FINANCIAL CONTROL OF A COMPETITIVE ECONOMY WITHOUT RANDOMNESS

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Financial Control of a Competitive Economy Without Randomness

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Abstract

The monetary and fiscal control of a simple economy without outside randomness is studied here from the micro-economic basis of a strategic market game. The government's bureaucracy is treated as a public good that provides services at a cost. A conventional public good is also considered.

1 Introduction

In this and two companion essays, macro-economic problems of the monetary and fiscal control of an economy are viewed building up from the microeconomic basis of a strategic market game. In previous works (cf. [4], [5], [6], [7]) we considered a highly simplified economy with either a representative agent or a continuum of small, strategically independent agents. We continue this approach here, with a more explicit consideration of government and the role of bureaucracy in policing the economy.

There is a vast macroeconomic literature on monetary and fiscal policy. Our contribution is complementary with much of this literature, in the sense that we present in explicit detail closed strategic models for the microeconomic processes of the economy. Our purpose is to illustrate at a high level of simplification some fundamental features of government control of taxation, bureaucratic enforcement of default conditions, and public debt creation. These are treated with emphasis on information conditions, on the equations of motion, on the logical necessity for default rules and their enforcement, and on the minimal requirements for the various institutions in the economy.

The simple, low-dimensional models treated here, like many such low-dimensional models studied in the literature (cf. Lucas [10], [11], Lucas and Stokey [12], [13] among others), are toy-like when compared with the large macro-economic models utilized by practitioners (cf. Fair [3]). These simple models may provide some insight into qualitative properties of an economy and raise questions about the logical completeness of the large models. However, they are not substitutes.

This essay is devoted to non-stochastic models, prior to considering the more difficult stochastic versions. An important feature of non-stochastic models with simultaneous moves by small homogeneous agents is that the symmetric equilibria coincide with the equilibria of representative agent models.

One of our motivations for treating the non-stochastic models is that, although the solutions are relatively simple, careful process modelling of the laws of motion for the agents requires laying out virtually all of the details required to study the stochastic models. Another reason to study these relatively simple models is that they often admit closed-form solutions, which is rarely the case for stochastic models with independent agents.

Outline. We shall consider five basic models. In each of the models there is a continuum of producer-consumer agents who seek to maximize their total discounted utility from consumption over a countably infinite sequence of time-periods. In Models 1 and 2 the government taxes the agents, and uses the revenue to support a bureaucracy that provides government services. In Models 3,4, and 5 a conventional public good is supported. In the first three models a central bank sets an interest rate and stands ready to make loans or accept deposits. In the fourth model, one-period bonds are introduced. Model 5 considers consols or perpetuities.

For each of the models we construct a type-symmetric stationary equilibrium. Some of the details of the proofs are in the Appendix.

2 Model 1: An economy with taxation to support a government bureaucracy

All of the models we consider will involve a government and a continuum $I = [0, 1]$ of producer/consumer agents. In each model there will also be a single private, perishable good, which serves both as a consumption and a production commodity.

The first model involves in addition another continuum $J = [0, \delta]$ of government bureaucrats. The size $\delta > 0$ of the bureaucracy is exogenously given. (The question of the optimal size of the bureaucracy has been considered elsewhere by Shubik and Smith [17].) The government taxes the producer/consumers in order to support the bureaucrats. The productivity of the bureaucracy is implicit in the model and can be interpreted as supplying the basic running of the government and the enforcement of the laws. For simplicity we assume that the bureaucrats obtain a tax-free income. This saves on some accounting but yields substantially the same problem. We turn now to a detailed description of the model.

At the beginning of every time period $n = 1, 2, \ldots$, every producer/consumer agent $\alpha \in I$ holds cash $m_n^\alpha \geq 0$ and a quantity $q_n^\alpha \geq 0$ of goods. Every bureaucrat $\gamma \in J$ holds cash $m_n^\gamma \geq 0$ but no goods. (The bureaucrats neither hold nor produce the private good.) Every agent α sells his goods in a market, so that the total amount of goods sold in period n is

$$Q_n = \int_I q_n^\alpha \, d\alpha \,. \tag{1}$$

In every period n, every agent $\alpha \in I$ bids cash b_n^α in the market in order to obtain goods for consumption and as input for production in the next period; likewise, every bureaucrat $\gamma \in J$ bids cash b_n^γ for goods. The total amount of cash bid is therefore

$$B_n = B_n^A + B_n^\Gamma = \int_I b_n^\alpha \, d\alpha + \int_J b_n^\gamma \, d\gamma \,; \tag{2}$$

thus the price of the good is formed endogenously, by supply and demand, as

$$p_n = \frac{B_n}{Q_n} \,. \tag{3}$$

Then each agent $\alpha \in I$ receives in cash the revenue $p_n q_n^\alpha$ from the sale of his goods. However, this income is taxed at a rate $\theta \in (0,1)$ set by the government, so that α's *net income* is $\bar{\theta} p_n q_n^\alpha$, where $\bar{\theta} := 1 - \theta$.

Prior to bidding in the private goods market, each agent $\alpha \in I$ can borrow from, or deposit into, a central bank, which sets the rate of interest $\rho > 0$. The bank predicts the price p_n to be $\widehat{p}_n$ and allows agent α to borrow up to $(\bar{\theta}\, \widehat{p}_n q_n^\alpha)/(1+\rho)$, the amount that α is expected to be able to repay with interest. Thus, α is able to bid any amount

$$b_n^\alpha \in \left[0,\, m_n^\alpha + \frac{\bar{\theta}\, \widehat{p}_n q_n^\alpha}{1+\rho} \right] . \tag{4}$$

In a rational expectations equilibrium, we shall have $\widehat{p}_n = p_n$.

The agent receives his bid's worth b_n^α / p_n in goods, then selects a portion $k_n^\alpha \in [0, b_n^\alpha / p_n]$ to be put into production. He consumes the remaining goods $x_n^\alpha = (b_n^\alpha / p_n) - k_n^\alpha$, and receives $u(x_n^\alpha)$ in utility, where $u : [0,\infty) \to [0,\infty)$ is a concave, increasing, differentiable *utility function*, such that $u(0) = 0$.

Each agent $\alpha \in I$ seeks to maximize the total discounted utility

$$\sum_{n=1}^{\infty} \beta^{n-1} u(x_n^\alpha)$$

from consumption of the perishable good, where $\beta \in (0,1)$ is the discount factor. Agent α begins period $n+1$ with cash

$$m_{n+1}^\alpha = (1+\rho)(m_n^\alpha - b_n^\alpha) + \bar{\theta} p_n q_n^\alpha. \tag{5}$$

The producer/consumer agents have a common *production function* $f : [0,\infty) \to [0,\infty)$ which is also assumed to be concave, increasing, and continuously differentiable, with $f(0) = 0$. We further assume that

$$f'(0) = +\infty \,, \qquad \lim_{k\to\infty} f'(k) = 0 \,. \tag{6}$$

The input k_n^α of agent α results in the output $f(k_n^\alpha)$. Agent α begins period $n+1$ with goods

$$q_{n+1}^\alpha = f(k_n^\alpha) + y. \tag{7}$$

The quantity $y \geq 0$ of goods can be viewed as an additional endowment. (It is included here, with an eye toward generalizing to a random variable Y in future work.)

The situation for a bureaucrat $\gamma \in J$ is similar, except that the bureaucrats do not hold goods, do not produce them, and pay no income tax. The bureaucrats receive as income the tax revenues collected from the producer/consumer agents. The total revenue collected in period n is $\theta B_n = \theta p_n Q_n$, and this amount is divided equally among the bureaucrats. Since the total size of the bureaucracy is δ, each of them receives the cash amount $\theta p_n Q_n/\delta$ at the end of the period. Prior to bidding in period n, the bureaucrat γ is allowed to borrow from the bank an amount based on his expected income and can thus bid any amount

$$b_n^\gamma \in \left[0,\, m_n^\gamma + \frac{\theta\, \widehat{p}_n Q_n}{\delta(1+\rho)}\right]. \tag{8}$$

As already mentioned above, the bank's predicted price $\widehat{p}_n$ will agree with the actual price p_n in rational expectations equilibrium. The bureaucrat γ then receives an amount $x_n^\gamma = b_n^\gamma/p_n$ of goods, consumes it all, and thereby gets $u(x_n^\gamma)$ in utility. Each bureaucrat γ seeks to maximize the total discounted utility

$$\sum_{n=1}^{\infty} \beta^{n-1} u(x_n^\gamma).$$

(Our analysis would be unchanged if the bureaucrats had a different utility function from that of the producer-consumer agents.) The bureaucrat γ begins period $n+1$ with cash

$$m_{n+1}^\gamma = (1+\rho)(m_n^\gamma - b_n^\gamma) + \frac{\theta p_n Q_n}{\delta}. \tag{9}$$

We are now ready to construct an equilibrium for this model. However, we pause first for a brief aside on solution concepts.

