ROBUSTNESS OF BOOTSTRAP IN INSTRUMENTAL VARIABLE REGRESSION

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Robustness of bootstrap in instrumental variable regression*

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Abstract

This paper studies robustness of bootstrap inference methods for instrumental variable regression models. In particular, we compare the uniform weight and implied probability bootstrap approximations for parameter hypothesis test statistics by applying the breakdown point theory, which focuses on behaviors of the bootstrap quantiles when outliers take arbitrarily large values. The implied probabilities are derived from an information theoretic projection from the empirical distribution to a set of distributions satisfying orthogonality conditions for instruments. Our breakdown point analysis considers separately the effects of outliers in dependent variables, endogenous regressors, and instruments, and clarifies the situations where the implied probability bootstrap can be more robust than the uniform weight bootstrap against outliers. Effects of tail trimming introduced by Hill and Renault (2010) are also analyzed. Several simulation studies illustrate our theoretical findings.

1 Introduction

Instrumental variable (IV) regression is one of the most widely used methods in empirical economic analysis. By introducing instruments which are orthogonal to a structural error component in a regression model and utilizing this orthogonality as moment conditions for slope parameters of interest, we can consistently estimate the slope parameters for endogenous regressors. There are numerous empirical examples and theoretical studies on IV regression. To analyze its theoretical properties, in modern econometrics it is common to invoke the framework of the generalized method of moments (GMM), which provides a unified approach for statistical inference in econometric models that are specified by

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some moment conditions (see, e.g., Hall, 2005, for a review on the GMM as well as IV regression). However, recent research indicates that there are considerable problems with the conventional IV regression technique particularly in its finite sample performance, and that approximations based on the asymptotic theory can yield poor results (see, e.g., the special issue of the *Journal of Business & Economic Statistics*, vol. 14, and Stock, Wright and Yogo, 2002).

A common way to refine the approximations for the distributions of the IV regression estimator and related test statistics is to employ a bootstrap method. In the IV regression context, there are at least two approaches to conduct bootstrap approximation: the uniform weight bootstrap and implied probability bootstrap. The uniform weight bootstrap draws resamples from the original sample with equal weights and uses quantiles of the resampled statistics to approximate the distribution of the original statistic of interest (see Efron and Tibshirani, 1993, for example). When the number of instruments exceeds the number of parameters (called over-identification), it is reasonable to impose the over-identified moment conditions to bootstrap resamples. Hall and Horowitz (1996) suggested to use the uniform weight bootstrap with recentered moment conditions and established higher-order refinements of their bootstrap inference over the first-order asymptotic approximation. On the other hand, the implied probability bootstrap, proposed by Brown and Newey (2002), draws resamples with unequal weights defined by the so-called implied probabilities from the moment conditions, and uses quantiles of the resampled statistics based on the moment conditions without recentering (see also Hall and Presnell, 1999).

In order to find reasonable bootstrap weights which satisfy moment conditions of interest (i.e., orthogonality of instruments in this paper), information theoretic argument plays a crucial role (see, Cover and Thomas, 1991, and Golan, Judge and Miller, 1996). In particular, we focus on an information projection from the empirical distribution (or equal weights on data points) to a set of distributions satisfying the moment conditions, and utilize the projection, called the implied probabilities in this paper, as weights for bootstrapping. These implied probabilities can be computed based on the Boltzmann-Shannon entropy yielding the exponential tilting weights (Kitamura and Stutzer, 1997, and Imbens, Spady and Johnson, 1998), Burg entropy yielding the empirical likelihood weights (Owen, 1988), Fisher information yielding the GMM-type weights (Back and Brown, 1993), or their variants (Smith, 1997, and Newey and Smith, 2004).

Brown and Newey (2002) argued that the implied probability bootstrap provides a higher-order refinement over the first-order asymptotic approximation under certain regularity conditions as well as the uniform weight bootstrap (with recentering). An important feature of implied probabilities emphasized in the literature is that they provide semiparametrically efficient estimators for the distribution function and its moments under the moment conditions (Back and Brown, 1993, and Brown and Newey, 1998). Antoine, Bonnal and Renault (2007) employed implied probabilities to construct an asymptotically efficient estimator for parameters in the moment conditions. Recently, Camponovo and Otsu (2010) introduced an alternative viewpoint based on robustness analysis against outliers to compare these bootstrap approaches. They extended the breakdown point theory for bootstrap quantiles (Singh, 1998) to the over-identified GMM setting and investigated robustness properties of the implied probability bootstrap. In particular, they analyzed behaviors of bootstrap quantiles of the uniform weight and implied probability bootstraps (using Back and Brown's, 1993, weight) when outliers take arbitrary large values, and compared the breakdown points for these bootstrap quantiles.

The purpose of this paper is to refine the breakdown point analysis of Camponovo and Otsu (2010) by focusing on the IV regression models. In contrast to Camponovo and Otsu (2010), who focused on developing a basic framework for breakdown point analysis and considered somewhat artificial examples such as the trimmed mean with prior information, this paper focuses on the IV regression which is one of the most popular econometrics models, and studies separately the effects of outliers in dependent variables, endogenous regressors, and instruments. As test statistics of interest to be resampled by bootstrapping, this paper considers commonly used statistics, such as the t-statistic based on the two-stage least squares and two-step GMM estimators, and their non-studentized versions. Therefore, this paper clarifies when researchers using IV regression should seriously think about adopting the implied probability bootstrap for their inference in practice. Another important feature of this paper is that in the IV regression setting, there are many cases where the conventional statistics have zero breakdown point (i.e., divergence of a single outlier implies divergence of the statistic), and the breakdown point analysis yields rather striking difference for the different bootstrap approaches. For example, consider the case where divergence of an outlier implies divergence of the t-value in a just-identified setting (one endogenous regressor and one instrument) with the sample size $1,000$. As shown in Table 2 below, divergence of a single outlier implies divergence of more than 63% of the uniform weight bootstrap resamples of the t-values. On the other hand, divergence of a single outlier implies divergence of around 1% of the implied probability bootstrap resamples of the t-values. In addition to the breakdown point analysis to the conventional statistics, we also study the effect of tail trimming proposed by Hill and Renault (2010) to the bootstrap inference and provide a striking simulation evidence to show the difference of the two bootstrap approaches.

There is vast literature on the breakdown point theory in robust statistics (see Hampel, 1971, and Donoho and Huber, 1983, for general definitions of breakdown points, and Singh, 1998, Salibian-Barrera, Van Aelst and Willems, 2007, and Camponovo, Scaillet and Trojani, 2010a, for the use of the breakdown point theory in bootstrap contexts). On the other hand, the literature of robustness study in the IV regression or GMM context is relatively thin and is currently under development. Ronchetti and Trojani (2001) extended robust estimation methods for (just-identified) estimating equations to the over-identified GMM setup. Gagliardini, Trojani and Urga (2005) proposed a robust GMM test for structural breaks. Hill and Renault (2010) proposed a GMM estimator with asymptotically vanishing tail trimming for robust estimation of dynamic moment condition models. Kitamura, Otsu and Evdokimov (2010) and Kitamura and Otsu (2010) studied local robustness against perturbations controlled by the Hellinger distance for point estimation and hypothesis testing, respectively, in moment condition models. Our breakdown point analysis studies global robustness of bootstrap methods in IV regression models when outliers take arbitrarily large values.

The rest of the paper is organized as follows. Section 2 studies a just-identified case, which can be a benchmark for our breakdown point analysis. Section 3 generalizes the analysis in Section 2 to an over-identified model. Section 4 discusses the breakdown point analysis for test statistics based on tail trimming, and remarks on extensions to the over-identifying restriction test, high-dimensional moment functions, and time series data. Section 5 illustrates the theoretical results by simulations. Section 6 concludes.

2 Just-identified case: a benchmark

2.1 Setup

Let $\{Y_i, X_i, Z_i\}_{i=1}^n$ be a random sample of size n from $(Y, X, Z) \in \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^k$, where $k \geq p$. We consider the linear model

$$Y_i \quad = \quad X_i'\theta_0 + U_i,$$

for $i = 1, \ldots, n$, where $\theta_0 \in \mathbb{R}^p$ is a vector of unknown parameters and $U_i \in \mathbb{R}$ is a zero-mean error term. We suspect that the regressors X_i have endogeneity (i.e., $E[X_i U_i] \neq 0$) and the OLS estimator cannot consistently estimate the parameter of interest θ_0. In such a situation, it is common to introduce instrumental variables Z_i, which are orthogonal to the error term U_i. Based on the orthogonality, our estimation problem for θ_0 reduces to the one from the moment condition model

$$E[g(W_i, \theta_0)] = E\left[Z_i\left(Y_i - X_i'\theta_0\right)\right] = 0, \tag{1}$$

where $W_i = (Y_i, X_i', Z_i')'$. When the number of instruments equals the number of regressors (i.e., $k = p$), the model is called just-identified. When the number of instruments exceeds the number of regressors (i.e., $k > p$), the model is called over-identified. To obtain an intuition of our breakdown point analysis, this section focuses on the case of $k = p = 1$, i.e., the model is just-identified and there is only one regressor.

Suppose we wish to test the null hypothesis $H_0 : \theta_0 = c$ for some given $c \in \mathbb{R}$ against the two-sided alternative $H_1 : \theta_0 \neq c$.[1] It is common to evaluate the difference between a point estimator of θ_0 and the hypothetical value c. Since we assume $k = p = 1$, we can apply the conventional method of moments to the condition in (1) and define the estimator as

$$\hat{\theta} = \frac{\sum_{i=1}^n Z_i Y_i}{\sum_{i=1}^n Z_i X_i}.$$

[1] For simplicity this paper focuses on two-sided testing. Our breakdown point analysis can be easily extended to one-sided testing by analyzing divergence properties of test statistics to positive and negative infinity separately.

