{Z_{n-1}(\alpha_n)}{B_n^2} \right), \frac{1}{2B_n^2} \right) \) imply

\[
\sqrt{n}(\hat{u}_n^*, \hat{\alpha}_n^R - \hat{\alpha}_n) = o_{P_{V\to Z\to}}(1), \ wpa1(P_{n,Z}^∞),
\]

i.e., the restricted bootstrap estimator \( \hat{\alpha}_n^R \) centers at \( \hat{\alpha}_n \), regardless of the local alternative. Thus

\[
\langle u_n^*, \hat{\alpha}_n^R - \hat{\alpha}_n \rangle = \langle u_n^*, \hat{\alpha}_n^R - \hat{\alpha}_n \rangle + \langle u_n^*, \hat{\alpha}_n - \hat{\alpha}_n \rangle = \langle u_n^*, \hat{\alpha}_n - \hat{\alpha}_n \rangle + o_{P_{V\to Z\to}}(n^{-1/2}), \ wpa1(P_{n,Z}^∞).
\]

This result and equation (C.2) \((i.e., Z_n(\alpha_n) + \langle u_n^*, \hat{\alpha}_n - \hat{\alpha}_n \rangle = o_{P_{V\to Z\to}}(n^{-1/2})\) imply that

\[
0.5 \left( \hat{Q}_n^B(\alpha_n) - \frac{Z_{n-1}(\alpha_n)}{B_n^2} \right) = \frac{Z_{n-1}(\alpha_n)}{B_n^2} \left( \frac{Z_{n-1}(\alpha_n)}{B_n^2} \right)^2 + \frac{1}{2B_n^2} + o_{P_{V\to Z\to}}(n^{-1}), \ wpa1(P_{n,Z}^∞)
\]

\[
= \frac{Z_{n-1}(\alpha_n)}{B_n^2} \left( Z_{n-1}(\alpha_n) + o_{P_{V\to Z\to}}(n^{-1/2}) \right) + \frac{1}{2B_n^2} + o_{P_{V\to Z\to}}(n^{-1}), \ wpa1(P_{n,Z}^∞)
\]

\[
= \frac{(Z_{n-1}(\alpha_n))^2}{2B_n^2} \times (1 + o_{P_{V\to Z\to}}(1)) \ wpa1(P_{n,Z}^∞).
\]

Following the proof of Result (1) of Theorem 5.3 step 3 with \( Z_{n-1}(\alpha_n) \) replacing \( Z_{n-1}^{-1} \), we obtain:

\[
\frac{QLR_n(\hat{\phi}_n)}{\sigma_n} = \left( \sqrt{n} \frac{Z_{n-1}(\alpha_n)}{\sigma_n \sqrt{B_n^2}} \right)^2 \times (1 + o_{P_{V\to Z\to}}(1)) = O_{P_{V\to Z\to}}(1), \ wpa1(P_{n,Z}^∞).
\]
This shows that, since for the bootstrap SCLR the “null hypothesis is \( \phi(\alpha) = \hat{\phi}_n \equiv \phi(\hat{\alpha}_n) \)”, it always centers correctly.

By similar calculations to those in the proof of Result (2) of Theorem 5.3, the law of \( \left( \frac{\hat{\alpha}_n - \alpha}{\sqrt{\frac{2\omega}{\sigma_n}} \sqrt{N}} \right)^2 \) is asymptotically (and \( wpa_1(P_n, Z = \infty) \)) equal to the law of \( \left( \frac{2}{||u_n^*||} \right)^2 \)

where \( Z \sim N(0, 1) \). This implies that the \( \alpha \)-th quantile of the distribution of \( \frac{\hat{\alpha}_n - \alpha}{\sqrt{\frac{2\omega}{\sigma_n}} \sqrt{N}} \) is uniformly bounded \( wpa_1(P_n, Z = \infty) \). Also, following the proof of Result (2) of Theorem 5.3 we obtain:

\[
\sup_{t \in \mathbb{R}} \left( \frac{QLR_n^B(\hat{\phi}_n)}{\sigma_n^2} \right) \leq t \mid Z^T = P_{Z \sim N(0, 1)} \left( \frac{QLR_n(\phi_0)}{\sigma_n^2} \right) \leq t \mid H_0 \right) = o_P(\frac{Z}{\sigma_n} (1) wpa(1, P_n, Z = \infty).
\]

This and Theorem A.1 (and the fact that \( ||u_n^*|| \leq c < \infty \)) immediately imply Results (2). Q.E.D.

**Proof of Theorem A.3** The proof is analogous to that of Theorems 4.2 and A.1 so we only present a sketch here.

Under our assumptions, Theorem 4.2 still holds under the local alternatives \( \alpha_n \).

Observe that, with \( \alpha_n = \alpha_0 + d_n \Delta_n \in \mathcal{N}_{ass} \) and \( d_n = o(1) \),

\[
T_n \equiv \sqrt{n} \frac{\phi(\hat{\alpha}_n) - \phi_0}{||u_n^*||_{sd}} = \sqrt{n} \frac{\phi(\hat{\alpha}_n) - \phi_0}{||u_n^*||_{sd}} \times (1 + o_P(n, Z = \infty(1))
\]

\[
= \sqrt{n}(u_n^*, \hat{\alpha}_n - \alpha_0) \times (1 + o_P(n, Z = \infty(1)) + o_P(n, Z = \infty(1))
\]

\[
= \left( -\sqrt{n}Z_n(\alpha_n) + \sqrt{n} \frac{d_n \kappa(1 + o(1))}{||u_n^*||_{sd}} \right) \times (1 + o_P(n, Z = \infty(1)) + o_P(n, Z = \infty(1),
\]

where the second line follows from assumption 3.5 the third line follows from equation C.2, and \( \hat{\alpha}_n(\alpha_n) \Rightarrow N(0, 1) \) under the local alternatives (i.e., assumption 3.6) under the alternatives.

For Result (1), under local alternatives with \( d_n = n^{-1/2}||u_n^*||_{sd} \) we have:

\[
T_n = -\left( \sqrt{n}Z_n(\alpha_n) - \kappa(1 + o(1)) \right) \times (1 + o_P(n, Z = \infty(1)) + o_P(n, Z = \infty(1)), \quad \text{and} \quad W_n \equiv (T_n)^2 \Rightarrow \chi_1^2(\kappa^2).
\]

For Result (2), under local alternatives with \( \sqrt{n} \frac{d_n}{||u_n^*||_{sd}} \to \infty \) we have:

\[
W_n \equiv (T_n)^2 = \left( O_{P_n, Z = \infty(1)} - \sqrt{n} \frac{d_n \kappa(1 + o(1))}{||u_n^*||_{sd}} \right) \times (1 + o_P(n, Z = \infty(1)) + o_P(n, Z = \infty(1) \to \infty wpa(1, P_n, Z = \infty).
\]

Q.E.D.

**Proof of Theorem A.4** For Result (1), following the proofs of Theorems 5.2 (1) and A.2 we have: under local alternatives \( \alpha_n \) defined in (A.2), for \( j = 1, 2 \),

\[
\hat{W}_{j,n}^B = -\sqrt{n} \frac{Z_n^{\omega-1}(\alpha_n)}{\sigma_n} + o_P(n, Z = \infty(1) wpa(1, P_n, Z = \infty).
\]

By similar calculations to those in the proof of Theorem 5.2 (1), the law of \( \sqrt{n} \frac{Z_n^{\omega-1}(\alpha_n)}{\sigma_n} \) is asymptotically (and \( wpa(1, P_n, Z = \infty) \)) equal to the law of \( Z^T \sim N(0, 1) \). Then under the
local alternatives \( \alpha_n \),

\[
\text{(C.5)} \quad \sup_{t \in \mathbb{R}} |P_{V \mid Z \geq 2} \left( \hat{W}_{j,n}^B \leq t \mid Z^n \right) - P_{Z \geq 2} \left( \hat{W}_n \leq t \right)| = o_{P_{V \mid \infty \mid Z \geq 2}}(1) \quad \text{wpa1}(P_{n,Z \geq 2}),
\]

where \( \lim_{n \to \infty} P_{Z \geq 2} \left( \hat{W}_{j,n}^B \leq t \right) = \Phi(t) \) (i.e., the standard normal cdf). Thus the \( \alpha \)-th quantile of the distribution of \( \left( \hat{W}_{j,n}^B \right)^2, \hat{c}_{j,n}(a) \), is uniformly bounded \( \text{wpa1}(P_{n,Z \geq 2}) \).

For **Result (2a)**, by Theorem \[A.3(2)\], Result (1) (i.e., equation \( \text{(C.5)} \)) and the continuous mapping theorem, we have:

\[
P_{n,Z \geq 2} \left( W_n \geq \hat{c}_{j,n}(1 - \tau) \right) = P_{V \mid Z \geq 2} \left( \left( \hat{W}_{j,n}^B \right)^2 \geq \hat{c}_{j,n}(1 - \tau) \mid Z^n \right)
\]

\[
= \Pr \left( \chi_1^2(\kappa^2) \geq \hat{c}_{j,n}(1 - \tau) \right) - \Pr \left( \chi_1^2 \geq \hat{c}_{j,n}(1 - \tau) \right) + o_{P_{V \mid \infty \mid Z \geq 2}}(1) \quad \text{wpa1}(P_{n,Z \geq 2}).
\]

Thus by the definition of \( \hat{c}_{j,n}(1 - \tau) \) we obtain:

\[
P_{n,Z \geq 2} \left( W_n \geq \hat{c}_{j,n}(1 - \tau) \right) = \tau + \Pr \left( \chi_1^2(\kappa^2) \geq \hat{c}_{j,n}(1 - \tau) \right) - \Pr \left( \chi_1^2 \geq \hat{c}_{j,n}(1 - \tau) \right)
\]

\[+ o_{P_{V \mid \infty \mid Z \geq 2}}(1) \quad \text{wpa1}(P_{n,Z \geq 2}).
\]

**Result (2b)** directly follows from Theorem \[A.3(2)\], equation \( \text{(C.5)} \) and the continuous mapping theorem. \( Q.E.D. \)

### C.3. Proofs for Section \[A.4\] on asymptotic theory under increasing dimension of \( \phi \)

**Lemma C.1** Let Assumption \[3.1(iv)\] hold. Then: there exist positive finite constants \( c,C \) such that

\[
c^2 I_{d(n)} \leq D_n^2 \leq C^2 I_{d(n)},
\]

where \( I_{d(n)} \) is the \( d(n) \times d(n) \) identity and for matrices \( A \leq B \) means that \( B - A \) is positive semi-definite.

**Proof of Lemma C.1** By Assumption \[3.1(iv)\], the eigenvalues of \( \Sigma_0(x) \) and \( \Sigma(x) \) are bounded away from zero and infinity uniformly in \( x \). Therefore, for any matrix \( A \),

\[
A'\Sigma^{-1}(x)\Sigma_0(x)\Sigma^{-1}(x)A \geq dA'\Sigma^{-1}(x)A
\]

and

\[
A'\Sigma^{-1}(x)\Sigma_0(x)\Sigma^{-1}(x)A \leq DA'\Sigma^{-1}(x)A
\]

for some finite constant \( 0 < d \leq D < \infty \), and for all \( x \). Taking expectations at both sides and choosing \( A' \equiv \frac{d m(x, \nu_0)}{d \nu_2} [\nu_0]' \), these displays imply that

\[
\Omega_{sd,n} \geq d \Omega_n \quad \text{and} \quad \Omega_{sd,n} \leq D \Omega_n
\]

Thus

\[
D_n^2 = \Omega_{sd,n}^{1/2} \Omega_n^{1/2} \Omega_{sd,n}^{1/2} \Omega_{sd,n}^{1/2} \geq d \Omega_{sd,n}^{1/2} \Omega_n^{1/2} \Omega_{sd,n}^{1/2} \geq d^2 \Omega_{sd,n}^{1/2} \Omega_n^{-1/2} \Omega_{sd,n}^{1/2} = d^2 I_{d(n)}.
\]

Similarly, \( d_n^2 \leq D^2 I_{d(n)} \). \( Q.E.D. \)
Lemma C.2 Let \( T_n^M \equiv \{ t \in \mathbb{R}^{d(n)} : ||t||_e \leq M_n n^{-1/2} \sqrt{d(n)} \} \). Then:

\[
||\Omega_{sd,n}^{-1/2} Z_n||_e = O_P \left( n^{-1/2} \sqrt{d(n)} \right) \quad \text{and} \quad \Omega_{sd,n}^{-1/2} Z_n \in T_n^M \text{ wpa} 1.
\]

Proof of Lemma C.2 Let \( \Omega_{sd,n}^{-1/2} Z_n \equiv n^{-1} \sum_{i=1}^{n} \zeta_{in} \) where \( \zeta_{in} \in \mathbb{R}^{d(n)} \). Observe that \( E[sd,n^t \zeta_{in}] = I_{d(n)} \). It follows that

\[
E_P[(\Omega_{sd,n}^{-1/2} Z_n)'](\Omega_{sd,n}^{-1/2} Z_n) = tr \left\{ E_P[\Omega_{sd,n}^{-1/2} Z_n Z_n' \Omega_{sd,n}^{-1/2}] \right\} = n^{-2} \sum_{i=1}^{n} tr \left\{ E_P[\zeta_{in} \zeta_{in}'] \right\} = n^{-1} d(n),
\]

and thus the desired result follows by the Markov inequality. Q.E.D.