2.1 An aside on solution concepts

For each of our models, we shall construct a type-symmetric Nash equilibrium. That is, we shall find initial conditions and strategies for the agents such that all agents of a given type use the same strategy and every agent is playing optimally, given that all other agents follow the prescribed strategies. These equilibria will be *almost subgame perfect*, in the sense that their restriction to any subgame that occurs along the equilibrium path will again be a Nash equilibrium. However, the strategies need not correspond to a Nash equilibrium at subgames which are off the equilibrium path. Thus, our equilibria need not be fully *subgame perfect* as defined by Selten [15].

For these equilibria, the phenomenon called "time inconsistency" by macro-economists (cf. [9], [2]) does not arise. From the viewpoint of formal game theory, this notion raises problems in the definition of the extensive form of a game and on whether non-binding language statements are formally modelled as moves in the game. Many years ago Schelling [14] launched a well-directed critique against formal game theory; but apart from several intuitively attractive observations on the problems of pre-commitment and threat, he offered no formal solution.

Given certain reasonable contexts, such as individual behavior in mass markets with or without the presence of a large (atomic) agent as a controller or government, the subgame perfect equilibrium solution can be regarded as attractive and is reflected in the mathematical formulation of parallel dynamic programs and rational expectations. The phrase "rational expectations" itself appears to be nothing more than the consistency of expectations required at any noncooperative equilibrium point [1].

2.2 An equilibrium for Model 1

To construct a type-symmetric equilibrium, we suppose that all producer/consumer agents $\alpha \in I$ begin play with the same amounts of cash $m^\alpha = m_A > 0$ and goods $q^\alpha = q_A > 0$, respectively; and that every bureaucrat $\gamma \in J$ begins play with cash $m^\gamma = m_\Gamma > 0$ at hand. Let

$$M := \int_I m^\alpha \, d\alpha + \int_J m^\gamma \, d\gamma = m_A + \delta \cdot m_\Gamma$$

be the total cash holdings of both types, and let

$$Q := \int_I q^\alpha \, d\alpha = q_A$$

be the total amount of goods available.

Next, let us assume that every agent $\alpha \in I$ bids the same amount of cash

$$b^\alpha = b_A = a_A \cdot M$$

for goods, and that every bureaucrat $\gamma \in J$ bids

$$b^\gamma = b_\Gamma = a_\Gamma \cdot M$$

[1] We have not even mentioned the difficulties posed by lack of common knowledge and incomplete knowledge concerning the rules of the game. These extra factors suggest the dangers of taking rational expectations too seriously. We work with it primarily because the model of an optimizing individual with limited intelligence, computational ability, and memory is far harder than *homo oeconomicus.*

for suitable positive constants a_A and a_Γ. The total bid is then

$$B = \int_I b^\alpha \, d\alpha + \int_J b^\gamma \, d\gamma = b_A + \delta b_\Gamma = (a_A + \delta a_\Gamma) M = aM \,,$$

where

$$a := a_A + \delta a_\Gamma \,.$$

The price of goods is formed as

$$p = \frac{B}{Q} = \frac{aM}{Q} \,.$$

We assume also that each agent $\alpha \in I$ inputs the same quantity k for production, and that

$$Q = q_A = f(k) + y \,.$$

Thus the quantity of goods will again be Q in the next period.

By the laws of motion (5) and (9), in the next period agents and bureaucrats will have cash holdings

$$\widetilde{m}_A = (1+\rho)(m_A - b_A) + \bar{\theta} p Q = (1+\rho)(m_A - b_A) + \bar{\theta} B \tag{10}$$

and

$$\widetilde{m}_\Gamma = (1+\rho)(m_\Gamma - b_\Gamma) + \frac{\theta p Q}{\delta} = (1+\rho)(m_\Gamma - b_\Gamma) + \frac{\theta B}{\delta} \,, \tag{11}$$

respectively. The total amount of cash in the next period will be

$$\widetilde{M} = \widetilde{m}_A + \delta \widetilde{m}_\Gamma = (1+\rho)(M - B) + B = (1+\rho)(M - aM) + aM = \tau M \,, \tag{12}$$

where

$$\tau := 1 + \rho - \rho a \tag{13}$$

is the *rate of inflation* (or deflation). If the agents and bureaucrats continue to bid the same proportions of their money, then the price in the next period will be

$$\widetilde{p} = \frac{a\widetilde{M}}{Q} = \frac{a\tau M}{Q} = \tau p \,. \tag{14}$$

Thus, prices will inflate at the same rate as the money supply.

Theorem 1: *Suppose that every producer/consumer agent $\alpha \in I$ begins play with cash $m^\alpha = \bar{\theta} M$ and goods $q^\alpha = f(k_1) + y$, where $M > 0$ and the input level k_1 satisfies* [2]

$$f'(k_1) \;=\; (1+\rho)/(\beta\bar{\theta}) \,;$$

[2] The existence of such a k_1 follows from (6). Indeed, our only use of (6) is to guarantee the existence of k_1.

and that every bureaucrat $\gamma \in J$ begins with cash $m^\gamma = \theta M/\delta$.

Then there is an equilibrium for Model 1, in which every agent $\alpha \in I$ inputs k_1 for production in every period, and all agents $\alpha \in I$ and bureaucrats $\gamma \in J$ bid the constant proportion a of their cash-at-hand in every period, where

$$a = \frac{(1+\rho)(1-\beta)}{\rho} . \qquad (15)$$

The proof of Theorem 1 is in the Appendix.

In the equilibrium of Theorem 1, we have $a_A = a\bar{\theta}$ and $a_B = a\theta/\delta$, with a given by (15). Thus, by (13) and (15), money and prices inflate at the same rate

$$\tau = (1+\rho) - \rho \cdot \frac{(1+\rho)(1-\beta)}{\rho} = \beta(1+\rho) , \qquad (16)$$

in agreement with the classical Fisher equation. Also, the quantity of goods produced stays fixed at

$$Q = \int_I q^\alpha \, d\alpha = f(k_1) + y. \qquad (17)$$

The government influences inflation and production through its choice of the tax rate θ and the interest rate ρ. Clearly, there is no inflation if the central bank sets

$$1 + \rho = \frac{1}{\beta} .$$

However, production is an increasing function of the input level

$$k_1 = (f')^{-1} \left(\frac{1+\rho}{\beta(1-\theta)} \right) .$$

Since f' is a decreasing function, we see that lower interest rates increase production. The same is true of lower tax rates. However, a lower tax rate also results in less money to support the bureaucracy and the government services it provides. This trade-off between private and public production will be made explicit in Model 3 below.

A simple example is illustrative.

Example 1: Consider the stationary equilibrium of Theorem 1 in the special case when the production function of the agents is $f(k) = 2\sqrt{k}$, and $y = 0$. Then

$$f'(k_1) = \frac{1}{\sqrt{k_1}} = \frac{1+\rho}{\beta(1-\theta)}$$

so that

$$k_1 = \left(\frac{\beta(1-\theta)}{1+\rho} \right)^2 \qquad \text{and} \qquad Q = f(k_1) = \frac{2\beta(1-\theta)}{1+\rho} .$$

In a period when the money supply is M, the consumer/producer agents bid $b_A = a\bar{\theta}M$, the bureaucrats bid $b_\Gamma = a\theta M/\delta$, and the price of the good is formed as $p = aM/Q$. Thus, in every period, the agents and bureaucrats consume, respectively,

$$x_A = \frac{b_A}{p} - k_1 = \bar{\theta}Q - k_1 = \frac{\beta(1-\theta)^2}{1+\rho}\left(2 - \frac{\beta}{1+\rho}\right)$$

and

$$x_\Gamma = \frac{b_\Gamma}{p} = \frac{2\beta\theta(1-\theta)}{\delta(1+\rho)}.$$

Their total discounted utilities are thus

$$\frac{u(x_A)}{1-\beta} \quad \text{and} \quad \frac{u(x_\Gamma)}{1-\beta},$$

respectively. It is easy to check that x_A increases as the tax rate θ and the interest rate ρ decrease. On the other hand, x_B increases as ρ decreases, but has a maximum in θ at $\theta = 1/2$.