Based on this estimator, we study robustness of two test statistics for H_0:

$$T_n = \sqrt{n}\left(\hat{\theta} - c\right) \quad \text{(non-studentized statistic)}$$

$$t_n = \frac{\hat{\theta} - c}{\hat{\sigma}} \quad \text{(studentized statistic)} \tag{2}$$

where $\hat{\sigma} = \sqrt{\left(\frac{1}{n}\sum_{i=1}^{n}\hat{U}_i^2\right)\left(\sum_{i=1}^{n} Z_i^2\right)/\left(\sum_{i=1}^{n} Z_i X_i\right)^2}$ is the standard error of $\hat{\theta}$ under homoskedasticity, and $\hat{U}_i = Y_i - X_i\hat{\theta}$ is the residual.[2] In practice, we commonly use the studentized statistic t_n, which converges to the standard normal distribution under the null hypothesis with certain regularity conditions. We also consider the non-studentized statistic T_n to analyze the effect of studentization to robustness. Indeed, the statistics T_n and t_n show different divergence properties against different types of outliers.

To implement hypothesis testing based on the above test statistics, we need to find approximations to the distributions of the test statistics under the null hypothesis H_0, which in turn give us critical vales for the tests. One way to approximate these distributions is to apply the uniform weight bootstrap approach. The uniform weight bootstrap draws many bootstrap resamples from the observations $\{W_i\}_{i=1}^n$ with the uniform weight $1/n$, and approximates the distributions of T_n and t_n by the resampled statistics.

Another way to apply the bootstrap approach in this context is to impose the moment condition $E\left[g\left(W_i, c\right)\right] = 0$ implied from the null hypothesis H_0, and draw bootstrap resamples using the implied probabilities (Back and Brown, 1993),

$$\pi_i = \frac{1}{n} - \frac{1}{n}\frac{\left(g\left(W_i, c\right) - \bar{g}\right)\bar{g}}{\frac{1}{n}\sum_{i=1}^{n} g\left(W_i, c\right)^2}, \tag{3}$$

for $i = 1, \ldots, n$, where $\bar{g} = \frac{1}{n}\sum_{i=1}^{n} g\left(W_i, c\right)$ (note: g is assumed to be scalar-valued in this section).[3] The second term in (3) can be interpreted as a penalty term for the deviation from H_0. If $\left|g\left(W_i, c\right)\right|$ becomes larger, then the second term tends to be negative (because $\left(g\left(W_i, c\right) - \bar{g}\right)$ and $\bar{g}$ tends to take the same sign) and the weight π_i tends to be smaller than the uniform weight $1/n$. Intuitively, if outliers in the sample yield large values of $\left|g\left(W_i, c\right)\right|$, then the implied probability bootstrap tends to draw those outliers less frequently and is expected to show different robustness properties than the uniform weight bootstrap. The next subsection formalizes this intuitive argument by using the breakdown point theory.[4]

[2] The results in this section do not change even if we use the heteroskedasticity robust standard error $\hat{\sigma} = \sqrt{\left(\sum_{i=1}^{n}\hat{U}_i^2 Z_i^2\right)/\left(\sum_{i=1}^{n} Z_i X_i\right)^2}$.

[3] Our breakdown point analysis assumes that all implied probabilities are non-negative. This assumption is typically justified when the sample size is sufficiently large. However, in finite samples, it is possible to have negative implied probabilities. In the simulation study below, we adopt a shrinkage-type modification suggested by Antoine, Bonnal and Renault (2007) to avoid negative implied probabilities.

[4] For the breakdown point analysis below, we focus on Back and Brown's (1993) implied probability in (3) because of its simplicity for analysis. Back and Brown's (1993) implied probability can be interpreted as an approximation to the Fisher information projection from the empirical distribution to the space of distributions satisfying the moment conditions. It is important to extend our analysis to other implied probabilities using different information projections based on the Boltzmann-Shannon entropy yielding the exponential tilting weights (Kitamura and Stutzer, 1997, and Imbens, Spady and Johnson, 1998) and Burg entropy yielding the empirical likelihood weights (Owen, 1988) for example.

2.2 Breakdown point analysis

Based on the above setup, we now introduce the breakdown point analysis. The first step is to define outliers in our context. Since the observations are multivariate vectors, we define the outliers based on the Euclidean norms. Let $\|W_{(1)}\| \leq \cdots \leq \|W_{(n)}\|$ be the ordered sample by the Euclidean norm $\|\cdot\|$. Hereafter we treat $W_{(n)}$ having the largest Euclidean norm as an outlier and consider the limiting situation where $\left(\|W_{(1)}\| \cdots \|W_{(n-1)}\|\right)$ are all finite but $\|W_{(n)}\| \to +\infty$.[5] Note that the sample size n is held fixed. Thus the breakdown point analysis is different from the conventional asymptotic analysis sending n to infinity. Since the outlier $W_{(n)}$ contains three elements $\left(Y_{(n)}, X_{(n)}, Z_{(n)}\right)$ in the case of $k = p = 1$, we consider seven cases that imply $\|W_{(n)}\| \to +\infty$. Those cases are summarized in Table 1 below.

Table 1: Cases such that $\|W_{(n)}\| \to +\infty$

Case	Diverge	Bounded
1	Z	$X,\ Y$
2	X	$Y,\ Z$
3	Y	$X,\ Z$
4	$X,\ Z$	$Y,\ \frac{X}{Z}$
5	$X,\ Y$	$Z,\ \frac{X}{Y}$
6	$Y,\ Z$	$X,\ \frac{Z}{Y}$
7	$X,\ Y,\ Z$	$\frac{Z}{Y},\ \frac{X}{Y}$

For the sake of brevity, in Table 1 (and also in Tables 3, 4, and 5 below) X, Y , and Z mean (the realizations of) $|X_{(n)}|$, $|Y_{(n)}|$, and $|Z_{(n)}|$, respectively. For example, the second row for Case 1 means that $|Z_{(n)}| \to +\infty$, but $|X_{(n)}|$ and $|Y_{(n)}|$ are bounded, while the sixth row for Case 5 means that $|X_{(n)}| \to +\infty$ and $|Y_{(n)}| \to +\infty$, but $|Z_{(n)}|$ and $\left|\frac{X_{(n)}}{Y_{(n)}}\right|$ are bounded. Of course this table is not exhaustive and there are other cases where the norm $\|W_{(n)}\|$ diverges. We mention that the same breakdown point analysis presented below goes through for such other cases.

For $i = 1, \ldots, n$, let $g_{(i)} = g\left(W_{(i)}, c\right)$ and $\pi_{(i)}$ be the implied probability associated with the observation $W_{(i)}$. Using some algebra, we can verify that for all the seven cases introduced in Table 1 we have $|g_{(n)}| \to +\infty$.[6] Therefore, the implied probability defined in (3) for the observation $W_{(n)}$ satisfies

$$\pi_{(n)} = \frac{1}{n} - \frac{1}{n} \frac{\left(1 - \frac{1}{n} - \frac{\bar{g}_-}{g_{(n)}}\right)\left(\frac{\bar{g}_-}{g_{(n)}} + \frac{1}{n}\right)}{\frac{\bar{v}_-}{g_{(n)}^2} + \frac{1}{n}} \to \frac{1}{n^2}, \tag{4}$$

[5] This is just one way to define outliers. For other definitions, basically the same breakdown point analysis goes through by analyzing limiting behaviors of the test statistics and bootstrap quantiles as the outliers diverge or converge to some points.

[6] For Cases 5 and 7 with $c = 1$, we introduce the additional condition $\left|\left(1 - \frac{Y_{(n)}}{X_{(n)}}\right) X_{(n)}\right| \to +\infty$.

as $\left|g_{(n)}\right| \to +\infty$, where $\bar{g}_- = \frac{1}{n}\sum_{i=1}^{n-1} g\left(W_{(i)}, c\right)$ and $\bar{v}_- = \frac{1}{n}\sum_{i=1}^{n-1} g\left(W_{(i)}, c\right)^2$. Again the sample size n is fixed here. Note that compared to the uniform weight n^{-1}, the implied probability bootstrap draws the outlier $W_{(n)}$ with a smaller weight converging to n^{-2} as $\left|g_{(n)}\right| \to +\infty$.

Now suppose that divergence of $\left\|W_{(n)}\right\|$ causes divergence of $|T_n|$ (the same argument applies for $|t_n|$). Let $T_n^{\#}$ and T_n^* denote the bootstrap counterparts of T_n based on the uniform weight and implied probability bootstraps, respectively. Letting $B(n,p)$ be a binomial random variable with parameters n and p, the probability that the uniform weight bootstrap counterpart $T_n^{\#}$ is free from the outlier $W_{(n)}$ is written as

$$p^{\#} = P\left(B\left(n, \frac{1}{n}\right) = 0\right).$$

Therefore, if $\left\|W_{(n)}\right\| \to +\infty$, then $100\left(1 - p^{\#}\right)\%$ of resamples of $\left|T_n^{\#}\right|$ will diverge to $+\infty$. In other words, the t-th bootstrap quantile $Q_t^{\#}$ of $\left|T_n^{\#}\right|$ will diverge to $+\infty$ for all $t > p^{\#}$ as $\left\|W_{(n)}\right\| \to +\infty$. On the other hand, from (4), the probability that the implied probability bootstrap counterpart T_n^* is free from the outlier $W_{(n)}$ converges to

$$p^* = P\left(B\left(n, \frac{1}{n^2}\right) = 0\right),$$

as $\left\|W_{(n)}\right\| \to +\infty$. Thus, the t-th bootstrap quantile Q_t^* of $|T_n^*|$ will diverge to $+\infty$ for all $t > p^*$ satisfying $p^* > p^{\#}$. Table 2 presents the values of $p^{\#}$ and p^* for several choices of the sample size n.

Table 2: Values of $p^{\#}$ and p^*

n	$p^{\#}$	p^*
10	0.349	0.904
20	0.358	0.951
50	0.364	0.980
100	0.366	0.990
500	0.368	0.998
1000	0.368	0.999

For example, when $n = 100$, divergence of a single outlier implies divergence of more than 63% of the uniform weight bootstrap resamples of $T_n^{\#}$. On the other hand, divergence of a single outlier implies divergence of around 1% of the implied probability bootstrap resamples of T_n^*.

