Supplementary Appendices to Rationalizing Choice with Multi-Self Models

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This document contains supplementary appendices to "Rationalizing Choice with Multi-Self Models" by Ambrus and Rozen. The main paper is referenced throughout as AR.

A Genericity

In this Appendix we formally establish that for any fixed scaling function $\phi(\alpha)$ the property that an additive and scale-invariant aggregator is not degenerate holds generically.

Let F^* denote the set of additive and scale-invariant aggregators. In order to define a topology on F^*, we transform the latter set of aggregators to a convenient representation. Note that for a fixed scaling function, specifying the aggregated utilities of n alternatives for selves in the n-dimensional simplex determines the aggregated utilities of n alternatives for all possible selves over n alternatives, since any self is a scalar multiple of exactly one self from the simplex. Hence, with respect to a grand set of alternatives with three elements, there is a natural bijection β between additive and scale-invariant aggregators, and the set of pairs of operators

$$\Omega = (O_1, O_2 | O_1 : \Delta_2 \to \mathbb{R}^2; O_2 : \Delta_3 \to \mathbb{R}^3),$$

where O_1 determines how a self's utilities get aggregated in pairs, and O_2 determines how a self's utilities get aggregated in the triple.

Define metric d on Ω such that the distance between (O_1, O_2) and (O_1', O_2') is defined as $\max_{i=1,2} \sup_{x \in \mathbb{R}^i} \, |O_i(x) - O_i'(x)|$.

Theorem 1. *Given the topology induced by d, the pairs of operators in Ω that are associated with non-degenerate aggregators in F^* is open and dense relative to Ω.*

Proof. For ease of exposition, let

$$\Gamma_1^l(f, v) = f(a, \{a, c\}, v) - f(c, \{a, c\}, v),$$

$$\Gamma_2^l(f,v) = f(a,\{a,b\},v) - f(b,\{a,b\},v) + f(b,\{b,c\},v) - f(c,\{b,c\},v),$$

$$\Gamma_2^l(f,v) = f(a,\{a,b,c\},v) - f(b,\{a,b,c\},v)] + [f(a,\{a,b,c\},v) - f(c,\{a,b,c\},v),$$

$$\Gamma_2^l(f,v) = [f(a,\{a,b\},v) - f(b,\{a,b\},v)] + [f(a,\{a,c\},v) - f(c,\{a,c\},v)],$$

for every $v \in F^*$. Note that $\Gamma_i^j(v)$ stands for side j of the equation in condition i in the definition of a degenerate aggregator, given aggregator f and self v.

1. (*Openness*). Suppose that for aggregator f there is a self u over a triple such that neither of the equalities in the definition of a degenerate aggregator hold with equality. Note that u cannot be an indifferent self. Let $\varepsilon_i = \Gamma_i^l(f,v) - \Gamma_i^r(f,v)$ for $i \in \{1,2\}$, and let $\varepsilon = \max(|\varepsilon_1|,|\varepsilon_2|)$. Next, for every $i,j \in \{a,b,c\}$ such that $i \neq j$, let α^{ij} be such that $\alpha^{ij}(u(i),u(j)) \in \Delta^2$. Note that the terms α^{ij} are uniquely defined. Similarly, let α^{abc} be such that $\alpha^{abc}(u(a),u(b),u(c)) \in \Delta^3$. Let $\alpha = \max(|\alpha^{ab}|,|\alpha^{ac}|,|\alpha^{bc}|,|\alpha^{abc}|)$. Since u is not an indifferent self, $\alpha > 0$. Then for $\delta < \frac{\varepsilon}{8\alpha}$ it holds that $\Gamma_i^l(f',v) \neq \Gamma_i^r(f',v)$ for $i \in \{1,2\}$ for every f' such that $|\beta(f) - \beta(f')| < \delta$, since each term given f' in the above inequalities can differ from the corresponding term given f by at most $\frac{\varepsilon}{8}$.

2. (*Denseness*). Let $\delta > 0$. Consider a self $u \in \Delta_3$ over $\{a,b,c\}$ such that $u(a) > u(b) > u(c)$. For every $i,j \in \{a,b,c\}$ such that $i \neq j$, let α^{ij} be such that $\alpha^{ij}(u(i),u(j)) \in \Delta^2$. Let $\alpha = \max(|\alpha^{ab}|,|\alpha^{ac}|,|\alpha^{bc}|)$.