Lemma C.3 Let Conditions for Lemma C.2 and Assumption A.3 hold. Denote \( \tilde{\gamma}_n \equiv \sqrt{\gamma_n}(1 + b_n) + a_n \). Then:

1. \( \| \Omega_{sd,n}^{-1/2} (Z_n + (\nu_n^*, \hat{\alpha}_n - \alpha_0)) \|_e = O_P(\sqrt{d(n)} \tilde{\gamma}_n) = o_P(n^{-1/2}) \);
2. further let Assumption A.3 hold. Then

\[
\| \Omega_{sd,n}^{-1/2} (Z_n + \phi(\tilde{\alpha}_n) - \phi(\alpha_0)) \|_e = o_P(n^{-1/2}).
\]

Proof of Lemma C.3 For Result (1), note that \( ||t||_e^2 = \sum_{i=1}^{d(n)} |t_i|^2 \) and if we obtain \( ||t||_e = O_P(\tilde{\gamma}_n) \) for \( \tilde{\gamma}_n \) uniformly over \( t \), then \( ||t||_e^2 = O_P(d(n) \tilde{\gamma}_n) \).

The rest of the proof follows closely the proof of Theorem 4.1 so we only present the main steps. By definition of the approximate PSMD estimator \( \hat{\alpha}_n \), and Assumption [A.3 1],

\[
0 \leq t' \Omega_{sd,n}^{-1/2} (Z_n + (\nu_n^*, \hat{\alpha}_n - \alpha_0)) + \frac{1}{2} t'B_n t + O_P(r^{-1}(t)).
\]

We now choose \( t = \sqrt{n} e \) where \( e \in \{ (1, 0, ..., 0), (0, 1, ..., 0), ..., (0, ..., 1) \} \), it is easy to see that this \( t \in T_n^M \), and thus the display above implies

\[
0 \leq e' \Omega_{sd,n}^{-1/2} (Z_n + (\nu_n^*, \hat{\alpha}_n - \alpha_0)) + O_P(\tilde{\gamma}_n).
\]

By changing the sign of \( t \), it follows that

\[
e' \Omega_{sd,n}^{-1/2} (Z_n + (\nu_n^*, \hat{\alpha}_n - \alpha_0)) = O_P(\tilde{\gamma}_n).
\]

Observe that the RHS holds uniformly over \( e \), thus, since \( e \in \{ (1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, ..., 1) \} \), it follows that

\[
\| \Omega_{sd,n}^{-1/2} (Z_n + (\nu_n^*, \hat{\alpha}_n - \alpha_0)) \|_e = O_P(\sqrt{d(n)} \tilde{\gamma}_n) = o_P(n^{-1/2}),
\]

where the second equal sign is due to Assumption [A.3 ii).

For Result (2). In view of Result (1), it suffices to show that

\[
\| \Omega_{sd,n}^{-1/2} (\phi(\tilde{\alpha}_n) - \phi(\alpha_0) - (\nu_n^*, \hat{\alpha}_n - \alpha_0)) \|_e = o_P(n^{-1/2}).
\]
Following the proof of Theorem A.4, we have:

\[ \langle \mathbf{w}_n^*, \hat{\alpha}_n - \alpha_0 \rangle = \frac{d\phi(\alpha_0)}{d\alpha} [\hat{\alpha}_n - \alpha_{0,n}] = \frac{d\phi(\alpha_0)}{d\alpha} [\hat{\alpha}_n - \alpha_0] - \frac{d\phi(\alpha_0)}{d\alpha} [\alpha_{0,n} - \alpha_0]. \]

Since Assumption A.2 (ii) (with \( t = 0 \)) implies that

\[ \left\| \Omega_{sd,n}^{1/2} \{ \phi(\hat{\alpha}_n) - \phi(\alpha_0) - \frac{d\phi(\alpha_0)}{d\alpha} [\hat{\alpha}_n - \alpha_0] + \frac{d\phi(\alpha_0)}{d\alpha} [\alpha_{0,n} - \alpha_0] \} \right\|_e = Op(c_n), \]

the desired result now follows from Assumption A.2 (iv) of \( c_n = o(n^{-1/2}) \). Q.E.D.

**Proof of Theorem A.5** Throughout the proof let \( \hat{W}_n \equiv n(\phi(\hat{\alpha}_n) - \phi(\alpha_0))\Omega_{sd,n}^{-1}(\phi(\hat{\alpha}_n) - \phi(\alpha_0)) \). By Lemma C.3 (2),

\[ T_n \equiv (\phi(\hat{\alpha}_n) - \phi(\alpha_0) + Z_n)\Omega_{sd,n}^{-1}(\phi(\hat{\alpha}_n) - \phi(\alpha_0) + Z_n) = Op(n^{-1}). \]

Observe that

\[ |(\phi(\hat{\alpha}_n) - \phi(\alpha_0))\Omega_{sd,n}^{-1}(\phi(\hat{\alpha}_n) - \phi(\alpha_0)) - (Z_n)\Omega_{sd,n}^{-1}(Z_n)| \leq T_n + 2|| (\phi(\hat{\alpha}_n) - \phi(\alpha_0) + Z_n)\Omega_{sd,n}^{-1/2} ||_e \times || \Omega_{sd,n}^{-1/2} Z_n ||_e \]

\[ = Op(n^{-1}) + 2|| (\phi(\hat{\alpha}_n) - \phi(\alpha_0) + Z_n)\Omega_{sd,n}^{-1/2} ||_e \times || \Omega_{sd,n}^{-1/2} Z_n ||_e = Op(n^{-1}) + Op(n^{-1} \sqrt{d(n)}) \]

where the last equality is due to Lemmas C.3 (2) and C.3 (2). Therefore we obtain Result (1):

\[ \hat{W}_n = (\sqrt{n}Z_n)\Omega_{sd,n}^{-1}(\sqrt{n}Z_n) + Op(\sqrt{d(n)}) \equiv W_n + Op(\sqrt{d(n)}). \]

Result (2) follows directly from Result (1) when \( d(n) = d \) is fixed and finite. Result (3) follows from Result (1) and the following property:

\[ \Xi_n \equiv (2d(n))^{-1/2}(W_n - d(n)) \Rightarrow N(0, 1) \]

where \( W_n \equiv (\sqrt{n}Z_n)\Omega_{sd,n}^{-1}(\sqrt{n}Z_n) \), or formally,

\[ \sup_{f \in BL_1(\mathbb{R})} |E[f(\Xi_n)] - E[f(\Xi)]| = o(1) \]

where \( Z \sim N(0, 1) \) and \( BL_1(\mathbb{R}) \) is the space of bounded (by 1) Lipschitz functions from \( \mathbb{R} \) to \( \mathbb{R} \).

By triangle inequality it suffices to show that

\[ \sup_{f \in BL_1(\mathbb{R})} |E[f(\Xi_n)] - E[f(\xi_n)]| = o(1) \]

and

\[ \sup_{f \in BL_1(\mathbb{R})} |E[f(\xi_n)] - E[f(Z)]| = o(1) \]

where \( \xi_n \equiv (2d(n))^{-1/2}(\Sigma_{j=1}^{d(n)} Z_j^2 - d(n)) \) with \( Z_j \sim N(0, 1) \) and independent across \( j = 1, \ldots, d(n) \). We now show that both equations hold.
Equation C.6. Let \( t \mapsto \nu_M(t) \equiv \min\{t', M\} \) for some \( M > 0 \). Observe

\[
E[f(\Xi_n)] - E \left[ f \left( (2d(n))^{-1/2}(\nu_M(\Omega_{sd,n}^{-1/2}\sqrt{Z}Z) - d(n)) \right) \right] = E \left[ f \left( (2d(n))^{-1/2}(\nu_M(\Omega_{sd,n}^{-1/2}\sqrt{Z}Z) - d(n)) \right) - f \left( (2d(n))^{-1/2}(\nu_M(\Omega_{sd,n}^{-1/2}\sqrt{Z}Z) - d(n)) \right) \right] = \int_{\{z:x^\top \Omega_{sd,n}^{-1} x > M\}} \left[ f \left( \frac{\nu_M(\Omega_{sd,n}^{-1/2}\sqrt{Z}Z) - d(n)}{\sqrt{2d(n)}} \right) - f \left( \frac{M - d(n)}{\sqrt{2d(n)}} \right) \right] P_{Z\sim}(dz) \leq 2P_{Z\sim}(\sqrt{\nu_M(\Omega_{sd,n}^{-1/2}\sqrt{Z}Z) > M}) \]

where the last line follows from the fact that \( f \) is bounded by 1. Therefore, by the Markov inequality, for any \( \epsilon \), there exists a \( M \) such that

\[
E[f(\Xi_n)] - E \left[ f \left( (2d(n))^{-1/2}(\nu_M(\Omega_{sd,n}^{-1/2}\sqrt{Z}Z) - d(n)) \right) \right] < \epsilon
\]

for sufficiently large \( n \). A similar result holds if we replace \( \Omega_{sd,n}^{-1/2}\sqrt{Z}Z \) by \( \zeta_n = (Z_1, \ldots, Z_d) \) with \( Z_j \sim N(0, 1) \) and independent across \( j = 1, \ldots, d(n) \). Therefore, in order to show equation C.6, it suffices to show

\[
\sup_{f \in B_{L_1}(R)} E[f(\Xi_{M,n})] - E[f(\xi_{M,n})] = o(1)
\]

where \( \Xi_{M,n} \equiv (2d(n))^{-1/2}(\nu_M(\Omega_{sd,n}^{-1/2}\sqrt{Z}Z) - d(n)) \) and \( \xi_{M,n} \equiv (2d(n))^{-1/2}(\nu_M(\zeta_n) - d(n)) \).

Since \( f \) is uniformly bounded and continuous, it is clear that in order to show the previous display, it suffices to show that

\[
(2d(n))^{-1/2}|\nu_M(\Omega_{sd,n}^{-1/2}\sqrt{Z}Z) - \nu_M(\zeta_n)| = o_P(1).
\]

It turns out that \( |\nu_M(t) - \nu_M(r)| \leq 2\sqrt{M}|t - r| \), so \( t \mapsto \nu_M(t) \) is Lipschitz (and uniformly bounded). So in order to show equation C.8, it is sufficient to show that for any \( \delta > 0 \), there exists a \( N(\delta) \) such that

\[
Pr \left( (2d(n))^{-1/2}|\Omega_{sd,n}^{-1/2}\sqrt{Z}Z - \zeta_n| > \delta \right) < \delta
\]

for all \( n \geq N(\delta) \). Note that \( \Omega_{sd,n}^{-1/2}\sqrt{Z}Z = \sum_{i=1}^{n} \Psi^\top_i(Z_i) \zeta_n, \) with \( \Psi^\top_i(Z_i) = \left( \frac{d\Psi_i(x; \alpha_0)}{dx} [\nu_M^{1/2}]^{\top} \right)^t \rho(z, \alpha_0), \) and that \( \zeta_n \) can be cast as \( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{n,i} \) with \( Z_{n,i} \sim N(0, I_d(n)) \), iid across \( i = 1, \ldots, n \). Following the arguments in Section 10.4 of [Pollard (2001)], we obtain: for any \( \delta > 0 \),

\[
Pr \left( \left\| \sqrt{n} \zeta_n \Omega_{sd,n}^{-1/2} \right\| > \frac{\delta}{\sqrt{n} \sqrt{2d(n)}} \right) \leq \chi_{d(n)} \left( \frac{\mu_{3,n} d(n)^{5/2}}{\delta \sqrt{n}} \right),
\]

for any \( n \), where \( x \mapsto \chi_{d(n)}(x) \equiv Cx \times (1 + \log(1/x)) / d(n) \) and \( \mu_{3,n} \equiv E \left[ \left\| \frac{d\Psi_i(x; \alpha_0)}{dx} [\nu_M^{1/2}]^{\top} \right\|_{L_1}^3 \right] \).

Therefore,

\[
Pr \left( (2d(n))^{-1/2}|\Omega_{sd,n}^{-1/2}\sqrt{Z}Z - \zeta_n| > \delta \right) \leq \chi_{d(n)} \left( \frac{\mu_{3,n} d(n)^{5/2}}{\delta \sqrt{n} \sqrt{2d(n)}} \right) \approx \chi_{d(n)} \left( \frac{n^{-1/2} d(n)^{1/2}}{\delta / 3 \sqrt{n}} \right) \to 0
\]

provided that \( d(n) = o(\sqrt{n}) \) which is assumed in the Theorem Result (3).
Equation C.7 Observe that \( \xi_n \equiv (2d(n))^{-1/2} (\sum_{j=1}^{d(n)} Z_j^2 - d(n)) \) with \( Z_j \sim N(0,1) \) i.i.d. across \( j = 1, \ldots, d(n) \). \( E[(Z_j^2 - 1)] = 0 \) and \( E[(Z_j^2 - 1)^2] = 2 \). Thus, \( \xi_n \Rightarrow N(0,1) \) by a standard CLT. Q.E.D.

In the following we recall that \( \alpha(t) = \alpha + \nu^*_n(\Omega_{sd,n})^{-1/2} t \) for \( t \in \mathbb{R}^{d(n)} \).