Given the above argument, the remaining task is to figure out when the statistics T_n and t_n defined in (2) diverge as $\left\|W_{(n)}\right\| \to +\infty$. Table 3 below analyzes the limiting behaviors of the statistics T_n and t_n for each of the seven cases listed in Table 1.[7]

[7] For Case 1, assume $\left|X_{(n)}\right|$ is bounded away from zero. For Cases 2 and 4, $|t_n|$ diverges only when $c \neq 0$.

Table 3: Limits of T_n and t_n as $\left\|W_{(n)}\right\| \to +\infty$

Case	Diverge	Bounded	Limit of $\lvert T_n \rvert$	Limit of $\lvert t_n \rvert$
1	Z	$X,\ Y$	bounded	bounded
2	X	$Y,\ Z$	bounded	$+\infty$
3	Y	$X,\ Z$	$+\infty$	bounded
4	$X,\ Z$	$Y,\ \frac{X}{Z}$	bounded	$+\infty$
5	$X,\ Y$	$Z,\ \frac{X}{Y}$	bounded	$+\infty$
6	$Y,\ Z$	$X,\ \frac{Z}{Y}$	$+\infty$	bounded
7	$X\ Y,\ Z$	$\frac{Z}{Y},\ \frac{X}{Y}$	bounded	$+\infty$

For example, the second row for Case 1 means that when $\left|Z_{(n)}\right| \to +\infty$ but $\left|X_{(n)}\right|$ and $\left|Y_{(n)}\right|$ are bounded, T_n and t_n are bounded. Also the sixth row for Case 5 means that when $\left|X_{(n)}\right| \to +\infty$ and $\left|Y_{(n)}\right| \to +\infty$ but $\left|Z_{(n)}\right|$ and $\left|\frac{X_{(n)}}{Y_{(n)}}\right|$ are bounded, T_n is bounded but $|t_n|$ diverges to infinity. From this table, we can regard the outliers of Cases 3 and 6 as the ones for the non-studentized statistic T_n and also regard the outliers of Cases 2, 4, 5 and 7 as the ones for the studentized statistic t_n.

The findings in this subsection are summarized as follows.

Proposition 1. *Consider the setup of this section. Let* $p^{\#} = P\left(B\left(n, \frac{1}{n}\right) = 0\right)$ *and* $p^{*} = P\left(B\left(n, \frac{1}{n^2}\right) = 0\right)$. *If* $\left\|W_{(n)}\right\| \to +\infty$, *the followings hold true.*

(i) *For Cases 3 and 6, the uniform weight bootstrap t-th quantile* $Q_t^{\#}$ *from the resamples* $\left|T_n^{\#}\right|$ *of* $|T_n|$ *will diverge to* $+\infty$ *for all* $t > p^{\#}$, *and the implied probability bootstrap t-th quantile* Q_t^{*} *from the resamples* $|T_n^{*}|$ *of* $|T_n|$ *will diverge to* $+\infty$ *for all* $t > p^{*}$.

(ii) *For Cases 2, 4, 5 and 7, the uniform weight bootstrap t-th quantile* $Q_t^{\#}$ *from the resamples* $\left|t_n^{\#}\right|$ *of* $|t_n|$ *will diverge to* $+\infty$ *for all* $t > p^{\#}$, *and the implied probability bootstrap t-th quantile* Q_t^{*} *from the resamples* $|t_n^{*}|$ *of* $|t_n|$ *will diverge to* $+\infty$ *for all* $t > p^{*}$.

For example, if there is an outlier in the endogenous regressor (i.e., Case 2), the uniform weight bootstrap resamples for the studentized statistic $|t_n|$ contain the outlier more often than the implied probability bootstrap resamples. Thus, the uniform weight bootstrap quantiles tend to take large values. On the other hand, we can expect that the bootstrap quantiles for the non-studentized statistic $|T_n|$ are relatively robust against such an outlier, since the non-studentized statistic does not diverge in this case. Similar comments apply to the other cases. In Section 5.1, we illustrate these findings by simulation.

Proposition 1 may be extended to the case where we have $m > 1$ outliers. Suppose that for each $j = 1, \ldots, m$, it holds $\frac{g_{(n-j+1)}}{g_{(n)}} \to 1$ as $\left\|W_{(n-j+1)}\right\| \to \infty$. Then, we have

$$\pi_{(n)} = \frac{1}{n} - \frac{1}{n} \frac{\left\{\left(1 - \frac{1}{n}\right) - \frac{1}{n}\left(\frac{g_{(n-1)} + \cdots + g_{(n-m+1)}}{g_{(n)}}\right) - \frac{\bar{g}_{m-}}{g_{(n)}}\right\}\left\{\frac{\bar{g}_{m-}}{g_{(n)}} + \frac{1}{n}\left(\frac{g_{(n)} + \cdots + g_{(n-m+1)}}{g_{(n)}}\right)\right\}}{\frac{\bar{v}_{m-}}{g_{(n)}^2} + \frac{1}{n}\left(\frac{g_{(n)}^2 + \cdots + g_{(n-m+1)}^2}{g_{(n)}^2}\right)} \to \frac{m}{n^2},$$

as $\left\|W_{(n-j+1)}\right\| \rightarrow \infty$ for all $j=1,\ldots,m$, where $\bar{g}_{m-}=\frac{1}{n}\sum_{i=1}^{n-m} g\left(X_{(i)},c\right)$ and $\bar{v}_{m-}=\frac{1}{n}\sum_{i=1}^{n-m} g\left(X_{(i)},c\right)^2$. By applying the same argument, we obtain $\pi_{(n-j+1)} \rightarrow \frac{m}{n^2}$, for all $j=1,\ldots,m$. Therefore, the probability that the implied probability bootstrap resample T_n^* is free from m outliers $\left(W_{(n-m+1)},\ldots,W_{(n)}\right)$ converges to

$$p_m^* = P\left(B\left(n,\left(\frac{m}{n}\right)^2\right)=0\right),$$

as $\left\|W_{(n-j+1)}\right\| \rightarrow \infty$ for all $j=1,\ldots,m$. On the other hand, the probability that the uniform weight bootstrap resample $T_n^{\#}$ is free from m outliers $\left(W_{(n-m+1)},\ldots,W_{(n)}\right)$ is written as

$$p_m^{\#} = P\left(B\left(n,\frac{m}{n}\right)=0\right).$$

Note that p_m^* is always larger than $p_m^{\#}$ for any m and n. We can extend the cases displayed in Table 3 to the m outliers setup, i.e., X, Y, and Z mean $\left(X_{(n-m+1)},\ldots,X_{(n)}\right)$, $\left(Y_{(n-m+1)},\ldots,Y_{(n)}\right)$, and $\left(Z_{(n-m+1)},\ldots,Z_{(n)}\right)$, respectively, and the ratio such as $\frac{X}{Z}$ means $\frac{X_{(n-j+1)}}{Z_{(n-j'+1)}}$ for all $j,j'=1,\ldots,m$. Then, we obtain the following result.

Proposition 2. *Consider the setup of this section. Let* $p_m^{\#}=P\left(B\left(n,\frac{m}{n}\right)=0\right)$ *and* $p_m^*=P\left(B\left(n,\left(\frac{m}{n}\right)^2\right)=0\right)$. *If* $\left\|W_{(n-j+1)}\right\| \rightarrow +\infty$ *for all* $j=1,\ldots,m$, *the followings hold true.*

(i) *For Cases 3 and 6, the uniform weight bootstrap t-th quantile $Q_t^{\#}$ from the resamples $\left|T_n^{\#}\right|$ of $|T_n|$ will diverge to $+\infty$ for all $t>p_m^{\#}$, and the implied probability bootstrap t-th quantile Q_t^* from the resamples $|T_n^*|$ of $|T_n|$ will diverge to $+\infty$ for all $t>p_m^*$.*

(ii) *For Cases 2, 4, 5 and 7, the uniform weight bootstrap t-th quantile $Q_t^{\#}$ from the resamples $\left|t_n^{\#}\right|$ of $|t_n|$ will diverge to $+\infty$ for all $t>p_m^{\#}$, and the implied probability bootstrap t-th quantile Q_t^* from the resamples $|t_n^*|$ of $|t_n|$ will diverge to $+\infty$ for all $t>p_m^*$.*

3 Over-identified case

3.1 Setup

We now extend our breakdown point analysis to an over-identified case with $k=2$ and $p=1$. Extensions to high dimension cases will be briefly discussed in Section 4.2. In this case, the moment conditions (1) can be written as

$$E\left[g\left(W_i,\theta_0\right)\right]=E\begin{bmatrix} g_1\left(W_i,\theta_0\right) \\ g_2\left(W_i,\theta_0\right)\end{bmatrix}=E\begin{bmatrix} Z_{1i}\left(Y_i-X_i\theta_0\right) \\ Z_{2i}\left(Y_i-X_i\theta_0\right)\end{bmatrix}=0.$$

where $W_i=\left(Y_i,X_i,Z_{1i},Z_{2i}\right)' \in \mathbb{R}^4$. Similar to the just-identified case, we consider the parameter hypothesis testing problem $H_0:\theta_0=c$ against the two-sided alternative $H_1:\theta_0\neq c$ using the non-studentized statistic $T_n=\sqrt{n}\left(\hat{\theta}-c\right)$ and the studentized statistic $t_n=\frac{\hat{\theta}-c}{\hat{\sigma}}$. The point estimator for