   If for an aggregator f neither of the equalities in the definition of a degenerate aggregator hold, then the aggregator is by definition non-degenerate, hence there is trivially a point in the δ-neighborhood of $\beta(f)$ that corresponds to a non-degenerate aggregator. Otherwise let $\varepsilon \in (0, \frac{\delta}{\alpha})$ be such that $\varepsilon \neq |\Gamma_i^l(f,v) - \Gamma_i^r(f,v)|$ for $i \in \{1,2\}$.

   Consider now any $f' \in F^*$ for which

   (a) For triples, f' is equivalent to f

   (b) For a pair $\{x,y\}$, given any utility function v over $\{x,y\}$ for which $v(x) \geq v(y)$, $f'(x,\{x,y\},v) = f(x,\{x,y\},v)$ and $f'(y,\{x,y\},v) = f(y,\{x,y\},v)$ if $v(x) - v(y) < u(a) - u(c)$, but $f'(x,\{x,y\},v) = f(x,\{x,y\},v) + \varepsilon$ and $f'(y,\{x,y\},v) = f(y,\{x,y\},v)$ if $v(x) - v(y) \geq u(a) - u(c)$.

   In words, with respect to selves for which the utility difference between the elements of the pair is at least $u(a) - u(c)$ the aggregated utility is $\varepsilon > 0$ higher than what f yields for the preferred alternative (while it is the same for the other alternative) - otherwise f' is equivalent to f. By construction, $|\beta(f') - \beta(f)| < \delta$. Also note that

$$\Gamma_1^l(f',v) = \Gamma_1^l(f,v) + \varepsilon,$$

$$\Gamma_1^r(f', v) = \Gamma_1^r(f, v),$$

$$\Gamma_2^l(f', v) = \Gamma_2^l(f, v),$$

and

$$\Gamma_2^r(f', v) = \Gamma_2^r(f, v) + \varepsilon.$$

Then $\varepsilon \neq |\Gamma_i^l(f, v) - \Gamma_i^r(f, v)|$ for $i \in \{1, 2\}$ implies that $\Gamma_i^l(f', v) \neq \Gamma_i^r(f', v)$ for $i \in \{1, 2\}$. Hence, f' is non-degenerate. ∎

B Approximate triple-solvability

For some aggregators a tighter upper bound can be provided for the minimum number of selves needed to rationalize a choice function, through a weakening of the triple-solvability requirement. In particular, it suffices for triple-solvability to hold only *approximately*, which can yield a triple-basis with a smaller number of selves. For ease of exposition we only state this property for additively separable aggregators.

Definition 1. *We say $\hat{U} \in \mathcal{U}(\{a, b, c\})$ is a (δ, ε)-approximate triple-basis for f with respect to $\{a, b, c\}$ if $f(a, \{a, b\}, \{a, b, c\}, \hat{U}) = f(b, \{a, b\}, \{a, b, c\}, \hat{U}) + \delta$ and*

$$|f(x, A, \{a, b, c\}, \hat{U}) - f(y, A, \{a, b, c\}, \hat{U})| < \varepsilon$$

for all other $A \subseteq \{a, b, c\}$ and $x, y \in A$.

That is, a collection of selves U is a (δ, ε)-approximate triple basis for f if given choice set $\{a, b\}$ the aggregated utility of U for a is exactly δ higher than the aggregated utility of b, while U is ε-indifferent among all alternatives given every other choice set.

We say that an aggregator f is *approximately triple-solvable with k selves* if there is $\overline{\delta} > 0$ such that exists a (δ, ε)-approximate triple-basis with k selves for every $\delta < \overline{\delta}$ and $\varepsilon > 0$. That is, for approximate triple-solvability we do not require that the collection of selves in the triple is exactly indifferent between all elements in choice sets other than $\{a, b\}$, only that they can be arbitrarily close to being indifferent.

AR-Theorem 1 can then be modified as follows.