Lemma C.4 Let all conditions for Theorem A.6(i) hold. Then there exists a \( t_n \) (possibly random) such that: (1) \( t_n \in T_n \) wpa1, (2) \( \alpha_n(t_n) = \{ \alpha \in A_{k(n)} : \phi(\alpha) = \phi_0 \} \) wpa1, and (3)

\[
0 \leq n(\tilde{Q}_n(\hat{\alpha}_n(t_n)) - \tilde{Q}_n(\hat{\alpha}_n)) \leq (\sqrt{n} \Omega_{sd,n}^{-1/2} Z_n)^t D_n (\sqrt{n} \Omega_{sd,n}^{-1/2} Z_n) + O_P(\sqrt{d(n)}).
\]

Proof of Lemma C.4 To show Parts (1) and (2), we define the following mappings:

\[
t \in \mathbb{R}^{d(n)} \rightarrow \varphi_n(t) \equiv \Omega_{sd,n}^{-1/2} \left\{ \phi(\hat{\alpha}_n(t)) - \phi(\alpha_0) - \frac{d\phi(\alpha_0)}{d\alpha} [\hat{\alpha}_n(t) - \alpha_0] \right\}
\]

and \( t \in \mathbb{R}^{d(n)} \rightarrow \tau_n(t) \equiv -D_n \Omega_{sd,n}^{-1/2} \left\{ \nu^*_n, \hat{\alpha}_n - \alpha_0 \right\} + \frac{d\phi(\alpha_0)}{d\alpha} [\hat{\alpha}_n(t) - \alpha_0] \Omega_{sd,n}^{1/2} t \). Under our assumptions, both mappings are continuous in \( t \) (a.s.) and thus \( \Phi_n \equiv \varphi_n \circ \tau_n \) is also continuous in \( t \) (a.s.). Given \( c_n = o(n^{-1/2}) \) satisfying assumption A.2(iv), we define \( T_n \equiv \{ t \in \mathbb{R}^{d(n)} : ||t|| \leq L_n c_n \} \) where \( (L_n)_{n} \) is a positive real valued sequence diverging to infinity slowly such that \( L_n c_n = o(n^{-1/2}) \) (such a sequence exists by assumption A.2(iv)).

Let \( S_n \equiv \{ Z^n : \text{sup}_{t \in T_n} ||\Phi_n(t)|| \leq L_n c_n \} \). By Lemmas C.1, C.2, and C.3(2), and assumption A.2(iii), we have that for any \( t \in T_n \),

\[
||\tau_n(t)|| = O_P(\sqrt{d(n)}(\gamma_n + n^{-1/2})) + O(c_n) + ||D_n t|| = O_P(n^{-1/2} \sqrt{d(n)}) + O(L_n c_n)
\]

where \( \gamma_n \equiv \sqrt{\text{Var}_n} (1 + b_n) + a_n = o(n^{-1/2}) \) (by assumption A.3(ii)). Hence \( \tau_n(t) \in T_n \) for all \( t \in T_n \). This implies, by assumption A.2(ii), \( P(S_n) = 1 \).

Moreover, these results imply that \( ||\Phi_n(t)|| \leq L_n c_n \) for all \( t \in T_n \) and \( Z^n \in S_n \).

This implies that \( \{ \Phi_n(t) : t \in T_n \} \subset T_n \) wpa1.

For any given \( n \), \( T_n \) is compact and convex in \( \mathbb{R}^{d(n)} \) and since \( \Phi_n \) is continuous and maps \( T_n \) into itself (wpa1), by Brouwer fixed point theorem, wpa1 there exists a \( \hat{t}_n \in T_n \) such that \( \Phi_n(\hat{t}_n) = \hat{t}_n \). Therefore,

\[
\hat{t}_n = \varphi_n \circ \tau_n(\hat{t}_n) = \Omega_{sd,n}^{-1/2} \left\{ \phi(\hat{\alpha}_n(\tau_n(\hat{t}_n))) - \phi(\alpha_0) - \frac{d\phi(\alpha_0)}{d\alpha} [\hat{\alpha}_n(\tau_n(\hat{t}_n)) - \alpha_0] \right\}.
\]

Since

\[
\hat{\alpha}_n(\tau_n(\hat{t}_n)) = \hat{\alpha}_n + \nu^*_n \Omega_{sd,n}^{-1/2} \tau_n(\hat{t}_n)
\]

and \( \Omega_{sd,n}^{-1/2} \tau_n(\hat{t}_n) = \Omega_{sd,n}^{-1/2} \hat{t}_n \), we obtain

\[
\frac{d\phi(\alpha_0)}{d\alpha} [\hat{\alpha}_n(\tau_n(\hat{t}_n)) - \alpha_0] = \frac{d\phi(\alpha_0)}{d\alpha} [\hat{\alpha}_n - \alpha_0] - \nu^* \nu^*_n [\nu^*_n, \hat{\alpha}_n - \alpha_0] + \frac{d\phi(\alpha_0)}{d\alpha} [\alpha_0 - \alpha_0] + \Omega_{sd,n}^{-1/2} \hat{t}_n.
\]
Since $\langle \mathbf{v}_n', \mathbf{v}_n' \rangle = \Omega_n$ we obtain

$$
\hat{t}_n = \Omega_{sd,n}^{-1/2} \left\{ \phi \left( \hat{\alpha}(n) \{\hat{\alpha}_n - \alpha_0 \} \right) - \phi(\alpha_0) - \frac{d\phi(\alpha_0)}{d\alpha} [\hat{\alpha}_n - \alpha_0] + \langle \mathbf{v}_n', \hat{\alpha}_n - \alpha_0 \rangle + \frac{d\phi(\alpha_0)}{d\alpha} [\alpha_0 n - \alpha_0] + \Omega_{sd,n}^{1/2} n \right\} 
$$

Thus $\Omega_{sd,n}^{-1/2} \{ \phi \left( \hat{\alpha}_n \{\hat{\alpha}_n - \alpha_0 \} \right) - \phi(\alpha_0) \} = 0$ wpa1 iff $\phi \left( \hat{\alpha}_n \{\hat{\alpha}_n - \alpha_0 \} \right) - \phi(\alpha_0) = 0$ wpa1.

Also, since $\alpha_n = \{\hat{\alpha}_n - \alpha_0 \}$, we can rewrite $\hat{t}_n$ as $\alpha_n = \{\hat{\alpha}_n - \alpha_0 \}$.

To show **Part (3)**, recall that $\hat{\alpha}_n \in \mathbb{N}_{\alpha_0}$ wpa1 and $\hat{\alpha}_n(t_n) \in A_{\alpha_0}$ by Parts (1) and (2) with $\alpha_n = \{\hat{\alpha}_n - \alpha_0 \}$. We can rewrite $\alpha_n$ as

$$
\alpha_n = \Omega_{sd,n}^{-1/2} \left\{ \langle \mathbf{v}_n', \hat{\alpha}_n - \alpha_0 \rangle + A_n(\hat{t}_n) \right\} \quad \text{with} \quad A_n(t) \equiv \frac{d\phi(\alpha_0)}{d\alpha} [\alpha_0 n - \alpha_0] + \Omega_{sd,n}^{1/2} n.
$$

Observe that $||t_n||_c = O_P(\sqrt{d(n)/n})$, so by Assumption A.3(i) and the definition of $\hat{\alpha}_n$,

$$
0 \leq n \left[ \hat{\alpha}_n \{\hat{\alpha}_n - \alpha_0 \} - \hat{\alpha}_n \{\hat{\alpha}_n - \alpha_0 \} \right] = n \left[ \langle \mathbf{v}_n', \hat{\alpha}_n - \alpha_0 \rangle + \{\hat{\alpha}_n - \alpha_0 \} \right] + n \times O_P(s_n + ||t_n||_c a_n + ||t_n||^2 b_n) 
$$

where the third line follows from the fact that sup \( |t||t|_c = 1 \) \( \{\mathbb{B}_n - \mathbb{D}_n^{-1} \} t = O_P(b_n) \) by assumption, and thus we have: $t' \mathbb{B}_n t \leq |t'| \{\mathbb{B}_n - \mathbb{D}_n^{-1} \} t + t' \mathbb{D}_n^{-1} t \leq ||t||^2 O_P(b_n) + t' \mathbb{D}_n^{-1} t$ uniformly over $t \in \mathbb{R}^d(t)$ with $||t||_c = 1$.

By the fact that $\Omega_{sd,n}^{-1/2} \mathbb{D}_n^{-1} \mathbb{D}_n \Omega_{sd,n}^{-1/2} = \Omega_{sd,n}^{-1/2} \mathbb{D}_n \Omega_{sd,n}^{-1/2} = \Omega_n^{-1}$, the definition of $t_n$ and straightforward algebra, the previous display implies that

$$
\Omega_{sd,n}^{-1/2} \mathbb{D}_n^{-1} \mathbb{D}_n \Omega_{sd,n}^{-1/2} = \Omega_{sd,n}^{-1/2} \mathbb{D}_n \Omega_{sd,n}^{-1/2} = \Omega_n^{-1}, \quad \text{the definition of} \quad t_n 
$$

where second line follows because $\langle \langle \mathbf{v}_n' \mathbf{v}_n, \hat{\alpha}_n - \alpha_0 \rangle \rangle \Omega_n^{-1} (\langle \mathbf{v}_n', \hat{\alpha}_n - \alpha_0 \rangle) \geq 0$ and straightforward algebra. Observe that

$$
\left[ \langle \mathbf{v}_n', \hat{\alpha}_n - \alpha_0 \rangle \right]^{1/2} = O_P \left( \left[ \langle \mathbf{v}_n', \hat{\alpha}_n - \alpha_0 \rangle \right]^{1/2} \right)
$$

where the first equation follows from Lemma C.7 and the last equality follows from
assumption \( A.2 (iii) \) and the results from Parts (1) and (2). Also
\[
\left[ (\mathbf{Z}_n)^\top \Omega_n^{-1} (\mathbf{Z}_n) \right]^{1/2} = \left( \Omega_{sd,n}^{-1/2} \mathbf{Z}_n \right)^\top \mathcal{D}_n (\Omega_{sd,n}^{-1/2} \mathbf{Z}_n) \right]^{1/2} = O_P (n^{-1/2} \sqrt{d(n)})
\]
by Lemmas \( C.1 \) and \( C.2 \). Thus \( (\mathbf{Z}_n)^\top \Omega_n^{-1} \mathbf{Z}_n = O_P (n^{-1/2}) O_P (n^{-1/2} \sqrt{d(n)}) = O_P (n^{-1} \sqrt{d(n)}) \).
Similarly, \( \left( (\mathbf{v}_n^\top, \hat{\alpha}_n - \alpha_0) + \mathbf{Z}_n \right)^\top \Omega_n^{-1} (\mathbf{Z}_n) = O_P (n^{-1} \sqrt{d(n)}) \) by Lemmas \( C.1 \) and \( C.3 (1) \). Since \( ||t_n||_e = O_P (\sqrt{d(n)} n^{-1/2}) \), we have \( n(s_n + (||t_n||_e \alpha_n + ||t_n||_e^2) b_n) = O_P (n s_n + \sqrt{d(n)} n^{1/2} a_n + \sqrt{d(n)} n^{-1/2} b_n) = O_P (n^{-1} \sqrt{d(n)}) \) under assumption \( A.3 (ii) \). Therefore,
\[
n \left[ \hat{Q}_n (\tilde{\Theta}_n(t_n)) - \hat{Q}_n (\hat{\Theta}_n) \right] \leq n (\mathbf{Z}_n)^\top \Omega_n^{-1} (\mathbf{Z}_n) + o_P (\sqrt{d(n)}) = O_P (\sqrt{d(n)} \mathbf{Z}_n)^\top \mathcal{D}_n (\sqrt{n} \Omega_{sd,n}^{-1/2} \mathbf{Z}_n) + o_P (\sqrt{d(n)})
\]
Q.E.D.

Proof of Theorem \( A.6 \): The proof is very similar to that of Theorem \( A.3 \) and we only provide main steps here.