θ_0 is either the two-stage least square estimator

$$\hat{\theta}_{2SLS} = \left[\left(\sum_{i=1}^{n} X_i Z_i\right)' \left(\sum_{i=1}^{n} Z_i Z_i'\right)^{-1} \left(\sum_{i=1}^{n} X_i Z_i\right)\right]^{-1} \left(\sum_{i=1}^{n} X_i Z_i\right)' \left(\sum_{i=1}^{n} Z_i Z_i'\right)^{-1} \left(\sum_{i=1}^{n} Z_i Y_i\right),$$

or the two-step GMM estimator

$$\hat{\theta}_{GMM} = \left[\left(\sum_{i=1}^{n} X_i Z_i\right)' \left(\sum_{i=1}^{n} \hat{U}_i^2 Z_i Z_i'\right)^{-1} \left(\sum_{i=1}^{n} X_i Z_i\right)\right]^{-1} \left(\sum_{i=1}^{n} X_i Z_i\right)' \left(\sum_{i=1}^{n} \hat{U}_i^2 Z_i Z_i'\right)^{-1} \left(\sum_{i=1}^{n} Z_i Y_i\right),$$

where $\hat{U}_i = Y_i - X_i \hat{\theta}_{2SLS}$ is the residual from the first-stage regression using $\hat{\theta}_{2SLS}$. Since we have the same breakdown point properties, in this section we indifferently denote $T_n = \sqrt{n}\left(\hat{\theta}_{2SLS} - c\right)$ or $\sqrt{n}\left(\hat{\theta}_{GMM} - c\right)$. For the standard error in the studentized statistic t_n, we consider

$$\hat{\sigma}_{2SLS} = \sqrt{\left(\frac{1}{n}\sum_{i=1}^{n} \hat{U}_i^2\right) \left[\left(\sum_{i=1}^{n} X_i Z_i\right)' \left(\sum_{i=1}^{n} Z_i Z_i'\right)^{-1} \left(\sum_{i=1}^{n} X_i Z_i\right)\right]^{-1}},$$

for the two-stage least square estimator and

$$\hat{\sigma}_{GMM} = \sqrt{\left[\left(\sum_{i=1}^{n} X_i Z_i\right)' \left(\sum_{i=1}^{n} \tilde{U}_i^2 Z_i Z_i'\right)^{-1} \left(\sum_{i=1}^{n} X_i Z_i\right)\right]^{-1}},$$

for the two-step GMM estimator, where $\tilde{U}_i = Y_i - X_i \hat{\theta}_{GMM}$. Since we have the same breakdown point properties, in this section we indifferently denote $t_n = \frac{\hat{\theta}_{2SLS} - c}{\hat{\sigma}_{2SLS}}$ or $\frac{\hat{\theta}_{GMM} - c}{\hat{\sigma}_{GMM}}$.

Similar to the last section, we compare the robustness properties of bootstrap quantiles based on the uniform weigh and implied probability bootstraps. In this case, since the moment function g is now a vector, Back and Brown's (1993) implied probability for the observation W_i from the moment condition $E\left[g\left(W_i, c\right)\right] = 0$ is written as

$$\pi_i = \frac{1}{n} - \frac{1}{n}\left(g\left(W_i, c\right) - \bar{g}\right)' \left[\frac{1}{n}\sum_{i=1}^{n} g\left(W_i, c\right) g\left(W_i, c\right)'\right]^{-1} \bar{g}. \qquad (5)$$

Although the implied probability takes more complicated form than the just-identified case considered in the last section, we can still apply the same breakdown point analysis to this setting.

3.2 Breakdown point analysis

Similar to the just-identified case, we order the sample as $\left\|W_{(1)}\right\| \leq \cdots \leq \left\|W_{(n)}\right\|$ based on the Euclidean norm, and consider the limiting situation where $\left(\left\|W_{(1)}\right\| \cdots \left\|W_{(n-1)}\right\|\right)$ are all finite but $\left\|W_{(n)}\right\| \to +\infty$. Since the outlier $W_{(n)}$ now contains four elements $\left(Y_{(n)}, X_{(n)}, Z_{1(n)}, Z_{2(n)}\right)$, we consider fifteen cases summarized in Table 4 below.

Table 4: Conditions such that $\|W_{(n)}\| \to +\infty$

Case	Diverge	Bounded	Case	Diverge	Bounded	Case	Diverge	Bounded
1	Z_1	$X,\ Y,\ Z_2$	6	$Z_1,\ X$	$Y,\ ,Z_2\ ,\frac{Z_1}{X}$	11	$Z_1,\ Z_2,\ X$	$Y,\ \frac{Z_1}{Z_2},\ \frac{Z_1}{X}$
2	Z_2	$X,\ Y,\ Z_1$	7	$Z_2,\ X$	$Y,\ ,Z_1\ ,\frac{Z_2}{X}$	12	$Z_1,\ Z_2,\ Y$	$X,\ \frac{Z_1}{Z_2},\ \frac{Z_1}{Y}$
3	X	$Z_1\ Z_2,\ Y$	8	$Z_1,\ Y$	$X,\ ,Z_2\ ,\frac{Z_1}{Y}$	13	$Z_1,\ Y,\ X$	$Z_2,\ \frac{Z_1}{X},\ \frac{Z_1}{Y}$
4	Y	$Z_1\ Z_2,\ X$	9	$Z_2,\ Y$	$X,\ ,Z_1\ ,\frac{Z_2}{Y}$	14	$Z_2,\ Y,\ X$	$Z_1,\ \frac{Z_2}{X},\ \frac{Z_2}{Y}$
5	$Z_1,\ Z_2$	$X,\ Y,\frac{Z_1}{Z_2}$	10	$X,\ Y$	$Z_1,\ ,Z_2\ ,\frac{Y}{X}$	15	$Z_1,\ Z_2,\ Y,\ X$	$\frac{Z_2}{X},\ \frac{Z_1}{X},\ \frac{Z_1}{Y}$

Let $g_{1(n)} = g_1\left(W_{(n)}, c\right)$ and $g_{2(n)} = g_2\left(W_{(n)}, c\right)$. Using some algebra we can verify that[8]

$$\left(\left|g_{1(n)}\right|, \left|g_{2(n)}\right|\right) \to \begin{cases} (+\infty, C) & \text{for Case 1} \\ (C', +\infty) & \text{for Case 2} \\ (+\infty, +\infty) & \text{for Cases 3-15} \end{cases}$$

as $\|W_{(n)}\| \to +\infty$. By applying the results in Camponovo and Otsu (2010) to the present setup, the limiting behavior of the implied probability $\pi_{(n)}$ for the outlier $W_{(n)}$ is characterized as follows

$$\pi_{(n)} \to \begin{cases} \frac{1}{n^2} + \frac{1}{n}\frac{\bar{g}_{2-}^2}{v_{22}} & \text{for Case 1} \\ \frac{1}{n^2} + \frac{1}{n}\frac{\bar{g}_{1-}^2}{v_{11}} & \text{for Case 2} \\ \frac{1}{n^2} + \frac{1}{n}\frac{(\bar{g}_{1-} - \bar{g}_{2-})^2}{v_{11}+v_{22}-2v_{12}} & \text{for Cases 3-15} \end{cases} \tag{6}$$

where $\bar{g}_{1-} = \frac{1}{n}\sum_{i=1}^{n-1} g_{1(i)}$, $\bar{g}_{2-} = \frac{1}{n}\sum_{i=1}^{n-1} g_{2(i)}$, $v_{11} = \frac{1}{n}\sum_{i=1}^{n-1} g_{1(i)}^2$, $v_{22} = \frac{1}{n}\sum_{i=1}^{n-1} g_{2(i)}^2$, and $v_{12} = \frac{1}{n}\sum_{t=1}^{n-1} g_{1(i)}g_{2(i)}$.

Unlike the just-identified case, in this setting the limit of the implied probability $\pi_{(n)}$ depends on the terms $\bar{g}_{1-}$, $\bar{g}_{2-}$, v_{11}, v_{22}, and v_{12}. Therefore, the implied probability bootstrap does not necessarily draw outliers with lower probability than the uniform weight bootstrap. Nevertheless, it should be noted that the terms $\bar{g}_{1-}$, $\bar{g}_{2-}$, v_{11}, v_{22}, and v_{12} do not contain the outlier $W_{(n)}$ and thus the second terms appearing in the limit (6) are typically small when the sample size n is large. Also we can empirically evaluate the second terms in (6) and assess the difference with the uniform weight $1/n$.

To conduct the breakdown point analysis, it remains to characterize the limits of the statistics T_n and t_n when $\|W_{(n)}\| \to +\infty$. For the sake of brevity, in Table 5 below we do not report all the results for the fifteen cases introduced in Table 4, but we report only the cases where at least one of the two test statistics diverges to infinity.[9]

[8]For Cases 10. 13, 14, and 15 with $c = 1$, we additionally assume that $\left|\left(1 - \frac{Y_{(n)}}{X_{(n)}}\right) X_{(n)}\right| \to +\infty$.

[9]For Cases 3, 6, 7 and 11, $|t_n|$ diverges to infinity only when $c \neq 0$.

Table 5: Limits of T_n and t_n as $\|W_{(n)}\| \to +\infty$

Case	Diverge	Bounded	Limit of $\lvert T_n \rvert$	Limit of $\lvert t_n \rvert$
3	X	$Z_1\ Z_2,\ Y$	bounded	$+\infty$
4	Y	$Z_1\ Z_2,\ X$	$+\infty$	bounded
6	$Z_1,\ X$	$Y,\ ,Z_2\ ,\frac{Z_1}{X}$	bounded	$+\infty$
7	$Z_2,\ X$	$Y,\ ,Z_1\ ,\frac{Z_1}{X}$	bounded	$+\infty$
8	$Z_1,\ Y$	$X,\ ,Z_2\ ,\frac{Z_1}{Y}$	$+\infty$	bounded
9	$Z_2,\ Y$	$X,\ ,Z_1\ ,\frac{Z_1}{Y}$	$+\infty$	bounded
10	$X,\ Y$	$Z_1,\ ,Z_2\ ,\frac{Y}{X}$	bounded	$+\infty$
11	$Z_1,\ Z_2,\ X$	$Y,\ \frac{Z_1}{Z_2},\ \frac{Z_1}{X}$	bounded	$+\infty$
12	$Z_1,\ Z_2,\ Y$	$X,\ \frac{Z_1}{Z_2},\ \frac{Z_1}{Y}$	$+\infty$	bounded
13	$Z_1,\ Y,\ X$	$Z_2,\ \frac{Z_1}{X},\ \frac{Z_1}{Y}$	bounded	$+\infty$
14	$Z_2,\ Y,\ X$	$Z_1,\ \frac{Z_2}{X},\ \frac{Z_2}{Y}$	bounded	$+\infty$
15	$Z_1,\ Z_2,\ Y,\ X$	$\frac{Z_2}{X},\ \frac{Z_1}{X},\ \frac{Z_1}{Y}$	bounded	$+\infty$

We can summarize the findings of this subsection in the following proposition.