3 Model 2: Unlimited lending with penalties for default

In Model 1 individuals were permitted to borrow up to a level limited by their expected income for the period. Instead of using this "secured lending" rule, we now consider an economy in which unsecured lending is allowed. But with unsecured lending there is a need for default penalties to guard against failure to repay, and the penalties require enforcement. (For this model one can view the contribution of the bureaucracy as being the enforcement of the laws, rules and instruments promoting efficient and honest trade [17].)

In most respects Model 2 is the same as Model 1. There is a continuum $I = [0, 1]$ of producer-consumer agents and a continuum $J = [0, \delta]$ of bureaucrats. The producer-consumers hold cash and goods, and are taxed to pay the bureaucrats who hold cash but no goods. The critical difference from Model 1 is that the bank no longer imposes any bound on lending to either agents or bureaucrats.

At the start of any period when loans are due, we assume in Model 2 that the bank can collect any liquid assets held by a debtor and enforce a penalty in units of negative utility, proportional to any debt that remains. The debt is then erased, and the individual is permitted to borrow again. When the penalty for default is sufficiently severe, the agents and bureaucrats will not wish to incur large debts and will themselves limit their borrowing.

To make these assumptions precise, consider first a producer/consumer agent $\alpha \in I$ who begins a period with cash $m^\alpha \geq 0$ and goods $q^\alpha \geq 0$. The agent is permitted to bid *any* amount $b^\alpha \geq 0$ for goods and input $k^\alpha \in [0, b^\alpha/p]$ for production.

The agent then consumes $x^\alpha = (b^\alpha/p) - k^\alpha$ and receives $u(x^\alpha)$ in utility. Here p is the price formed in the market as in Model 1.

Agent α begins the next period at $(\widetilde{m}^\alpha, \widetilde{q}^\alpha)$, where

$$\widetilde{m}^\alpha = (1+\rho)(m^\alpha - b^\alpha) + \bar{\theta} p q^\alpha \qquad \text{and} \qquad \widetilde{q}^\alpha = f(k^\alpha) + y\,. \tag{18}$$

If $\widetilde{m}^\alpha \geq 0$, play continues as before. However, if $\widetilde{m}^\alpha < 0$, the agent is in default and is punished in utility by the amount

$$\zeta \cdot \frac{\widetilde{m}^\alpha}{\widetilde{p}}\,, \tag{19}$$

where ζ is a positive parameter and $\widetilde{p}$ is the price of the good in the period when the punishment takes place.

The situation of a bureaucrat $\gamma \in J$ with cash $m^\gamma \geq 0$ is analogous. The bureaucrat can bid any amount $b^\gamma \geq 0$ for goods, consume $x^\gamma = b^\gamma/p$, receive $u(x^\gamma)$ in utility, and begin the next period in cash-position

$$\widetilde{m}^\gamma = (1+\rho)(m^\gamma - b^\gamma) + \frac{\theta p Q}{\delta}\,.$$

If $\widetilde{m}^\gamma < 0$, the bureaucrat is punished in the amount $\zeta\,(\widetilde{m}^\gamma/\widetilde{p})$, and then plays from zero-cash position.

If the punishment parameter ζ is sufficiently large, then there is an equilibrium in which no bankruptcy occurs. Indeed, the equilibrium is the same as that of Model 1.

Theorem 2: *If $\zeta \geq u'(0)$, then the equilibrium of Theorem 1 is also an equilibrium for Model 2.*

The proof of Theorem 2 is in the Appendix.

If the default penalty is sufficiently small, the equilibrium of Theorem 1 for Model 1 need not be an equilibrium for Model 2.

Example 2: *Let $u(x) = x$ and $\zeta = 1/2$. A producer/consumer α who bids $b > m^\alpha + (\bar{\theta} p q^\alpha)/(1+\rho)$ will have marginal utility*

$$\frac{1}{p}\left[u'\left(\frac{b}{p} - k^\alpha\right) - \zeta\right] = \frac{1}{p}\left[1 - \frac{1}{2}\right] > 0\,,$$

corresponding to the difference between the additional marginal utility from consumption and the disutility from the default penalty. Thus agent α will prefer to go bankrupt rather than follow the strategy of Theorem 1.

The penalty as defined in (19) is adjusted for inflation. In reality, it may take decades to correct for inflation in many laws involving the worth of property. Thus, the crime of stealing \$20 in 1880 should be reclassified as a tort in 2000, but may remain on the books as a crime.

Consider a variation of Model 2, call it Model 2′, in which the penalty for default is not indexed for inflation. Thus, if a player begins a period with a negative cash position $\widetilde{m}$, the player is punished by the amount $\zeta \cdot \widetilde{m}$ regardless of the price level. In all other respects, Model 2′ is the same as Model 2.

In the presence of inflation the effective penalty increases over time and, as in Model 2, the equilibrium for Model 1 is also an equilibrium for Model 2′.

Theorem 3: *Assume $\beta(1+\rho) \geq 1$, and let p_1 be the price for goods at the first stage in the equilibrium of Theorem 1. If $\zeta \geq u'(0)/p_1$, then the equilibrium of Theorem 1 for Model 2, is also an equilibrium for Model 2′.*

The proof is similar to that for Theorem 2 as given in the Appendix.

Note that a type-symmetric equilibrium with active bankruptcy cannot occur for the deterministic economy of Model 2′. If it were advantageous to default, agents would increase their bids to the point at which the advantage would disappear. Active bankruptcy can occur for a model with exogenous uncertainty (cf. Geanakoplos et al. [4]).

4 Model 3: An economy with a generic public good

In Models 1 and 2, the producer/consumer agents were taxed in order to support a bureaucracy that provided government services such as law enforcement. In Model 3 the agents are taxed in order to provide a physical public good, such as highways, that may depreciate. It is more conventional in economic theory to consider models with physical public goods. Nevertheless, the services of government bureaucracy are perhaps no less important than physical public goods such as infrastructure.

As was the case for Models 1 and 2, the model of this section, Model 3, will posit again a continuum of producer/consumer agents $\alpha \in I$ who hold cash $m_n^\alpha \geq 0$ and perishable goods $q_n^\alpha \geq 0$ at the start of each period n.

However, we shall assume here that the government now provides a quantity $G_n \geq 0$ of a generic *public good* in every period n. For simplicity, Model 3 will not include the bureaucrats. Instead, we shall assume that the government spends all its tax revenues on the production and maintenance of the public good.

Now for the details: As in Model 1, each agent α makes a cash bid

$$b_n^\alpha \in \left[0,\, m_n^\alpha + \frac{\bar{\theta}\widehat{p}_n q_n^\alpha}{1+\rho}\right].$$

in period n for the private good. As before, $\widehat{p}_n$ will denote the bank's estimate of the price p_n of the private good and, in rational expectations equilibrium, $\widehat{p}_n = p_n$. Although default is possible out of equilibrium, we shall not model it explicitly.

In period n each agent α sells his goods q_n^α in a market at price p_n and pays the amount $\theta p_n q_n^\alpha$ in taxes. The government spends its total tax revenues

$$B_n^G = \int_I \theta p_n q_n^\alpha \, d\alpha = \theta p_n \int_I q_n^\alpha \, d\alpha = \theta p_n Q_n$$

in the private goods market. Thus, the price p_n for this private good is formed as

$$p_n = \frac{B_n + B_n^G}{Q_n}, \qquad \text{where} \qquad B_n := \int_I b_n^\alpha \, d\alpha$$

is the total bid by the agents. Each agent α purchases the quantity b_n^α / p_n of private goods, and puts $k_n^\alpha \in [0, b_n^\alpha / p_n]$ into production for such goods. Agent α's cash m_{n+1}^α and goods q_{n+1}^α in the next period are thus given by (5) and (7), as in Model 1.

The government, on the other hand, acquires the quantity

$$k_n^G = \frac{B_n^G}{p_n}$$

of the private good, all of which is used for the production or maintenance of the public good.

Suppose $G_n \geq 0$ is the quantity of the public good available at the beginning of period n, and that $\eta \in (0, 1]$ is the depreciation rate. Then the amount of the public good in the next period is

$$G_{n+1} = (1 - \eta) G_n + F(k_n^G), \tag{20}$$

where $F(\cdot)$ is the government's production function for the public good.

The utility of an agent α in period n is

$$u(x_n^\alpha, G_n),$$

a concave increasing function of the agent's private consumption

$$x_n^\alpha = \frac{b_n^\alpha}{p_n} - k_n^\alpha$$

and the public good G_n provided by the government. For fixed values of G, we assume that $u(\cdot, \cdot)$ is concave and differentiable, and increasing in each of its variables when the other is held fixed in $[0, \infty)$. As before, each agent α seeks to maximize the total discounted utility

$$\sum_{n=1}^{\infty} \beta^{n-1} u(x_n^\alpha, G_n)$$

from consumption of the private good.

