PRIZES VERSUS WAGES WITH ENVY AND PRIDE

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Prizes versus Wages with Envy and Pride

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Abstract

We show that if agents are risk neutral, prizes outperform wages when there is sufficient pride and envy relative to the noisiness of performance. If agents are risk averse, prizes are a necessary supplement to wages (as bonuses).

Keywords: Envy, Pride, Wages, Prizes, Bonus

JEL Classification: C72, D01, D23, L14.

1 Introduction

Suppose all your fellow workers got a pay raise of \$1 but you didn't. You would likely feel worse off.

Suppose instead everyone, including you, took a pay cut of \$$\alpha$. Again you would feel worse off.

How high would α need to be to make you feel indifferent between the two scenarios?

Alternatively, suppose all your fellow workers took a pay cut of \$1 but you didn't. You would likely feel better off.

Suppose instead everyone, including you, got a pay raise of \$$\beta$. Again you would feel better off. How high would β need to be to make you indifferent?

For people whose utility does not involve comparisons with others, $\alpha = \beta = 0$. But for many people, α and β would need to be near 1; indeed several of our colleagues gave numbers even higher. The numbers α and β roughly measure how powerful feelings of envy and pride are. People typically care not only about their direct personal reward, but also about their standing vis-a-vis others, feeling envy when

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their reward is lower, and pride when it is higher. Such attitudes appear to be quite prevalent, especially when agents compete in a common work environment.[1]

Consider budding athletes. Many make extraordinary sacrifices to undergo training in the hope of winning a rare prize, for example an Olympic medal or a spot on a professional sports team. Only a minute percentage achieve success.

A spot on an NBA basketball team can be worth a great deal, say a million dollars, and so it is not surprising that such a prize elicits effort. But if that money were divided up among the hundred people competing for the spot as wages, payable conditional on the same effort they undertook for the prize, would the effort be forthcoming? We think not, and not by a long shot. Why is a one in a hundred chance of $1 million worth so much more than $10,000 for sure?

Status might be the answer. Prizes involve the explicit ranking of performances, and are, by their very nature, public. The player who makes the team is universally lauded for beating his ninety-nine competitors. This confers on him the pride of status, over and above the direct utility of the $1 million (at the same time as it inflicts envy on the losers). In contrast, wages are paid based on individual output, regardless of how others perform, and are quite often secret.

Suppose, however, that wages were made as public as prizes. Could they not also unleash feelings of pride and envy, and intensify the competition among workers? Would wages then inspire more effort than prizes?

In fact, wages have an advantage over prizes in that they are flexible. A prize is given to a worker who produces more than all the others, independent of the levels. In contrast, piece-rate wages and productivity-indexed bonuses increase with an individual worker's output. This linkage between output and remuneration appears to be an important advantage of wages.

The motivating power of wages versus prizes has been considered before, most famously in Lazear and Rosen (1981), who showed that wages can do at least as well as prizes from the principal's point-of-view. (Neither pride nor envy was postulated to exist in their model.) In a follow-up paper, Green and Stokey (1983) showed that, if agents are risk-averse and if their productivities are sufficiently correlated via a common random shock, then prizes (or, as they called them, tournaments) outperform wages. The reason is that the incentives provided by wages are reduced on account of the shock and the risk aversion; while, on the other hand, the incentives generated by prizes are invariant of the shock because it is common. Even with risk

[1]There is a large empirical literature, starting from Easterlin (1975), which has shown that happiness depends not just on absolute, but also on relative, consumption. This externality has been formally modeled along two different lines. The cardinal approach makes utility depend on the difference between an individual's consumption and others' consumption (see, e.g., Duesenberry (1949), Pollak (1976), and Fehr and Schmidt (1999)). The ordinal approach makes utility depend on the individual's rank in the distribution of consumption (see, e.g., Frank (1985), Direr (2001), and Hopkins and Kornienko (2004)). Our model is in the cardinal tradition. The ordinal approach is examined in Dubey and Geanakoplos (2004, 2005). There is a big difference between the two approaches. Our results here on the efficacy of prizes rely heavily on cardinal envy and pride.

neutrality, if worker outputs are correlated, one can imagine learning from the output of one worker something about the effort of another worker, justifying prizes based on relative performance. Index-linked incentive contracts are very common for this reason. But in the absence of superior publicity for prizes, and without risk aversion, or correlation in production, a puzzle remains as to how prizes could ever outperform wages.

We shall explain the superiority of prizes over wages entirely on the basis of envy and pride. In our model, agents are homogeneous and their outputs are independent; furthermore, wages are as public as prizes. We show that if agents are risk neutral, and the feelings of envy and pride exceed a threshold dependent on the noisiness in output, then a single prize will generate strictly more effort than any (even non-linear) wage schedule. If the agents are risk averse, then under similar conditions, any wage schedule can be improved (giving higher incentives with less expected payments) by reducing the wages and substituting a prize (bonus) based on relative performance.

Bonuses based on relative performance are widespread in the marketplace and have been justified in many ways. Our explanation via envy and pride appears to be new.[2]

When there is no noise or envy or pride, wages give the same incentives as prizes to risk neutral agents, and outperform prizes for risk averse agents. With the introduction of noise (randomness) in output, wages strictly dominate prizes in either case: since outputs are independent, compensating a worker on the basis of relative performance only distorts his incentives (the shirker wins the prize with positive probability just because of luck).

The situation dramatically reverses with envy and pride. Section 2 examines the case of two risk neutral agents whose intensities of envy and pride are equal, i.e. $\alpha = \beta$. We show that for any level of noise σ below some upper bound $\bar{\sigma}$, there is a threshold $\alpha(\sigma)$ of envy-pride such that if $\alpha = \beta > \alpha(\sigma)$, prizes will outperform wages, while if $\alpha = \beta < \alpha(\sigma)$, wages outperform prizes. For noise $\sigma > \bar{\sigma}$, no amount of envy-pride can restore the superiority of prizes.

In Section 3 we show that, no matter how large the noise and how small the envy-pride, the superiority of prizes is restored with a large enough group of competitors, since a shirker will very rarely be lucky enough to pass so many hard-working rivals.

It is true that envy-pride lowers the wages an employer needs to pay his agents, because if one of them works less he will not only get a lesser wage, but also envy those who are working and getting paid more. But the motivating power of envy-pride is even stronger with prizes. The shirker not only reduces his (expected) prize, he increases the (expected) prize of his rivals, generating more envy than in the wage

[2] In this paper we assume that the reward based on relative performance takes the form of a prize (or bonus) with a fixed value. If the relative performance bonus were variable, paying the winner according to the margin of victory, it might become still more attractive. We are content to show that even without this flexibility, adding bonuses to wages will improve worker incentives, if there is envy and pride.

scenario.

In Section 4 we disentangle envy from pride. Were envy to disappear, pride by itself would create more incentives for prizes than for wages. Pride does not really enter the picture with wages, because a hard-worker will rarely beat his equally talented, hard-working rivals by much, even if there is noise. However, when everybody strives hard for a prize, a single person always enjoys the pride of victory, no matter how close the race. We passed over this earlier, when we took $\alpha = \beta$, because then the pride in victory is exactly neutralized by the envy in defeat. If $\alpha = 0$, but $\beta > 0$, prizes continue to be bolstered by pride, while wages retain only the direct utility of money. ("To play for pride" is a time honored expression, as is "to work for wages".) We show that for any level of noise σ below some reasonably large upper bound, there is a threshold $\beta(\sigma)$ of pride such that if $\alpha \leq \beta$ and $\beta > \beta(\sigma)$, prizes will outperform wages.

In Section 5 we observe that any prize-based incentive scheme imposes a big risk on workers, namely that they might not win the prize even when they work hard. This is a terrible disadvantage for prizes if workers are risk-averse. Paying workers entirely by way of a single prize to the victor is therefore untenable. But that does not mean there is no role for prizes. It suggests a combination of wages and prizes. Such mixed contracts are not necessary when agents are risk neutral, since a pure wage or a pure prize is always at least as good as any mixture. But, without risk neutrality, mixed contracts cannot be ignored.

We show that for any level of noise σ below some reasonably large upper bound, there is a threshold $\alpha(\sigma)$ of envy-pride such that if $\alpha = \beta > \alpha(\sigma)$, *every optimal mixed contract will entail a positive prize*. Furthermore for any given envy-pride $\alpha = \beta > 0$, there exists a threshold of noise $\sigma(\alpha) > 0$ such that this need for prizes remains whenever $\sigma < \sigma(\alpha)$.

The intuition is roughly as follows. Suppose there are N workers earning only wages. When there is no noise, a worker knows the wage w he will earn for sure. Assuming differentiable utilities, he is nearly risk neutral for small variations in consumption. So consider reducing the wage w by ε, and instead awarding a prize of $N\varepsilon$. Then the expected utility for a worker from consumption stays almost the same. But as we argued before with risk neutrality, the incentive created by envy-pride is greater for the prize than the wage. Thus a small bonus increases incentives without increasing total expected payout for the employer.

The efficacy of a single prize relies on each agent believing he has roughly as good a chance as anyone else to win if he works. If agents had such disparate abilities that only a few believed they could win, the others would despair and not work. In this event it would be necessary to create handicaps to generate more competition. Though handicaps are an important fact of life, we do not deal with them here. We assume instead that the competitors think they are fairly evenly matched when their efforts are equal. This assumption is most plausible when the agents do not know their relative abilities. (For example, teenage athletes may realize they are gifted, but not know how they would stack up against competition after a long period of arduous training, leading them to suppose that they are equal till proved otherwise.) We could have allowed for significant heterogeneity in the disutility of effort and the feelings of envy and pride. Our analysis would essentially remain intact. For details see the longer version of this paper (Dubey, Geanakoplos, and Haimanko (2005)).