Proposition 3. *Consider the setup of this section. Let* $p^{\#} = P\left(B\left(n, \frac{1}{n}\right) = 0\right)$, *and* $p^* = P\left(B\left(n, \frac{1}{n^2} + \frac{1}{n}\frac{(\bar{g}_{1-} - \bar{g}_{2-})^2}{v_{11} + v_{22} - 2v_{12}}\right) = 0\right)$. *If* $\|W_{(n)}\| \to +\infty$, *the followings hold true.*

(i) *For Cases 4, 8, 9 and 12, the uniform weight bootstrap t-th quantile* $Q_t^{\#}$ *from the resamples* $\left|T_n^{\#}\right|$ *of* $|T_n|$ *will diverge to* $+\infty$ *for all* $t > p^{\#}$, *and the implied probability bootstrap t-th quantile* Q_t^* *from the resamples* $|T_n^*|$ *of* $|T_n|$ *will diverge to* $+\infty$ *for all* $t > p^*$.

(ii) *For Cases 3, 6, 7, 10, 11, 13, 14 and 15, the uniform weight bootstrap t-th quantile* $Q_t^{\#}$ *from the resamples* $\left|t_n^{\#}\right|$ *of* $|t_n|$ *will diverge to* $+\infty$ *for all* $t > p^{\#}$, *and the implied probability bootstrap t-th quantile* Q_t^* *from the resamples* $|t_n^*|$ *of* $|t_n|$ *will diverge to* $+\infty$ *for all* $t > p^*$.

For example, if there is an outlier in the endogenous regressor (i.e., Case 3) and it holds $\frac{1}{n^2} + \frac{1}{n}\frac{(\bar{g}_{1-} - \bar{g}_{2-})^2}{v_{11} + v_{22} - 2v_{12}} < \frac{1}{n}$, then the uniform weight bootstrap resamples for the studentized statistic t_n contain the outlier more often than the implied probability bootstrap resamples. Thus the uniform weight bootstrap quantiles tend to take large values. On the other hand, we can expect that the bootstrap quantiles for the non-studentized statistic T_n are relatively robust against such an outlier, since the non-studentized statistic does not diverge in this case. Similar comments apply to the other cases. In Section 5.2, we illustrate these findings by simulation.

Similar to the just-identified case, Proposition 3 may be extended to the case where we have $m > 1$ outliers. Suppose that for each $j = 1, \ldots, m$, it holds $\frac{g_{1(n-j+1)}}{g_{1(n)}} \to 1$ and $\frac{g_{2(n-j+1)}}{g_{2(n)}} \to 1$ as $\|W_{(n-j+1)}\| \to \infty$. Then, using the results in Camponovo and Otsu (2010), we have

$$\pi_{(n)} \rightarrow \begin{cases} \frac{m}{n^2} + \frac{1}{n} \frac{\bar{g}_{2-,m}^2}{v_{22,k}} & \text{for Case 1} \\ \frac{m}{n^2} + \frac{1}{n} \frac{\bar{g}_{1-,m}^2}{v_{11,k}} & \text{for Case 2} \\ \frac{m}{n^2} + \frac{1}{n} \frac{\left(\bar{g}_{1-,m} - \bar{g}_{2-,k}\right)^2}{v_{11,m} + v_{22,m} - 2v_{12,m}} & \text{for Cases 3-15} \end{cases}$$

as $\|W_{(n-j+1)}\| \rightarrow \infty$ for all $j = 1, \ldots, m$, where $\bar{g}_{1-,m} = \frac{1}{n} \sum_{i=1}^{n-m} g_{1(i)}$, $\bar{g}_{2-,m} = \frac{1}{n} \sum_{i=1}^{n-m} g_{2(i)}$, $v_{11,m} = \frac{1}{n} \sum_{t=1}^{n-m} g_{1(m)}^2$, $v_{22,m} = \frac{1}{n} \sum_{i=1}^{n-m} g_{2(i)}^2$, and $v_{12,m} = \frac{1}{n} \sum_{i=1}^{n-m} g_{1(i)} g_{2(i)}$. By applying the same argument, we can see that $\pi_{(n-j+1)}$ has the same limit for all $j = 1, \ldots, m$. We extend the cases displayed in Table 5 to the m outlier setup, i.e., X, Y, Z_1, and Z_2 mean $\left(X_{(n-m+1)}, \ldots, X_{(n)}\right)$, $\left(Y_{(n-m+1)}, \ldots, Y_{(n)}\right)$, $\left(Z_{1(n-m+1)}, \ldots, Z_{1(n)}\right)$, and $\left(Z_{2(n-m+1)}, \ldots, Z_{2(n)}\right)$, respectively, and the ratio such as $\frac{X}{Z_1}$ means $\frac{X_{(n-j+1)}}{Z_{1(n-j'+1)}}$ for all $j, j' = 1, \ldots, m$. Then we obtain following result.

Proposition 4. *Consider the setup of this section. Let* $p_m^{\#} = P\left(B\left(n, \frac{m}{n}\right) = 0\right)$ *and* $p_m^* = P\left(B\left(n, \left(\frac{m}{n}\right)^2 + \frac{m}{n} \frac{\left(\bar{g}_{1-,m} - \bar{g}_{2-,m}\right)^2}{v_{11,m} + v_{22,m} - 2v_{12,m}}\right) = 0\right)$. *If* $\|W_{(n-j+1)}\| \rightarrow +\infty$ *for all* $j = 1, \ldots, m$, *the followings hold true.*

(i) *For Cases 4, 8, 9 and 12, the uniform weight bootstrap t-th quantile* $Q_t^{\#}$ *from the resamples* $\left|T_n^{\#}\right|$ *of* $|T_n|$ *will diverge to* $+\infty$ *for all* $t > p_m^{\#}$, *and the implied probability bootstrap t-th quantile* Q_t^* *from the resamples* $|T_n^*|$ *of* $|T_n|$ *will diverge to* $+\infty$ *for all* $t > p_m^*$.

(ii) *For Cases 3, 6, 7, 10, 11, 13, 14 and 15, the uniform weight bootstrap t-th quantile* $Q_t^{\#}$ *from the resamples* $\left|t_n^{\#}\right|$ *of* $|t_n|$ *will diverge to* $+\infty$ *for all* $t > p_m^{\#}$, *and the implied probability bootstrap t-th quantile* Q_t^* *from the resamples* $|t_n^*|$ *of* $|t_n|$ *will diverge to* $+\infty$ *for all* $t > p_m^*$.

4 Discussions

4.1 Tail trimming

To deal with possibly heavy tailed data, Hill and Renault (2010) recently introduced the tail trimming GMM estimator. The idea of this approach consists of trimming an asymptotically vanishing sample portion of the estimating equations. The application of the tail trimming method to the IV regression model analyzed in this paper delivers interesting robustness properties. We briefly describe Hill and Renault's (2011) approach in our context and extend the breakdown point analysis to the bootstraps for the statistics based on the tail trimming GMM estimator.

We first consider the just-identified case with $k = p = 1$ and the test for the null $H_0 : \theta_0 = c$. Let $\left|g\left(W_{[1]}, c\right)\right| \leq \cdots \leq \left|g\left(W_{[n]}, c\right)\right|$ be the ordered estimating equations evaluated at $\theta = c$, and define the trimmed sample as $\check{W}_i = \left(\check{Y}_i, \check{X}_i, \check{Z}_i\right)' = W_i \times I\left\{\left|g\left(W_i, c\right)\right| < \left|g\left(W_{[n-r+1]}, c\right)\right|\right\}$ for some $r < n$, where $I\{\cdot\}$ is the indicator function (note: $g(\cdot)$ is scalar-valued if $k = p = 1$). The tail trimming IV regression estimator of θ_0 is defined as

$$\check{\theta} = \frac{\sum_{i=1}^n \check{Z}_i \check{Y}_i}{\sum_{i=1}^n \check{Z}_i \check{X}_i}.$$

Based on this estimator, we introduce the non-studentized statistic $\check{T}_n = \sqrt{n}\left(\check{\theta} - c\right)$ and the studentized statistic $\check{t}_n = \frac{\check{\theta}-c}{\check{\sigma}}$, where $\check{\sigma} = \sqrt{\left(\frac{1}{n}\sum_{i=1}^n \check{U}_i^2\right)\left(\sum_{i=1}^n \check{Z}_i^2\right) / \left(\sum_{i=1}^n \check{Z}_i \check{X}_i\right)^2}$ and $\check{U}_i = \check{Y}_i - \check{X}_i\check{\theta}$. Hill and Renault (2011) assumed $r/n \to 0$ as $n \to +\infty$ and studied the asymptotic properties of the tail trimming GMM estimator in a general setting. Here we fix the sample size n (and also r) and consider the situation where $\left\|W_{(n-j+1)}\right\| \to +\infty$ for all $j = 1, \ldots, m$.

