Supplement to
"Foundations of Intrinsic Habit Formation"

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This supplement contains additional results that combine with Lemmas 1-22 in Appendix B to prove Theorem 2.[1]

Sufficiency: Monotonicity properties of u under Axiom GM

Lemma 23. *$\forall\, T \in \mathbb{N}$, $\bar{x} = (x_0, x_1, \ldots, x_T) \in \mathbf{R}^{T+1}$, $\exists\, h_{\bar{x},T} \in H$ such that*

$$(x_0, x_1, \ldots, x_T, 0, 0, \ldots) \in C^*_{h_{\bar{x},T+1}}.$$

Proof. For arbitrary h, define c^h by $c_0^h = x_0 + \varphi(h)$, $c_t^h = x_t + \varphi(hc_0^h c_1^h \cdots c_{t-1}^h)$ for all $1 \le t \le T$, and $c_t^h = \varphi(hc_0^h c_1^h \cdots c_{t-1}^h)$ for $t > T$. φ is strictly increasing, so we may choose $h_{\bar{x},T} \in H$ sufficiently large so that $(c_0^{h_{\bar{x},T}}, c_1^{h_{\bar{x},T}}, \ldots c_T^{h_{\bar{x},T}})$ is nonnegative. But if $(c_0^{h_{\bar{x},T}}, c_1^{h_{\bar{x},T}}, \ldots c_T^{h_{\bar{x},T}})$ is nonnegative, then so is ${}^{T+1}c^{h_{\bar{x},T}}$. Moreover, the stream is ultimately weakly decreasing. Therefore $c^{h_{\bar{x},T}} \in C$. □

Lemma 24. *Under Axiom GM the period-utility u is an increasing function.*

Proof. Suppose u is not increasing. Because it is continuous, there exist some $x \in \mathbf{R}$ and $\alpha > 0$ such that $\forall\, \alpha' \in (0, \alpha]$, $u(x + \alpha') < u(x)$.

Let T be arbitrary for the moment. Note that by Lemma 23 there is h' such that $(x, x, \ldots, x, 0, 0, \ldots) \in C^*_{h'}$ (where x is repeated $T+1$ times). Again by Lemma 23 there is h'' such that $(x + \alpha, x, x, \ldots, x, 0, 0, \ldots) \in C^*_{h''}$ (where x by itself is repeated T times). Let $h \ge h', h''$, and recall that the $C^*_{\hat{h}}$ are nested. Using the representation for $\succeq^*$ and the fact that $u(x + \alpha) < u(x)$,

$$u(x) + \sum_{t=1}^{T} \delta^t u(x) + \sum_{t=T+1}^{\infty} \delta^t u(0) > u(x + \alpha) + \sum_{t=1}^{T} \delta^t u(x) + \sum_{t=T+1}^{\infty} \delta^t u(0). \quad (33)$$

[1] To ease cross-referencing, we continue the enumeration of results and equations in the main paper.

Since $(x, x, \ldots, x, 0, 0, \ldots) \in C_h^*$, there is $c \in C$ with $g(h, c) = (x, x, \ldots, x, 0, 0, \ldots)$. Clearly $c + \alpha \in C$, and by GM we know $c + \alpha \succ_h c$. Moreover, $g(h, c + \alpha)$ is

$$(x+\alpha, x+\alpha(1-\lambda_1), \ldots, x+\alpha(1-\sum_{k=1}^{T}\lambda_k), \alpha(1-\sum_{k=1}^{T+1}\lambda_k), \alpha(1-\sum_{k=1}^{T+2}\lambda_k), \ldots), \tag{34}$$

where x appears $T+1$ times. Therefore, by the representation theorem for $\succeq^*$,

$$\begin{aligned} u(x+\alpha)+&\sum_{t=1}^{T}\delta^t u\big(x+\alpha(1-\sum_{k=1}^{t}\lambda_k)\big) + \sum_{t=T+1}^{\infty}\delta^t u\big(\alpha(1-\sum_{k=1}^{t}\lambda_k)\big) \\ &> \sum_{t=0}^{T}\delta^t u(x) + \sum_{t=T+1}^{\infty}\delta^t u(0). \end{aligned} \tag{35}$$