2 Pride and Envy

2.1 The Basic Model

We first consider two identical agents with utility

$$u(A, B, e) = A + \beta \max(A - B, 0) - \alpha \max(B - A, 0) - ce,$$

where A is the money the agent gets, B is the money his rival gets, and e is the effort he exerts. The parameters $\beta \geq 0$ and $\alpha \geq 0$ correspond to pride and envy, and $c > 0$ is the marginal disutility of effort.[3] In this Section we shall assume that $\alpha = \beta$, merging the effects of pride and envy. In Section 4, we shall disentangle the two effects, taking $\alpha \neq \beta$, and often letting one of them be zero. Thus for now write

$$u(A, B, e) = A + \alpha(A - B) - ce, \tag{1}$$

and call α the *envy-pride* parameter.[4]

Let $\mathcal{E} \subset [0, 1]$ be the set of effort levels available to each agent, with $0 \in \mathcal{E}$ and $1 \in \mathcal{E}$. (Thus $\mathcal{E}$ can be discrete or continuous. All we require is that it contain two special levels: $0 \equiv$ "shirking", and $1 \equiv$ "working at full capacity".) If agent $i \in \{1, 2\}$ chooses effort level $e_i \in \mathcal{E}$, he produces $e_i + \varepsilon_i^\sigma$ units of output, where ε_1^σ and ε_2^σ are random noises (i.i.d. nonatomic random variables with mean zero), parameterized by a scalar $\sigma > 0$ measuring their noisiness[5]. We denote by G^σ the cumulative distribution function of the random variable $\varepsilon_1^\sigma - \varepsilon_2^\sigma$. Clearly, since ε_1^σ and ε_2^σ have positive variance and are nonatomic i.i.d., we have $G^\sigma(0) = 1/2$. We suppose that as noise disappears, $\lim_{\sigma \to 0} G^\sigma(t) = 0$ for every $t < 0$, and as noise goes to infinity, $\lim_{\sigma \to \infty} G^\sigma(t) = 1/2$ for every t. We also assume that G^σ is continuous and convex on $[-1, 0]$ (i.e., G^σ possesses a density function which is nondecreasing on $[-1, 0]$). To

[3] Fehr and Schmidt (1999) studied the same utility function, but curiously they took β negative, implying that people feel compassion when they are ahead. In conjunction with envy from being behind, their formulation amounts to "inequity aversion".

[4] Utility functions of this particular form were considered, e.g., in Fershtman et al (2003), Bolle (2000), and Kirchsteiger (1994).

[5] Our model allows for negative outputs of the agents. This might make sense in certain contexts (think of money managers who make losses). But the case of exclusively non-negative outputs can be incorporated by putting a positive lower bound on effort levels and a suitably small upper bound on the support of the random noise.

include deterministic output in our analysis, we allow for $\sigma = 0$, in which case both ε_1^0 and ε_2^0 are fixed at zero.

If each ε_i^σ is normally distributed, with mean zero and standard deviation σ, then $\varepsilon_1^\sigma - \varepsilon_2^\sigma$ is also normally distributed, with mean zero and standard deviation $\sqrt{2}\sigma$; thus, $G^\sigma(x) = \frac{1}{2\sigma\sqrt{\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{4\sigma^2}} dt$.

If the ε_i^σ are uniformly distributed on $[-\sigma, \sigma]$, then[6]

$$G^\sigma(x) = \begin{cases} 0, & \text{if } x \leq -2\sigma, \\ \frac{1}{8\sigma^2}(x+2\sigma)^2, & \text{if } -2\sigma \leq x \leq 0, \\ 1 - \frac{1}{8\sigma^2}(-x+2\sigma)^2, & \text{if } 0 \leq x \leq 2\sigma, \\ 1, & \text{if } x \geq 2\sigma. \end{cases}$$

It is easy to check that all our hypotheses are satisfied for the normal and uniform noise terms.

2.2 The Wage and Prize Games

We will compare two types of contracts that the principal may write. The first is a piece-rate wage contract: each agent is paid rq, when the piece-rate is r and his output is q. In the second contract, a prize P is awarded to the agent with the highest output; in case of ties, a fair coin is tossed to decide who gets the prize. There is always one winner.

Each of these contracts induces, in an obvious manner, a non-cooperative game in which agents' strategies are to choose effort levels. Denote these games with wages, prizes by $\Gamma_\alpha^\sigma(r)$, $\tilde{\Gamma}_\alpha^\sigma(P)$.

The principal wishes to elicit maximal effort from the agents (i.e., $e_1 = e_2 = 1$) at minimal expected cost to himself. Let

$$M_\alpha^\sigma = 2\min\{r \mid (e_1 = 1, e_2 = 1) \text{ is a Nash equilibrium of } \Gamma_\alpha^\sigma(r)\},$$

$$\tilde{M}_\alpha^\sigma = \min\left\{P \mid (e_1 = 1, e_2 = 1) \text{ is a Nash equilibrium of } \tilde{\Gamma}_\alpha^\sigma(P)\right\}.$$

Clearly $M_\alpha^\sigma, \tilde{M}_\alpha^\sigma$ is the minimal expected payment by the principal needed to elicit maximal effort via wages, prizes[7].

Define, in either the prize or the wage game, the *incentive to work* for an agent i to be the increase in his payoff when he switches from shirk ($e_i = 0$) to work ($e_i = 1$), ignoring his disutility of effort and assuming that his rival j is working ($e_j = 1$). Then agent i will not want to unilaterally deviate from work to shirk iff his incentive to work is no less than his disutility of effort $e_i = 1$. It turns out that if this simple

[6] See (9) and (10) in Example 2.

[7] We have assumed a *single* prize for the best-performing agent. If the loser were also awarded, incentives to exert maximal effort would become smaller. Thus a single prize will, in fact, always be preferred by the principal.

criterion is met for both agents, then $(1,1)$ is a Nash equilibrium (NE) of the relevant game. In other words, if the deviation from 1 to 0 is not profitable, then neither is the deviation to any $e_i \in \mathcal{E}$. The proof of this, as will be seen, relies on the linearity of costs and the convexity of the distribution G^σ on $[-1,0]$.

Our first proposition uses the incentive criterion to establish explicit formulae for M_α^σ and $\tilde{M}_\alpha^\sigma$.

Proposition 1.

$$M_\alpha^\sigma = \frac{2c}{1+\alpha} \tag{2}$$

and

$$\tilde{M}_\alpha^\sigma = \frac{c}{\frac{1}{2} - G_\sigma(-1)} \cdot \frac{1}{1+2\alpha} \tag{3}$$

Proof. In the game $\Gamma_\alpha^\sigma(r)$ the expected utility of agent i, when he chooses effort level e_i and his rival j chooses effort level e_j, is $re_i + \alpha(re_i - re_j) - ce_i = (1+\alpha)re_i - \alpha re_j - ce_i$. Thus, in order for $(e_1 = 1, e_2 = 1)$ to be a Nash equilibrium of $\Gamma_\alpha^\sigma(r)$, it is necessary and sufficient that r satisfy the incentive constraint: $(1+\alpha)r - \alpha r - c \geq (1+\alpha)re_i - \alpha r - ce_i$ for every $e_i \in \mathcal{E}$. When $e_i = 1$ we have an equality, and when $e_i = 0$ the incentive constraint reduces to the condition:

$$r \geq \frac{c}{1+\alpha}.$$

By linearity, this condition guarantees the incentive constraint for every $e_i \in \mathcal{E}$. Since each agent gets $re_i = r$ (when $e_i = 1$), (2) follows.

Next consider the prize game $\tilde{\Gamma}_\alpha^\sigma(P)$. When agent i chooses effort level e_i and his rival j chooses effort level e_j, i wins the prize for sure if and only if $e_i + \varepsilon_i^\sigma > e_j + \varepsilon_j^\sigma$, i.e., $\varepsilon_j^\sigma - \varepsilon_i^\sigma < e_i - e_j$. Since noise is nonatomic, the probability of this event is $G^\sigma(e_i - e_j)$. Hence i's payoff is[8] $G^\sigma(e_i - e_j)(1+\alpha)P - [1 - G^\sigma(e_i - e_j)]\,\alpha P - ce_i$. Thus, in order to implement $(e_1 = 1, e_2 = 1)$ as a Nash equilibrium of $\tilde{\Gamma}_\alpha^\sigma(P)$, it is necessary and sufficient that P satisfy

$$G^\sigma(0)(1+\alpha)P - [1 - G^\sigma(0)]\,\alpha P - c \geq G^\sigma(e_i - 1)(1+\alpha)P - [1 - G^\sigma(e_i - 1)]\,\alpha P - ce_i$$

for every $e_i \in \mathcal{E}$. Recalling that $G^\sigma(0) = 1/2$, and letting $\tau = (e_i - 1) \in \mathcal{E}_{-1}$ (where $\mathcal{E}_{-1}$ denotes the set $\{e - 1 \mid e \in \mathcal{E}\}$) we find that P must be bounded from below by

$$\frac{c \sup_{\tau \in \mathcal{E}_{-1} \setminus \{0\}} \frac{\tau}{G^\sigma(\tau) - \frac{1}{2}}}{1+2\alpha}.$$

[8] The formula remains valid when $\sigma = 0$. We take $G^0(0) = \frac{1}{2}$, since an anonymous tie-breaking rule would allocate the prize to either agent with equal probability. And, of course, we take $G^0(x) = 0$ if $x < 0$ and $G^0(x) = 1$ if $x > 0$.

Since G^σ is convex on $[-1, 0]$, the supremum in this expression is attained for $\tau = -1$, which leads to (3).■

Theorem 1 (**Wages outperform Prizes without Envy or Pride**). *If there is no envy or pride, and even a little noise, then wages outperform prizes:* $M_0^\sigma = \tilde{M}_0^\sigma$ *if* $G^\sigma(-1) = 0$, *and* $M_0^\sigma < \tilde{M}_0^\sigma$ *if* $G^\sigma(-1) > 0$.