The government is regarded as a controller, and moves first in the game. Its actions are specified by its selection of its control variables, the interest rate ρ and the tax rate θ. The tax rate determines the income level of the government, and thereby also its production of the public good. Furthermore, the objectives of the government are assumed known to the agents. For example, the government may wish to supply some level of the public good, subject to some condition on inflation.

Remark: An alternative model would have the government finance the public good with the earnings of the central bank. (The bank has a positive return when the agents borrow and pay back with interest. This occurs in equilibrium if $\beta(1+\rho) > 1$.) The government could use its bank earnings instead of, or in addition to, its income from taxes. We will not explore these alternative models in this paper.

4.1 An equilibrium for Model 3

We shall construct a type-symmetric equilibrium, in which money and prices may inflate, but consumption and production remain constant.

Suppose that all agents begin with the same amount of cash $m^\alpha = M$ and goods $q^\alpha = Q = f(k) + y$. Assume each agent makes the same bid $b^\alpha = b = aM$ for the private good, and inputs the same amount $k^\alpha = k$ for its production. Since the government bids its total income, namely θpQ, the price of the private good is given by

$$p = \frac{\int_I b^\alpha \, d\alpha + \theta pQ}{Q} = \frac{aM}{Q} + \theta p$$

and hence

$$p \;=\; (aM)/(\bar{\theta}Q)\,.$$

The amount of private good consumed by each agent in the period is

$$x^\alpha = \frac{b^\alpha}{p} - k^\alpha = \frac{aM}{p} - k = \bar{\theta}Q - k. \tag{21}$$

The government inputs

$$k^G = \frac{\theta pQ}{p} = \theta Q$$

for production of the public good, thus producing the quantity $F(\theta Q)$. Assume that, in each period n. the government holds the quantity G_n of the public good equal to a constant G. Then by (20) we have $G = (1-\eta)G + F(\theta Q)$, and thus

$$G = \frac{1}{\eta} F(\theta Q) \tag{22}$$

is the amount of the public good provided by the government in every period. By (21) and (22) the utility received by each agent in every period is

$$u(x^\alpha, G) \;=\; u\left(\bar{\theta}Q - k\,,\, \frac{1}{\eta}F(\theta Q)\right)\,. \tag{23}$$

Theorem 4: *Suppose that every agent $\alpha \in I$ begins with cash $m^\alpha = M > 0$ and goods $q^\alpha = Q = f(k_1) + y$, where k_1 satisfies*

$$f'(k_1) \;=\; (1+\rho)/(\beta\bar{\theta})\,.$$

Assume also that the government initially provides the quantity $G = F(\theta Q)/\eta$ of the public good.

Then there is an equilibrium for Model 3, in which all agents bid the proportion

$$a \;=\; \frac{(1+\rho)(1-\beta)}{\rho} \tag{24}$$

of their cash for the private good, and input k_1 for its production in every period. In this equilibrium, the government inputs $k^G = \theta Q$ for production of the public good in every period, thereby holding a constant quantity of the public good $G_n = G$ for all n.

The proof is in the Appendix.

In the equilibrium of Theorem 4, money and prices inflate at the rate $\tau = \beta(1+\rho)$ as they did in the equilibrium of Theorem 1. To see this, suppose that all the agents begin at $m^\alpha = M$ and $q^\alpha = Q$ and play the strategy of the theorem. Then at the next stage, each agent has cash

$$\begin{aligned}\widetilde{M} = \widetilde{m}^\alpha &= (1+\rho)(m^\alpha - am^\alpha) + \bar{\theta}pq^\alpha \\ &= (1+\rho)(M - aM) + \bar{\theta}\cdot\frac{aM}{\bar{\theta}Q}\cdot Q = \tau M\,.\end{aligned}$$

The price at the next stage is

$$\widetilde{p} = \frac{a\widetilde{M}}{\bar{\theta}Q} = \tau\frac{aM}{\bar{\theta}Q} = \tau p.$$

4.2 A control problem for the government

A benevolent government will try to maximize the welfare of the agents through its choice of the values of its control variables, the interest rate ρ, and the tax rate θ. Observe that, in the equilibrium of Theorem 4, the total discounted utility of every agent is, by (21) and (22),

$$\sum_{n=1}^{\infty}\beta^{n-1}u(x_n^\alpha, G_n) \;=\; \frac{1}{1-\beta}\cdot u\left(\bar{\theta}Q - k_1, \frac{1}{\eta}F(\theta Q)\right)\,,$$

where

$$k_1 = \left(f'\right)^{-1}\left(\frac{1+\rho}{(1-\theta)\rho}\right) \qquad \text{and} \qquad Q = f(k_1) + y\,.$$

Thus the utility of an agent can be written as a function $\varphi(\theta, \rho)$ of the government's control variables θ and ρ.

We illustrate the government's optimization with a simple example.

Example 3: *Suppose* $f(k) = 2\sqrt{k},\ F(k) = k,\ \beta = 1/2,\ y = 0,\ \eta = 1,\ u(x, G) = \log(xG)$ *. Then*

$$f'(k_1) = \frac{1}{\sqrt{k_1}} = \frac{1+\rho}{\beta\bar{\theta}} = \frac{2(1+\rho)}{1-\theta},$$

so the equilibrium values are

$$k_1 = \frac{(1-\theta)^2}{4(1+\rho)^2}, \qquad Q = f(k_1) = \frac{1-\theta}{1+\rho},$$

and

$$G = F(k^G) = k^G = \theta Q = \frac{\theta(1-\theta)}{1+\rho}.$$

The utility to maximize is $\varphi(\theta, \rho)$*, where*

$$(1-\beta)\cdot\varphi(\theta,\rho) = u\big(\bar{\theta}Q - k_1, \theta Q\big) = u\left(\frac{(1-\theta)^2}{1+\rho} - \frac{(1-\theta)^2}{4(1+\rho)^2}, \frac{\theta(1-\theta)}{1+\rho}\right).$$

Since $u(x, G) = \log(xG)$*, trivial algebra shows that*

$$\varphi(\theta,\rho) = 3\log(1-\theta) + \log\theta + \log(3+4\rho) - 3\log(1+\rho) + C,$$

where C *does not depend on* θ *or* ρ *, thus*

$$\frac{\partial\varphi}{\partial\theta}(\theta,\rho) = \frac{-3}{1-\theta} + \frac{1}{\theta} = 0$$

if $\theta = 1/4$*. This is the optimal tax rate. It is also easy to see that*

$$\frac{\partial\varphi}{\partial\rho}(\theta,\rho) < 0$$

holds for all $\rho > 0$*, so that small positive values of* ρ *are better for the agents than large ones.* [3]

This example is fairly typical, in that there is an optimal tax rate which is an interior point of (0,1). This is intuitively obvious, since a tax rate $\rho = 0$ results in no production of the public good, and a rate $\rho = 1$ means that the agents have no incentive to produce the private good. The example is typical also insofar the agents' utility is decreasing in the interest rate. However, this is misleading because the bank might (and usually does) have multiple goals. In particular, it may wish to maximize the welfare of the agents subject to a constraint on the rate of inflation. In

[3] There are problems when $\rho = 0$, Theorem 4 does not apply.

the equilibrium of our deterministic model it is possible to reduce inflation to zero by setting $1+\rho = 1/\beta$. When there is exogenous uncertainty, this need not be the case (cf. [7]). Even without uncertainty, though, this is not always true if some part of the population, such as pensioners, is living off its capital; for a discussion of such issues, see [6].

5 Model 4: Financing with national debt, via one-period bonds

In Models 1, 2 and 3 above, a public good or a bureaucracy was financed via an income tax. In Models 4 and 5 we shall consider financing the government through the introduction of a public debt. To do so it is necessary to introduce the instrument of a *government bond*.

In Model 4, the subject of this section, we consider the simplest possible case, that of a bond that has only a one-period duration. To make the situation even simpler, we shall assume that there is no income tax in the model. The government sets an interest rate $\rho > 0$ for its one-period bonds, and sells them at face value. We can assume, with little loss of generality, that there is no bank available to accept deposits. If deposits paid a higher (respectively, a lower) rate of interest than bonds, then everyone (respectively, no one) would use the bank rather than buy bonds.