**Step 1.** Similar to Steps 1 and 2 in the proof of Theorem \( A.3 \) by the definitions of \( \hat{\Theta}_n^R \) and \( \hat{\Theta}_n \) and Assumption \( A.6 (i) \), it follows that for any (possibly random) \( t \in \mathcal{T}_M \),
\[
0.5 Q L R_n (\phi_0) \geq 0.5 n \left( \hat{Q}_n (\hat{\Theta}_n^R) - \hat{Q}_n (\hat{\Theta}_n (t)) \right) - o_P (1)
\]
\[
= -n \left( t' \Omega_{sd,n}^{-1/2} \mathbf{Z}_n + (\mathbf{v}_n^\top, \hat{\alpha}_n - \alpha_0) \right) + 0.5 t' \mathbb{B}_n t
\]
\[
+ O_P (s_n n + ||t_n||_e \alpha_n + ||t_n||_e^2 b_n).
\]

By Assumption \( A.2 (i) (ii) \),
\[
\left\| \Omega_{sd,n}^{-1/2} \left( \hat{\Theta}_n^R - \hat{\Theta}_n \right) - \frac{d \phi (\alpha_0)}{d \alpha} (\hat{\Theta}_n - \alpha_0) \right\|_e = O_P (c_n).
\]

Hence, by Assumption \( A.2 (iii) \),
\[
(C.9) \left\| \Omega_{sd,n}^{-1/2} (\mathbf{v}_n^\top, \hat{\alpha}_n - \alpha_0) \right\|_e = O_P (c_n).
\]

Since \( \sup_t ||t||_e = 1 \) \( t' \{ \mathbb{B}_n - \mathbb{D}_n^{-1} \} t = O_P (b_n) \) by assumption, we have:
\[
t' \mathbb{B}_n t \leq ||t'||_e^2 O_P (b_n) + \max \left( \frac{t'}{||t'||_e} \right) \leq ||t'||_e^2 O_P (b_n) + \max \left( \frac{t'}{||t'||_e} \right) \leq \mathbb{B}_n t \leq ||t'||_e^2 O_P (b_n) + \max \left( \frac{t'}{||t'||_e} \right) \leq \mathbb{B}_n t \leq \mathbb{B}_n t
\]
uniformly over \( t \in \mathbb{R}^d(n) \) with \( ||t||_e = 1 \). This, Assumption \( A.3(i) \) and equation \( (C.9) \), together imply that
\[
0.5 Q L R_n (\phi_0) \geq -n \left( t' \Omega_{sd,n}^{-1/2} \mathbf{Z}_n + 0.5 \mathbb{D}_n^{-1} t + O_P (s_n n + ||t||_e (a_n + c_n) + ||t||_e^2 b_n) \right).
\]

In the above display we let \( t' = -\mathbf{Z}_n^\top \Omega_{sd,n}^{-1/2} \mathbb{D}_n, \) which, by Lemmas \( C.1 \) and \( C.2 \), is an admissible choice and \( ||t||_e = O_P \left( n^{-1/2} \sqrt{d(n)} \right) \). Observe that \( t' \Omega_{sd,n}^{-1/2} \mathbf{Z}_n = -\mathbf{Z}_n^\top \Omega_{sd,n}^{-1/2} \mathbb{D}_n \Omega_{sd,n}^{-1/2} \mathbf{Z}_n \) and \( t' \mathbb{D}_n^{-1} t = \mathbf{Z}_n^\top \Omega_{sd,n}^{-1/2} \mathbb{D}_n \Omega_{sd,n}^{-1/2} \mathbf{Z}_n \), we obtain:
\[
0.5 Q L R_n (\phi_0) \geq 0.5 (\sqrt{n} \Omega_{sd,n}^{-1/2} \mathbf{Z}_n)^\top \mathcal{D}_n (\sqrt{n} \Omega_{sd,n}^{-1/2} \mathbf{Z}_n) + O_P (s_n n + n^{1/2} \sqrt{d(n)} (a_n + c_n) + \sqrt{d(n)} b_n)
\]
where the last equal sign is due to Assumptions \( A.2 (iv) \) and \( A.3 (ii) \).
By following the proof of Lemma C.3(ii) of Chen and Pouzo (2012a), we have that
\[ M_{\alpha}(Z^n) \equiv \nu_{\alpha}|Z| \sup_{N_{\alpha,n}} \left\{ \frac{\tau_n}{n} \sum_{i=1}^{n} \left\| \bar{m}^B(X_i, \alpha) - \hat{m}(X_i, \alpha) - \bar{m}^B(X_i, \alpha_0) \right\|_2^2 \geq M | Z^n \right\} \]
\[ \leq \nu_{\alpha}|Z| \left( \sup_{N_{\alpha,n}} \left\{ \frac{\tau_n}{n} \sum_{i=1}^{n} \left\| \bar{m}^B(X_i, \alpha) - \hat{m}(X_i, \alpha) - \bar{m}^B(X_i, \alpha_0) - \hat{m}(X_i, \alpha_0) \right\|_2^2 \geq \frac{M}{2} | Z^n \right\} \right) \]
\[ + \nu_{\alpha}|Z| \left( \sup_{N_{\alpha,n}} \left\{ \frac{\tau_n}{n} \sum_{i=1}^{n} \left\| \hat{m}(X_i, \alpha) - \hat{m}(X_i, \alpha) - \hat{m}(X_i, \alpha_0) \right\|_2^2 \geq \frac{M}{2} | Z^n \right\} \right) \]
\[ \equiv (M_{1,\alpha}(Z^n) + M_{2,\alpha}(Z^n)). \]
we have: for all \( \delta > 0 \), there is a \( M(\delta) > 0 \) such that for all \( M \geq M(\delta) \),
\[ P_{Z^\infty} (M_{\alpha}(Z^n) \geq 2\delta) \leq P_{Z^\infty} (M_{1,\alpha}(Z^n) \geq \delta) + P_{Z^\infty} (M_{2,\alpha}(Z^n) \geq \delta). \]
By following the proof of Lemma C.3(ii) of Chen and Pouzo (2012a), we have that
\[ P_{Z^\infty} (M_{2,\alpha}(Z^n) \geq \delta) < \delta/2 \text{ eventually. Thus, to establish Result (1), it suffices to bound} \]
\[ P_{Z^\infty} \left( \left\{ M_{1,\alpha}(Z^n) \geq \delta \right\} \cap \left\{ \lambda_{\text{min}}((P^n)/n) > c \right\} \right) + P_{Z^\infty} (\lambda_{\text{min}}((P^n)/n) \leq c). \]
By Assumption A.4(ii)(iii) and theorem 1 in Newey (1997) \( \lambda_{\text{min}}((P^n)/n) \geq c > 0 \) with probability \( P_{Z^\infty} \) approaching one, hence \( P_{Z^\infty} (\lambda_{\text{min}}((P^n)/n) \leq c) < \delta/4 \text{ eventually.} \)
To bound the term corresponding to $M_{1,n}$, we note that
\[
\sum_{i=1}^{n} \|\hat{m}^B(X_i, \alpha) - \hat{m}(X_i, \alpha) - \{\hat{m}^B(X_i, \alpha_0) - \hat{m}(X_i, \alpha_0)\}\|_2^2 \\
= \sum_{i=1}^{n} \Delta\zeta^B(\alpha) P(\mathcal{P}P') - p^B(X_i)P'(X_i)^\top(P'P) - P'\Delta\zeta^B(\alpha) \\
= \Delta\zeta^B(\alpha) P(\mathcal{P}P') - P'\Delta\zeta^B(\alpha) \\
\leq \frac{1}{\lambda_{\text{min}}(\mathcal{P}P') \alpha} [n^{-1} \Delta\zeta^B(\alpha) P(\mathcal{P}P') \Delta\zeta^B(\alpha)];
\]
where $\Delta\zeta^B(\alpha) = (\omega_1 - 1)\Delta\rho(Z_1, \alpha), \ldots, (\omega_n - 1)\Delta\rho(Z_n, \alpha)$, with $\Delta\rho(Z, \alpha) \equiv \rho(Z, \alpha) - \rho(Z, \alpha_0)$. It is thus sufficient to show that, for large enough $n$,
\[
(P_{\hat{V}} \approx \delta M_{1,n} \approx \sup_{\mathcal{X}_{\alpha_0}} \frac{\tau_n}{n^2} \|\Delta\zeta^B(\alpha) P(\mathcal{P}P') \Delta\zeta^B(\alpha) \geq M | Z^n \| \geq \delta) < \delta,
\]
which is established in Lemma 5.5.

For Result (2), recall that $\ell_m^B(x, \alpha) \equiv \hat{m}(x, \alpha) + \hat{m}^B(x, \alpha_0)$. By similar calculations to those in [Ai and Chen 2003] (p. 1824) it follows
\[
E_{P_{\hat{V}}} \left[ n^{-1} \sum_{i=1}^{n} \|\hat{m}^B(X_i, \alpha_0)\|_2^2 \right] \\
= E_{P_{\hat{V}}} \left[ p^B(X_i)^\top(P'P) - p^B(X_i)^\top \right] \left[ \rho(\alpha_0) \rho^B(\alpha_0)^\top | X^n \right] P(\mathcal{P}P') - p^B(X_i) \\
\]
where $\rho^B(\alpha) \equiv (\rho^B(V_1, \alpha), \ldots, \rho^B(V_n, \alpha))^\top$ with $\rho^B(V_i, \alpha) \equiv \omega_i \rho(Z_i, \alpha)$. Note that
\[
E_{P_{\hat{V}} | X^n} [\rho^B(V_i, \alpha_0) \rho^B(V_j, \alpha_0)^\top | X^n] = E_{P_{\hat{V}}} [\omega_i \omega_j E_{P_{\hat{V}} | X^n} [\rho(Z_i, \alpha_0) \rho(Z_j, \alpha_0)^\top | X_i, X_j]] \\
= 0 \quad \text{for all } i \neq j,
\]
and
\[
E_{P_{\hat{V}} | X^n} [\rho^B(V_i, \alpha_0) \rho^B(V_i, \alpha_0)^\top | X^n] = \sigma^2 \Sigma_0(X_i).
\]
So under Assumption Boot.1 or Boot.2, Assumptions 3.1(iv) and 4.4(ii), applying Markov inequality we obtain, for all $\delta > 0$, there is a $M(\delta)$ such that for all $M \geq M(\delta)$,
\[
P_{\hat{V}} \approx \left( \frac{J_n}{n} \right) \left( \frac{1}{n} \sum_{i=1}^{n} \|\hat{m}^B(X_i, \alpha_0)\|_2^2 \right) \geq M | Z^n \| \geq \delta < \delta.
\]
To establish Result (2), with $(\tau_n)^{-1} = \max\{\frac{d_n}{n}, \frac{b_n^2}{n}, (M_n \delta_n)^2\}$, it remains to show

\[\text{\footnotesize ...}\]
that
\[(C.11) \quad P_{Z} \approx \left( \sup_{N_{\alpha n}} E_{P_{X}} \left[ \left\| \tilde{m}(X, \alpha) \right\|_{L}^{2} \right] \geq M \right) < \delta. \]

By Lemma SM.1 of [Chen and Pouzo (2012b)], under Assumptions [A.4] and [A.5(i)], we have: there are finite constants \(c, c' > 0\) such that, for all \(\delta > 0\), there is a \(N(\delta)\), such that for all \(n \geq N(\delta)\),
\[P_{Z} \approx \left( \forall \alpha \in N_{\alpha n} \colon c E_{P_{X}} \left[ \left\| \tilde{m}(X, \alpha) \right\|_{L}^{2} \right] \leq \frac{1}{n} \sum_{i=1}^{n} \left\| \tilde{m}(X, \alpha) \right\|_{L}^{2} \leq c' E_{P_{X}} \left[ \left\| m(X, \alpha) \right\|_{L}^{2} \right] \right) > 1 - \delta. \]

Thus to show (C.11), it suffices to show that
\[\sup_{N_{\alpha n}} E_{P_{X}} \left[ \left\| \tilde{m}(X, \alpha) \right\|_{L}^{2} \right] = O(1). \]

By Assumption [A.4(ii)] it follows
\[\sup_{\alpha \in N_{\alpha n}} E_{P_{X}} \left[ \left\| \tilde{m}(X, \alpha) \right\|_{L}^{2} \right] \leq \sup_{N_{\alpha n}} \left\{ E_{P_{X}} \left[ \left\| \tilde{m}(X, \alpha) - m(X, \alpha) \right\|_{L}^{2} \right] + E_{P_{X}} \left[ \left\| m(X, \alpha) \right\|_{L}^{2} \right] \right\} \leq \text{const.} \sup_{\alpha \in N_{\alpha n}} \max \left\{ b_{m, n, \tau} \left\| \alpha - \alpha_{0} \right\|^{2} \right\} = O\left( (\tau_{n})^{-1} \right), \]

where the last inequality follows from Assumptions [A.4(ii)(iii) and (iv) and (3.4)]. We thus obtain Result (2).

For Result (3), we note that
\[\frac{1}{n} \sum_{i=1}^{n} \left\| \tilde{m}^{B}(X, \alpha) \right\|_{L}^{2} - \frac{1}{n} \sum_{i=1}^{n} \left\| \ell^{B}(X, \alpha) \right\|_{L}^{2} = R_{1}^{B}(\alpha) + 2 R_{2}^{B}(\alpha), \]

where
\[R_{1}^{B}(\alpha) \equiv \frac{1}{n} \sum_{i=1}^{n} \left\| \tilde{m}^{B}(X, \alpha) - \ell^{B}(X, \alpha) \right\|_{L}^{2}, \quad R_{2}^{B}(\alpha) \leq \sqrt{R_{1}^{B}} \left( \frac{1}{n} \sum_{i=1}^{n} \left\| \ell^{B}(X, \alpha) \right\|_{L}^{2} \right). \]

By Result (1) and Assumption [A.4(iii)], we have:
\[P_{Z} \approx \left( P_{Z_{\infty}} \left( \sup_{N_{\alpha n}} \tau_{n} R_{1}^{B}(\alpha) \geq M \mid Z^{n} \right) \right) < \delta \]

with \(\tau_{n}^{-1} = \delta_{n}^{2}(M_{n} \delta_{n})^{2} C_{n}\). By Results (1) and (2), and Assumption [A.4(iii)], we have:
\[P_{Z} \approx \left( P_{Z_{\infty}} \left( \sup_{N_{\alpha n}} \tau_{n} R_{2}^{B}(\alpha) \geq M \mid Z^{n} \right) \right) < \delta \]

with \(\tau_{n}^{-1} \equiv M_{n} \delta_{n}^{2}(M_{n} \delta_{n})^{2} C_{n}\). By Assumption [A.5(iii)] and the fact that \(L_{n}\) diverges, we obtain the desired result. \(Q.E.D.\)

In the following we state Lemma [C.5] and its proof.
Lemma C.5 Let Assumptions \(A.4(i),(ii), A.5(i)(ii)\) and either Boot.1 or Boot.2 hold. Then: for all \(\delta > 0\), there is a \(M(\delta) > 0\) such that for all \(M \geq M(\delta)\),

\[
P_{Z \neq \emptyset} \left( P_{V \neq \emptyset} \left| Z \neq \emptyset \right. \right) = \sup_{N_{\alpha \in N}} \frac{\tau_n}{n^2} \Delta \zeta^B(\alpha') PP' \Delta \zeta^B(\alpha) \geq M \mid Z^n \right) \geq \delta \right) < 0.5 \delta
\]
eventually, with \(\tau_n^{-1} \equiv (\delta_n)^2 (M_n \delta_{s,n})^{2n} C_n\), where \(\Delta \zeta^B(\alpha) = ((\omega_1 - 1)\Delta \rho(Z_1, \alpha), \ldots, (\omega_n - 1)\Delta \rho(Z_n, \alpha))'\) and \(\Delta \rho(Z, \alpha) \equiv \rho(Z, \alpha) - \rho(Z, \alpha_0)\).