Proof. Immediate from Proposition 1.■

The intuition behind Theorem 1 is straightforward. If agent i works ($e_i = 1$) in the prize game and so does his rival, i's expected share of the prize is exactly $P/2$. If he shirks ($e_i = 0$) and his rival still works, his expected payoff does not fall to zero, since with noise he may, with a stroke of luck, win anyway. His expected payoff is $G^\sigma(-1)P$, which is positive if there is enough noise. On net his incentive to work is $P(1/2 - G^\sigma(-1))$. When the wage rate is set equal to $P/2$, his incentive to work in the wage game is $P/2$, no matter what the noise. But if $G^\sigma(-1) > 0$, then $P(1/2 - G^\sigma(-1)) < P/2$. Hence the prize P will need to be more than twice the optimal wage r if $G^\sigma(-1) > 0$, and will never be less.

2.3 The Power of Envy and Pride

Envy makes it easier to motivate the agents to work, via wages or prizes.[9] For wages, this is because shirking entails not only a lesser payment, but also the envy of those who are working and getting paid more.

But the motivating power of envy-pride is even stronger with prizes than with wages. An agent who shirks not only reduces his (expected) prize, he *increases* the (expected) prize of his rival, generating still more envy.[10]

This can most succinctly be seen if there is no noise. From (2) and (3) we see at once that without noise, the principal needs to pay out total wages $M_\alpha^0 = 2c/(1+\alpha)$ to motivate both agents to work, but a prize of only $\tilde{M}_\alpha^0 = 2c/(1+2\alpha)$. Clearly the required wage bill and the prize become smaller as the envy-pride factor α rises. When $\alpha = 0$, $M_0^0 = \tilde{M}_0^0 = 2c$ whereas both M_α^0 and $\tilde{M}_\alpha^0$ converge to zero as $\alpha \to \infty$. With enough envy-pride, the principal hardly needs to expend any money at all. But the point is, he expends less on prizes than on wages for any degree of envy-pride $\alpha > 0$. The maximum savings in absolute terms from using prizes instead of wages occur when $\alpha = 1/\sqrt{2} \approx .70$, yielding a savings of $M_{1/\sqrt{2}}^0 - \tilde{M}_{1/\sqrt{2}}^0 \approx 1.17c - .83c = .34c$. The presence of envy-pride reduces the wage bill from $2c$ to $1.17c$, or about 41%, and

[9] As envy-pride increases, agents are obviously more easily motivated to work in the prize game (see Proposition 1). This fact was noted in Grund and Sliwka (2005). However, they did not compare prizes to wages.

[10] Notice that this effect relies on the cardinal approach to envy: envy increases as the gap grows bigger.

the switch from wages to prizes reduces the total payments from $1.17c$ to $.83c$, or another 29%.

To put the matter another way, if the principal has a fixed pot of money P to spend, but is unsure of the disutility c his employees feel about working, he should offer a prize instead of wages. The maximum disutility c which can be overcome with a prize of P is $P(1+2\alpha)/2$, while with a total wage bill of P, it is only $P(1+\alpha)/2$. The advantage of the prize is $\alpha P/2$.

The general picture, with the possibility of noise, goes as follows.

Theorem 2 (**Prizes outperform Wages iff Envy-Pride exceeds Noise Threshold**). *Suppose the noise is not too large:* $G^{\sigma}(-1) < 1/4$*. Define the noise-dependent threshold* $\alpha^* = 2G^{\sigma}(-1)/(1-4G^{\sigma}(-1))$*. If envy-pride is greater than the threshold, then prizes outperform wages:* $\tilde{M}^{\sigma}_{\alpha} < M^{\sigma}_{\alpha}$ *if* $\alpha > \alpha^*$*. In particular, if there is no noise (and so* $G^{\sigma}(-1) = G^{0}(-1) = 0$*), then* $\alpha^* = 0$*, and so prizes outperform wages with the slightest envy-pride. If envy-pride is below the threshold, then wages outperform prizes:* $M^{\sigma}_{\alpha} < \tilde{M}^{\sigma}_{\alpha}$ *if* $\alpha < \alpha^*$*.*

Proof. Immediate from Proposition 1.■

Another way of expressing the same idea is:

Theorem 2′ (**Prizes elicit Effort from a wider Range of Workers**). *Consider a fixed pot of money* P*. The principal can elicit full effort from the agents*[11] *via wages if their disutility* $c \leq (1+\alpha)P/2$ *(≡incentive to work in the wage game). With a prize, he can elicit full effort if* $c \leq (1+2\alpha)P/2 - (1+2\alpha)PG^{\sigma}(-1)$ *(≡incentive to work in the prize game). If the noise term* $G^{\sigma}(-1)$ *is small, prizes elicit full effort from a wider range of workers.*

Example 1. Suppose ε^{σ}_1 and ε^{σ}_2 are normally distributed with mean zero and standard deviation σ. Then Theorem 2 applies whenever $\sigma \leq 1$, since then $G^{\sigma}(-1) \leq G^{1}(-1) \approx 0.24 < \frac{1}{4}$. For instance, if $\sigma = 1/2$, then $G^{\sigma}(-1) \approx 0.08$ and $\alpha^* \approx 0.23 < 1$ turns out to be quite low, i.e. when agents care one fourth as much about the gap in payments as about their own payment, prizes dominate wages.[12]

Our third Theorem states that for any positive envy-pride parameter (however small), prizes outperform wages provided that the random component in agents' outputs is sufficiently low.

[11] I.e., $(e_1 = 1, e_2 = 1)$ is a Nash equilibrium in the corresponding game.

[12] Note that, with normally distributed noise, the principal *collects* money from an agent with positive probability in a wage contract (whenever the agent produces negative output, i.e., a "loss" - see footnote 5). With prizes, he only *hands out* money. Inspite of this, the principal prefers prizes to wages when the level of envy-pride is sufficiently high.

Theorem 3 (Prizes outperform Wages with any Envy-Pride, if Noise is small). *Given $\alpha > 0$, there exists $\sigma' > 0$ such that whenever $\sigma \leq \sigma'$, $\tilde{M}_{\alpha}^{\sigma} < M_{\alpha}^{\sigma}$.*

Proof. Since $\lim_{\sigma \to 0} G^{\sigma}(-1) = 0$, by Proposition 1 $\lim_{\sigma \to 0} \tilde{M}_{\alpha}^{\sigma} = \frac{2c}{1+2\alpha} < \frac{2c}{1+\alpha} = M_{\alpha}^{\sigma}$.■

Our next result emphasizes one drawback of prizes: too much noise destroys their efficacy, even if there is envy-pride. When wages are based on a noisy measure of output, a worker may be overpaid or underpaid w.r.t. his effort. But as long as the noise is unbiased, and wages are linear, his expected wage is correct. In contrast, when prizes are based on a noisy measure of relative output, the expected payment a worker gets is biased toward $P/2$, diminishing the expected payment to the hard worker and increasing the expected payment to the shirker. Theorem 4 shows that when noise is high, prizes are worse than wages no matter how much envy-pride there is.

Theorem 4 (If Noise is too large, Wages outperform Prizes no matter how much Envy-Pride there is). *If $G^{\sigma}(-1) \geq 1/4$, then $M_{\alpha}^{\sigma} < \tilde{M}_{\alpha}^{\sigma}$ for every $\alpha \geq 0$.*

Proof. By Proposition 1, $\tilde{M}_{\alpha}^{\sigma} \geq \frac{4c}{1+2\alpha}$, and this is always above $M_{\alpha}^{\sigma} = \frac{2c}{1+\alpha}$.■

2.4 Non-linear Wages

We have shown that prizes outperform wages when noise is small. Now we show that this continues to hold even if we allow for non-linear wages.

A non-linear wage is given by a function w, defined for all possible outputs. These functions are assumed to be nondecreasing, and to have the property that expected wages are nonnegative even with zero effort, i.e. $Ew(\varepsilon_i^{\sigma}) \geq 0$ for $i = 1, 2$. This guarantees that agents do not get expected negative wages under any level of effort. Denote by $\bar{M}_{\alpha}^{\sigma}(w)$ the expected payment by the principal under wage contract w, when both agents make effort 1. Also let $\bar{M}_{\alpha}^{\sigma} \leq M_{\alpha}^{\sigma}$ be the infimum of $\bar{M}_{\alpha}^{\sigma}(w)$ over all (non-linear) w which implement maximal effort by both agents in Nash equilibrium.