We shall assume for this model that

$$\beta(1+\rho) > 1\,,$$

so agents will have an incentive to buy bonds. (Notice that when $\beta(1+\rho) < 1$ in Model 3, the quantity of (24) is greater than 1. This means that, in the equilibrium of Theorem 4, agents borrow from the bank and make no deposits. The income tax nevertheless allows the government to provide the public good.)

As in Model 3, the government provides here again a quantity $G_n \geq 0$ of a generic public good in each period n. As in all the previous models, there is a continuum $I = [0,1]$ of producer/consumer agents, and each agent $\alpha \in I$ holds cash $m_n^\alpha \geq 0$ and goods $q_n^\alpha \geq 0$ at the start of each period n. The goods are sold in a market, and each agent α bids $b_n^\alpha \in [0, m_n^\alpha]$ to purchase an amount of goods b_n^α/p_n. Here p_n is the price of private goods formed in period n, as explained below. (There is no default in this model, since the agents' bids are limited to their cash holdings.)

Any excess cash $m_n^\alpha - b_n^\alpha$ is used to purchase bonds which mature at the end of the period. Thus, agent α begins the next period with cash

$$m_{n+1}^\alpha = (1+\rho)(m_n^\alpha - b_n^\alpha) + p_n q_n^\alpha\,. \tag{25}$$

As in previous models, each agent α selects a quantity of goods $k_n^\alpha \in [0,\, b_n^\alpha/p_n]$ to input for production, consumes

$$x_n^\alpha = \frac{b_n^\alpha}{p_n} - k_n^\alpha\,, \tag{26}$$

and begins the next period with goods

$$q_{n+1}^{\alpha} = f(k_n^{\alpha}) + y\,. \tag{27}$$

The government bids in the private goods market all its income obtained from the sale of bonds, namely

$$B_n^G := \int_I (m_n^{\alpha} - b_n^{\alpha})\, d\alpha = M_n - B_n\,, \tag{28}$$

where $B_n = \int_I b_n^{\alpha}\, d\alpha$ is the total bid of the agents and $M_n = \int_I m_n^{\alpha}\, d\alpha$ their total cash holdings. The price of the private good in period n is thus

$$p_n = \frac{B_n + B_n^G}{Q_n} = \frac{M_n}{Q_n}\,, \tag{29}$$

where $Q_n = \int_I q_n^{\alpha}\, d\alpha$ is the total amount of private goods sold in the market.

The amount of private goods purchased by the government, namely $k_n^G = B_n^G/p_n$, is used as input to produce the quantity $F(k_n^G)$ of the public good. If G_n is the quantity of the public good available at the beginning of period n and $\eta \in (0,1]$ is the rate of depreciation, then

$$G_{n+1} = (1-\eta)G_n + F(k_n^G)\,. \tag{30}$$

As in Model 3, each agent α seeks to maximize

$$\sum_{n=1}^{\infty} \beta^{n-1} u(x_n^{\alpha}, G_n), \tag{31}$$

where the utility function $u(\cdot,\cdot)$ is concave and differentiable, and increasing in each of its variables when the other is held fixed in $[0,\infty)$.

5.1 An equilibrium for Model 4

To construct a type-symmetric equilibrium, assume that every agent $\alpha \in I$ begins with cash $m^{\alpha} = M > 0$ and goods $q^{\alpha} = Q = f(k) + y$; bids $b^{\alpha} = b = aM$ for goods; and inputs $k^{\alpha} = k$ for production. In the equilibrium constructed below, the proportion a will satisfy $0 < a < 1$; we set $\bar{a} := 1 - a$.

After bidding, each agent spends the remaining cash $M - aM = \bar{a}M$ to purchase bonds. The government thus receives an income of $\bar{a}M$, spends all of it in the private goods market, and then inputs all of the private goods to produce the public good. The price of the private good is

$$p = \frac{aM + \bar{a}M}{Q} = \frac{M}{Q} \tag{32}$$

and

$$k^G = \frac{\bar{a}M}{p} = \bar{a}Q$$

is the government's input for production of the public good. As in Model 3, we assume that the government holds the quantity of the public good G_n equal to a constant G so that $G = (1-\eta)G + F(\bar{a}Q)$ or, equivalently,

$$G = \frac{1}{\eta}F(\bar{a}Q).$$

Each agent α purchases the quantity

$$\frac{b}{p} = \frac{aM}{p} = aQ$$

of the private good, consumes the difference $x = aQ - k$, and receives utility $u(aQ - k, F(\bar{a}Q)/\eta)$ in the period.

Theorem 5: *With $\beta(1+\rho) > 1$, suppose that every agent $\alpha \in I$ begins with cash $m^\alpha = M > 0$ and goods $q^\alpha = Q = f(k_2) + y$, where*

$$f'(k_2) \;=\; (1+\rho)/\beta\,,$$

and that the government initially provides the quantity $G = F(\bar{a}Q)/\eta$ of the public good. Here $\bar{a} = 1 - a$ and

$$a \;:=\; (1-\beta) + \frac{1}{1+\rho} \;\in\; (0,1)\,. \tag{33}$$

Then there is an equilibrium for Model 4, in which all agents bid the proportion a as in (33) of their cash, and input k_2 for production in every period. In this equilibrium the government inputs $k^G = \bar{a}Q$ for production of the public good in every period, thereby holding a constant quantity $G_n = G = F(\bar{a}Q)/\eta$ of the public good, for all n.

The proof is in the Appendix.

In the equilibrium of Theorem 5, money and prices inflate at the rate $\tau = \beta(1+\rho) > 1$. (Recall that the rate of inflation was the same for the previous models in which ρ was the interest rate set by the central bank for deposits and loans.) To see this, suppose that all agents begin with cash M and goods Q, and follow the strategy of Theorem 5. Then, at the next stage, every agent will have cash

$$\begin{aligned}\widetilde{M} = (1+\rho)(M - aM) + pQ &= (1+\rho)(M - aM) + M \\ &= [2 + \rho - (1+\rho)a]M = \tau M,\end{aligned}$$

where the last equality is by (33). The price at the next stage will be

$$\widetilde{p} = \frac{a\widetilde{M}}{Q} = \tau \cdot \frac{aM}{Q} = \tau p.$$

5.2 Another control problem for the government

The only control variable of the government in Model 4 is the interest rate ρ paid on the one-period bonds. To find an optimal value for ρ, we consider the total discounted utility of an agent α in the equilibrium of Theorem 5:

$$\sum_{n=1}^{\infty} \beta^{n-1} u(x_n^\alpha, G_n) = \frac{1}{1-\beta} \cdot u\left(aQ - k_2,\, \frac{1}{\eta} F(\bar{a}Q)\right).$$

Since $k_2 = (f')^{-1}(\frac{1+\rho}{\beta})$ and $Q = f(k_2)$, this utility is a function $\varphi(\rho)$ of ρ.

To illustrate the optimization we take another look at Example 3 recast in the context of Model 4.

Example 4: *As in Example 3, let $f(k) = 2\sqrt{k}$, $F(k) = k$, $\beta = 1/2$, $y = 0$, $\eta = 1$, and $u(x, G) = \log(xG)$. Setting*

$$f'(k_2) = \frac{1}{\sqrt{k_2}} = \frac{1+\rho}{\beta} = 2(1+\rho)$$

with $1 + \rho > (1/\beta) = 2$, we see that

$$k_2 = \frac{1}{4(1+\rho)^2}$$

and thus

$$Q = f(k_2) = \frac{1}{1+\rho}.$$

Also, by (30) and (33), we have

$$G = F(k^G) = k^G = \bar{a}Q = \frac{\bar{a}}{1+\rho} = \frac{1}{1+\rho} \cdot \left(\frac{1}{2} - \frac{1}{1+\rho}\right),$$

and

$$x = aQ - k_2 = \frac{1}{1+\rho} \cdot \left(\frac{1}{2} + \frac{1}{1+\rho}\right) - \frac{1}{4(1+\rho)^2} = \frac{1}{2(1+\rho)} + \frac{3}{4} \cdot \frac{1}{(1+\rho)^2}.$$

Since $\varphi(\rho) = \log(xG) = \log(x) + \log(G)$, an elementary calculation shows that $\varphi(\rho)$ achieves its maximum at $\rho = 1.85$.

Unlike Model 3, here the agents' utility is not a monotonically decreasing function of the interest rate ρ. Indeed, $\varphi(\rho)$ typically has an interior maximum as in the example. While the agents' consumption of the private good is decreasing in ρ, this is not true of the public good since the government finances its production with income based on ρ.