Combine the RHS of (33) and the LHS of (35); and rearrange by subtracting the RHS of (33) from all sides of the inequalities. This obtains

$$\begin{aligned} \sum_{t=1}^{T}\delta^t\big[u\big(x+\alpha(1-\sum_{k=1}^{t}\lambda_k)\big) - u(x)\big] + \sum_{t=T+1}^{\infty}\delta^t\big[u\big(\alpha(1-\sum_{k=1}^{t}\lambda_k)\big) - u(0)\big] \\ > u(x) - u(x+\alpha), \end{aligned} \tag{36}$$

which is strictly positive. Since each $\lambda_k > 0$ and $\sum_{k=1}^{\infty}\lambda_k \leq 1$, we know that the value $\alpha(1-\sum_{k=1}^{t}\lambda_k) \in [0, \alpha)$ for every t, and is in fact strictly positive as $t < \infty$. The assumption that u dips below $u(x)$ just to the right of x implies that $\sum_{t=1}^{T}\delta^t\big[u\big(x+\alpha(1-\sum_{k=1}^{t}\lambda_k)\big) - u(x)\big] < 0$. This sum decreases in T. By continuity, u is bounded on $[0, \alpha]$. Choose T large enough so that $\sum_{t=T+1}^{\infty}\delta^t\big[u\big(\alpha(1-\sum_{k=1}^{t}\lambda_k)\big) - u(0)\big]$ is small enough to bring about the contradiction $0 > 0$ from (36). This is possible because Lemma 23 permits us to find h large enough so that the constructed streams are in C^*. □

Lemma 25. *Assume Axiom GM. If* $\sum_{k=1}^{\infty}\lambda_k < 1$ *then* $u(\cdot)$ *is strictly increasing on* $(0, \infty)$*; and if* $\sum_{k=1}^{\infty}\lambda_k = 1$ *then there is* a *with* $0 < a \leq \infty$ *such that* $u(\cdot)$ *is strictly increasing either on* $(-a, \infty)$ *or on* $(-\infty, a)$*.*

Proof. By Lemma 24 we know that $u(\cdot)$ is an increasing function. To prove it is strictly increasing on the relevant ranges we will consider the two cases separately. *Case (i):* $\sum_{k=1}^{\infty}\lambda_k = 1$. First we will show that $u(\cdot)$ is strictly increasing in some interval around 0. To complete the proof, we will show that there cannot exist $x > 0 > y$ such that $u(\cdot)$ does not increase strictly at both x and y. To see the first point, take any $q > 0$ and let $h = (\ldots, q, q)$ and $c = (q, q, \ldots)$. Then $g(h, c) = (0, 0, \ldots)$ and for small α, both $c + \alpha \succ_h c$ and $c \succ_h c - \alpha$ by Axiom GM. Using the representation for $\succeq^*$,

$$\sum_{t=0}^{\infty} \delta^t u\big(\alpha(1 - \sum_{k=1}^{t} \lambda_k)\big) > \sum_{t=0}^{\infty} \delta^t u(0) > \sum_{t=0}^{\infty} \delta^t u\big(-\alpha(1 - \sum_{k=1}^{t} \lambda_k)\big).$$

By monotonicity of $u(\cdot)$ it must be that $u(\cdot)$ increases strictly in a neighborhood of 0. For the second point, suppose by contradiction that there exist $x > 0 > y$ such that $u(\cdot)$ does not increase strictly at both x and y. By continuity and monotonicity of $u(\cdot)$ there is $\alpha > 0$ such that $u(\cdot)$ is constant on $(x, x + \alpha)$ and on $(y, y + \alpha)$. Without loss of generality suppose that x, y are rational (else take some rational x, y inside the interval). Since x, y are rational there exist m, n such that $mx = -ny$. Let $c^* = (x^m, y^n, x^m, y^n, \ldots)$ (i.e., x is repeated m times, then y is repeated n times, etc). Because the compensating streams are constant, we may use the characterization (31) in Lemma 21 to find $h \in H$ large enough so that there is $c \in C$ satisfying $g(h, c) = c^*$. Observe by GM that $c + \frac{\alpha}{2} \succ_h c$, a contradiction to the assumption that $u(\cdot)$ is constant on $(x, x + \alpha)$ and $(y, y + \alpha)$.