First suppose that there is no random noise at all: agent i's output precisely equals his effort e_i. It is easy to see that there is an optimal w achieving $\bar{M}_{\alpha}^{0}$. This w pays zero for all output levels below 1 (i.e. $w(x) = 0$ for $x < 1$), and $w(1)$ is the minimal payoff under which no agent i prefers $e_i = 0$ to $e_i = 1$ given that his opponent j is choosing $e_j = 1$. As in the computation of M_{α}^{σ} in the proof of Proposition 1, $w(1) = c/(1+\alpha)$, and so

$$\bar{M}_{\alpha}^{0} = \frac{2c}{1+\alpha} = M_{\alpha}^{0}.$$

Since $G^0(-1) = 0$, (3) implies

$$\tilde{M}_\alpha^0 = \frac{2c}{1+2\alpha}$$

Hence we have

$$\tilde{M}_\alpha^0 < \bar{M}_\alpha^0 \tag{4}$$

for all $\alpha > 0$. Thus, when there is no noise, prizes outperform *all* wage contracts for any given positive level of envy-pride. This continues to hold when the noise is sufficiently low:

Theorem 5 (Prizes outperform Non-linear Wages with small Noise). *Given $\alpha > 0$, there exists $\sigma' > 0$ such that $\tilde{M}_\alpha^\sigma < \bar{M}_\alpha^\sigma$ whenever $\sigma \le \sigma'$.*

Proof. Suppose that the assertion is false. Then one can find a sequence $(\sigma_k)_{k=1}^\infty$, and a sequence $(w_k)_{k=1}^\infty$ of wage contracts such that $\sigma_k \to 0$ as $k \to \infty$,

$$\bar{M}_\alpha^{\sigma_k}(w_k) \le \tilde{M}_\alpha^{\sigma_k} \tag{5}$$

for all k, and w_k implements maximal effort by both agents in Nash equilibrium when agents' outputs are affected by noises $\varepsilon_1^{\sigma_k}$, $\varepsilon_2^{\sigma_k}$. However, from (3) and the fact that $G^{\sigma_k}(-1) \to 0$ as $\sigma_k \to 0$, we obtain

$$\tilde{M}_\alpha^{\sigma_k} = \frac{c}{\frac{1}{2} - G^{\sigma_k}(-1)} \cdot \frac{1}{1+2\alpha} \longrightarrow_{k\to\infty} \frac{2c}{1+2\alpha} = \tilde{M}_\alpha^0. \tag{6}$$

On the other hand, we claim

$$\bar{M}_\alpha^{\sigma_k}(w_k) \ge \bar{M}_\alpha^0 \text{ for every } k. \tag{7}$$

Indeed, for every k construct a wage function $\overline{w}_k$ where $\overline{w}_k(x) = Ew_k(\varepsilon_1^{\sigma_k})$ for $x < 1$, and $\overline{w}_k(1) = Ew_k(1+\varepsilon_1^{\sigma_k})$. It is clear that, since agent i prefers $e_i = 1$ to $e_i = 0$ given $e_j = 1$ when there is noise $\varepsilon_i^{\sigma_k}$ that affects his (and independently his opponent's) output under wage function w_k, he also would prefer $e_i = 1$ to $e_i = 0$ under $\overline{w}_k$ when there is no noise. Thus $\bar{M}_\alpha^{\sigma_k}(w_k) = \bar{M}_\alpha^0(\bar{w}_k) \ge \bar{M}_\alpha^0$. However, the combination of (5), (6), and (7) contradicts (4).■

3 Multiple Agents

When there are many agents, the scope for envy and pride increases. Coming first (or last) among one hundred contestants gives more pleasure (or pain) than beating a single opponent. The principal can take advantage of this situation to pay less, whether he uses wages or prizes. We suppress this effect, and assume that agents care only about the *average* of others' receipts.

But multiple agents bring another benefit to prizes alone. With two contestants, an agent who shirks might get lucky and beat the other agent who works. However, with ninety-nine other agents working, the shirker is almost sure to come behind one of them. Thus sufficiently many agents tend to ameliorate the drawback of noise, helping prizes to become more efficacious than wages as long as there is some envy-pride.

Suppose that there are n identical agents. We assume that if agents $1, ..., n$ get $A_1, ..., A_n$ and agent i is exerting effort e_i, then i's utility is

$$u_i(A_1, ..., A_n, e_i) = A_i + \alpha(A_i - \frac{\sum_{j \neq i} A_j}{n-1}) - ce_i.$$

As in Section 2, one can see that

$$M_\alpha^\sigma = \frac{nc}{1+\alpha}$$

and

$$\tilde{M}_\alpha^\sigma = \frac{c}{1 + \frac{n}{n-1}\alpha} \sup_{\tau \in \mathcal{E}_{-1} \setminus \{0\}} \frac{\tau}{G_n^\sigma(\tau) - \frac{1}{n}},$$

where G_n^σ is the cumulative distribution function of the random variable

$$\max_{j \neq i} \varepsilon_j^\sigma - \varepsilon_i^\sigma,$$

and $\varepsilon_1^\sigma, ..., \varepsilon_n^\sigma$ are the i.i.d. noise terms affecting agents' outputs. It is easy to verify the following analogues of Theorems 4 and 2.

Theorem 4*. *If*

$$\sup_{\tau \in \mathcal{E}_{-1} \setminus \{0\}} \frac{\tau}{G_n^\sigma(\tau) - \frac{1}{n}} \geq \frac{n^2}{n-1},$$

then $\tilde{M}_\alpha^\sigma > M_\alpha^\sigma$ *for every* $\alpha \geq 0$.

Theorem 2*. *Suppose*

$$\sup_{\tau \in \mathcal{E}_{-1} \setminus \{0\}} \frac{\tau}{G_n^\sigma(\tau) - \frac{1}{n}} < \frac{n^2}{n-1},$$

Define

$$\alpha^*(n) = \frac{\sup_{\tau \in \mathcal{E}_{-1} \setminus \{0\}} \frac{\tau}{G_n^\sigma(\tau) - \frac{1}{n}} - n}{\frac{n^2}{n-1} - \sup_{\tau \in \mathcal{E}_{-1} \setminus \{0\}} \frac{\tau}{G_n^\sigma(\tau) - \frac{1}{n}}}. \tag{8}$$

If $\alpha > \alpha^*(n)$ *then* $\tilde{M}_\alpha^\sigma < M_\alpha^\sigma$, *and if* $\alpha < \alpha^*(n)$ *then* $\tilde{M}_\alpha^\sigma > M_\alpha^\sigma$.

Now we show that the premise of Theorem 2* holds for large n and furthermore $\alpha^*(n) \to 0$ as $n \to \infty$. In other words, no matter how slight the envy-pride, and how big the individual noise, prizes always outperform wages when there are sufficiently many competitors.

It will be useful to first consider a concrete example.

Example 2. Assume that $(\varepsilon_k^\sigma)_{k=1}^\infty$ is a sequence of i.i.d. random variables with the uniform distribution on $[-\sigma, \sigma]$, for some fixed $\sigma \geq 1$. For any n, let us denote the random noise terms in agents' outputs by $\varepsilon_1^\sigma, ..., \varepsilon_n^\sigma$. It is clear that for any $t \leq 0$

$$\Pr(\max_{1\leq j\leq n, j\neq i} \varepsilon_j^\sigma - \varepsilon_i^\sigma \leq t \mid \varepsilon_i^\sigma = y) = \begin{cases} 0, & \text{if } \ y \leq -\sigma - t; \\ \frac{1}{(2\sigma)^{n-1}} (y + t + \sigma)^{n-1}, & \text{if } \ -\sigma - t \leq y \leq \sigma. \end{cases}$$

Thus, for every $-1 \leq t \leq 0$, G_n^σ is given by

$$G_n^\sigma(t) = \int_{-\sigma-t}^{\sigma} \Pr(\max_{1\leq j\leq n, j\neq i} \varepsilon_j^\sigma - \varepsilon_i^\sigma \leq t \mid \varepsilon_i^\sigma = y) \frac{1}{2\sigma} dy \tag{9}$$

$$= \frac{1}{(2\sigma)^n} \int_{-\sigma-t}^{\sigma} (y + t + \sigma)^{n-1} \, dy = \frac{1}{(2\sigma)^n n} (t + 2\sigma)^n . \tag{10}$$

The density function of G_n^σ is clearly increasing on $[-1, 0]$, and therefore G_n^σ is convex on this interval, which implies

$$\sup_{\tau \in \mathcal{E}_{-1} \setminus \{0\}} \frac{\tau}{G_n^\sigma(\tau) - \frac{1}{n}} = \frac{1}{\frac{1}{n} - G_n^\sigma(-1)} = \frac{n}{1 - \left(\frac{2\sigma-1}{2\sigma}\right)^n}.$$

Clearly $\frac{n}{1-\left(\frac{2\sigma-1}{2\sigma}\right)^n} < \frac{n^2}{n-1}$ for all sufficiently large n, and thus Theorem 2* can be applied. It follows from (8) that

$$\alpha^*(n) = \frac{\frac{n}{1-\left(\frac{2\sigma-1}{2\sigma}\right)^n} - n}{\frac{n^2}{n-1} - \frac{n}{1-\left(\frac{2\sigma-1}{2\sigma}\right)^n}} = \frac{\left(\frac{2\sigma-1}{2\sigma}\right)^n}{\frac{n}{n-1}\left(1 - \left(\frac{2\sigma-1}{2\sigma}\right)^n\right) - 1}$$

$$= \frac{(n-1)\left(\frac{2\sigma-1}{2\sigma}\right)^n}{n\left(1 - \left(\frac{2\sigma-1}{2\sigma}\right)^n\right) - (n-1)} = \frac{(n-1)\left(\frac{2\sigma-1}{2\sigma}\right)^n}{-n\left(\frac{2\sigma-1}{2\sigma}\right)^n + 1} \to_{n\to\infty} 0$$

Thus *the minimal level of envy-pride required for prizes to outperform wages becomes vanishingly small as the number of competitors increases.*

The reader can check that only two properties of the uniform distribution play a role in establishing the above result:

$$G_n^\sigma \text{ is convex on } [-1,0] \text{ and } n^2 G_n^\sigma(-1) \to 0 \text{ as } n \to \infty. \tag{11}$$