5.3 A remark on government debt

In Model 4 the government is explicitly financing the public good by selling bonds, thereby creating public debt that grows geometrically. Indeed, if we measure government debt by the number of bonds outstanding, then, in the equilibrium of Theorem 5, the debt in period n is $\bar{a}M_n = \bar{a}\tau^{n-1}M_1$. Thus debt increases at the rate of inflation, and, if we correct for inflation, debt remains constant.

There is debt implicit in the earlier models when there is inflation; that is, when $\beta(1+\rho) > 1$. This is because agents are making deposits in the bank and the deposits earn interest. The amount of interest paid increases geometrically at the rate of inflation τ.

6 Model 5: Financing with national debt, via perpetuities or consols

For our final model, we assume that the government finances a public good through its sale of *perpetuities* or *consols*. These are bonds sold at face value, which pay interest at a rate $\rho > 0$ in every period in the future. (An interesting discussion of the use of consols in England was given by Keynes [8].) In a complex economy there is a time-structure of interest rates, stretching forward over many years, and the shape of the yield curve is watched closely by those setting policy. The flexibility of the economy is enhanced by this complex multidimensional control system. As a first step towards treating the multi-period control possibilities, the perpetuity picks up long-term financing but does not pick up the flexibility obtained with the full array of maturities of all lengths.

Model 5 differs from Model 4 in that the consols held by an agent do not mature at the end of each period, but are held by the agent indefinitely. In a stochastic model we would expect that some agents would find it advantageous to sell consols to others in a private market. However, in a type-symmetric deterministic equilibrium, either all the agents would wish to sell or they would all wish to buy. Thus in the equilibrium constructed below there is no private market for the consols.

As in Model 4, we assume that $\beta(1+\rho) > 1$ so that agents will have an incentive to buy bonds. Each agent $\alpha \in I$ holds cash $m_n^\alpha \geq 0$, goods $q_n^\alpha \geq 0$, and a quantity $c_n^\alpha \geq 0$ of consols at the beginning of each period n. Also, as in the previous model, the goods are sold in a market and each agent α bids $b_n^\alpha \in [0, m_n^\alpha]$ to purchase goods b_n^α/p_n. We assume that each agent α spends the remaining cash $m_n^\alpha - b_n^\alpha$ to purchase an equal quantity of consols; thus, agent α begins the next period with consols

$$c_{n+1}^\alpha = c_n^\alpha + m_n^\alpha - b_n^\alpha \,. \tag{34}$$

Agent α's cash in the next period comes from the interest on the consols together with the profits from the sale of goods:

$$m_{n+1}^{\alpha} = \rho c_{n+1}^{\alpha} + p_n q_n^{\alpha}\,. \tag{35}$$

As in all our models, each agent α chooses an amount of goods $k_n^{\alpha} \in [0, b_n^{\alpha}/p_n]$ as input for production, consumes the difference $x_n^{\alpha} = (b_n^{\alpha}/p_n) - k_n^{\alpha}$, and begins the next period with goods

$$q_{n+1}^{\alpha} = f(k_n^{\alpha}) + y.$$

The total bid B_n of the agents, and the government's bid B_n^G, are defined exactly as in Model 4; consequently, the price p_n is given by (29). Likewise, the quantity of the public good G_n provided by the government in period n is given by (30), and each agent α seeks to maximize total discounted utility as in (31).

6.1 An equilibrium for Model 5

As in Model 4, we suppose that every agent $\alpha \in I$ begins play with cash $m^{\alpha} = M > 0$ and goods $q^{\alpha} = Q = f(k) + y$; bids $b^{\alpha} = B = aM$; and inputs $k^{\alpha} = k$ for production. For Model 5 we further assume that every agent starts with the same quantity $c^{\alpha} = c = \gamma M$ of consols in his portfolio. Here a and γ are positive constants. The price p of the private good is given by (32).

At the next stage of play, every agent will have cash

$$\widetilde{M} = \rho(M - aM + c) + pQ = \big(\rho(1 - a + w) + 1\big)\cdot M$$

and consols

$$\widetilde{c} = c + M - aM = (w + 1 - a)\,M\,.$$

In equilibrium we expect the Fisher equation $\widetilde{M} = \beta(1+\rho)M$ to hold, and also that $\widetilde{c} = w\widetilde{M}$. If this is so, then we have

$$\rho(1 - a + w) + 1 = \beta(1+\rho) \qquad \text{and} \qquad w + 1 - a = w\beta(1+\rho).$$

Solving for w and a, we obtain

$$w = \frac{1}{\rho}\cdot\left(1 - \frac{1}{\tau}\right) \qquad \text{and} \qquad a = 1 - \frac{1}{\rho\tau}\cdot\big(\tau - 1\big)^2. \tag{36}$$

Here $\tau = \beta(1+\rho)$ is the inflation rate; recall that $\tau > 1$ by our assumption, thus $w > 0$ and $a < 1$.

Theorem 6: *With $\tau = \beta(1+\rho) > 1$, let w and a be given by (36) and assume that $a > 0$. Suppose that every agent $\alpha \in I$ begins with cash $m^{\alpha} = M > 0$, consols $c^{\alpha} = wM$, and goods $q^{\alpha} = f(k_2) + y$ where*

$$f'(k_2) = (1+\rho)/\beta\,.$$

Then there is an equilibrium for Model 5, in which all agents bid the proportion a of their cash and input k_2 for production in every period. In this equilibrium the government provides the quantity $G = F(\bar{a}Q)/\eta$ of the public good in every period, where $\bar{a} = 1 - a$.

The proof is in the Appendix.

6.2 The public debt

In the equilibrium of Theorem 6, we can measure government debt by the quantity of consols outstanding. If c_n is the quantity in period n, then $c_n = wM_n = w\tau^{n-1}M_1$. Thus, as in Model 4, the debt increases at the rate of inflation.

6.3 Non-inflationary finance

If conservation of the amount of money in the economy is desired, then some device is required which removes the money injected by the growth of national debt payments. In essence, the initial increase is ρc. This can be offset by a tax rate θ such that $\theta pq = \rho c$. The introduction of taxes will, however, result in a change in the optimal strategy of the agents.

7 Further remarks

7.1 The Fisher equation

In all of our models the classical Fisher equation $\tau = \beta(1+\rho)$ was seen to hold in equilibrium. (Indeed, the equation was key to our proofs.) We do not expect the equation to hold for stochastic models. Consider, for example, representative agent models in which the endowment y is replaced by a random variable Y, the same for all agents. The rate of inflation τ will then be a random variable and we conjecture that the Fisher equation will be replaced, just as it was in [7], by a "harmonic Fisher equation" of the form

$$E\left(\frac{1}{\tau}\right) = \frac{1}{\beta(1+\rho)}\,.$$

7.2 A comment on equilibrium and equations of motion

For models without uncertainty, it is often easier to analyze the equilibrium state without paying close attention to the details of the laws of motion required for fully specified dynamic programs. This is not so when exogenous uncertainty and disequilibrium positions need to be considered.

When a full dynamic model is specified, the distinction between establishing and maintaining a public good must be made. Here we are not examining the financing of the construction of such a public good, but are concerned with financing by national debt, taxation, or other means for maintenance.

7.3 Inflation or national debt or taxes?

Politics, or at least political economy, not economics, is the dominant force in selecting among the array of government control weapons, in order to achieve the goals that the political process selects. This can be seen when we consider yet another alternative means of financing, the use of the printing press and inflation. The government could finance economic activity by actually paying for goods and services in newly printed bills. The constraints on this scheme lie with the psychology and the political economy of acceptance, not the economics or the accounting.

7.4 Transaction costs and financing

Although the key stress in the selection of means of financing is often political, intertwined with this concern is the question of transaction costs and bureaucratic costs. They require a level of micro-economic detail not considered here, but often critical in specific applications.

7.5 A question of Barro

A stimulating article by Barro [1] raised the question "Are government Bonds Net Wealth?". The immediate answer from a game-theoretic viewpoint is that of course they are, in either a fully specified general equilibrium model or in a strategic market game with no unemployment or transactions costs, if the government is assumed to be serving the society. The question is how much net wealth they represent. As was partially indicated by Barro's article, the increment of wealth depends on the details of the structure assumed. The introduction of government bonds, and their acceptance in competitive markets, represent an enlargement of the strategy sets available to society. The bonds can be interpreted as a new instrument for facilitating finance in much the same way that a new process for making steel may facilitate its production. It is well known that, with incomplete markets, it is possible that an enlargement of the strategy sets can make some individuals worse off [4], thus one cannot give an unqualified comment without a full specification of the detailed assumptions.