The test statistics $\check{T}_n$ and $\check{t}_n$ based on the trimmed sample are more robust to outliers than the test statistics T_n and t_n based on the original sample. With slight modifications, the breakdown point analysis in the last section can be adapted to this context.[10]

Proposition 5. *Consider the setup of Section 2. Let* $p_m^{r\#} = P\left(B\left(n, \frac{m}{n}\right) \leq r\right)$ *and* $p_m^{r*} = P\left(B\left(n, \left(\frac{m}{n}\right)^2\right) \leq r\right)$. *If* $\left\|W_{(n-j+1)}\right\| \to +\infty$ *for all* $j = 1, \ldots, m$, *the followings hold true.*

(i) *For Cases 3 and 6 of Section 2.2, the uniform weight bootstrap t-th quantile* $\check{Q}_t^{\#}$ *from the resamples* $\left|\check{T}_n^{\#}\right|$ *of* $\left|\check{T}_n\right|$ *will diverge to* $+\infty$ *for all* $t > p_m^{r\#}$, *and the implied probability bootstrap t-th quantile* $\check{Q}_t^*$ *from the resamples* $\left|\check{T}_n^*\right|$ *of* $\left|\check{T}_n\right|$ *will diverge to* $+\infty$ *for all* $t > p_m^{r*}$.

(ii) *For Cases 2, 4, 5 and 7 of Section 2.2, the uniform weight bootstrap t-th quantile* $\check{Q}_t^{\#}$ *from the resamples* $\left|\check{t}_n^{\#}\right|$ *of* $\left|\check{t}_n\right|$ *will diverge to* $+\infty$ *for all* $t > p_m^{r\#}$, *and the implied probability bootstrap t-th quantile* $\check{Q}_t^*$ *from the resamples* $\left|\check{t}_n^*\right|$ *of* $\left|\check{t}_n\right|$ *will diverge to* $+\infty$ *for all* $t > p_m^{r*}$.

Note that as r increases (i.e., trim more samples), both $p_m^{r\#}$ and p_m^{r*} increase. Thus, tail trimming robustifies the both bootstrap quantiles. Also note that it always holds $p_m^{r\#} < p_m^{r*}$, and the implied probability bootstrap shows analogous robustness properties over the uniform weight bootstrap.

Analogously, for the over-identified case considered in Section 3, let $\left|g_1\left(W_{[1]}, c\right)\right| \leq \cdots \leq \left|g_1\left(W_{[n]}, c\right)\right|$ and $\left|g_2\left(W_{[1]}, c\right)\right| \leq \cdots \leq \left|g_2\left(W_{[n]}, c\right)\right|$ be the ordered estimating equations for each component. Then, the trimmed sample is defined as

$$\check{W}_i = \left(\check{Y}_i, \check{X}_i, \check{Z}_i\right)' = W_i \times I\left\{\left|g_1\left(W_i, c\right)\right| < \left|g_1\left(W_{[n-r+1]}, c\right)\right|, \ \left|g_2\left(W_i, c\right)\right| < \left|g_2\left(W_{[n-r+1]}, c\right)\right|\right\},$$

for some $r < n$, and the tail trimming two-stage least square estimator is written as[11]

$$\check{\theta} = \left[\left(\sum_{i=1}^n \check{X}_i \check{Z}_i\right)' \left(\sum_{i=1}^n \check{Z}_i \check{Z}_i'\right)^{-1} \left(\sum_{i=1}^n \check{X}_i \check{Z}_i\right)\right]^{-1} \left(\sum_{i=1}^n \check{X}_i \check{Z}_i\right)' \left(\sum_{i=1}^n \check{Z}_i \check{Z}_i'\right)^{-1} \left(\sum_{i=1}^n \check{Z}_i \check{Y}_i\right).$$

We can consider the non-studentized statistic $\check{T}_n = \sqrt{n}\left(\check{\theta} - c\right)$ and the studentized one $\check{t}_n = \frac{\check{\theta}-c}{\check{\sigma}}$, where $\check{U}_i = \check{Y}_i - \check{X}_i\check{\theta}$ and

$$\check{\sigma} = \sqrt{\left(\frac{1}{n}\sum_{i=1}^n \check{U}_i^2\right)\left[\left(\sum_{i=1}^n \check{X}_i \check{Z}_i\right)' \left(\sum_{i=1}^n \check{Z}_i \check{Z}_i'\right)^{-1} \left(\sum_{i=1}^n \check{X}_i \check{Z}_i\right)\right]^{-1}}.$$

For the over-identified case, we obtain analogous breakdown point properties.

[10] Note that $\left(W_{[n-m+1]}, \ldots, W_{[n]}\right) = \left(W_{(n-m+1)}, \ldots, W_{(n)}\right)$ as $\left\|W_{(n-j+1)}\right\| \to +\infty$ for all $j = 1, \ldots, m$ (because $\left|g\left(W_i, c\right)\right| \to +\infty$ when $\left\|W_i\right\| \to +\infty$).

[11] The results do not change even if we replace the estimator with the tail trimming two-stage GMM estimator.

Proposition 6. *Consider the setup of Section 3. Let* $p_m^{r\#} = P\left(B\left(n, \frac{m}{n}\right) \le r\right)$ *and* $p_m^{r*} = P\left(B\left(n, \left(\frac{m}{n}\right)^2 + \frac{m}{n}\frac{(\bar{g}_{1-,m} - \bar{g}_{2-,m})^2}{v_{11,m}+v_{22,m}-2v_{12,m}}\right) \le r\right)$. *If* $\left\|W_{(n-j+1)}\right\| \to +\infty$ *for all* $j = 1, \ldots, m$, *the followings hold true.*

(i) *For Cases 4, 8, 9 and 12 of Section 3.2, the uniform weight bootstrap t-th quantile* $\check{Q}_t^{\#}$ *from the resamples* $\left|\check{T}_n^{\#}\right|$ *of* $\left|\check{T}_n\right|$ *will diverge to* $+\infty$ *for all* $t > p_m^{r\#}$, *and the implied probability bootstrap t-th quantile* $\check{Q}_t^{*}$ *from the resamples* $\left|\check{T}_n^{*}\right|$ *of* $\left|\check{T}_n\right|$ *will diverge to* $+\infty$ *for all* $t > p_m^{r*}$.

(ii) *For Cases 3, 6, 7, 10, 11, 13, 14 and 15 of Section 3.2, the uniform weight bootstrap t-th quantile* $\check{Q}_t^{\#}$ *from the resamples* $\left|\check{t}_n^{\#}\right|$ *of* $\left|\check{t}_n\right|$ *will diverge to* $+\infty$ *for all* $t > p_m^{r\#}$, *and the implied probability bootstrap t-th quantile* $\check{Q}_t^{*}$ *from the resamples* $\left|\check{t}_n^{*}\right|$ *of* $\left|\check{t}_n\right|$ *will diverge to* $+\infty$ *for all* $t > p_m^{r*}$.

Similar to the just-identified case, both $p_m^{r\#}$ and p_m^{r*} increase as r increases. Thus, tail trimming helps to robustify bootstrap quantiles. Also, when n is large, p_m^{r*} is typically larger than $p_m^{r\#}$ and the implied probability bootstrap tends to be more robust than the uniform weight bootstrap.

The results of this subsection show that Hill and Renault's (2010) trimming approach improves robustness of the both bootstrap methods against outliers. In this setting, we can still observe desirable robustness properties of the implied probability bootstrap in the sense that p_m^{r*} tends to be larger than $p_m^{r\#}$. In Section 5.3, we provide a striking numerical example which shows that the improvement of robustness by tail trimming for the uniform weight bootstrap is rather small comparing to the one for the implied probability bootstrap.

4.2 Remarks

Over-identifying restriction test. Another important issue for over-identified IV regression models is to check the validity of instruments (i.e., test $H_0 : E\left[Z_i\left(Y_i - X_i'\theta\right)\right] = 0$ for some θ against $H_1 : E\left[Z_i\left(Y_i - X_i'\theta\right)\right] \neq 0$ for any θ). This problem is called the over-identifying restriction test. In the GMM context, the over-identifying restriction test statistic (so-called Hansen's J-statistic) is defined as

$$J_n = \left(\sum_{i=1}^{n} Z_i'\left(Y_i - X_i'\hat{\theta}_{GMM}\right)\right)\left(\sum_{i=1}^{n} \hat{U}_i^2 Z_i Z_i'\right)^{-1}\left(\sum_{i=1}^{n} Z_i\left(Y_i - X_i'\hat{\theta}_{GMM}\right)\right),$$

where $\hat{U}_i = Y_i - X_i'\hat{\theta}_{2SLS}$. Hall and Horowitz (1996) and Brown and Newey (2002) demonstrated higher order refinements of the uniform weight bootstrap with recentered moments and implied probability bootstrap, respectively, over the first-order asymptotic approximation which relies on the χ^2 distribution. To apply the breakdown point analysis to the test statistic J_n, we investigate the limiting behaviors of J_n for all cases displayed in Table 4. However, we find that the statistic J_n remains bounded for all the cases. Once we find the case where $J_n \to \infty$ (e.g., the instruments diverge at different rates), we can apply the same argument in the last section to derive the breakdown point properties for bootstrap quantiles.

High dimension moment functions. We can also extend our robustness analysis to the case of $k > 2$. The main issue consists in computing the limit of the implied probability $\pi_{(n)}$ defined in (5). As pointed out in Camponovo and Otsu (2010), if each element of $g\left(W_{(n)}, c\right)$ takes a different limit as $\left\|W_{(n)}\right\| \to +\infty$, it is necessary to explicitly evaluate the limit of the inverse $\left[\frac{1}{n}\sum_{i=1}^{n} g\left(W_i, c\right) g\left(W_i, c\right)'\right]^{-1}$. Consequently, the result may become more complicated and less intuitive. To obtain a comprehensible result, it would be reasonable to consider the case where all elements of $g\left(X_{(n)}, c\right)$ take only two limiting values. In this case, we can split $g\left(X_{(n)}, c\right)$ into two sub-vectors and apply the partitioned matrix inverse formula for $\left[\frac{1}{n}\sum_{i=1}^{n} g\left(X_i, c\right) g\left(X_i, c\right)'\right]^{-1}$ to derive the limit of the implied probability $\pi_{(n)}$.