Proof of Lemma C.5 Denote

\[M'_1(Z^n) \equiv P_{V \neq \emptyset} \mid Z \neq \emptyset \left( \sup_{N_{\alpha \in N}} \frac{\tau_n}{n^2} \Delta \zeta^B(\alpha') PP' \Delta \zeta^B(\alpha) \geq M \mid Z^n \right) \].

By the Markov inequality

\[M'_1(Z^n) \leq M^{-1} E P_{V \neq \emptyset} \mid Z \neq \emptyset \left( \sup_{N_{\alpha \in N}} \frac{\tau_n}{n^2} \Delta \zeta^B(\alpha') PP' \Delta \zeta^B(\alpha) \right). \]

Hence it is sufficient to bound

\[
P_{Z \neq \emptyset} \left( M'_1(Z^n) \geq \delta \right) \leq \frac{1}{M \delta} E P_{V \neq \emptyset} \left( \sup_{N_{\alpha \in N}} \frac{\tau_n}{n^2} \Delta \zeta^B(\alpha') PP' \Delta \zeta^B(\alpha) \right)
\]

\[
= \frac{\tau_n}{nM \delta} \sum_{j=1}^{J_n} E P_{V \neq \emptyset} \left( \sup_{N_{\alpha \in N}} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\omega_i - 1) f_j(Z_i, \alpha) \right)^2 \right)
\]

where the first inequality follows from the law of iterates expectations and the Markov inequality, and the second equality is due to the notation \(f_j(z, \alpha) \equiv p_j(x) \rho(z, \alpha) - \rho(z, \alpha_0)\).

Under assumption Boot.1 \(\{ (\omega_i - 1) f_j(Z_i, \alpha) \}_{i=1}^{n} \) are independent, and thus, by proposition A.1.6 in Van der Vaart and Wellner [1996] (VdV-W),

\[
\frac{\tau_n}{nM \delta} \sum_{j=1}^{J_n} E P_{V \neq \emptyset} \left( \sup_{N_{\alpha \in N}} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\omega_i - 1) f_j(Z_i, \alpha) \right)^2 \right)
\]

\[
\leq \frac{\tau_n}{nM \delta} \sum_{j=1}^{J_n} \left( E P_{V \neq \emptyset} \left( \sup_{N_{\alpha \in N}} \left| n^{-1/2} \sum_{i=1}^{n} (\omega_i - 1) f_j(Z_i, \alpha) \right| \right)^2 \right) + \sqrt{nM \delta} \sup_{N_{\alpha \in N}} E \left( |(\omega_i - 1) f_j(Z_i, \alpha)|^2 \right)^2.
\]

The second term in the RHS is bounded above by

\[
\sqrt{nn^{-1} E P_{V \neq \emptyset} \left( |(\omega_i - 1)^2 f_j(Z_i, \alpha)|^2 \right) \leq \sqrt{E P_{V \neq \emptyset} \left( |(\omega_i - 1)^2 f_j(Z_i, \alpha)|^2 \right) \sup_{N_{\alpha \in N}} \left( f_j(Z_i, \alpha) \right)^2} = O((M_n \delta_{s,n})^n)
\]

by Assumptions A.4(iii), A.5(ii) and Boot.1. Hence, under assumption Boot.1 we need to control

\[
(C.12) \left( \frac{\tau_n J_n}{nM \delta} (M_n \delta_{s,n})^{2n} \right).
\]

Under Assumption Boot.2 \((\omega_i - 1) f_j(Z_i, \alpha)\) are not independent. So we need to take some additional steps to arrive to an equation of the form of (C.12). Under
By a slight adaptation of lemma 3.6.6 in VdV-W (allowing for square of the norm), it follows

\[
\frac{\tau_n}{M} \sum_{j=1}^{J_n} E_{P_{\infty}} \left[ \sup_{N_{\alpha \in \mathcal{N}}} \left( n^{-1} \sum_{i=1}^{n} (\omega_i - 1) f_j(Z_i, \alpha) \right)^2 \right] = \frac{\tau_n}{M} \sum_{j=1}^{J_n} E_{P_{\infty}} \left[ \sup_{N_{\alpha \in \mathcal{N}}} \left( n^{-1} \sum_{i=1}^{n} \omega_i f_j(Z_i, \alpha) - n^{-1} \sum_{i=1}^{n} f_j(Z_i, \alpha) \right)^2 \right] = \frac{\tau_n}{M} \sum_{j=1}^{J_n} E_{P_{\infty} \times P_{\infty}} \left[ \sup_{N_{\alpha \in \mathcal{N}}} \left( n^{-1} \sum_{i=1}^{n} (\delta_{\tilde{Z}_i} - \mathbb{P}[f_j(\cdot, \alpha)]) \right)^2 \right],
\]

where the last line follows from the fact that \( \omega_i \) are the number of times the variable \( Z_i \) appear on the bootstrap sample. Thus, the distribution of \( \omega_i \delta_{\tilde{Z}_i} \) is bounded above by

\[
\left( \tilde{Z}_i - \mathbb{P}[f_j(\cdot, \alpha)] \right)^2.
\]

By Assumption Boot.2, it follows

\[
\frac{\tau_n}{M} \sum_{j=1}^{J_n} E_{P_{\infty}} \left[ \sup_{N_{\alpha \in \mathcal{N}}} \left( n^{-1} \sum_{i=1}^{n} (\delta_{\tilde{Z}_i} \mathbb{I}(\delta_{\tilde{Z}_i}) - \mathbb{P}[f_j(\cdot, \alpha)]) \right)^2 \right] \leq \frac{\tau_n}{M} \sum_{j=1}^{J_n} E_{P_{\infty} \times P_{\infty}} \left[ \sup_{N_{\alpha \in \mathcal{N}}} \left( n^{-1} \sum_{i=1}^{n} (\tilde{N}_i \delta_{\tilde{Z}_i}, f_j(\cdot, \alpha)) \right)^2 \right],
\]

where \( \tilde{N}_i = N_i - N'_i \) with \( N_i \) and \( N'_i \) being iid Poisson variables with parameter 0.5 (\( P_{\tilde{N}_{\infty}} \) is the corresponding probability). Note that now, \( \{\tilde{N}_i f_j(Z_i, \alpha)\}_{i=1}^{n} \) are independent. So by proposition A.1.6 in VdV-W,

\[
\frac{\tau_n}{nM} \sum_{j=1}^{J_n} E_{Q} \left[ \sup_{N_{\alpha \in \mathcal{N}}} \left( n^{-1/2} \sum_{i=1}^{n} \tilde{N}_i f_j(Z_i, \alpha) \right)^2 \right] \leq \frac{\tau_n}{nM} \sum_{j=1}^{J_n} E_{Q} \left[ \sup_{N_{\alpha \in \mathcal{N}}} \left( n^{-1/2} \sum_{i=1}^{n} \tilde{N}_i f_j(Z_i, \alpha) \right)^2 \right] + \sqrt{\sum_{i=1}^{n} \mathbb{E}[\max_{\alpha \in \mathcal{N}_{\alpha}} (\sup_{N_{\alpha \in \mathcal{N}}}|n^{-1/2} \tilde{N}_i f_j(Z_i, \alpha)|)^2],
\]

where \( Q \equiv P_{\infty} \times P_{\tilde{N}_{\infty}} \). By Cauchy-Schwarz inequality, the second term in the RHS is bounded above by

\[
\sqrt{nM} \sum_{i=1}^{n} \mathbb{E}[f_j(Z, \alpha)]^2 \leq \sqrt{E_{P_{\tilde{N}_{\infty}}} \sup_{N_{\alpha \in \mathcal{N}}} |f_j(Z, \alpha)|^2} \leq O((M_\alpha \delta_{\alpha, n})^\alpha)
\]

by Assumptions A.4(iii) and A.5(ii) and \( E[|N|^2] < \infty \). Therefore, under Assumption \( \text{Boot.2} \) we need to control

\[
(C.13) \quad \frac{\tau_n}{nM} \sum_{j=1}^{J_n} \left( E_{Q} \left[ \sup_{N_{\alpha \in \mathcal{N}}} \left( n^{-1/2} \sum_{i=1}^{n} \tilde{N}_i f_j(Z_i, \alpha) \right)^2 \right] \right)^2 + O \left( \frac{\tau_n J_n}{nM} (M_\alpha \delta_{\alpha, n})^{2\alpha} \right).
\]

Applying lemma 2.9.1 of VdV-W we can bound the leading terms in equations (C.12).
and \( \text{(C.13)} \) respectively as follows,

\[
\frac{\tau_n}{n M \delta} \sum_{j=1}^{J_n} E_{\mathcal{P}_V} \left[ \sup_{n_{ox,n}} n^{-1/2} \sum_{i=1}^{n} (\omega_i - 1) \delta_{Z_i} [f_j(\cdot, \alpha)] \right]
\]

(C.14) \[
\leq \frac{\tau_n}{n M \delta} \sum_{j=1}^{J_n} \left\{ \int_{0}^{\infty} \sqrt{P(|\omega - 1| \geq t)} dt \right\} \sup_{n_{ox,n}} \left[ n^{-1/2} \sum_{i=1}^{n} \epsilon_i \delta_{Z_i} [f_j(\cdot, \alpha)] \right]
\]

and

\[
\frac{\tau_n}{n M \delta} \sum_{j=1}^{J_n} E_{\mathcal{P}_G} \left[ \sup_{n_{ox,n}} n^{-1/2} \sum_{i=1}^{n} \bar{N}_i \delta_{Z_i} [f_j(\cdot, \alpha)] \right]
\]

(C.15) \[
\leq \frac{\tau_n}{n M \delta} \sum_{j=1}^{J_n} \left\{ \int_{0}^{\infty} \sqrt{P(|\bar{N}| \geq t)} dt \right\} \sup_{n_{ox,n}} \left[ n^{-1/2} \sum_{i=1}^{n} \epsilon_i \delta_{Z_i} [f_j(\cdot, \alpha)] \right]
\]

where \( (\epsilon_i)_{i=1}^{\infty} \) is a sequence of Rademacher random variables. Note that \( \left\{ \int_{0}^{\infty} \sqrt{P(|\omega - 1| \geq t)} dt \right\} < \infty \) (under Assumption Boot.1), and also \( \left\{ \int_{0}^{\infty} \sqrt{P(|\bar{N}| \geq t)} dt \right\} \leq 2\sqrt{2} \) (see VdV-W p. 351). Hence in both cases we need to bound

\[
\frac{\tau_n}{n M \delta} \sum_{j=1}^{J_n} \left( \max_{1 \leq i \leq n} E_{\mathcal{P}_G} \times \mathcal{P}_V \left[ \sup_{n_{ox,n}} \left| n^{-1/2} \sum_{i=1}^{n} \epsilon_i \delta_{Z_i} [f_j(\cdot, \alpha)] \right| \right] \right)^2
\]

(C.16) \[
\leq 2T_{1,n} + 2T_{2,n},
\]

where \( f_j(\cdot, \alpha) = f_j(\cdot, \alpha) - E[f_j(Z, \alpha)] \),

\[
T_{1,n} = \frac{\tau_n}{n M \delta} \sum_{j=1}^{J_n} \left( \max_{1 \leq i \leq n} E_{\mathcal{P}_G} \times \mathcal{P}_V \left[ \sup_{n_{ox,n}} \left| n^{-1/2} \sum_{i=1}^{n} \epsilon_i \delta_{Z_i} [f_j(\cdot, \alpha)] \right| \right] \right)^2
\]

and

\[
T_{2,n} = \frac{\tau_n}{n M \delta} \sum_{j=1}^{J_n} \left( \max_{1 \leq i \leq n} E_{\mathcal{P}_G} \times \mathcal{P}_V \left[ \sup_{n_{ox,n}} \left| n^{-1/2} \sum_{i=1}^{n} \epsilon_i E_{\mathcal{P}_G} [f_j(Z, \alpha)] \right| \right] \right)^2.
\]
To bound the term $T_{2,n}$, we note that

$$
\max_{1 \leq j \leq n} \max_{1 \leq n \leq n} E_{P_n} \sup_{N_{osn}} \left| l^{-1/2} \sum_{i=1}^{l} \epsilon_i E_{P_n} [f_j(Z, \alpha)] \right|
$$

$$=
\max_{1 \leq j \leq n} \max_{1 \leq n \leq n} \sup_{N_{osn}} \left| E_{P_n} [f_j(Z, \alpha)] E_{P_n} \left| l^{-1/2} \sum_{i=1}^{l} \epsilon_i \right| \right|
$$

$$\leq
\max_{1 \leq j \leq n} \max_{1 \leq n \leq n} \sup_{N_{osn}} \left| E_{P_n} \left| m(X, \alpha) \right| \right| \sqrt{E_{P_n} \left| \left( l^{-1/2} \sum_{i=1}^{l} \epsilon_i \right) \right|^2} \right|
$$

$$\leq
\max_{1 \leq j \leq n} \max_{1 \leq n \leq n} \left( \sqrt{E_{P_n} \left| m(X, \alpha) \right|^2} \sup_{N_{osn}} \sqrt{E_{P_n} \left| \Delta m(X, \alpha) \right|^2} \sup_{N_{osn}} \left| l^{-1} \sum_{i=1}^{l} (\epsilon_i^2) \right| \right)
$$

$$= O(M_n \delta_n),$$

where $\Delta m(X, \alpha) \equiv m(X, \alpha) - m(X, \alpha_0)$ and the inequality follows from Cauchy-Schwarz and the fact that $\epsilon_i$ are independent, and the last two equal signs are due to Assumptions (3.4)(ii) and (A.iii). Thus $T_{2,n} \leq const. \times (M_n \delta_n)^2 n M \delta$.

