NOTES ON COMPUTATIONAL COMPLEXITY
OF GE INEQUALITIES

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Abstract

Recently, Cheryche, et al. (2011) proved the important negative result that deciding the **strong** feasibility of the Marshallian equilibrium inequalities, introduced by Brown and Matzkin (1996), is NP-complete. Here, I show that the **weak** feasibility of the equivalent Hicksian equilibrium inequalities, introduced by Brown and Shannon (2000), can be decided in oracle-polynomial time.

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Introduction

We consider a pure exchange economy with N price-taking consumers, where in each of S periods we observe the market prices, the income distribution and the aggregate endowments of L goods and services. Consumer demands are unobservable.

In these notes, we propose an **oracle-polynomial time** algorithm for deciding the **weak feasibility** of the general equilibrium inequalities introduced by Brown and Shannon (2000). This system of multivariate polynomial inequalities is feasible iff there exists an **indirect utility function** for each consumer and utility maximizing demands, subject to her budget constraints, that define a competitive equilibrium in each observation. That is, in each observation, the endogenous aggregate demand of goods and services of consumers at the observed market prices and income distribution equals the exogenous aggregate supply of goods and services.

These notes are intended as a complement to Brown and Shannon.

Refutable Theories of Value

(1) Rationalization

(2) Weak Axiom of Revealed Preference (*WARP*)

(3) Afriat's Theorem

(4) Tarski–Seidenberg Theorem

Rationalization

The non-satiated utility function $U(x)$ rationalizes the market data

$$D \equiv \{x_j, p_j\}_{j=1}^{j=S}, \text{ where } x_j, p_j \in R_{++}^L$$

if for all j and for all $x \in R_{++}^L$:

$$p_j \cdot x \leq p_j \cdot x_j \Rightarrow U(x_j) \geq U(x)$$

Weak Axiom of Revealed Preference (WARP)

(1)

$$p_1 \cdot x_2 \leq p_1 \cdot x_1 \Rightarrow p_2 \cdot x_1 > p_2 \cdot x_2$$

(2) *WARP* is necessary but not sufficient for rationalizing D.

(3) *GARP* is a combinatorial restriction on the market data D that is necessary and sufficient for rationalizing D.

(4) The Generalized Axiom of Revealed Preference *GARP*, due to Varian (1982), is a generalization of the Strong Axiom of Revealed Preference (*SARP*).

Afriat's Theorem

The following conditions are equivalent:

(a) D is rationalized by a non-satiated utility function u.

(b) The "Afriat Inequalities":

$$u_i \leq u_j + \lambda_j p_j \cdot (x_i - x_j)$$

for $i, j = 1, ..., S$ are solvable.
The unknowns are the utility levels u_j and marginal utilities of income λ_j.
The parameters are the market data p_j and x_j.

(c) D satisfies $GARP$.

(d) D is rationalized by a non-satiated, **concave** utility function U.

Tarski–Seidenberg Theorem

(1) Semi-algebraic sets are solutions of a finite family of multivariate polynomial inequalities.

(2) The projection of a semi-algebraic set is a semi-algebraic set — see the proof in Steinhorn (2008).

(3) Example:

$$ax^2 + bx + c = 0$$

has a real solution iff

$$b^2 - 4ac \geq 0.$$

(4) Example: In Afriat's Theorem

(b) iff (c).

The General Equilibrium (GE) Manifold

(1) If 0 is a regular value of the smooth market excess demand function

$$F(p, \omega_1, \omega_2, .., \omega_N),$$

then M is a smooth manifold, where

$$M \equiv \{(p, \omega_1, \omega_2, .., \omega_N) : F(p, \omega_1, \omega_2, .., \omega_N) = 0\}$$

(2) 0 is a regular value of the smooth market excess demand function for almost all

$$(\omega_1, \omega_2, .., \omega_N)$$

(3) If $D \subseteq M$, then the GE manifold M rationalizes the market data D, where

$$D = \{(p_r, \omega_{r,1}, \omega_{r,2}, .., \omega_{r,N})_{r=1}^{r=S}\}$$

The Marshallian GE Inequalities

(1) The Afriat inequalities for each consumer, where individual demands are not observed, the budget constraints for each consumer in each observation and the market clearing conditions in each observation.

(2) Brown and Matzkin (1996) prove that a finite family of observations of aggregate endowments, market prices and the income distribution can be rationalized with a GE manifold, iff the Marshallian GE inequalities are feasible.

(3) Recently, Cherchye et al. (2011) proved the important negative result that deciding the **strong** feasibility of the Marshallian GE inequalities is a NP complete problem. That is, if $P \neq NP$ then there is no "efficient" (polynomial time) method for deciding the **strong** feasibility of the Marshallian GE inequalities.