Theorem 2. *Suppose f satisfies P1-P6 and P9, and is approximately triple-solvable with k_f selves. Then, for any finite set of alternatives X, and any choice function $c : P(X) \to X$ that exhibits at most $\frac{n-1}{k_f}$ IIA-violations, f can rationalize c with n selves.*

Proof. The only difference compared to the proof of AR-Theorem 1 is in the construction of selves. Recall the definition of $(I_j)_{j=1,...,j^*}$ from the proof of AR-Theorem 1. Let $\delta_1 \in (0, \overline{\delta})$. Define iteratively δ_j for $j \in \{2, ..., j^* + 1\}$ such that $\delta_j \in (0, \frac{\delta_{j-1}}{IIA(c)+1})$. Define a self u^X such that u^X is δ_{j^*+1}-indifferent and the preference ordering of the self is $c(X) \succ c(X \setminus \{c(X)\}) \succ$ ... Let

$$\delta^{**} = \min_{x \neq y \in X,\ A \ni x,y} |f(x, A, X, u^X)| - |f(y, A, X, u^X)|.$$

Finally, let $\varepsilon \in (0, \frac{\delta^{**}}{|X|})$. Then for every $j \in \{1, ..., j^*)$ and $A \in I_j$ construct a collection of selves $U^A \in \mathcal{U}(X)$ the following way: take a (δ_j, ε)-approximate triple-basis U, and define U^A by defining, for each $u_i \in U$, a self $u_i^A \in U^A$ by

$$u_i^A(x) = \begin{cases} u_i(a) & x = c(A) \\ u_i(b) & x \in A \setminus \{c(A)\} \\ u_i(c) & x \in X \setminus A. \end{cases}$$

Proving the collection of selves consisting of u^X and U^A for each $A \in \bigcup_{j=1}^{j^*} I_j$ rationalizes c is analogous to the proof in AR-Theorem 1. ∎

C Relaxing P6

Our main results can be extended to aggregators violating P6, that is, to aggregators that depend in a nontrivial way on alternatives unavailable in a given choice set. However, the appropriate definition of triple-solvability is more complicated.

The main complication arising in the absence of P6 is that triple-solvability needs to be defined on a general X, as opposed to just a triple $\{a, b, c\}$. It is convenient to introduce the following notation: for any triple $\{a, b, c\}$, any basic set of alternatives $X \supset \{a, b, c\}$, and any self u defined on $\{a, b, c\}$, define the set $E(u, X) = \{\widehat{u} : X \to \{u(a), u(b), u(c)\} | \widehat{u}(x) = u(x)\ \forall\ x \in \{a, b, c\}\}$. In words, $E(u, X)$ is the set of extensions of u from $\{a, b, c\}$ to X for which each element in $X/\{a, b, c\}$ receives the same utility as either a or b or c. Similarly, for any $U = (u_1, ..., u_m) \in \mathcal{U}(\{a, b, c\})$, let $E(U, X) = \{(\widehat{u}_1, ..., \widehat{u}_m) | \widehat{u}_i \in E(u_i, X)$ for all $i \in \{1, ..., m\}\}$.

Definition 2. *We say $U \in \mathcal{U}(\{a, b, c\})$ is a universal triple-basis for f if for any $X \supset \{a, b, c\}$ the following holds: for all $\hat{U} \in E(U, X)$, $f(a, \{a, b\}, X, \hat{U}) > f(b, \{a, b\}, X, \hat{U})$, and $f(\cdot, A, X, \hat{U})$ is constant for all other $A \subseteq \{a, b, c\}$.*

A universal triple-basis solves the triple $\{a, b, c\}$ whenever the utilities of unattainable elements don't differ from utilities of elements in $\{a, b, c\}$, for all selves in the triple-basis. An aggregator f is *universally triple-solvable* if the following condition is satisfied.

Condition (Universal triple-solvability of f) There exists a triple $\{a, b, c\}$ and $k \in Z_+$ such that for every $\delta > 0$ there is a δ-indifferent $U \in \mathcal{U}^k(\{a, b, c\})$ constituting a universal triple-basis for f with respect to $\{a, b, c\}$.

It is easy to see that for aggregators satisfying P6, universal triple-solvability is equivalent to triple-solvability. If f satisfying P1-P5 is universally triple-solvable with k selves, then the same construction can be applied as in the proof of AR-Theorem 1 to obtain an analogous lower bound on the set of choice functions that f can rationalize with a given number of selves. The proof of this result is analogous to the proof of AR-Theorem 1 and hence omitted.

Theorem 3. *Suppose f satisfies P1-P5 and is universally triple-solvable wrt to X with k_f selves. Then, using n selves, f can rationalize any choice function, on any grand set of alternatives X, that exhibits at most $\frac{n-1}{k_f}$ IIA-violations.*