To bound the term $T_{1,n}$, we note that by the “desymmetrization lemma” 2.3.6 in VdV-W (note that $f_j(Z, \alpha)$ are centered),

$$T_{1,n} \leq const. \times \frac{T_n}{n M \delta} \sum_{j=1}^{J_n} \max_{1 \leq n \leq n} \left( E_{P_n} \sup_{N_{osn}} \left| l^{-1/2} \sum_{i=1}^{l} f_j(Z_i, \alpha) \right| \right)^2 .$$

By Van der Vaart and Wellner (1996) theorem 2.14.2, we have (up to some omitted constant), for all $j$,

$$E_{P_n} \sup_{N_{osn}} \left| l^{-1/2} \sum_{i=1}^{l} f_j(Z_i, \alpha) \right|
$$

$$\leq \left( (M_n \delta_n)^n \int_0^1 \sqrt{1 + \log N_{1/w} \left( w (M_n \delta_n)^n, E_{osn}, ||\cdot||_{L^2(f)} \right) dw} \right)$$

where $E_{osn} = \{ p_j(\cdot)(\rho(\cdot, \alpha) - \rho(\cdot, \alpha_0)) - E[p_j(\cdot)(\rho(\cdot, \alpha) - \rho(\cdot, \alpha_0))]) : \alpha \in N_{osn} \}$. Given any $w > 0$, let $\{ g_{m}^{n} \} \in N_{osn}$ be the $||\cdot||_{L^2(f)}$-norm brackets of $O_{osn}$. If $\{ \rho(\cdot, \alpha) - \rho(\cdot, \alpha_0) \} \in O_{osn}$ belongs to a bracket $\{ g_{m}^{n}, \rho_{n}^{n} \}$, then, since $|p_j(x)| < const < \infty$ by Assumption (A.iii),

$$g_{m}^{n}(Z) \leq p_j(X) \{ \Delta \rho(Z, \alpha) \} \leq g_{m}^{n}(Z)$$

(whence $\{ g_{m}^{n}, \rho_{n}^{n} \}$ are transformations of the original ones, given by $1\{ p_j > 0 \} g_{m}^{n} + 1\{ p_j \leq 0 \} g_{m}^{n}$ and $1\{ p_j > 0 \} g_{m}^{n} + 1\{ p_j \leq 0 \} g_{m}^{n}$ and since $|p_j(x)| < const < \infty$ the $||\cdot||_{L^2(f)}$-norm of the new brackets is given by $\delta \times 2const$. We keep the same notation and omit the constant “$2const$” to ease the notational burden, and from the previous calculations it is easy to see that

$$\{ g_{m}^{n}(Z) - E[g_{m}^{n}(Z)] \} \leq p_j(X) \{ \Delta \rho(Z, \alpha) - E[p_j(X) \Delta \rho(Z, \alpha)] \} \leq \{ g_{m}^{n}(Z) - E[g_{m}^{n}(Z)] \}. $$
Hence (up to some omitted constants)

\[ |m| \] (C.19)

and does not depend on \( \text{\it where the last inequality follows from assumption A.5(ii). Notice that the above RHS} \]

\[ T^\alpha : \text{Denote} \]

\[ \{ \text{so functions of the form } (g^n(Z) - E[g^n(Z)]), (g^n(Z) - E[g^n(Z)]) \}_{m=1, \ldots, N(w)} \text{ form } \| \cdot \|_{L^2(f_{zj})} \text{ norm brackets on } E_{\varphi/\alpha}. \text{ By construction, } N_{\|w, E_{\varphi/\alpha}, \|_2 \| L^2(f_{zj}) \leq N(w).} \]

Hence (up to some omitted constants)

\[ E_{P_{zj}} \left[ \sup_{\alpha \in N_{\alpha_n}} \left| l^{-1/2} \sum_{i=1}^{l} f_j(Z_i, \alpha) \right| \right] \]

\[ \leq (M_n \delta_{x,n})^\alpha \max_{j=1, \ldots, j_n} \left\{ \int_0^1 \sqrt{1 + \log N_{\alpha}^2 (w(M_n \delta_{x,n})^\alpha, O_{\alpha_n}, \| \cdot \|_{L^2(f_{zj})} )} dw \right\} \]

\[ \leq (M_n \delta_{x,n})^\alpha \sqrt{C_n}, \]

where the last inequality follows from assumption \( \text{\it A.3(ii). Notice that the above RHS does not depend on } \alpha \text{ nor on } j, \text{ so we obtain} \]

(C.17) \[ \max_{1 \leq j \leq j_n} \max_{1 \leq l \leq n} E_{P_{zj}} \left[ \sup_{\alpha \in N_{\alpha_n}} \left| l^{-1/2} \sum_{i=1}^{l} f_j(Z_i, \alpha) \right|^2 \right] \leq \text{const.} \times (M_n \delta_{x,n})^{2\alpha} C_n \]

and hence \( T_{1,n} \leq \text{const.} \times (M_n \delta_{x,n})^{2\alpha} C_n \).

Note that \( \max \left\{ (M_n \delta_{x,n})^2, (M_n \delta_{x,n})^{2\alpha} \right\} = (M_n \delta_{x,n})^{2\alpha} \) (by assumption) and that \( \tau_{\alpha_n}^{-1} \equiv \frac{\delta_n}{n} (M_n \delta_{x,n})^{2\alpha} C_n \), the desired result follows. \( Q.E.D. \)

**Proof of Lemma A.3**

Denote

\[ T_{n,l}^\beta \equiv \sup_{N_{\alpha_n}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right)^\prime \tilde{\Sigma}(X_i)^{-1} e_{l}^{\beta} (X_i, \alpha) - \frac{1}{n} \sum_{i=1}^{n} \left( \frac{dm(X_i, \alpha_0)}{d\alpha} [u_n^*] \right)^\prime \Sigma(X_i)^{-1} e_{l}^{\beta} (X_i, \alpha), \right\}, \]

and

\[ T_{n,l}^\beta \equiv \sup_{N_{\alpha_n}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( \frac{dm(X_i, \alpha_0)}{d\alpha} [u_n^*] \right)^\prime \Sigma(X_i)^{-1} e_{l}^{\beta} (X_i, \alpha) - \langle Z_n^\alpha + \langle u_n^*, \alpha - \alpha_0 \rangle \rangle \right\}. \]

It suffices to show that for all \( \delta > 0 \), there is \( N(\delta) \) such that for all \( n \geq N(\delta), \)

(C.18) \[ P_{Z_n} \left( P_{V^\infty} \left| Z_n \right| \geq \delta \right) < \delta \]

and

(C.19) \[ P_{Z_n} \left( P_{V^\infty} \left| Z_n \right| \geq \delta \right) < \delta. \]

We first verify equation (C.18). Note that

\[ \frac{T_{n,l}^\beta}{N_{\alpha_n}} \leq \sup_{N_{\alpha_n}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] - \frac{dm(X_i, \alpha_0)}{d\alpha} [u_n^*] \right)^\prime \tilde{\Sigma}(X_i)^{-1} e_{l}^{\beta} (X_i, \alpha) \right\} \]

\[ + \sup_{N_{\alpha_n}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( \frac{dm(X_i, \alpha_0)}{d\alpha} [u_n^*] \right)^\prime \left\{ \tilde{\Sigma}(X_i)^{-1} - \Sigma(X_i)^{-1} \right\} e_{l}^{\beta} (X_i, \alpha) \right\} \]

\[ \equiv T_{n,l}^{\beta_a} + T_{n,l}^{\beta_b}. \]
By Assumption 4.1(iii) and the Cauchy-Schwarz inequality, it follows that, for some $C \in (0, \infty)$,

$$P_{Z_n} \leq P_{Z_n} \left( \sqrt{nT_n^2} \geq \delta \mid Z_n \right) \geq \delta$$

$$\leq P_{Z_n} \left( \sup_{N_{\text{min}}} \left\{ \sum_{i=1}^{n} \left\| \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] - \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\}_e \right\} \geq C \delta \sqrt{n} \right) \geq \delta$$

$$+ P_{Z_n} \left( \lambda_{\text{min}}(\hat{X}(X)) < c \right).$$

The second term in the RHS vanishes eventually, so we focus on the first term. It follows

$$P_{Z_n} \leq P_{Z_n} \left( \sup_{N_{\text{min}}} \left\{ \sum_{i=1}^{n} \left\| \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] - \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\}_e \right\} \geq C \delta \sqrt{n} \right) \geq \delta$$

$$\leq P_{Z_n} \left( \sup_{N_{\text{min}}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left\| \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] - \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\}_e \right\}_e \geq C \delta \sqrt{n} \right) \geq \delta$$

$$+ P_{Z_n} \left( \sup_{N_{\text{min}}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left\| \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\}_e \right\}_e \geq C \delta \sqrt{n} \right) \geq \delta$$

By Lemma A.2(2) the second term on the RHS is less than 0.5δ eventually (with $(\tau_n^*)^{-1} = \text{const.}(M, \delta)^2$). Regarding the first term, note that

$$\sup_{N_{\text{min}}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left\| \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] - \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\}_e \right\}_e \geq \sqrt{M} \mid Z_n \right) \geq 0.5\delta$$

$$\leq \sup_{N_{\text{min}}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left\| \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] - \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\}_e \right\}_e \times \frac{n}{\tau_n}$$

$$+ \sup_{N_{\text{min}}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left\| \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\}_e \right\}_e \times \frac{n}{\tau_n}$$

by the LS projection property and the definition of $\tilde{m}$, as well as by the Markov inequality and Assumption A.6(i). Next, by the Markov inequality and Assumption A.7(ii), we have:

$$P_{Z_n} \left( \sup_{N_{\text{min}}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left\| \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] - \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\}_e \right\}_e \geq \frac{n}{\tau_n} \geq 0.5\delta \right)$$

$$\leq \frac{2}{\delta} \left( \sup_{N_{\text{min}}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left\| \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] - \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\}_e \right\}_e \times \frac{n}{\tau_n} \rightarrow 0.$$
Thus, we established that
\[
PZ \approx P_{\lambda \rightarrow Z}^2 \left( \sqrt{n}T_{\alpha,1}^B \geq \delta \mid Z^n \right) \geq \delta \quad \text{eventually.}
\]

By similar arguments, Assumptions 4.1(iii) and A.5(iv), Lemma A.2(2), and that \( \frac{1}{n} \sum_{i=1}^{n} \left\| \frac{dm(X_i, u_n)}{d\alpha} [u_n^*] \right\|_2 \) is bounded in probability, it can be shown that
\[
PZ \approx P_{\lambda \rightarrow Z}^2 \left( \sqrt{n}T_{\alpha,11}^B \geq \delta \mid Z^n \right) \geq \delta 
\quad \text{eventually.}
\]

Therefore, we establish equation (C.18).