A Local Characterization of the GE Manifold

(1) Ekeland and Chiappori (1999) proposed a local characterization of the GE manifold, using the exterior calculus and the consumer's smooth, convex indirect utility function $V(p, I)$.

(2) If p are the market prices and I is the consumer's income, then they express the consumer's Marshallian demand as:

$$\frac{I[\nabla_p V(p, I)]}{[p \cdot \nabla_p V(p, I)]}$$

The Dual Afriat Inequalities

Theorem

Let $(p^r, x^r), r = 1, \ldots, S$ be given and let $I^r = p^r \cdot x^r$ for each r. There exists a utility function rationalizing this data that is strictly quasiconcave and monotone if and only if there exist numbers V^i, λ^i, and vectors $q^i \in R^L$, $i = 1, \ldots, S$ such that:
(a) for $i \neq j$,

$$V^i - V^j \geq q^j \cdot \left(\frac{p^i}{I^i} - \frac{p^j}{I^j} \right) \text{ for } i, j = 1, \ldots, S.$$

(b) $\lambda^j > 0$, $q^j \ll 0$, $j = 1, \ldots, S$.

(c) $\frac{q^j}{I^j} = -\lambda^j x^j$, $j = 1, \ldots, S$.

Remark: Solutions of (a) define a convex indirect utility function $V(\frac{P}{I})$.

The Dual Afriat Inequalities (continued)

Conditions (a) and (b) constitute the "dual Afriat inequalities." Condition (c) is an expression of Roy's identity in this context. To see this, note that if (c) holds for some $\lambda^j > 0$, then $\frac{p^j \cdot q^j}{I^j} = -\lambda^j(p^j \cdot x^j) = -\lambda^j I^j$, i.e., $\lambda^j = -\frac{p^j \cdot q^j}{(I^j)^2}$, which implies that the vector $(\frac{q^j}{I^j}, \lambda^j)$ corresponds to the gradient of the rationalizing indirect utility function V evaluated at (p^j, I^j). This is essentially the content of (a). More precisely, (a) says that q^j is the derivative of V with respect to the income normalized price vector $\frac{p}{I}$ evaluated at (p^j, I^j). Thus $\frac{\partial V}{\partial p}(p^j, I^j) = \frac{q^j}{I^j}$ and $\frac{\partial V}{\partial I}(p^j, I^j) = -\frac{p^j \cdot q^j}{(I^j)^2}$. If $\frac{q^j}{I^j} = -\lambda^j x^j$, then x^j is the demand at the price-income pair (p^j, I^j) by Roy's identity.

A Global Characterization of the GE Manifold

(1) Brown and Shannon (2000) proposed a global characterization of the GE manifold, using the theory of revealed preference and the consumer's smooth, convex indirect utility function, $V(\frac{p}{I})$.

(2) That is, they introduced the Hicksian GE inequalities,consisting of the first order conditions for minimizing a smooth convex indirect utility function, $V(\frac{p}{I})$, subject to a budget constraint, i.e., the dual Afriat inequalities for each consumer, the budget constraints for each consumer and the market clearing conditions in each observation.

(3) Brown and Shannon proved that a finite family of observations of aggregate endowments, market prices and the income distribution can be rationalized with a GE manifold iff the Hicksian GE inequalities are feasible.

Complexity of GE Inequalities

(1) Brown and Shannon (2000) proved that the Hicksian GE inequalities are feasible iff the Marshallian GE inequalities are feasible.

(2) The Cherchye et al. result suggests that the Hicksian equilibrium inequalities are also *NP*-complete. That is, there does not exist a polynomial-time algorithm for deciding the **strong** feasibility of the Hicksian GE inequalities.

(3) In fact, we show there exits an oracle-polynomial time algorithm for deciding the **weak** feasibility of the Hicksian GE inequalities. That is, there exists an oracle-polynomial time algorithm for deciding if the aggregate endowment in each observation is "ε-near" the aggregate demand.

(4) Hence there exists an oracle-polynomial time algorithm for deciding the **weak** feasibility of the Marshallian GE inequalities.

Membership Oracles

(1) A convex body is a compact convex set with nonempty interior. A strong membership oracle for a convex body K asserts for any rational $y \in R^L$ that $y \in K$ or $y \notin K$. See 2.1.5 in in Grotschel, Lovasz and Schrijver $[GLS]$.

(2) A convex body K is centered if there explicitly exists $\alpha_0 \in K$ and $r \in R$ such that $B_r(\alpha_0) \subset K$.