Case (ii): $\sum_{k=1}^{\infty} \lambda_k < 1$. In this case, for any $q \in Q$, if we set $h = (\ldots, q, q)$ and $c = (q, q, \ldots)$, then $g(h, q) = (q[1 - \sum_{k=1}^{\infty} \lambda_k], q[1 - \sum_{k=1}^{\infty} \lambda_k], \ldots)$. As q is arbitrary, for any $x \geq 0$, $(x, x, x, \ldots) \in C^*$. Suppose to the contrary that $u(\cdot)$ is not increasing from the right at x. Since $u(\cdot)$ is continuous and weakly increasing, this implies that there exists some $\beta^+ > 0$ such that for every $0 < \beta \leq \beta^+$, $u(x + \beta) = u(x)$. Take h, c such that $g(h, c) = (x, x, x, \ldots)$. By GM, $c + \beta \succ_h c$. Then the representation says that $\sum_{t=0}^{\infty} \delta^t u\big(x + \beta(1 - \sum_{k=1}^{t} \lambda_k)\big) > \sum_{t=0}^{\infty} \delta^t u(x)$. Since $0 < \beta \leq \beta^+$ and $\sum_{k=1}^{t} \lambda_k < 1$, $u\big(x + \beta(1 - \sum_{k=1}^{t} \lambda_k)\big) = u(x)$ for every $t \geq 0$, a contradiction. □

Necessity: Why u need not be strictly increasing everywhere under GM

Lemma 26. *Suppose that $\sum_{k=1}^{\infty} \lambda_k < 1$. Then,*

1. *For any $\gamma > 0$, there are no $c \in C$, $h \in H$ such that $c \geq (\gamma, \gamma, \ldots)$ and $g(h, c) \leq (0, 0, \ldots)$.*

2. *For any $\gamma < 0$, there are no $c \in C$, $h \in H$ such that $g(h, c) \leq (\gamma, \gamma, \ldots)$.*

Proof. To see (i), we first note that if $g(h, c) \leq (0, 0, \ldots)$ then $c_0 \leq \varphi(h)$, $c_1 \leq \varphi(hc_0)$, $c_2 \leq \varphi(hc_0c_1)$, etc. Using the monotonicity of φ and recursive substitution, we see that $c_1 \leq \varphi(h\varphi(h))$, $c_2 \leq \varphi(h\varphi(h)\varphi(h\varphi(h)))$, etc. But by Lemma 8, the compensating streams $(\varphi(h), \varphi(h\varphi(h)), \varphi(h\varphi(h)\varphi(h\varphi(h))), \dots)$ tend to zero.

Similarly, to see (ii), note that if $g(h, c) \leq (\gamma, \gamma, \dots)$ then $c_0 \leq \varphi(h) + \gamma$, $c_1 \leq \varphi(hc_0) + \gamma \leq \varphi(h\varphi(h)) + \lambda_1\gamma + \gamma$. But since $\gamma < 0$, we may drop the term $\lambda_1\gamma$ to obtain $c_1 \leq \varphi(h\varphi(h)) + \gamma$. In this manner, $c_2 \leq \varphi(h\varphi(h)\varphi(h\varphi(h))) + \gamma$, and so on. The stream $(\varphi(h), \varphi(h\varphi(h)), \varphi(h\varphi(h)\varphi(h\varphi(h))), \dots)$ tends to zero asymptotically, and $\gamma < 0$ is fixed, implying c is eventually negative, a contradiction. $\square$

When $\sum_{k=0}^{\infty} \lambda_k < 1$, part (i) in Lemma 26 means the argument of u cannot always be strictly negative when the consumption stream is bounded from zero (we cannot shift down a stream using GM to conclude u is increasing in the negative range). Part (ii) means the argument of u cannot be bounded below zero (we cannot shift up a stream using GM to conclude u is increasing in the negative range). It suffices that u is sensitive on the nonnegative domain to satisfy GM. To see why it suffices that for some $0 < a \leq \infty$, u is only strictly increasing either on $(-\infty, a)$ or $(-a, \infty)$ when $\sum_{k=0}^{\infty} \lambda_k = 1$, use Lemma 20. By (31), there cannot exist h and c such that $g(h, c)$ is always positive and bounded from zero (c would be unbounded), or always negative and bounded from zero (c would violate nonnegativity).