For equation (C.19), let \( g(X, u_n^*) \equiv \left( \frac{dm(X, u_n)}{d\alpha} [u_n^*] \right)' \Sigma^{-1}(X) \). Then
\[
T_{n,11}^B \leq \sup_{N_n, \alpha_n} \left\{ \frac{1}{n} \sum_{i=1}^{n} g(X_i, u_n^*) \tilde{m}(X, \alpha) - \langle u_n^*, \alpha - \alpha_0 \rangle \right\} + \frac{1}{n} \sum_{i=1}^{n} g(X_i, u_n^*) \tilde{m}^B(X_i, \alpha_0) - Z_n^\omega
\]
\[
\equiv T_{n,11a} + T_{n,11b}.
\]

Thus to show equation (C.19) it suffices to show that \( \sqrt{n}T_{n,11a} = o_p(1) \) and that
\[
PZ \approx P_{\lambda \rightarrow Z}^2 \left( \sqrt{n}T_{n,11a} \geq \delta \mid Z^n \right) \geq \delta 
\quad \text{eventually.}
\]

First we consider the term \( T_{n,11a} \). This part of proof is similar to those in Ai and Chen (2003), Ai and Chen (2007) and Chen and Pouzo (2009) for their regular functional case, and hence we shall be brief. By the orthogonality properties of the LS projection and the definition of \( \tilde{m}(X, \alpha) \) and \( \tilde{g}(X, u_n^*) \), we have:
\[
n^{-1} \sum_{i=1}^{n} g(X_i, u_n^*) \tilde{m}(X_i, \alpha) = n^{-1} \sum_{i=1}^{n} \tilde{g}(X_i, u_n^*) m(X_i, \alpha).
\]

By Cauchy-Schwarz inequality,
\[
\sup_{N_n, \alpha_n} \left\{ \frac{1}{n} \sum_{i=1}^{n} \{ \tilde{g}(X_i, u_n^*) - g(X_i, u_n^*) \} \{ m(X_i, \alpha) - m(X_i, \alpha_0) \} \right\} \leq \left( \frac{1}{n} \sum_{i=1}^{n} \left\| \tilde{g}(X_i, u_n^*) - g(X_i, u_n^*) \right\|^2 \sup_{N_n, \alpha_n} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left\| m(X_i, \alpha) - m(X_i, \alpha_0) \right\|^2 \right\}.
\]

By assumption A.6(iii),
\[
\sqrt{n} \sup_{N_n, \alpha_n} \frac{1}{n} \sum_{i=1}^{n} \{ \| m(X_i, \alpha) - m(X_i, \alpha_0) \|^2 - EP_X [ \| m(X_1, \alpha) - m(X_1, \alpha_0) \|^2 ] \} = o_p(1).
\]

Thus, since \( \sup_{N_n, \alpha_n} EP_X [ \| m(X_1, \alpha) - m(X_1, \alpha_0) \|^2 ] = O(M_\alpha^2 \delta_n^2) \), it follows
\[
\sup_{N_n, \alpha_n} \frac{1}{n} \sum_{i=1}^{n} \| m(X_i, \alpha) - m(X_i, \alpha_0) \|^2 = O_P(1) \left( (M_\alpha^2 \delta_n^2) + o_P(n^{-1/2}) \right).
\]
This, Assumption A.6(ii) and \( \delta_n = o(n^{-1/4}) \) (by assumption A.5(iv)) imply that
\[
\sup_{N_{0,n}} \frac{1}{n} \sum_{i=1}^{n} \left| \bar{g}(X_i, u_n^*) - g(X_i, u_n^*) \right| \{m(X_i, \alpha) - m(X_i, \alpha_0)\}
\leq \alpha_{P_{\infty}} \left( \frac{1}{\sqrt{n}M_n \delta_n} \right) \times O_{P_{\infty}} \left( \sqrt{(M_n \delta_n)^2 + o(n^{-1/2})} \right) = o_{P_{\infty}} (n^{-1/2})
\]
Therefore,
\[
\sqrt{n}T_{nIIa} = \sqrt{n} \sup_{N_{0,n}} \frac{1}{n} \sum_{i=1}^{n} \left| g(X_i, u_n^*)m(X_i, \alpha) - \langle u_n^*, \alpha - \alpha_0 \rangle \right| + o_{P_{\infty}} (n^{-1/2}).
\]
By assumption A.6(iv), \( \sqrt{n} \sup_{N_{0,n}} \frac{1}{n} \sum_{i=1}^{n} \left| g(X_i, u_n^*)m(X_i, \alpha) - E_{P_{\infty}} \left[ g(X_i, u_n^*) \{m(X_i, \alpha) - m(X_i, \alpha_0)\} \right] \right| = o_{P_{\infty}} (1) \). Thus, by Assumption A.7(iv), we conclude that \( \sqrt{n}T_{nIIa} = o_{P_{\infty}} (1) \).

Next we consider the term \( T_{nIIb} \). By the orthogonality properties of the LS projection,
\[
n^{-1} \sum_{i=1}^{n} \left( g(X_i, u_n^*) - \bar{g}(X_i, u_n^*) \right) \hat{m}^{B}(X_i, \alpha_0) = n^{-1} \sum_{i=1}^{n} \left( g(X_i, u_n^*) - \bar{g}(X_i, u_n^*) \right) \rho^{B}(V_i, \alpha_0),
\]
where \( \rho^{B}(V_i, \alpha_0) \equiv \omega_{i,n} \rho(Z_i, \alpha_0) \) and \( \{\omega_{i,n}\}_{i=1}^{n} \) is independent of \( \{Z_i\}_{i=1}^{n} \).

Hence, by applying the Markov inequality twice, it follows that
\[
E_{P_{\infty}} \left[ n^{-1} \sum_{i=1}^{n} \left( g(X_i, u_n^*) - \bar{g}(X_i, u_n^*) \right) \rho^{B}(V_i, \alpha_0) \right] \leq \delta^{-4} E_{P_{\infty}} \left[ \left( n^{-1} \sum_{i=1}^{n} \left( g(X_i, u_n^*) - \bar{g}(X_i, u_n^*) \right) \rho^{B}(V_i, \alpha_0) \right)^2 \right].
\]
Regarding the cross-products terms where \( i \neq j \), note that
\[
E_{P_{\infty}} \left[ \left( g(X_j, u_n^*) - \bar{g}(X_j, u_n^*) \right) \left( g(X_i, u_n^*) - \bar{g}(X_i, u_n^*) \right) \rho^{B}(V_j, \alpha_0) \right] = E_{P_{\infty}} \left[ \left( g(X_j, u_n^*) - \bar{g}(X_j, u_n^*) \right) \left( g(X_i, u_n^*) - \bar{g}(X_i, u_n^*) \right) \right] E_{P_{\infty} | X^n} \left[ \rho^{B}(V_j, \alpha_0) \right] = 0,
\]
since \( E_{P_{\infty} | X^n} \left[ \rho(Z_i, \alpha_0) \rho(Z_j, \alpha_0) | X^n \right] = E_{P_{\infty} | X} [ \rho(Z_i, \alpha_0) | X] E_{P_{\infty} | X} [ \rho(Z_j, \alpha_0) | X] = 0 \) for \( i \neq j \). Thus, it suffices to study
\[
\delta^{-4} E_{P_{\infty}} \left[ n^{-1} \sum_{i=1}^{n} \left( g(X_i, u_n^*) - \bar{g}(X_i, u_n^*) \right)^2 \right] \leq \delta^{-4} \sum_{i=1}^{n} E_{P_{\infty}} \left[ \left( g(X_i, u_n^*) - \bar{g}(X_i, u_n^*) \right)^2 \right] E_{P_{\infty} | X^n} \left[ \left( \omega_i \rho(Z_i, \alpha_0) \right)^2 \right].
\]
By the original-sample \( \{Z_i\}_{i=1}^{n} \) being i.i.d., \( \{\omega_{i,n}\}_{i=1}^{n} \) being independent of \( \{Z_i\}_{i=1}^{n} \), Assumption A.1(iv) and the fact that \( \sigma^2_\omega < \infty \), we can majorize the previous expression (up to an omitted constant) by
\[
\delta^{-4} E_{P_{\infty}} \left[ \left( g(X_i, u_n^*) - \bar{g}(X_i, u_n^*) \right)^2 \right] = O(1),
\]
where the last equality is due to Assumption A.6(ii). Hence we established equation
Proof of Lemma A.5

By Markov inequality and the LS projection properties, the LHS of the previous equation can be bounded above by

\[ P_{\tilde{Z}} \left( \sup_{N_{\alpha n}} \left\| n^{-1} \sum_{i=1}^{n} \frac{d^2 \tilde{m}(X_i, \alpha)}{d\alpha^2}[u_n^*, u_n^*] \right\|_{\ell}^2 \geq \frac{\delta}{M_n \delta_n} \right) < \delta. \]

By Lemma A.2, it suffices to show that

\[ P_{\tilde{Z}} \left( \sup_{N_{\alpha n}} \left\| n^{-1} \sum_{i=1}^{n} \frac{d^2 \tilde{m}(X_i, \alpha)}{d\alpha^2}[u_n^*, u_n^*] \right\|_{\ell}^2 \geq \frac{\delta}{M_n \delta_n} \right) < \delta. \]

By Markov inequality and the LS projection properties, the LHS of the previous equation can be bounded above by

\[ \frac{M^2 \delta^2}{\delta} E_{\tilde{Z}} \left[ \sup_{N_{\alpha n}} \left\| n^{-1} \sum_{i=1}^{n} \frac{d^2 \tilde{m}(X_i, \alpha)}{d\alpha^2}[u_n^*, u_n^*] \right\|_{\ell}^2 \right] \leq \frac{M^2 \delta^2}{\delta} E_{\tilde{Z}} \left[ \sup_{N_{\alpha n}} \left\| \frac{d^2 m(X, \alpha)}{d\alpha^2}[u_n^*, u_n^*] \right\|_{\ell}^2 \right] < \delta \]

eventually, which is satisfied given Assumption A.7(iii). The desired result follows. Q.E.D.

Proof of Lemma A.4

By the Cauchy-Schwarz inequality and Assumption 4.1(iii), it suffices to show that

\[ P_{\tilde{Z}} \left( \sup_{N_{\alpha n}} \left\| n^{-1} \sum_{i=1}^{n} \frac{d^2 \tilde{m}(X_i, \alpha)}{d\alpha^2}[u_n^*, u_n^*] \right\|_{\ell}^2 \geq \frac{\delta}{M_n \delta_n} \right) < \delta. \]

Result (1), we first want to show that

\[ \sup_{N_{\alpha n}} \left\| \frac{d^2 \tilde{m}(X, \alpha)}{d\alpha^2}[u_n^*, u_n^*] \right\|_{\ell}^2 \leq T_{n,l} + T_{n,II} + T_{n,III} = o_{P_{\tilde{Z}}} (1) \]

where

\[ T_{n,l} = \sup_{N_{\alpha n}} \left\| n^{-1} \sum_{i=1}^{n} \left( \frac{d\tilde{m}(X_i, \alpha)}{d\alpha}[u_n^*] \right)^2 \right\|_{\ell_1} - \left\| \frac{d\tilde{m}(X, \alpha_0)}{d\alpha}[u_n^*] \right\|_{\ell_1} \leq T_{n,l} + T_{n,II} + T_{n,III} = o_{P_{\tilde{Z}}} (1) \]

\[ T_{n,II} = \left\| n^{-1} \sum_{i=1}^{n} \left( \frac{d\tilde{m}(X_i, \alpha)}{d\alpha}[u_n^*] \right)^2 \right\|_{\ell_1} - \left\| \frac{d\tilde{m}(X, \alpha_0)}{d\alpha}[u_n^*] \right\|_{\ell_1} \]

\[ T_{n,III} = \left\| n^{-1} \sum_{i=1}^{n} \left( \frac{d\tilde{m}(X, \alpha_0)}{d\alpha}[u_n^*] \right)^2 \right\|_{\ell_1} - \left\| \frac{d\tilde{m}(X, \alpha_0)}{d\alpha}[u_n^*] \right\|_{\ell_1} \]

Therefore, to prove equation (C.21), it suffices to show that

\[ T_{n,j} = o_{P_{\tilde{Z}}} (1) \text{ for } j \in \{I, II, III\}. \]

Note that for \(|a|_{L^2(P_n)} \) with \(P_n\) being the empirical measure, \(|a|_{L^2(P_n)} - |b|_{L^2(P_n)}| \leq |a - b|_{L^2(P_n)} + 2|b, a - b|_{L^2(P_n)}| \).

Now, let \(a \equiv \frac{d\tilde{m}(X, \alpha)}{d\alpha}[u_n^*] \) and \(b \equiv \frac{d\tilde{m}(X, \alpha_0)}{d\alpha}[u_n^*] \). In
order to show $T_{n,l} = o_{P_2}(1)$, under Assumption 4.1(iii), it suffices to show
\[
\sqrt{n^{-1} \sum_{i=1}^n \left\| \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\|_e^2} \leq \sqrt{n^{-1} \sum_{i=1}^n \left\| \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n] - \frac{\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\|_e^2} = o_{P_2}(1).
\]
By the property of LS projection, we have:
\[
n^{-1} \sum_{i=1}^n \left\| \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\|_e^2 \leq n^{-1} \sum_{i=1}^n \left\| \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] \right\|_e^2 = O_{P_2}(1)
\]
due to iid data, Markov inequality, the definition of $E_{P_2}$ and Assumption 3.1(iv). Next, by the property of LS projection, we have:
\[
\sup \mathcal{N}_{n,m} \left\{ \frac{1}{n} \sum_{i=1}^n \left\| \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] - \frac{\tilde{m}(X_i, \alpha)}{d\alpha} [u_n^*] \right\|_e^2 \right\} \leq \sup \mathcal{N}_{n,m} \left\{ \frac{1}{n} \sum_{i=1}^n \left\| \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] - \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] \right\|_e^2 \right\} = o_{P_2}(1)
\]
due to iid data, Markov inequality and Assumption 3.1(iv). Thus we established $T_{n,l} = o_{P_2}(1)$.

By similar algebra as before, in order to show $T_{n,l} = o_{P_2}(1)$, given Assumption 4.1(iii), it suffices to show
\[
\sqrt{n^{-1} \sum_{i=1}^n \left\| \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] \right\|_e^2} \leq \sqrt{n^{-1} \sum_{i=1}^n \left\| \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n] - \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] \right\|_e^2} = o_{P_2}(1).
\]
The term $n^{-1} \sum_{i=1}^n \left\| \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] \right\|_e^2 = O_{P_2}(1)$ is due to iid data, Markov inequality, the definition of $E_{P_2}$ and Assumption 3.1(iv). The term $n^{-1} \sum_{i=1}^n \left\| \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n] - \frac{dm(X_i, \alpha)}{d\alpha} [u_n^*] \right\|_e^2 \leq o_{P_2}(1)$ is due to iid data, Markov inequality and Assumption 3.1(iv). Thus $T_{n,l} = o_{P_2}(1)$.

Finally, $T_{n,l,l} = o_{P_2}(1)$ follows from the fact that $n^{-1} \sum_{i=1}^n \left\| \frac{dm(X_i, \alpha)}{d\alpha} [u_n] \right\|_e^2 = O_{P_2}(1)$ and Assumption 4.1(iii). We thus established equation (C.21). Since
\[
E_{P_2} \left[ n^{-1} \sum_{i=1}^n \left\| \frac{d\tilde{m}(X_i, \alpha)}{d\alpha} [u_n] \right\|_{\Sigma^{-1}}^2 \right] = E_{P_X} \left[ g(X, u_n^*) \Sigma(X) g(X, u_n^*) \right] \leq C < \infty,
\]
we obtain Result (1).