These two properties are common, however, to all distributions of the random noise variables $(\varepsilon_k^\sigma)_{k=1}^\infty$ which have bounded support $[-\sigma,\sigma]$ and possess a continuously differentiable and positive density function f^σ on it. Indeed, for every $-1 \le t \le 0$, G_n^σ is given by

$$G_n^\sigma(t) = \int_{-\sigma-t}^{\sigma} \Pr(\max_{1\le j\le n, j\ne i} \varepsilon_j^\sigma - \varepsilon_i^\sigma \le t \mid \varepsilon_i^\sigma = y) f^\sigma(y)\,dy = \int_{-\sigma-t}^{\sigma} F^\sigma(y+t)^{n-1} f^\sigma(y)\,dy, \tag{12}$$

where F^σ denotes the cumulative distribution function of the random noise variable. Using (12),

$$\begin{aligned}
&\frac{\partial}{\partial t} G_n^\sigma(t)\\
&= \int_{-\sigma-t}^{\sigma} (n-1) F^\sigma(y+t)^{n-2} \frac{\partial}{\partial t} F^\sigma(y+t) f^\sigma(y)\,dy\\
&\quad + F^\sigma((-\sigma-t)+t)^{n-1} f^\sigma(-\sigma-t)\\
&= \int_{-\sigma-t}^{\sigma} (n-1) F^\sigma(y+t)^{n-2} f^\sigma(y+t) f^\sigma(y)\,dy,
\end{aligned}$$

and (for $n \ge 3$)

$$\begin{aligned}
\frac{\partial}{\partial^2 t} G_n^\sigma(t) &= \int_{-\sigma-t}^{\sigma} (n-1)(n-2) F^\sigma(y+t)^{n-3} f^\sigma(y+t)^2 f^\sigma(y)\,dy\\
&\quad + \int_{-\sigma-t}^{\sigma} (n-1) F^\sigma(y+t)^{n-2} \frac{\partial}{\partial t} f^\sigma(y+t) f^\sigma(y)\,dy\\
&\quad + (n-1) F^\sigma((-\sigma-t)+t)^{n-2} f^\sigma((-\sigma-t)+t) f^\sigma(-\sigma-t)\\
&= (n-1)\int_{-\sigma-t}^{\sigma} F^\sigma(y+t)^{n-3}\left[(n-2) f^\sigma(y+t)^2 + F^\sigma(y+t)\frac{\partial}{\partial t} f^\sigma(y+t)\right] f^\sigma(y)\,dy.
\end{aligned}$$

Since $\min_{y\in[-\sigma,\sigma]} f^\sigma(y) > 0$, it is clear that

$$(n-2) f^\sigma(y+t)^2 + F^\sigma(y+t)\frac{\partial}{\partial t} f^\sigma(y+t) > 0$$

for every $y \in [-\sigma - t, \sigma]$ and for all sufficiently large n. We conclude that $\frac{\partial}{\partial^2 t} G_n^\sigma(t) > 0$ and thus the function G_n^σ is convex on $[-1,0]$ for all sufficiently large n. Next, (12) implies that

$$G_n^\sigma(-1) \le F^\sigma(\sigma-1)^{n-1}.$$

Since $F^{\sigma}(\sigma-1)<1$,

$$\lim_{n\to\infty} n^2 G_n^{\sigma}(-1)=0.$$

Both conditions in (11) therefore hold, and this allows us to generalize the conclusion of Example 2 as Theorem 6 below: the minimal level of envy-pride required for prizes to outperform wages becomes vanishingly small as the number of competitors increases, even if the distribution of random noises is not uniform.

Theorem 6 (Prizes outperform Wages even with large Noise, if there are enough Competitors). *Suppose that $(\varepsilon_k^{\sigma})_{k=1}^{\infty}$ is a sequence of i.i.d. random variables with zero mean and values in $[-\sigma,\sigma]$ for some $\sigma\geq 1$, that possess a positive and continuously differentiable density function. For any n, suppose the random noise terms in agents' outputs are given by $\varepsilon_1^{\sigma},...,\varepsilon_n^{\sigma}$. Then $\lim_{n\to\infty}\alpha^*(n)=0$.*

4 Pride vs Envy

We return to the utility function

$$u(A,B,e)=A+\beta\max(A-B,0)-\alpha\max(B-A,0)-ce,$$

disentangling pride β from envy α.

Now it becomes possible to ask whether it is pride or envy that makes prizes a better incentive mechanism than wages.

One indication that pride is the real motivator is that people say they "play for pride". That is much less often said about regular work. Indeed workers are more likely to say they need the money, or they are afraid of the embarrassment of being unemployed, or of falling behind their peers.

This is revealed in our mathematics. The incentive to work for wage rate r, when there is no noise, is

$$r+\alpha r.$$

This is the sum of the money incentive r, and the envy αr of earning zero while the rival works and receives r.

The payoff to an agent who works in the prize game, excluding the disutility of work, is $\frac{1}{2}(P+\beta P)+\frac{1}{2}(-\alpha P)$; if he shirks, he gets $-\alpha P$. Thus the incentive to work with prize P is

$$\frac{1}{2}P+\frac{1}{2}\alpha P+\frac{1}{2}\beta P.$$

Setting the prize fund P equal to the total wage bill $2r$, we see that prizes provide an extra incentive of βr, independent of α. Thus no matter how large or small envy

α is, the slightest inclination towards pride $\beta > 0$ will cause prizes to outperform wages. Without pride (but with envy), prizes and wages are equivalent.

We now present a more precise analysis. Even when there is noise, prizes outperform wages provided there is enough pride. To show this, we first establish a variant of Proposition 1 below, replacing α by the vector (α, β) in our previous notation:

Proposition 1$'$. *Let*

$$\varphi_\sigma \equiv \frac{1}{2} E\left(\varepsilon_1^\sigma - \varepsilon_2^\sigma \mid \varepsilon_1^\sigma - \varepsilon_2^\sigma \geq 0\right)$$

and

$$\psi_\sigma \equiv E\left(\varepsilon_1^\sigma - \varepsilon_2^\sigma - 1 \mid \varepsilon_1^\sigma - \varepsilon_2^\sigma \geq 1\right) \Pr\left(\varepsilon_1^\sigma - \varepsilon_2^\sigma \geq 1\right)$$

(note that $0 < \Delta_\sigma \equiv \varphi_\sigma - \psi_\sigma \leq \frac{1}{2}$ if $\sigma > 0$*). Then*

$$M_{\alpha,\beta}^\sigma \geq \frac{2c}{1 + \alpha + (\beta - \alpha)\left(\varphi_\sigma - \psi_\sigma\right)} \tag{13}$$

and

$$\tilde{M}_{\alpha,\beta}^\sigma = \frac{c}{\frac{1}{2} - G_\sigma(-1)} \cdot \frac{1}{1 + \alpha + \beta}. \tag{14}$$

Proof. In the game $\Gamma_{\alpha,\beta}^\sigma(r)$ the expected utility of agent i, when he chooses effort level e_i and his rival j chooses effort level e_j, is

$$re_i + \alpha r E\left(e_i + \varepsilon_i^\sigma - e_j - \varepsilon_j^\sigma \mid e_i + \varepsilon_i^\sigma - e_j - \varepsilon_j^\sigma \leq 0\right) \Pr\left(e_i + \varepsilon_i^\sigma - e_j - \varepsilon_j^\sigma \leq 0\right)$$

$$+\beta r E\left(e_i + \varepsilon_i^\sigma - e_j - \varepsilon_j^\sigma \mid e_i + \varepsilon_i^\sigma - e_j - \varepsilon_j^\sigma \geq 0\right) \Pr\left(e_i + \varepsilon_i^\sigma - e_j - \varepsilon_j^\sigma \geq 0\right) - ce_i.$$

In order for $(e_1 = 1, e_2 = 1)$ to be a Nash equilibrium of $\Gamma_{\alpha,\beta}^\sigma(r)$, it is necessary that (under the piece-rate r) effort level 1 is not less attractive to an agent than effort level 0, given that his rival chooses effort level 1. Thus, we must have

$$r + \alpha r E\left(\varepsilon_i^\sigma - \varepsilon_j^\sigma \mid \varepsilon_i^\sigma - \varepsilon_j^\sigma \leq 0\right) \Pr\left(\varepsilon_i^\sigma - \varepsilon_j^\sigma \leq 0\right)$$

$$+\beta r E\left(\varepsilon_i^\sigma - \varepsilon_j^\sigma \mid \varepsilon_i^\sigma - \varepsilon_j^\sigma \geq 0\right) \Pr\left(\varepsilon_i^\sigma - \varepsilon_j^\sigma \geq 0\right) - c$$

$$\geq \alpha r E\left(\varepsilon_i^\sigma - \varepsilon_j^\sigma - 1 \mid \varepsilon_i^\sigma - \varepsilon_j^\sigma \leq 1\right) \Pr\left(\varepsilon_i^\sigma - \varepsilon_j^\sigma \leq 1\right)$$

$$+\beta r E\left(\varepsilon_i^\sigma - \varepsilon_j^\sigma - 1 \mid \varepsilon_i^\sigma - \varepsilon_j^\sigma \geq 1\right) \Pr\left(\varepsilon_i^\sigma - \varepsilon_j^\sigma \geq 1\right),$$

i.e.,

$$r(1 - \alpha\varphi_\sigma + \beta\varphi_\sigma) - c \geq r(-\alpha\left(\psi_\sigma + 1\right) + \beta\psi_\sigma).$$

(Here we use the obvious fact that $E\left(\varepsilon_i^\sigma-\varepsilon_j^\sigma-1 \mid \varepsilon_i^\sigma-\varepsilon_j^\sigma\leq 1\right)\Pr\left(\varepsilon_i^\sigma-\varepsilon_j^\sigma\leq 1\right)+\psi_\sigma=-1$.) This implies

$$r\geq\frac{c}{1+\alpha\left(1+\psi_\sigma-\varphi_\sigma\right)+\beta\left(\varphi_\sigma-\psi_\sigma\right)},$$

and (13) follows.