(3) A weak membership oracle for a centered, convex body K asserts for any positive rational δ and any rational $y \in R^L$ that y is "ε-near" K or $y \notin K$ — see Lemma 4.3.3 in $[GLS]$.

(4) There exists an oracle-polynomial time algorithm for weak membership in $K_1 + K_2$, given oracle-polynomial time algorithms for weak membership in the centered convex bodies K_1 and K_2. See section 4.7 in $[GLS]$.

The Perspective Map

(1) The perspective map

$$P : R^K \times R_{++} \to R^K,$$

where

$$P(x, x_{K+1}) \equiv \frac{x}{x_{K+1}}.$$

(2) If

$$(x, x_{K+1}), (y, y_{K+1}) \in R^{K+1},$$

then the perspective image of the interval.

$$[(x, x_{K+1}), (y, y_{K+1})] \subset R^{K+1}$$

is the interval

$$[P(x, x_{K+1}), P(y, y_{K+1})] \subset R^K.$$

(3) Hence, the image and pre-image of a convex set is a convex set under the perspective map. See section 2.3.3 in Boyd and Vandenberghe (2004) for proof.

Perspective Image of the Dual Afriat Inequalities

We define a weak membership oracle for the centered convex body derived from the perspective image of solutions of the dual Afriat inequalities, where the utility level in each observation is mapped into the marginal utility of income in that observation. That is,

$$(q_{t,j}, V_{t,j}) \to \left(q_{t,j}, \frac{p_j}{I_{t,j}} \cdot q_{t,j}\right).$$

Hence

$$P(q_{t,j}, \frac{p_j}{I_{t,j}} \cdot q_{t,j}) = \frac{I_{t,j} q_{t,j}}{p_j \cdot q_{t,j}}.$$

Centered Convex Bodies of Demand

(1) A sequence of S non-negative vectors in R^L_+, bounded by the sequence of S aggregate endowments, are the images of the perspective map iff they satisfy the linear Afriat inequalities for the observed market prices and income distribution.

(2) Solutions of the strict Afriat inequalities constitute the interior of the convex body of demands defined as the intersection of the **closure** of the perspective image of the convex set of marginal indirect utilities and marginal utilities of income and the convex interval in the positive orthant defined by the origin and the aggregate endowments.

(3) Since any strict smooth concave utility function satisfies the strict Afriat inequalities, we can center each convex body of demands.

A Weak Feasibility Oracle for the GE Inequalities

(1) It follows from a theorem of Yudin and Nemirovskii (1976) — see section 4.3 in [GLS] — that in each observation there exists an oracle-polynomial time algorithm for weak membership of the aggregate endowment in the sum of the centered convex bodies of demand, derived from solutions of the dual Afriat inequalities.

(2) That is, there exists an oracle-polynomial time algorithm for the weak feasibility of the Hicksian (Marshallian) GE inequalities.

Strong and Weak Feasibility of the GE Inequalities

(1) The apparent contradiction between the negative result of Cherchye et al. on *NP*-completeness of the Marshallian GE inequalities and our positive result on deciding feasibility of the Hicksian GE inequalities in oracle-polynomial time derives from two different notions of feasibility.

(2) We use the notion of **weak** feasibility, common in convex optimization and Cherchye et al. use the notion of **strong** feasibility, common in combinatorial optimization.

(3) These notions of feasiblity are equivalent for rational polyhedra but not for convex bodies in general. See Chapter 6 in [*GLS*].

The Weak Constrained Convex Minimization Problem

(1) This is Theorem 4.3.13 in [*GLS*].

(2) There exists an oracle-polynomial time algorithm that solves the following problem.

(3) Given a rational number $\varepsilon > 0$, find a vector y "ε-near" the centered, convex body K given by a weak membership oracle such that

$$f(y) \leq f(x) + \varepsilon$$

for all x "ε-deep" in K, where f is a convex function given by an oracle that for all x and $\delta > 0$ returns a rational number t such that $\|f(x) - t\| \leq \delta$.

(4) We compute t, using Dirichlet's Theorem (Theorem 5.1.3 in [*GLS*]) or continued fractions in section 5.1 in [*GLS*].

Market Clearing and Infeasibility of GE Inequalities

(1) Infeasibility of the aggregate endowment in the Hicksian GE inequalities is a measure of the lack of market clearing. We propose the minimization of a convex measure of the lack of market clearing over each consumer's family of piece-wise linear indirect utility functions.

(2) We minimize the maximum Euclidean distance between the aggregate endowment and the aggregate demands in the sum of the centered demand-convex bodies. It follows from Theorem 4.3.13 in $[GLS]$ that this optimization problem can be solved in oracle-polynomial time. The optimal value of this problem is a measure of the lack of market clearing.

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