Result (2) immediately follows from equation (C.21) and Assumption [1] Q.E.D.
APPENDIX D: SIEVE SCORE STATISTIC AND SCORE BOOTSTRAP

In the main text we present the sieve Wald, SQLR statistics and their bootstrap versions. Here we consider sieve score (or LM) statistic and its bootstrap version. Both the sieve score test and score bootstrap only require to compute the original-sample restricted PSMD estimator of $\alpha_0$, and hence are computationally attractive.

Recall that $\hat{\alpha}_n^R$ is the original-sample restricted PSMD estimator \[10]. Let $\tilde{\nu}_n^R$ be computed in the same way as $\tilde{\nu}_n^\alpha$ in Subsection \[2] except that we use $\hat{\alpha}_n^R$ instead of $\hat{\alpha}_n$. And

$$||\tilde{\nu}_n^R||_{n, sd}^2 = -\sum_{i=1}^n \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}_n^R] \right)' \tilde{\Sigma}_i^{-1} \rho(Z_i, \hat{\alpha}_n^R) \rho(Z_i, \hat{\alpha}_n^R) \tilde{\Sigma}_i^{-1} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}_n^R] \right),$$

Denote

$$\tilde{S}_n \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}_n^R] / ||\tilde{\nu}_n^R||_{n, sd} \right)' \tilde{\Sigma}_i^{-1} \tilde{m}(X_i, \hat{\alpha}_n^R),$$

$$\tilde{S}_{1n} \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}_n^R] / ||\tilde{\nu}_n^R||_{n, sd} \right)' \tilde{\Sigma}_i^{-1} \rho(Z_i, \hat{\alpha}_n^R),$$

and

$$\tilde{S}_B^B \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}_n^R] / ||\tilde{\nu}_n^R||_{n, sd} \right)' \tilde{\Sigma}_i^{-1} \{ \tilde{m}^B(X_i, \hat{\alpha}_n^R) - \hat{m}(X_i, \hat{\alpha}_n^R) \},$$

$$\tilde{S}_{1B}^B \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}_n^R] / ||\tilde{\nu}_n^R||_{n, sd} \right)' \tilde{\Sigma}_i^{-1} \{ (\omega_{i,n} - 1) \rho(Z_i, \hat{\alpha}_n^R) \}.$$ 

Then

$$\text{Var} \left( \tilde{S}_{1n}^B | Z^n \right) = \frac{\sigma^2 \sum_{i=1}^n \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}_n^R] \right)' \tilde{\Sigma}_i^{-1} \rho(Z_i, \hat{\alpha}_n^R) \rho(Z_i, \hat{\alpha}_n^R) \tilde{\Sigma}_i^{-1} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}_n^R] \right)}{n ||\tilde{\nu}_n^R||_{n, sd}^2} = \sigma^2.$$

which coincides with that of $\tilde{S}_{1n}$ (once adjusted by $\sigma^2$).

Following the results in Subsection \[2], one can compute $\tilde{\nu}_n^R$ in closed form, $\tilde{\nu}_n^R = \tilde{\nu}^{k(n)}(\cdot) \tilde{D}_n \tilde{F}_n$ where

$$\tilde{F}_n = \frac{d\phi(\hat{\alpha}_n^R)}{dx} [\tilde{\nu}^{k(n)}(\cdot)], \quad \tilde{D}_n = n^{-1} \sum_{i=1}^n \left[ \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}^{k(n)}(\cdot)'] \right]' \tilde{\Sigma}_i^{-1} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}^{k(n)}(\cdot)'] \right),$$

And $||\tilde{\nu}_n^R||_{n, sd}^2 = \tilde{F}'_n \tilde{D}_n \tilde{D}_n \tilde{D}_n \tilde{F}_n$ with

$$\tilde{S}_n = \frac{1}{n} \sum_{i=1}^n \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}^{k(n)}(\cdot)'] \right)' \tilde{\Sigma}_i^{-1} \rho(Z_i, \hat{\alpha}_n^R) \rho(Z_i, \hat{\alpha}_n^R) \tilde{\Sigma}_i^{-1} \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}^{k(n)}(\cdot)'] \right).$$

Therefore, the bootstrap sieve score statistic $\tilde{S}_{1n}^B$ can be expressed as

$$\tilde{S}_{1n}^B = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}_n^R] / ||\tilde{\nu}_n^R||_{n, sd} \right)' \tilde{\Sigma}_i^{-1} (\omega_{i,n} - 1) \rho(Z_i, \hat{\alpha}_n^R).$$

$$= \left( \tilde{F}'_n \tilde{D}_n \tilde{D}_n \tilde{D}_n \tilde{F}_n \right)^{-1/2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{d\hat{m}(X_i, \hat{\alpha}_n^R)}{dx} [\tilde{\nu}^{k(n)}(\cdot)'] \right)' \tilde{\Sigma}_i^{-1} (\omega_{i,n} - 1) \rho(Z_i, \hat{\alpha}_n^R).$$
For the case of IID weights, this expression is similar to that proposed in Kline and San- 

Chen (2003). Assumption D.1(ii) can be established by controlling the entropy, as in 

VdV-W Chapter 2.11 and (2003). However, Assumption D.1(ii) can also be obtained by weaker conditions, yielding a (Υ(ζ)2)2 that is slower than O(n−1/2) provided that Υ(ζm)Mnζm = o(n−1/2). In the proof we show that |ζmR − u∗m| = oP(1); faster rates of convergence will relax the conditions needed to show part (ii).

**Theorem D.1** Let ζmR be the restricted PSMD estimator (1.10), and conditions for 

Lemma 3.3 and Proposition 1.1 hold. Let Assumptions 1.3.4(a) hold. Then, under the null hypothesis of φ(μ0) = φ0,
4.2, except that we use \( \hat{\alpha} \) and 

\[ \phi_{\text{Boot.3(i)}} \]

hypothesis in two steps. This and Assumption 3.1(iv) imply that 

\[ \phi_{\text{Boot.3(ii)}} \]

hold. Also, by Proposition B.1 we have \( \hat{\alpha} \):

Proof of Theorem D.1 We first note that by Lemma 5.1 Assumptions 3.6(i) and 

\[ \phi_{\text{Boot.3(i)}} \]

hold. Also, by Proposition 3.3 we have \( \hat{\alpha}_R \) under the null hypothesis of \( \phi(\alpha_0) = \phi_0 \). Under the null hypothesis, and Assumption 3.5 we also have

(see Step 1 in the proof of Theorem 4.3):

\[ \sqrt{n}(u_n^*, \hat{\alpha}_R - \alpha_0) = o_{P_2}(1). \]

For Result (1), we show that \( \hat{S}_n \) is asymptotically standard normal under the null hypothesis in two steps.

**Step 1.** We first show that 

\[ \frac{\|\tilde{v}_n^* - v_n^*\|}{\|v_n^*\|} = O_{P_2}(1), \quad \frac{\|\tilde{v}_n^* - v_n^*\|}{\|v_n^*\|} = O_{P_2}(1), \quad \sup_{v_n \in \mathbf{V}_n} \frac{\langle u_n^* - \tilde{v}_n^*, v_n \rangle}{\|v_n\|} = o_{P_2}(1). \]

This and Assumption 3.1(iv) imply that 

\[ \frac{\|\tilde{v}_n^* - v_n^*\|}{\|v_n^*\|} = O_{P_2}(1) \]

and 

\[ \frac{\|\tilde{v}_n^* - v_n^*\|}{\|v_n^*\|} = O_{P_2}(1). \]

Thus 

\[ \frac{\|\tilde{v}_n^* - v_n^*\|}{\|v_n^*\|} = O_{P_2}(1). \]

**Step 2.** We show that under the null hypothesis,

\[ \hat{S}_n = \sqrt{n}Z_n + o_{P_2}(1) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{d \Phi(X_i, \alpha)}{d \alpha} u_n^* \right) \Sigma(X_i)^{-1} \rho(Z_i, \alpha_0) + o_{P_2}(1). \]
By Step 1, it suffices to show that under the null hypothesis,
\[
\mathcal{S}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{d\hat{m}(X_i, \alpha_n^R)}{d\alpha} \hat{u}_n^R \right)' \hat{\Sigma}^{-1}(X_i) \hat{m}(X_i, \alpha_n^R) = \sqrt{n} \mathcal{Z}_n + o_{P, \infty}(1).
\]
Recall that \( \ell_n(x, \alpha) \equiv \hat{m}(x, \alpha_0) + \hat{m}(x, \alpha) \). We have:
\[
\mathcal{S}_n - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{d\hat{m}(X_i, \alpha_n^R)}{d\alpha} \hat{u}_n^R \right)' \hat{\Sigma}^{-1}(X_i) \ell_n(X_i, \alpha_n^R)
\leq \sqrt{n} \left( \sum_{i=1}^{n} \left\| \hat{\Sigma}^{-1/2}(X_i) \frac{d\hat{m}(X_i, \alpha_n^R)}{d\alpha} \hat{u}_n^R \right\|^2 \right)^{1/2} \left( \sum_{i=1}^{n} \left\| \ell_n(X_i, \alpha_n^R) \right\|^2 \right)^{1/2}.
\]
By Lemma A.2(1) and the assumption that \( n\sigma_n^2(M_n \delta_x, n)^{2\kappa} C_n = o(1) \), we have:
\[
\sqrt{n} \sum_{i=1}^{n} \left\| \ell_n(X_i, \alpha_n^R) \right\|^2 \leq \sigma_n^2 (n\delta_x, n)^{2\kappa} C_n = o(1),
\]
which implies:
\[
\mathcal{S}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{d\hat{m}(X_i, \alpha_n^R)}{d\alpha} \hat{u}_n^R \right)' \hat{\Sigma}^{-1}(X_i) \ell_n(X_i, \alpha_n^R) + o_{P, \infty}(1).
\]
Assumption D.1(iii) implies that
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{d\hat{m}(X_i, \alpha_n^R)}{d\alpha} \hat{u}_n^R \right)' \hat{\Sigma}(X_i)^{-1} \ell_n(X_i, \alpha_n^R)
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{d\hat{m}(X_i, \alpha_n^R)}{d\alpha} \hat{u}_n^R \right) \hat{\Sigma}(X_i)^{-1} \ell_n(X_i, \alpha_n^R) + o_{P, \infty}(1),
\]
where the second equality is due to \( \left\| \hat{u}_n^R - u_n^* \right\| = o_{P, \infty}(\zeta_n) \) (Assumption D.1(ii)) and Assumption D.1(iii).

Since \( \alpha_n^R \in \mathcal{N}_{\text{runo}} \), under the null hypothesis, \( \sqrt{n}(u_n^*, \hat{\alpha}_n^R - \alpha_0) = o_{P, \infty}(1) \), and by analogous calculations to those in the proof of Lemma A.3, we obtain:
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{d\hat{m}(X_i, \alpha_n^R)}{d\alpha} \hat{u}_n^R \right)' \hat{\Sigma}(X_i)^{-1} \ell_n(X_i, \alpha_n^R) = \sqrt{n} \mathcal{Z}_n + o_{P, \infty}(1),
\]
and hence equation (D.1) holds. By Assumption 3.6(ii) we have: \( \hat{S}_n \Rightarrow N(0, 1) \) under the null hypothesis.

For Result (2), we now show that \( \hat{S}^B_n \) also converges weakly (in the sense of Bootstrap Section 5) to a standard normal under the null hypothesis. It suffices to show that

\[
(D.2) \quad \hat{S}^B_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\omega_i - 1) \left( \frac{dm(X_i, \hat{\alpha}_R)}{d\alpha} [u^*_n] \right)' \Sigma(X_i)^{-1} \rho(Z_i, \alpha_0) + o_{P_{\upsilon|Z=\infty}}(1) \text{ wpa1}(P_{Z=\infty}).
\]

Note that \( \ell^B(X_i, \hat{\alpha}_R) - \ell_n(X_i, \hat{\alpha}_n) = \hat{m}^B(X_i, \alpha_0) - \hat{m}(X_i, \alpha_0) \), and that \( n^{-1} \sum_{i=1}^{n} ||\hat{m}^B(X_i, \alpha_0) - \hat{m}(X_i, \alpha_0)||^2 = O_{P_{\upsilon|Z=\infty}}(J_n/n) \text{ wpa1}(P_{Z=\infty}) \) (see the proof of Lemma A.2). We have, by calculations similar to Step 2,

\[
\left| \hat{S}^B_n - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{dm(X_i, \hat{\alpha}_R)}{d\alpha} [u^*_n] \right)' \Sigma(X_i)^{-1} \{ \ell^B_n(X_i, \hat{\alpha}_R) - \ell_n(X_i, \hat{\alpha}_R) \} \right| = o_{P_{\upsilon|Z=\infty}}(1) \text{ wpa1}(P_{Z=\infty}).
\]

By analogous calculations to those in the proof of Lemma A.3 we obtain equation (D.2). This and Result (1) and Assumption Boot.3(ii) now imply that under the null and conditional on the data, \( \sigma^{-1} \hat{S}_n \) is also asymptotically standard normally distributed. The last part of Result (2) can be established in the same way as that of Theorem 5.2(1), and is omitted. Q.E.D.