Time series data. For time series data, the bootstrap methods discussed in this paper need to be modified to reflect dependence of the data generating process. Combining the ideas of Kitamura (1997) and Brown and Newey (2002), in a recent study Allen, Gregory and Shimotsu (2010) proposed an extension of the implied probability bootstrap to a time series context by using block averages of moment functions. We expect that the breakdown point analysis of this paper can be adapted to such a modified bootstrap method (see Camponovo, Scaillet and Trojani, 2010b, for the breakdown point analysis of resampling methods in time series data).

5 Simulation

To illustrate the breakdown point properties of the bootstrap methods described above, we conduct Monte Carlo experiments. To ensure that all the implied probabilities are non-negative, we employ a shrinkage approach suggested by Antoine, Bonnal and Renault (2007) (i.e., $\tilde{\pi}_i = \frac{1}{1+\epsilon_n}\pi_i + \frac{\epsilon_n}{1+\epsilon_n}\frac{1}{n}$ with $\epsilon_n = -n \min\left\{\min_{1\le i\le n} \pi_i, 0\right\}$). As pointed out in Antoine, Bonnal and Renault (2007) this approach preserves the order of the implied probabilities, has no impact when the imply probabilities are already non-negative, and assigns zero probability only to the observation associated to the smallest probability when it is negative.

5.1 Just-identified case

We consider iid samples of size $n = 50$ generated from $Y_i = X_i\theta_0 + U_i$ and $X_i = Z_i\pi + V_i$, where $Z_i \sim N(0,1)$, $\begin{pmatrix} U_i \\ V_i \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.2 \\ 0.2 & 1 \end{pmatrix}\right)$, and $\pi = 1$. The true parameter value is set as $\theta_0 = 0$. We are interested in testing the null hypothesis $H_0 : \theta_0 = 0$ against the alternative $H_1 : \theta_0 \neq 0$. To study the robustness properties of the bootstrap methods, we consider two situations: (i) $(W_1, \ldots, W_n)$ are generated from the above model (No contamination), and (ii) $\left(\tilde{W}_1, \ldots, \tilde{W}_n\right)$ with $\tilde{W}_{(i)} = W_{(i)}$ for $i = 1, \ldots, 49$ and $\tilde{W}_{(50)} = \left(\tilde{Y}_{(50)}, \tilde{X}_{(50)}, \tilde{Z}_{(50)}\right) = \left(C, X_{(50)}, Z_{(50)}\right)$ with $C = 5$ and 10 (Contamination on Y). This setup for the contamination corresponds to Case 3 in Section 2.2 and Proposition 1 says that as $\left\|W_{(n)}\right\| \to \infty$, the uniform weight bootstrap t-th quantile $Q_t^{\#}$ from the resamples $\left|T_n^{\#}\right|$ of the non-studentized statistic T_n will diverge to $+\infty$ for all $t > p^{\#} = P\left(B\left(n, \frac{1}{n}\right) = 0\right)$, and the implied probability bootstrap t-th quantile Q_t^* from the resamples $|T_n^*|$ of T_n will diverge to $+\infty$ for all $t > p^* = P\left(B\left(n, \frac{1}{n^2}\right) = 0\right)$. On the other hand, the bootstrap quantiles for the studentized statistic t_n does not show such divergence properties.

Tables 1 and 2 report the Monte Carlo means of the uniform weight bootstrap quantiles $Q_t^{\#}$ and implied probability bootstrap quantiles Q_t^* for t =0.9, 0.95, and 0.99 for the statistics $|T_n|$ and $|t_n|$, respectively. The number of bootstrap replications is 99 for each Monte Carlo sample. The number of Monte Carlo replications is $1,000$.

For non-contaminated samples, both the uniform weight and implied probability bootstrap quantiles are very close to the true quantile values. As expected from Proposition 1, the presence of outliers decreases the accuracy of the bootstrap approximations of the non-studentized statistic distribution. Because of the higher probability to select outliers, the uniform weight bootstrap resamples tend to contain the outlier $W_{(50)}$ more frequently. Therefore, the uniform weight bootstrap quantiles tend to be larger than the true quantile values. In contrast, since the implied probability $\pi_{(50)}$ for the outlier $W_{(50)}$ is very small, the implied probability bootstrap is able to mitigate this robustness problem. For example, when $Y_{(n)} = 10$, for the non-studentized statistic $|T_n|$ with the true .99-th quantile $Q_{.99} = 4.8683$, the uniform weight bootstrap quantile is $Q_{.99}^{\#} = 6.3541$, while the implied probability bootstrap quantile is $Q_{.99}^* = 4.2179$. As expected, Table 2 shows that both bootstrap methods provide valid approximations for the distribution of the studentized statistic $|t_n|$ against this particular contamination.

5.2 Over-identified case

We consider iid samples of size $n = 50$ generated from $Y_i = X_i\theta_0 + U_i$ and $X_i = Z_i'\pi + V_i$, where $Z_i = \begin{pmatrix} Z_{1i} \\ Z_{2i} \end{pmatrix} \sim N\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$, $\begin{pmatrix} U_i \\ V_i \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.2 \\ 0.2 & 1 \end{pmatrix}\right)$, and $\pi = (0.5, 0.5)'$. The true parameter value is set as $\theta_0 = 1$. We are interested in testing the null hypothesis $H_0 : \theta_0 = 1$ against the alternative $H_1 : \theta_0 \neq 1$. We consider two situations: (i) $(W_1, \ldots, W_n)$ are generated from the above model (No contamination), and (ii) $\left(\tilde{W}_1, \ldots, \tilde{W}_n\right)$ with $\tilde{W}_{(i)} = W_{(i)}$ for $i = 1, \ldots, 49$ and $\tilde{W}_{(50)} = \left(\tilde{Y}_{(50)}, \tilde{X}_{(50)}, \tilde{Z}_{1(50)}, \tilde{Z}_{2(50)}\right) = \left(Y_{(50)}, C, Z_{1(50)}, Z_{2(50)}\right)$ with $C = 10$ and 15 (Contamination on X). This setup for the contamination corresponds to Case 3 in Section 3.2, and Proposition 3 says that as $\left\|W_{(n)}\right\| \to \infty$, the uniform weight bootstrap t-th quantile $Q_t^{\#}$ from the resamples $\left|t_n^{\#}\right|$ of the studentized statistic t_n will diverge to $+\infty$ for all $t > p^{\#} = P\left(B\left(n, \frac{1}{n}\right) = 0\right)$, and the implied probability bootstrap t-th quantile Q_t^* from the resamples $|t_n^*|$ of t_n will diverge to $+\infty$ for all $t > p^* = P\left(B\left(n, \frac{1}{n^2} + \frac{1}{n}\frac{(\bar{g}_{1-} - \bar{g}_{2-})^2}{v_{11} + v_{22} - 2v_{12}}\right) = 0\right)$. On the other hand, the bootstrap quantiles for the non-studentized statistic T_n does not show such divergence properties.

Tables 3 and 4 report the Monte Carlo means of the uniform weight bootstrap quantiles $Q_t^{\#}$ and implied probability bootstrap quantiles Q_t^* for t =0.9, 0.95, and 0.99 for the statistics $|T_n|$ and $|t_n|$, respectively. The number of bootstrap replications is 99 for each Monte Carlo sample. The number of Monte Carlo replications is $1,000$.

Similar to the just-identified case, without contamination, both the uniform weight and implied probability bootstraps are accurate for the true quantiles. It should be noted that the implied probability bootstrap quantiles Q_t^* provide reasonable approximations to the true quantiles Q_t even under contaminations. On the other hand, the uniform weight bootstrap quantiles $Q_t^{\#}$ tend to be larger than the true ones. For example, when $X_{(n)} = 15$, for the studentized statistic $|t_n|$ with the true .99-th quantile $Q_{.99} = 3.0748$, the bootstrap approximations are $Q_{.99}^{\#} = 4.8735$ and $Q_{.99}^* = 3.2908$. Finally, as expected from the theoretical results in Section 3.2, the bootstrap quantiles for the non-studentized statistic $|T_n|$ are relatively stable and do not show such divergence properties for this particular contamination.

5.3 Tail trimming

We finally consider the effect of tail trimming introduced by Hill and Renault (2010) to bootstrap quantiles. We set $r = 2$ for tail trimming. Table 5 reports the results for the non-studentized statistic $\check{T}_n$ with tail trimming under the data generating process in Section 5.1 (just-identified case). In particular, we consider two situations: (i) $(W_1, \ldots, W_n)$ are generated according to the true data generating process (No contamination), and (ii) $\left(\tilde{W}_1, \ldots, \tilde{W}_n\right)$ with $\tilde{W}_{(i)} = W_{(i)}$ for $i = 1, \ldots, 49$ and $\tilde{W}_{(50)} = \left(\tilde{Y}_{(50)}, \tilde{X}_{(50)}, \tilde{Z}_{(50)}\right) = \left(C, X_{(50)}, C\right)$ with $C = 5$ and 10 (Contamination on Y and Z). Table 6 reports the results for the the studentized statistic $\check{t}_n$ with tail trimming under the data generating process in Section 5.2 (over-identified case). We consider two situations: (i) $(W_1, \ldots, W_n)$ are generated according to the true data generating process (No contamination), and (ii) $\left(\tilde{W}_1, \ldots, \tilde{W}_n\right)$ with $\tilde{W}_{(i)} = W_{(i)}$ for $i = 1, \ldots, 49$ and $\tilde{W}_{(50)} = \left(\tilde{Y}_{(50)}, \tilde{X}_{(50)}, \tilde{Z}_{1(50)}, \tilde{Z}_{2(50)}\right) = \left(Y_{(50)}, C, C, Z_{2(50)}\right)$ with $C = 10$ and 15 (Contamination on X and Z_1). Tables 5 and 6 report the Monte Carlo means of the uniform weight bootstrap quantiles $\check{Q}_t^{\#}$ and implied probability bootstrap quantiles $\check{Q}_t^*$ for t =0.9, 0.95, and 0.99 for the statistics $\left|\check{T}_n\right|$ and $\left|\check{t}_n\right|$, respectively. The number of bootstrap replications is 99 for each Monte Carlo sample. The number of Monte Carlo replications is $1,000$.