Next consider the prize game $\tilde{\Gamma}^\sigma_{\alpha,\beta}(P)$. Here the expected utility of agent i, when he chooses effort level e_i and his rival j chooses effort level e_j, is $G^\sigma(e_i-e_j)(1+\beta)P-[1-G^\sigma(e_i-e_j)]\,\alpha P-ce_i$ (and, if $e_i=e_j$ and $\sigma=0$, replace $G^\sigma(e_i-e_j)$ by $\frac{1}{2}-$ see Footnote 8). Thus, in order to implement $(e_1=1,e_2=1)$ as a Nash equilibrium of $\tilde{\Gamma}^\sigma_{\alpha,\beta}(P)$, it is necessary and sufficient that P satisfy

$$G^\sigma(0)(1+\beta)P-[1-G^\sigma(0)]\,\alpha P-c\geq G^\sigma(e_i-1)(1+\beta)P-[1-G^\sigma(e_i-1)]\,\alpha P-ce_i \tag{15}$$

for every $e_i\in\mathcal{E}$. As in the proof of Proposition 1, the minimal P that satisfies (15) for every $e_i\in\mathcal{E}$ is:

$$\frac{c\sup_{\tau\in\mathcal{E}_{-1}\setminus\{0\}}\frac{\tau}{G^\sigma(\tau)-\frac{1}{2}}}{1+\alpha+\beta}.$$

Since G^σ is convex on $[-1,0]$, the supremum in this expression is attained for $\tau=-1$, which leads to (14).■

Theorem 7 (Prizes outperform Wages with Pride alone). *Suppose the noise is not too large: $G^\sigma(-1)<(1-\Delta_\sigma)/4$. (This holds, for instance, when $G^\sigma(-1)<1/8$.) Then there exists $\beta'>0$ such that whenever $\beta\geq\beta'$ and $\alpha\leq\beta$ prizes outperform wages: $\tilde{M}^\sigma_{\alpha,\beta}<M^\sigma_{\alpha,\beta}$.*

Proof. If $\alpha\leq\beta$ then, using Proposition 1′,

$$\frac{\tilde{M}^\sigma_{\alpha,\beta}}{M^\sigma_{\alpha,\beta}}\leq\frac{1+\alpha+(\beta-\alpha)\,\Delta_\sigma}{2(\frac{1}{2}-G_\sigma(-1))\,(1+\alpha+\beta)}=\frac{\frac{1}{\alpha+\beta}+\frac{\alpha}{\alpha+\beta}(1-\Delta_\sigma)+\frac{\beta}{\alpha+\beta}\Delta_\sigma}{2(\frac{1}{2}-G_\sigma(-1))\left(1+\frac{1}{\alpha+\beta}\right)}$$

$$\leq\frac{\frac{1}{\beta}+\frac{1}{2}(1-\Delta_\sigma)+\Delta_\sigma}{2(\frac{1}{2}-G_\sigma(-1))\left(1+\frac{1}{2\beta}\right)}.$$

The expression on the right converges to $\frac{\frac{1}{2}(1-\Delta_\sigma)+\Delta_\sigma}{2(\frac{1}{2}-G_\sigma(-1))}=\frac{\frac{1}{2}(1+\Delta_\sigma)}{2(\frac{1}{2}-G_\sigma(-1))}$ when $\beta\to\infty$, which is below 1 given that $G^\sigma(-1)<(1-\Delta_\sigma)/4$. Thus, there exists $\beta'>0$ such that whenever $\beta\geq\beta'$ and $\alpha\leq\beta$, $\frac{\tilde{M}^\sigma_{\alpha,\beta}}{M^\sigma_{\alpha,\beta}}<1$, implying that prizes outperform wages.■

In Theorem 7 we could have allowed for higher levels of envy, not only those below the pride parameter. What is important is that the envy parameter should not exceed some multiple of the pride parameter: $\alpha \leq K\beta$ for some fixed $K > 1$. Theorem 7 will remain valid, by similar arguments, provided the assumption of low noise is strengthened: $G^\sigma(-1) < (1-\Delta_\sigma)/2(K+1)$. (Since $\lim_{\sigma\to 0} G^\sigma(-1) = \lim_{\sigma\to 0}\Delta_\sigma = 0$, this inequality holds for all sufficiently small σ.)

It turns out that, with noise, envy alone can sometimes ensure that prizes are better than wages. In Section 2, with $\alpha = \beta$, we found that noise had no effect on the incentives provided by wages (since by risk neutrality we could replace each agent's random wage with its expected value), while it hurt prizes (since it occasionally enables the shirker to win). But when $\alpha > 0$ and $\beta = 0$, noise can hurt wage incentives more than prize incentives.

Example 3. Just as in the proof of Theorem 7, one can check that

$$\frac{\tilde{M}^\sigma_{\alpha,\beta}}{M^\sigma_{\alpha,\beta}} \leq \frac{\frac{1}{\alpha} + (1-\Delta_\sigma) + \frac{1}{2}\Delta_\sigma}{2(\frac{1}{2} - G_\sigma(-1))\left(1+\frac{1}{2\alpha}\right)}$$

if $\beta \leq \alpha$. Since $\lim_{\alpha\to\infty} \frac{\frac{1}{\alpha}+(1-\Delta_\sigma)+\frac{1}{2}\Delta_\sigma}{2(\frac{1}{2}-G_\sigma(-1))\left(1+\frac{1}{2\alpha}\right)} = \frac{1-\frac{1}{2}\Delta_\sigma}{2(\frac{1}{2}-G_\sigma(-1))}$, the condition

$$G^\sigma(-1) < \Delta_\sigma/4 \tag{16}$$

ensures the existence of $\alpha' > 0$ such that whenever $\alpha \geq \alpha'$ and $\beta \leq \alpha$ prizes outperform wages: $\tilde{M}^\sigma_{\alpha,\beta} < M^\sigma_{\alpha,\beta}$. While there is no universal condition that would guarantee (16) (unlike the analogous condition in Theorem 7, which was implied by $G^\sigma(-1) < \frac{1}{8}$), (16) holds for normally distributed random noises for some values of the standard deviation σ. For instance, when $\sigma = 0.4$,

$$G^\sigma(-1) - \Delta_\sigma/4$$

$$= \frac{1}{2\cdot 0.4\sqrt{\pi}}\int_{-\infty}^{-1} e^{-\frac{t^2}{4\cdot 0.4^2}}\,dt - \frac{1}{4}\left(\frac{1}{2\cdot 0.4\sqrt{\pi}}\int_0^\infty te^{-\frac{t^2}{4\cdot 0.4^2}}\,dt - \frac{1}{2\cdot 0.4\sqrt{\pi}}\int_1^\infty (t-1)e^{-\frac{t^2}{4\cdot 0.4^2}}\,dt\right)$$

$$\approx -.0\,156\,8 < 0,$$

and thus in this case prizes outperform wages for all sufficiently large levels of envy.

5 When Agents are not Risk Neutral: Wages Plus Prizes and the Need for a Bonus

The biggest objection to prizes is that they force agents to face a huge uncertainty about who will get the prize, even if they work hard. It comes to the fore when agents are risk averse. But even here we find that wages supplemented by prizes are always an improvement on wages alone. These supplementary prizes are common in practice, where they are known as bonuses.

Let the utility function of each of the two agents be given by

$$u(A, B, e) = U(A) - \alpha V(\max\{B - A, 0\}) + \beta V(\max\{A - B, 0\}) - ce,$$

where, as before, A is the amount paid to the agent, B the amount paid to his rival, α his envy parameter, β his pride parameter, e his choice of effort level, and $c > 0$ the disutility from effort.[13] We assume that U and V are continuously differentiable, and that their derivatives are strictly positive[14]; furthermore both U and V vanish at zero.

We do *not* need to assume that either U or V is concave, though that is the case we mostly have in mind. In that case, beating a rival with nearly the same income by a dollar confers a lot of pride, but if you are a king and he is a pauper, an extra dollar of disparity does not add much more to your pride.

For simplicity, we will assume $\alpha = \beta$. The i.i.d. random noises ε_1^σ and ε_2^σ are now taken to be supported on a compact interval $[-\lambda, \lambda]$, for all σ. As before, G^σ denotes the cumulative distribution function of $\varepsilon_1^\sigma - \varepsilon_2^\sigma$, and we assume that G^σ is convex on $[-1, 0]$.

Up until now we only considered "pure" contracts which could take the form of either a prize P or a piece-rate wage r. Now we allow for *mixed* contracts (P, r) : each agent is paid rq when his output is q, plus a prize (bonus) P if his output is more than his rival's (tossing a coin in case of ties). The contract (P, r) induces[15] a game $\Gamma_\alpha^\sigma(P, r)$ in the obvious manner.

Let Π_α^σ denote the set of mixed contracts which elicit full effort, i.e.,

$$\Pi_\alpha^\sigma = \left\{(P, r) \in R_+^2 \mid (e_1 = 1, e_2 = 1) \text{ is a Nash equilibrium of } \Gamma_\alpha^\sigma(P, r)\right\}.$$

The principal's payout is $P + 2r$ when $(e_1 = 1, e_2 = 1)$ is played in $\Gamma_\alpha^\sigma(P, r)$. Thus the set of *optimal* contracts is

$$\Pi_\alpha^{*\sigma} = \arg\min\{P + 2r \mid (P, r) \in \Pi_\alpha^\sigma\}.$$

With risk neutral agents, there was no need to consider Π_α^σ because pure contracts are just as good as any mixture: there always exists $(P, r) \in \Pi_\alpha^{*\sigma}$ such that either $P = 0$ or $r = 0$.[16]

[13] We could have more generally considered $u(A, B, e) = U(A) - \alpha V_{envy}(\max\{B - A, 0\}) + \beta V_{pride}(\max\{A - B, 0\}) - ce$ instead of supposing $V = V_{pride} = V_{envy}$. Similar results would obtain but at the cost of more notation.

[14] U is defined on R, while V on R_+.

[15] The underlying components c, U, V of the utility are held fixed, while (P, r) and $\sigma, \alpha = \beta$ vary.

[16] This is obvious from the linearity of the optimization problem when agents are risk neutral. See the appendix in Dubey, Geanakoplos, and Haimanko (2005) for the details.