Two different but important questions concern the individual and societal implications on wealth of the differences between various combinations of national debt, and taxation funding of the same economic program. In order to answer both of these

[4] One need go no further than a two-person, one-shot game with a single action for each player yielding a payoff to each player of 10 (say). By adding one further action for each player the game can be converted into a prisoner's dilemma with equilibrium payoffs less than 10.

questions, models involving both independent individual agents, as well as a representative agent, need to be studied. In the absence of exogenous uncertainty, a model with independent but symmetric agents yields the same results as a representative agent model. When uncertainty is present, the differences are considerable. We plan to consider models with exogenous uncertainty in a subsequent paper.

8 Appendix

8.1 The proof of Theorem 1

We need to show that the strategies described in Theorem 1 are feasible and optimal for each agent and each bureaucrat, when all other agents and bureaucrats follow these same strategies.

Consider first a producer/consumer $\alpha \in I$ with cash m and goods q, when the price is p. Then α faces a dynamic programming problem in which the optimal return $V^\alpha(m,q,p)$ satisfies the Bellman equation

$$V^\alpha(m,q,p) = \sup_{\substack{0\le b\le m+\frac{\bar\theta pq}{1+\rho} \\ 0\le k\le \frac{b}{p}}} \left[u\left(\frac{b}{p}-k\right) + \beta\, V^\alpha\left((1+\rho)(m-b)+\bar\theta pq, f(k)+y, \tau p\right)\right]$$

Assuming an interior solution, the Euler equations for this problem are:

$$\frac{1}{p}\,u'\left(\frac{b}{p}-k\right) = \frac{\beta(1+\rho)}{\widetilde p}\cdot u'\left(\frac{\widetilde b}{\widetilde p}-\widetilde k\right) \tag{37}$$

and

$$u'\left(\frac{b}{p}-k\right) = \frac{\bar\theta\beta}{1+\rho}\,f'(k)\cdot u'\left(\frac{\widetilde b}{\widetilde p}-\widetilde k\right), \tag{38}$$

where $\widetilde p$, $\widetilde b$, and $\widetilde k$ are the price, the agent's bid, and the agent's input for the next period. Under the hypothesized strategy for the agent, we have

$$\frac{\widetilde b}{\widetilde p} = \frac{a\bar\theta\widetilde m}{\widetilde p} = \frac{a\bar\theta\tau m}{\tau p} = \frac{a\bar\theta m}{p} = \frac{b}{p} \qquad \text{and} \qquad \widetilde k = k = k_1\,.$$

Thus

$$\frac{\beta(1+\rho)}{\widetilde p} = \frac{\beta(1+\rho)}{\tau p} = \frac{1}{p} \qquad \text{and} \qquad \frac{b}{p}-k = \frac{\widetilde b}{\widetilde p}-\widetilde k$$

so that (37) and (38) are satisfied when $k = k_1$. The appropriate transversality condition is trivial, because the quantities consumed and input for production are the same in every period.

We must also verify that the given solution is interior. That is, we need

$$0 < b < m + \frac{\bar{\theta} pq}{1+\rho} \tag{39}$$

and

$$0 < k < \frac{b}{p} \tag{40}$$

when $q = f(k_1) + y$, $m = \bar{\theta} M$, $b = a\bar{\theta} M$ and $k = k_1$. Since $pq = aM$, the inequalities in (39) follow from the definition of a in (15) and from our assumptions that $\theta < 1$ and $\beta < 1$. The first inequality in (40) is trivial; the second follows from a calculation:

$$\begin{aligned}
\frac{b}{p} &= \frac{\bar{\theta} a M}{\frac{aM}{Q}} = \bar{\theta} Q \\
&= \bar{\theta}\left(f(k_1) + y\right) \\
&\geq \bar{\theta} \int_0^{k_1} f'(x)dx \\
&\geq \bar{\theta} k_1 f'(k_1) \\
&= \bar{\theta} k_1 \frac{(1+\rho)}{\beta \bar{\theta}} \\
&> k_1 \, .
\end{aligned}$$

The proof of optimality for agent $\alpha \in I$ is now complete.

Consider now a bureaucrat $\gamma \in J$ with cash m, when the price is p; he has only one control variable, the bid for goods. Consequently, the optimal return $V^\gamma(m,p)$ for γ satisfies the simpler Bellman equation

$$V^\gamma(m,p) = \max_{0 \leq b \leq m + \frac{\theta p Q}{\delta(1+\rho)}} \left[u\left(\frac{b}{p}\right) + \beta V^\gamma\left((1+\rho)(m-b) + \frac{\theta p Q}{\delta}, \tau p\right)\right] \tag{41}$$

and the Euler equation

$$\frac{1}{p} u'\left(\frac{b}{p}\right) = \frac{\beta(1+\rho)}{\widetilde{p}} \cdot u'\left(\frac{\widetilde{b}}{\widetilde{p}}\right) ,$$

where

$$\frac{\widetilde{b}}{\widetilde{p}} = \frac{\frac{a\theta \widetilde{M}}{\delta}}{\widetilde{p}} = \frac{\frac{a\theta\tau M}{\delta}}{\tau p} = \frac{b}{p}$$

and, as before, $\beta(1+\rho)/\widetilde{p} = 1/p$. Thus, the Euler equation is satisfied. As before it is easy to check that bids are interior and that the transversality condition is satisfied. Hence, the given strategy is optimal for $\gamma \in J$.

8.2 The proof of Theorem 2

Assume that all agents $\alpha \in I$ and bureaucrats $\gamma \in J$ play the strategies described in the statement of Theorem 1. We must show that the given strategy is optimal for any single player. Consider a producer/consumer agent $\alpha \in I$ who begins a period with cash m and goods q, and suppose that the price in the period is p. If $m \geq 0$, then the Bellman equation for agent α takes the form

$$V^\alpha(m,q,p) = \sup_{\substack{b\geq 0 \\ 0\leq k \leq \frac{b}{p}}} \left[u\left(\frac{b}{p} - k\right) + \beta V^\alpha\left((m-b)(1+\rho) + \bar{\theta} pq, f(k) + y, \tau p\right)\right].$$

However, if $m < 0$ so that the agent has defaulted, then the Bellman equation is

$$V^\alpha(m,q,p) = \zeta \cdot \frac{m}{p} + V^\alpha(0,q,p). \tag{42}$$

This second form of the Bellman equation reflects our rule that the agent is punished in proportion to the amount by which he has defaulted, and is then allowed to continue the game from a position of zero cash.

In the equilibrium of Theorem 1, every agent $\alpha \in I$ begins with cash $m^\alpha \equiv m = \theta M$ and goods $q^\alpha \equiv q = f(k_1) + y$. They each bid $b^\alpha \equiv b = \bar{\theta} a M$, where a is given by (15), and input k_1 for production.

We must show that their selections remain optimal in our present model. It suffices to show that it is never desirable for an agent α to choose a bid that exceeds the quantity

$$m_* := m + \frac{\bar{\theta} pq}{1+\rho}$$

because, if the agent chooses a bid in the range $[0, m_*]$, he is, in effect, playing in Model 1. Let

$$\Psi(b) := u\left(\frac{b}{p} - k\right) + \beta\, V^\alpha\left((m-b)(1+\rho) + \bar{\theta}pq, f(k) + y, \tau p\right).$$

If $b > m_*$, then we have $\widetilde{m}(b) < 0$, where

$$\widetilde{m}(b) = (1+\rho)(m-b) + \bar{\theta}pq$$

is the agent's cash position in the next period as in (18). Hence, by (42),

$$\Psi(b) = u\left(\frac{b}{p} - k\right) + \beta\left(\frac{\zeta \widetilde{m}(b)}{\tau p} + V^\alpha\big(0, f(k) + y, \tau p\big)\right)$$

Recall that $\tau = \beta(1+\rho)$. Hence,

$$\begin{aligned} \Psi'(b) &= \frac{1}{p} u'\left(\frac{b}{p} - k\right) - \frac{\beta\zeta(1+\rho)}{\beta(1+\rho)p} \\ &= \frac{1}{p}\left[u'\left(\frac{b}{p} - k\right) - \zeta\right] \leq \frac{1}{p}\big(u'(0) - \zeta\big) \leq 0. \end{aligned}$$

Thus the optimal bid of a producer-consumer α in Model 1 remains optimal in Model 2.

A similar argument shows that it also optimal for the bureaucrats to play their equilibrium strategy from Theorem 1.