Again without contamination, both bootstrap quantiles are very close to the true values. Under contaminations, the implied probability bootstrap provides reasonable approximations to the true quantiles for all cases. However, as Table 5 shows, the uniform weight bootstrap could be severely biased. For example, when $Y_{(n)} = Z_{(n)} = 10$, for the non-studentized statistic $\left|\check{T}_n\right|$ with the true .99-th quantile $\check{Q}_{.99} = 3.0221$, the implied probability bootstrap provides $\check{Q}_{.99}^{\#} = 3.9878$, but the uniform weight bootstrap provides $\check{Q}_{.99}^{\#} = 196.3904$, which is a strikingly large value. This simulation suggests that the implied probability bootstrap is particularly attractive to approximate the distributions of the statistics with tail trimming.

5.4 Summary

The Monte Carlo experiments show that in absence of contamination both the uniform weight and implied probability bootstraps provide accurate approximations of the test statistic distributions. The presence of contamination reduces the reliability of the bootstrap methods. Since the uniform weight bootstrap selects outliers with high probability, the uniform weight bootstrap quantiles tend to be larger than the true quantiles. The implied probability bootstrap is in part able to overcome this lack of robustness. Nevertheless, in some cases the implied probabilities of outliers are very small. In such situations the implied probability bootstrap quantiles are smaller than the true quantiles and tend to be close to the true quantiles without contamination.

The tail trimming approach introduced in Hill and Renault (2010) increases the robustness of the test statistics. Researchers need to be cautious for applying the uniform weight bootstrap for tail trimming statistics since its bootstrap reamples tend to contain more outliers than the original sample. In contrast, the implied probability bootstrap provides reasonable approximations to the tail trimming statistics.

Unreported Monte Carlo results for other types of outliers also confirm these conclusions.

6 Conclusion

This paper studies robustness of the uniform weight and implied probability bootstrap inference methods for instrumental variable regression models. In particular, we analyze the breakdown point properties of the quantiles of those bootstrap methods for studentized and non-studentized test statistics for parameter hypotheses. Simulation studies illustrate the theoretical findings. Our breakdown point analysis can be an informative guideline for applied researchers to decide which bootstrap method should be applied under existence of outliers. It is important to extend our analysis to dependent data setups, where different bootstrap methods, such as blocked bootstrap, need to be employed. Also, it is interesting to analyze the breakdown point properties for other implied probabilities, such as the exponential tilting weights obtained from the information projection by the Boltzmann-Shannon entropy.

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$\lvert T_n \rvert$		No Con.	Contamination	
			$Y_{(n)} = 5$	$Y_{(n)} = 10$
True	$Q_{.9}$	1.7059	2.0870	2.9616
	$Q_{.95}$	2.0968	2.5265	3.5943
	$Q_{.99}$	2.9004	3.4971	4.8683
Uniform	$Q^{\#}_{.9}$	1.7635	2.3302	3.2836
	$Q^{\#}_{.95}$	2.1732	2.8532	3.9836
	$Q^{\#}_{.99}$	3.1127	4.2318	6.3541
Implied	$Q^{*}_{.9}$	1.7511	2.0108	2.3405
	$Q^{*}_{.95}$	2.1525	2.5083	3.0213
	$Q^{*}_{.99}$	3.1083	3.7109	4.2179

Table 1: **Quantiles of the uniform weight and implied probability bootstrap.** "No Con." means "No Contamination". The rows labelled "True" report the simulated quantiles of the distribution of the non-studentized statistic $\lvert T_n \rvert$ based on 20,000 realizations. The rows labelled "Uniform" report the uniform weight bootstrap quantiles. The rows labelled "Implied" report the implied probability bootstrap quantiles.

$\lvert t_n \rvert$		No Con.	Contamination	
			$Y_{(n)} = 5$	$Y_{(n)} = 10$
True	$Q_{.9}$	1.6683	1.6430	1.6268
	$Q_{.95}$	1.9409	1.9348	1.9216
	$Q_{.99}$	2.5713	2.5588	2.5311
Uniform	$Q^{\#}_{.9}$	1.6278	1.6250	1.8365
	$Q^{\#}_{.95}$	1.9521	2.0006	2.2963
	$Q^{\#}_{.99}$	2.7366	2.8185	3.0861
Implied	$Q^{*}_{.9}$	1.6085	1.6079	1.6364
	$Q^{*}_{.95}$	1.9217	1.9251	1.9963
	$Q^{*}_{.99}$	2.6977	2.7184	2.8990

Table 2: **Quantiles of the uniform weight and implied probability bootstrap.** "No Con." means "No Contamination". The rows labelled "True" report the simulated quantiles of the distribution of the studentized statistic $\lvert t_n \rvert$ based on 20,000 realizations. The rows labelled "Uniform" report the uniform weight bootstrap quantiles. The rows labelled "Implied" report the implied probability bootstrap quantiles.

$\lvert T_n \rvert$		No Con.	Contamination	
			$X_{(n)} = 10$	$X_{(n)} = 15$
	$Q_{.9}$	1.0114	1.4327	1.8345
True	$Q_{.95}$	1.2082	1.6697	2.0805
	$Q_{.99}$	1.5963	2.1081	2.5421
	$Q^{\#}_{.9}$	1.0028	1.3595	1.7055
Uniform	$Q^{\#}_{.95}$	1.2085	1.6581	2.0077
	$Q^{\#}_{.99}$	1.7108	2.2955	2.6912
	$Q^{*}_{.9}$	0.9813	1.2654	1.5707
Implied	$Q^{*}_{.95}$	1.1835	1.5579	1.8782
	$Q^{*}_{.99}$	1.6799	2.0177	2.2108

Table 3: **Quantiles of the uniform weight and implied probability bootstrap.** "No Con." means "No Contamination". The rows labelled "True" report the simulated quantiles of the distribution of the non-studentized statistic $\lvert T_n \rvert$ based on 20,000 realizations. The rows labelled "Uniform" report the uniform weight bootstrap quantiles. The rows labelled "Implied" report the implied probability bootstrap quantiles.

$\lvert t_n \rvert$		No Con.	Contamination	
			$X_{(n)} = 10$	$X_{(n)} = 15$
	$Q_{.9}$	1.6905	1.8376	2.0544
True	$Q_{.95}$	2.0099	2.1775	2.3962
	$Q_{.99}$	2.6506	2.8162	3.0748
	$Q^{\#}_{.9}$	1.6901	2.1882	2.8206
Uniform	$Q^{\#}_{.95}$	2.0451	2.7615	3.3728
	$Q^{\#}_{.99}$	2.8885	3.9035	4.8735
	$Q^{*}_{.9}$	1.6981	1.8443	1.9642
Implied	$Q^{*}_{.95}$	2.0337	2.2425	2.3451
	$Q^{*}_{.99}$	2.8524	3.1438	3.2908

Table 4: **Quantiles of the uniform weight and implied probability bootstrap.** "No Con." means "No Contamination". The rows labelled "True" report the simulated quantiles of the distribution of the studentized statistic $\lvert t_n \rvert$ based on 20,000 realizations. The rows labelled "Uniform" report the uniform weight bootstrap quantiles. The rows labelled "Implied" report the implied probability bootstrap quantiles.

$\lvert \check{T}_n \rvert$		No Con.	Contamination	
			$Y_{(n)} = Z_{(n)} = 5$	$Y_{(n)} = Z_{(n)} = 10$
True	$\check{Q}_{.9}$	1.7341	1.8426	1.8426
	$\check{Q}_{.95}$	2.1321	2.2214	2.2214
	$\check{Q}_{.99}$	2.9980	3.0221	3.0221
Uniform	$\check{Q}^{\#}_{.9}$	1.8225	6.3818	29.3635
	$\check{Q}^{\#}_{.95}$	2.2266	10.2179	53.4920
	$\check{Q}^{\#}_{.99}$	3.0895	27.1615	196.3904
Implied	$\check{Q}^{*}_{.9}$	1.8389	2.3933	2.3488
	$\check{Q}^{*}_{.95}$	2.2574	2.9153	2.7979
	$\check{Q}^{*}_{.99}$	3.1003	4.1754	3.9878

Table 5: **Quantiles of the uniform weight and implied probability bootstrap.** "No Con." means "No Contamination". The rows labelled "True" report the simulated quantiles of the distribution of the non-studentized statistic $\lvert \check{T}_n \rvert$ based on 20,000 realizations. The rows labelled "Uniform" report the uniform weight bootstrap quantiles. The rows labelled "Implied" report the implied probability bootstrap quantiles.

$\lvert \check{t}_n \rvert$		No Con.	Contamination	
			$X_{(n)} = Z_{1(n)} = 10$	$X_{(n)} = Z_{1(n)} = 15$
True	$\check{Q}_{.9}$	1.8655	1.9475	1.9566
	$\check{Q}_{.95}$	2.2176	2.2947	2.3068
	$\check{Q}_{.99}$	2.9203	2.9700	2.9812
Uniform	$\check{Q}^{\#}_{.9}$	1.9246	4.0810	6.8627
	$\check{Q}^{\#}_{.95}$	2.3272	5.2062	8.7074
	$\check{Q}^{\#}_{.99}$	3.1718	7.5485	12.2239
Implied	$\check{Q}^{*}_{.9}$	1.9737	2.2310	2.2038
	$\check{Q}^{*}_{.95}$	2.3262	2.6594	2.6351
	$\check{Q}^{*}_{.99}$	3.1456	3.7534	3.6774

Table 6: **Quantiles of the uniform weight and implied probability bootstrap.** "No Con." means "No Contamination". The rows labelled "True" report the simulated quantiles of the distribution of the studentized statistic $\lvert \check{t}_n \rvert$ based on 20,000 realizations. The rows labelled "Uniform" report the uniform weight bootstrap quantiles. The rows labelled "Implied" report the implied probability bootstrap quantiles.