But if agents are *not* risk-neutral, mixed contracts may well beat pure contracts. We leave the exploration of the exact structure of optimal mixed contracts for future research. But we shall delineate two scenarios in which *any optimal mixed contract must necessarily entail a positive bonus*, i.e., $P > 0$ for every $(P, r) \in \Pi_{\alpha}^{*\sigma}$. In the first scenario (Theorem 8 below) envy-pride is fixed at an arbitrary positive level. Bonuses are needed, provided the noise is sufficiently small. In the second scenario (Theorem 9 below), the noise is fixed and not too large. Again bonuses are needed, for sufficiently high envy-pride.

The intuition for Theorem 8 is roughly as follows. Suppose the two agents are earning only wages. When there is no noise, a hard-working agent knows the wage $w = r1$ he will earn for sure. Assuming differentiable utilities, he is nearly risk neutral for small variations in consumption. So consider reducing the piece-rate by ε, and instead awarding a prize of 2ε to the highest performance. Then the expected consumption utility of a hard-working agent stays almost the same. But as we argued before with risk neutrality, the incentive created by envy-pride is greater for the prize than the wage. Thus a small bonus increases incentives without increasing the total expected payout of the principal.

This argument demonstrates the need for a bonus but sheds no light on its optimal size, which may be quite big. The argument, moreover, only works for small prizes and small noise. As the prize gets larger, risk aversion kicks in and the prize becomes a less attractive substitute for wages. As noise increases, the luckiest worker, who already has the highest wage and therefore the lowest marginal utility for money, will get the prize, reducing its ex ante consumption utility.

Theorem 8 (Bonus is needed with sufficiently low Noise, given any fixed Envy-Pride). *Assume that: (i) $\mathcal{E}$ is a finite set; (ii) there exist $B < \infty$ and $b > 0$ such that $U'(x) \leq B$ for every $x \in R$ and $b \leq V'(x) \leq B$ for every $x \in R_+$. Then, given $\alpha > 0$, there exists $\sigma' > 0$ such that $P > 0$ for every $(P, r) \in \Pi_{\alpha}^{*\sigma}$ whenever $\sigma \leq \sigma'$.*

Proof. Fix $\alpha > 0$. Suppose to the contrary that there exists a vanishing sequence $\{\sigma_k\}_{k=1}^{\infty}$ of positive numbers and $(P_k^*, r_k^*) \in \Pi_{\alpha}^{*\sigma_k}$ such that $P_k^* = 0$ (and, w.l.o.g., $r^* \equiv \lim_{k\to\infty} r_k^*$ exists and $0 < r^* < \infty$). Consider $(P_{\varepsilon,\delta}^k, r_{\varepsilon,\delta}^k) = \left(\varepsilon, r_k^* - \frac{1}{2}\varepsilon(1+\delta)\right)$. We shall show that there exist small enough $\varepsilon > 0$ and $\delta > 0$ such that $(P_{\varepsilon,\delta}^k, r_{\varepsilon,\delta}^k) \in R_+^2$ elicits full effort from both agents in a Nash equilibrium of $\Gamma_{\alpha}^{\sigma_k}(P_{\varepsilon,\delta}^k, r_{\varepsilon,\delta}^k)$ when k is large (and the noise parameter σ_k is small). Since

$$P_k^* + 2r_k^* = 2r_k^* > 2r_k^* - \varepsilon\delta = P_{\varepsilon,\delta}^k + 2r_{\varepsilon,\delta}^k,$$

it will follow that (P_k^*, r_k^*) are not optimal when k is large, a contradiction.

Now we turn to establishing the existence of the requisite $(P_{\varepsilon,\delta}^k, r_{\varepsilon,\delta}^k)$. First notice that in order to implement $(e_1 = 1, e_2 = 1)$ as a Nash equilibrium of $\Gamma_\alpha^{\sigma_k}(P_{\varepsilon,\delta}^k, r_{\varepsilon,\delta}^k)$, it is necessary and sufficient for the following incentive conditions to hold:

$$\frac{1}{2}E\left(\left[\begin{array}{c} U(P_{\varepsilon,\delta}^k + r_{\varepsilon,\delta}^k(1+\varepsilon_i^{\sigma_k})) \\ +\alpha V(P_{\varepsilon,\delta}^k + r_{\varepsilon,\delta}^k(\varepsilon_i^{\sigma_k} - \varepsilon_j^{\sigma_k})) \end{array}\right] \mid \varepsilon_i^{\sigma_k} > \varepsilon_j^{\sigma_k}\right)$$
$$+\frac{1}{2}E\left(\left[\begin{array}{c} U(r_{\varepsilon,\delta}^k(1+\varepsilon_i^{\sigma_k})) \\ -\alpha V(P_{\varepsilon,\delta}^k + r_{\varepsilon,\delta}^k(\varepsilon_j^{\sigma_k} - \varepsilon_i^{\sigma_k})) \end{array}\right] \mid \varepsilon_i^{\sigma_k} < \varepsilon_j^{\sigma_k}\right) - c$$
$$\geq G^{\sigma_k}(e_i - 1)$$
$$\cdot E\left(\left[\begin{array}{c} U(P_{\varepsilon,\delta}^k + r_{\varepsilon,\delta}^k(e_i+\varepsilon_i^{\sigma_k})) \\ +\alpha V(P_{\varepsilon,\delta}^k + r_{\varepsilon,\delta}^k(e_i + \varepsilon_i^{\sigma_k} - 1 - \varepsilon_j^{\sigma_k})) \end{array}\right] \mid e_i + \varepsilon_i^{\sigma_k} > 1 + \varepsilon_j^{\sigma_k}\right)$$
$$+ (1 - G^{\sigma_k}(e_i - 1))$$
$$\cdot E\left(\left[\begin{array}{c} U(r_{\varepsilon,\delta}^k(e_i+\varepsilon_i^{\sigma_k})) \\ -\alpha V(P_{\varepsilon,\delta}^k + r_{\varepsilon,\delta}^k(1 + \varepsilon_j^{\sigma_k} - e_i - \varepsilon_i^{\sigma_k})) \end{array}\right] \mid e_i + \varepsilon_i^{\sigma_k} < 1 + \varepsilon_j^{\sigma_k}\right) - ce_i$$

for every $e_i \in \mathcal{E}\setminus\{1\}$. Denote by $I_k(\varepsilon, \delta, e_i)$ the difference between the left-hand side and the right-hand side of the above inequality. Thus, each of the above incentive conditions is equivalent to

$$I_k(\varepsilon, \delta, e_i) \geq 0. \tag{17}$$

Observe that the derivative of I_k with respect to ε, evaluated at $\varepsilon = 0$, is given by

$$\frac{1}{2}E\left(\left[\begin{array}{c} U'(r_k^*(1+\varepsilon_i^{\sigma_k}))\left(1 - \frac{1}{2}(1+\delta)(1+\varepsilon_i^{\sigma_k})\right) \\ +\alpha V'(r_k^*(\varepsilon_i^{\sigma_k} - \varepsilon_j^{\sigma_k}))\left(1 - \frac{1}{2}(1+\delta)(\varepsilon_i^{\sigma_k} - \varepsilon_j^{\sigma_k})\right) \end{array}\right] \mid \varepsilon_i^{\sigma_k} > \varepsilon_j^{\sigma_k}\right)$$
$$+\frac{1}{2}E\left(\left[\begin{array}{c} U'(r_k^*(1+\varepsilon_i^{\sigma_k}))\left(-\frac{1}{2}(1+\delta)(1+\varepsilon_i^{\sigma_k})\right) \\ -\alpha V'(r_k^*(\varepsilon_j^{\sigma_k} - \varepsilon_i^{\sigma_k}))\left(1 - \frac{1}{2}(1+\delta)(\varepsilon_j^{\sigma_k} - \varepsilon_i^{\sigma_k})\right) \end{array}\right] \mid \varepsilon_i^{\sigma_k} < \varepsilon_j^{\sigma_k}\right)$$
$$-G^{\sigma_k}(e_i - 1)$$
$$\cdot E\left(\left[\begin{array}{c} U'(r_k^*(e_i+\varepsilon_i^{\sigma_k}))\left(1 - \frac{1}{2}(1+\delta)(e_i+\varepsilon_i^{\sigma_k})\right) \\ +\alpha V'(r_k^*(e_i - 1 + \varepsilon_i^{\sigma_k} - \varepsilon_j^{\sigma_k}))\left(1 - \frac{1}{2}(1+\delta)(e_i - 1 + \varepsilon_i^{\sigma_k} - \varepsilon_j^{\sigma_k})\right) \end{array}\right] \mid e_i + \varepsilon_i^{\sigma_k} > 1 + \varepsilon_j^{\sigma_k}\right)$$
$$-(1 - G^{\sigma_k}(e_i - 1))$$
$$\cdot E\left(\left[\begin{array}{c} U'(r_k^*(e_i+\varepsilon_i^{\sigma_k}))\left(-\frac{1}{2}(1+\delta)(e_i+\varepsilon_i^{\sigma_k})\right) \\ -\alpha V'(r_k^*(1 - e_i + \varepsilon_j^{\sigma_k} - \varepsilon_i^{\sigma_k}))\left(1 - \frac{1}{2}(1+\delta)(1 - e_i + \varepsilon_j^{\sigma_k} - \varepsilon_i^{\sigma_k})\right) \end{array}\right] \mid e_i + \varepsilon_i^{\sigma_k} < 1 + \varepsilon_j^{\sigma_k}\right),$$

for every $e_i \in \mathcal{E}\setminus\{1\}$. Since the random noises belong to a bounded interval by our assumption in this section, they converge to zero in probability as $\sigma \to 0$. Bearing in mind that $G^\sigma(t) \to_{\sigma\to 0} 0$ for $t < 1$ and that U', V' are continuous and bounded, as $k \to \infty$ the above expression converges to:

$$\frac{1}{2}\left(U'(r^*)\left(1 - \frac{1}{2}(1+\delta)\right) + \alpha V'(0)\right)$$

$$+\frac{1}{2}\left(U'(r^*)\left(-\frac{1}{2}(1+\delta)\right)-\alpha V'(0)\right)$$

$$-U'(r^*e_i)\left(-\frac{1}{2}(1+\delta)e_i\right)+\alpha V'(r^*(1-e_i))\left(1-\frac{1}{2}(1+\delta)(1-e_i)\right)$$

$$=-\frac{1}{2}U'(r^*)\delta-U'(r^*e_i)\left(-\frac{e_i}{2}(1+\delta)\right)+\alpha V'(r^*(1-e_i))\left(1-\frac{1-e_i}{2}(1+\delta)\right).$$