8.3 Sketch of the proof of Theorem 4

As explained in the paragraph after the statement of Theorem 4, money and prices inflate at the rate $\tau = 1 + \rho - \rho a = \beta(1+\rho)$ when agents play the given strategy. We must show that the strategy is optimal for a single agent $\alpha \in I$ when all others follow it. The Bellman equation for such an agent α is

$$V^\alpha(m,q,p) = \sup_{\substack{0 \le b \le m + \frac{\bar\theta pq}{1+\rho} \\ 0 \le k \le \frac{b}{p}}} \left[u\left(\frac{b}{p} - k, G\right) + \beta V^\alpha\left((m-b)(1+\rho) + \bar\theta pq, f(k) + y, \tau p\right)\right]. \tag{43}$$

Except for the dependence of the agent's utility on the public good G, the Euler equations are the same as in Model 1 (see (37) and (38)):

$$\begin{aligned} \frac{1}{p} u'\left(\frac{b}{p} - k, G\right) &= \frac{\beta(1+\rho)}{\tilde p} \cdot u'\left(\frac{\tilde b}{\tilde p} - \tilde k, G\right), \\ u'\left(\frac{b}{p} - k, G\right) &= \frac{\bar\theta\beta}{1+\rho} \cdot f'(k) \cdot u'\left(\frac{\tilde b}{\tilde p} - \tilde k, G\right). \end{aligned}$$

Here $u'(\cdot,\cdot)$ denotes differentiation with respect to the first coordinate. As in the proof of Theorem 1 the Euler equations are satisfied when the given strategy is followed. Also as in the proof of Theorem 1 it is not difficult to check that the solution is interior and satisfies a transversality condition.

8.4 The proof of Theorem 5

The Bellman equation for a single agent $\alpha \in I$ with cash m and goods q, when the price is p and others follow the proposed strategy, is

$$V^\alpha(m,q,p) = \sup_{\substack{0 \le b \le m \\ 0 \le k \le \frac{b}{p}}} \left[u\left(\frac{b}{p} - k, G\right) + \beta\, V^\alpha\left((m-b)(1+\rho) + pq, f(k) + y, \tau p\right)\right].$$

The Euler equations are

$$\begin{aligned} \frac{1}{p} u'\left(\frac{b}{p} - k, G\right) &= \frac{\beta(1+\rho)}{\tau p}\, u'\left(\frac{\widetilde b}{\widetilde p} - \tilde k, G\right), \\ u'\left(\frac{b}{p} - k, G\right) &= \frac{\beta f'(k)}{1+\rho}\, u'\left(\frac{\widetilde b}{\widetilde p} - \widetilde k, G\right). \end{aligned}$$

It is straightforward to verify that these equations are satisfied when the agent follows the proposed strategy, and that the actions are interior. As before, the appropriate transversality condition also holds.

8.5 The proof of Theorem 6

Consider the situation of an agent $\alpha \in I$ with cash m, goods q, and consols c, when the price of the private good is p and all other agents follow the strategy described in Theorem 6. We must show that the same strategy is optimal for α when she begins in the same position as the others.

The Bellman equation for agent α now takes the form

$$V^\alpha(m,q,c,p) = \sup_{\substack{0\le b\le m \\ 0\le k\le \frac{b}{p}}} \Big[u\Big(\frac{b}{p}-k,G\Big) + \beta\, V^\alpha\big(\rho(m-b+c)+pq, f(k)+y, c+m-b, \tau p\big)\Big].$$

Suppose agent α begins in the same position (m,q,c,p) as the other agents so that $q=f(k_2)+y, c=\gamma m$, and $p=m/q$. Suppose also that α, like the other agents, plays the strategy described in Theorem 6. Then α consumes the quantity

$$x = \frac{am}{p} - k$$

of the private good in the first period of the game. Let $(\tilde m, \tilde q, \tilde c, \tilde p)$ be the agent's position in the next period. Thus the agent consumes

$$\tilde x = \frac{a\tilde m}{\tilde p} - \tilde k$$

in the second period. Since $\tilde m = \tau m$, $\tilde k = k = k_2$, and $\tilde p = \tau p$, we see that $\tilde x = x$. It follows that α consumes the same quantity x in every period.

For an interior solution, standard arguments yield the following envelope equations:

$$\frac{\partial V^\alpha}{\partial m}(m,q,c,p) = \frac{1}{p}\,u'\left(\frac{b}{p}-k_2,G\right) = \frac{1}{p}\,u'(x,G)\,, \tag{44}$$

$$\frac{\partial V^\alpha}{\partial q}(m,q,c,p) = \beta p\,\frac{\partial V^\alpha}{\partial m}(\tilde m,\tilde q,\tilde c,\tilde p) = \frac{\beta p}{\tilde p}\,u'\left(\frac{\tilde b}{\tilde p}-k_2,G\right) = \frac{1}{1+\rho}\,u'(x,G)\,, \tag{45}$$

$$\begin{aligned}\frac{\partial V^\alpha}{\partial c}(m,q,c,p) &= \beta\rho\,\frac{\partial V^\alpha}{\partial m}(\tilde m,\tilde q,\tilde c,\tilde p) + \beta\,\frac{\partial V^\alpha}{\partial c}(\tilde m,\tilde q,\tilde c,\tilde p)\\ &= \frac{\beta\rho}{\tilde p}\,u'(x,G) + \beta\,\frac{\partial V^\alpha}{\partial c}(\tilde m,\tilde q,\tilde c,\tilde p)\,.\end{aligned} \tag{46}$$

In these equations and below we write $u'(x,G)$ for $\frac{\partial u}{\partial x}(x,G)$.

The value function V^α for the dynamic programming problem of agent $\alpha \in I$ has the homogeneity property:

$$V^\alpha(\tilde{m}, \tilde{q}, \tilde{c}, \tilde{p}) = V^\alpha(\tau m, q, \tau c, \tau p) = V(m, q, c, p).$$

Hence,

$$\frac{\partial V^\alpha}{\partial c}(\tilde{m}, \tilde{q}, \tilde{c}, \tilde{p}) = \frac{1}{\tau}\,\frac{\partial V^\alpha}{\partial c}(m, q, c, p).$$

Now substitute for the left-hand-side in (46) and, after some algebra, we have

$$\frac{\partial V^\alpha}{\partial c}(m, q, c, p) = \frac{1}{p}\,u'(x, G). \tag{47}$$

We can now differentiate the expression in brackets on the right side of the Bellman equation, to get the Euler equations. Differentiating with respect to the variable b, and using (44) and (47), we have

$$\frac{1}{p}\,u'\Big(\frac{b}{p} - k, G\Big) = \beta\rho\frac{\partial V^\alpha}{\partial m}(\tilde{m}, \tilde{q}, \tilde{c}, \tilde{p}) + \beta\frac{\partial V^\alpha}{\partial c}(\tilde{m}, \tilde{q}, \tilde{c}, \tilde{p}) = \frac{\beta(1+\rho)}{\tau p}u'(x, G). \tag{48}$$

This holds since, for α's strategy, $\tau = \beta(1+\rho)$ and $x = (am/p) - k_2 = (b/p) - k$. Now differentiate with respect to the variable k and use (45), to get

$$u'\Big(\frac{b}{p} - k, G\Big) = \beta f'(k)\,\frac{\partial V^\alpha}{\partial q}(\tilde{m}, \tilde{q}, \tilde{c}, \tilde{p}) = \frac{\beta f'(k)}{1+\rho}\,u'(x, G). \tag{49}$$

This also holds for α's strategy, since $k = k_2 = (f')^{-1}\big((1+\rho)/\beta\big)$.

To check that α's strategy is interior, first notice that $0 < b = am < m$ since $0 < a < 1$. We also require that $0 < k_2 < b/p = aQ$. The first inequality is clear. For the second, calculate as follows:

$$\begin{aligned} aQ = a(f(k_2) + y) \geq\ & af(k_2) \\ &= a\int_0^{k_2} f'(t)\,dt \geq ak_2 f'(k_2) = ak_2 \cdot \frac{1+\rho}{\beta}\,; \end{aligned}$$

therefore, we have $k_2 < aQ$, if

$$\frac{a(1+\rho)}{\beta} > 1.$$

Substitute the expression in (36) for a and use the equality $\beta(1+\rho) = \tau$ to see that this inequality is equivalent to

$$(1 + 3\rho + \rho^2)\beta^2 - (2 + 3\rho + \rho^2)\beta + 1 < 0\,.$$

The expression on the left factors to give $(\beta - 1)((\rho^2 + 3\rho + 1)\beta - 1)$ which is negative, since $\beta < 1$ and $(\rho^2 + 3\rho + 1)\beta > (\rho + 1)\beta > 1$.

Finally, the appropriate transversality condition holds because, as in the other models, consumption and input for production are the same in every period.

The proof of Theorem 6 is now complete.

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