The last expression is bounded from below by

$$-\frac{1}{2}U'(r^*)\delta+\alpha V'(r^*(1-e_i))\left(1-\frac{1-e_i}{2}(1+\delta)\right)$$

$$\geq -\frac{1}{2}B\delta+\alpha b\left(\frac{1}{2}(1-\delta)\right).$$

This is positive for $\delta^* \equiv \frac{\alpha b}{2(\alpha b+B)}$, and so $\frac{\partial}{\partial\varepsilon}I_k(\varepsilon,\delta^*,e_i)\mid_{\varepsilon=0}>0$ for all large enough k. Thus, since the incentive constraint (17) for any given $e_i\in\mathcal{E}\setminus\{1\}$ holds for $(P_k^*,r_k^*)=(P_{0,\delta^*}^k,r_{0,\delta^*}^k)$, it also holds for $(P_{\varepsilon,\delta^*}^k,r_{\varepsilon,\delta^*}^k)$ for all large enough k and some $\varepsilon=\varepsilon(k)>0$. Since $\mathcal{E}$ is finite, there are only finitely many incentive constraints, and thus all of them hold simultaneously for $(P_{\varepsilon,\delta^*}^k,r_{\varepsilon,\delta^*}^k)$ for all large enough k and some $\varepsilon=\varepsilon^*(k)>0$. Therefore $(P_{\varepsilon^*(k),\delta^*}^k,r_{\varepsilon^*(k),\delta^*}^k)$ elicits full effort from both agents in a Nash equilibrium of $\Gamma_\alpha^{\sigma_k}(P_{\varepsilon^*(k),\delta^*}^k,r_{\varepsilon^*(k),\delta^*}^k)$. As was said, this contradicts the optimality of (P_k^*,r_k^*) when k is large.■

A complementary theorem shows that for any fixed noise below some reasonably large upper bound, a bonus is again needed if there is enough envy-pride. The intuition for this result is that as envy-pride gets very large, the optimal piece-rate (assuming no prize) goes to zero. Since the noise is bounded, the final consumption, being the product of the piece-rate and output, also goes to zero. Thus consumption is practically certain, and the agents become nearly risk-neutral. Hence, as in the previous theorem, reducing the wage a tiny bit and substituting a tiny prize is necessarily beneficial.

Theorem 9 (Bonus is needed with sufficiently high Envy-Pride, given any fixed Noise). *Suppose that* $G^\sigma(-1)<\frac{1}{4}$. *Then there exists* $\alpha'>0$ *such that, if* $\alpha>\alpha'$, *then* $P>0$ *for every* $(P,r)\in\Pi_\alpha^{*\sigma}$.

Actually, more is true. Even a pure prize contract can do better than a wage contract under the conditions of Theorem 9:

Theorem 10 (Prizes outperform Wages even without Risk Neutrality, provided there is sufficient Envy-Pride). *Suppose that* $G^\sigma(-1)<\frac{1}{4}$. *Then there exists* $\alpha'>0$ *such that* $\tilde{M}_\alpha^\sigma<M_\alpha^\sigma$ *if* $\alpha>\alpha'$.

Proof. Recalling that $\alpha = \beta$, it will be convenient to write

$$u(A, B, e) = U(A) + \alpha W(A - B) - ce$$

where $W(x) = sign(x)V(|x|)$. Denote by r_α the minimal piece-rate at which, in the wage game Γ_α^σ, effort level 1 is not less attractive to an agent than effort level 0, given that his rival chooses effort level 1. Thus r_α is the *smallest* among all non-negative numbers r that satisfy the inequality

$$\begin{aligned} &EU(r(1+\varepsilon_i^\sigma)) + \alpha EW(r(1+\varepsilon_i^\sigma) - r(1+\varepsilon_j^\sigma)) - c \\ &\geq EU(r\varepsilon_i^\sigma) + \alpha EW(r\varepsilon_i^\sigma - r(1+\varepsilon_j^\sigma)), \end{aligned} \tag{18}$$

or

$$E\left[U(r(1+\varepsilon_i^\sigma)) - U(r\varepsilon_i^\sigma)\right] + \alpha E\left[W(r(1+\varepsilon_i^\sigma) - r(1+\varepsilon_j^\sigma)) - W(r\varepsilon_i^\sigma - r(1+\varepsilon_j^\sigma))\right] \geq c.$$

Let

$$K \equiv \min[\min_{-\lambda \leq x \leq 1+\lambda} U'(x), \min_{0 \leq x \leq 1+2\lambda} V'(x)] > 0.$$

Using the definition of W, for all $r \leq 1$

$$\left[U(r(1+\varepsilon_i^\sigma)) - U(r\varepsilon_i^\sigma)\right] + \alpha\left[W(r(1+\varepsilon_i^\sigma) - r(1+\varepsilon_j^\sigma)) - W(r\varepsilon_i^\sigma - r(1+\varepsilon_j^\sigma))\right] \tag{19}$$

$$\geq K(1+\alpha)r. \tag{20}$$

Consequently, for all large enough α, substituting $r = \frac{c}{K(1+\alpha)} \leq 1$ into (18) turns it into a valid inequality by (19)-(20), and hence $r_\alpha \leq \frac{c}{K(1+\alpha)}$. (In particular, $\lim_{\alpha\to\infty} r_\alpha = 0$.) Substituting $r = r_\alpha$ in (18), we can therefore use the first-order (linear) approximation $U'(0) \cdot x$ for $U(x)$, and $V'(0) \cdot x$ for $W(x)$, around 0, to derive an existence of $\gamma_\alpha \geq 0$ such that

$$U'(0) \cdot r_\alpha + \alpha V'(0) \cdot 0 - c + \gamma_\alpha \geq U'(0) \cdot 0 + \alpha V'(0) \cdot (-r_\alpha)$$

holds for every α, and $\lim_{\alpha\to\infty} \gamma_\alpha = 0$. Thus, $(U'(0) + \alpha V'(0)) \cdot r_\alpha \geq c - \gamma_\alpha$, or $r_\alpha \geq \frac{c-\gamma_\alpha}{U'(0)+\alpha V'(0)}$. The minimal piece rate that implements $(e_1 = 1, e_2 = 1)$ as a Nash equilibrium in the wage game Γ_α^σ should therefore be at least $\frac{c-\gamma_\alpha}{U'(0)+\alpha V'(0)}$. Consequently,

$$M_\alpha^\sigma \geq \frac{2(c-\gamma_\alpha)}{U'(0)+\alpha V'(0)} \tag{21}$$

for all sufficiently large α.

Arguing as in the proof of Proposition 1, one can show that the minimal prize $\tilde{M}_\alpha^\sigma$ that implements $(e_1 = 1, e_2 = 1)$ as a Nash equilibrium in the prize game $\tilde{\Gamma}_\alpha^\sigma$ satisfies

$$\frac{1}{2}U(\tilde{M}_\alpha^\sigma) - c = G^\sigma(-1)(U(\tilde{M}_\alpha^\sigma) + \alpha V(\tilde{M}_\alpha^\sigma)) - \left[1 - G^\sigma(-1)\right]\alpha V(\tilde{M}_\alpha^\sigma). \tag{22}$$

It follows that $\tilde{M}_\alpha^\sigma = F_\alpha^{-1}\left(\frac{c}{\frac{1}{2}-G^\sigma(-1)}\right)$, where $F_\alpha(x) \equiv U(x) + 2\alpha V(x)$. Since $2\alpha V \leq F_\alpha$ on R_+, $\tilde{M}_\alpha^\sigma \leq V^{-1}\left(\frac{1}{2\alpha}\frac{c}{\frac{1}{2}-G^\sigma(-1)}\right) \leq \frac{\tilde{K}}{2\alpha}$for some $\widetilde{K} > 0$ and for all large enough α (and in particular $\lim_{\alpha\to\infty} \tilde{M}_\alpha^\sigma = 0$). We can therefore use (22) and the linear approximation $U'(0) \cdot x$ for $U(x)$, and $V'(0) \cdot x$ for $V(x)$, around 0, to derive the existence of a $\delta_\alpha \geq 0$ such that

$$\frac{1}{2}U'(0) \cdot \tilde{M}_\alpha^\sigma - c - \delta_\alpha \leq G^\sigma(-1)(U'(0) + \alpha V'(0)) \cdot \tilde{M}_\alpha^\sigma - [1 - G^\sigma(-1)]\,\alpha V'(0) \cdot \tilde{M}_\alpha^\sigma$$

holds for every α and $\lim_{\alpha\to\infty} \delta_\alpha = 0$. Thus

$$\tilde{M}_\alpha^\sigma \leq \frac{c + \delta_\alpha}{(U'(0) + 2\alpha V'(0))} \cdot \frac{1}{\frac{1}{2} - G^\sigma(-1)} \tag{23}$$

for all sufficiently large α.

Since $G^\sigma(-1) < \frac{1}{4}$, it follows from (21) and (23) that $\limsup_{\alpha\to\infty} \frac{\tilde{M}_\alpha^\sigma}{M_\alpha^\sigma} \leq \frac{1}{4} \cdot \frac{1}{\frac{1}{2}-G^\sigma(-1)} < 1$, and thus indeed $\tilde{M}_\alpha^\sigma < M_\alpha^\sigma$ for all sufficiently large α.■

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