RANDOMIZED SIGN TESTS FOR DEPENDENT OBSERVATIONS ON DISCRETE CHOICE UNDER RISK

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Randomized Sign Tests for Dependent Observations on Discrete Choice under Risk*

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Abstract

This paper proposes nonparametric statistical procedures for analyzing discrete choice models of affective decision making. We make two contributions to the literature on behavioral economics. Namely, we propose a procedure for eliciting the existence of a Nash equilibrium in an intrapersonal, potential game as well as randomized sign tests for dependent observations on game-theoretic models of affective decision making. This methodology is illustrated in the context of a hypothetical experiment — the Casino Game.

Keywords: Behavioral economics, Affective decision making, Intrapersonal potential games, Randomized sign tests, Dependent observations, Adapted sequences, Martingale-difference sequences

JEL Classification: C12, C32, C35, C72, C91, D11, D81

1 Introduction

Casinos are natural laboratories for studying affective decision making (ADM). Here we see individuals engaged in risk taking behavior, where each outcome has both a monetary and an affective or emotional payoff. Probably the best examples are the video slot machines. Despite knowing that the payback for these machines may, on average, be about 75 to 80 percent — see Royer (2003) – these machines together with reel slots make up 80 percent of casinos' revenues. In Nevada alone this amounts to $7.2 billion a year. Why? Why pay a dollar to get, on average, 75 or 80 cents in return? The answer is simple. These games are fun to play — see Krieger and Reber (2005) or the discussion of video slots in Royer (2003). That is, the affective payoff more than compensates for the potential monetary loss, the "casino affect."[1]

The interplay between emotion and cognition in decision making is a well-studied topic in cognitive psychology — see Mellers and Schwartz (1999) and the references therein–and more recently in cognitive neuroscience— see Damasio (1994), Ledoux (1996), and Rolls (1999). In the cognitive psychology literature, the descriptive model of individual decision making under risk is prospect theory, due to Kahneman and Tversky (1979). The cognitive neuroscience literature attempts to correlate this behavior with neural activity in different regions of the brain, using neuroimaging techniques — see Breiter et al. (2001) and Ernst et al. (2004).

Prospect theory is only one of the many variants of expected utility theory seeking to describe choice under risk – see Camerer's survey article, 8.III, in Hagel and Roth (1995) and Starmer's survey article, Chapter 4, in Camerer, Loewenstein and Rabin (2004). Common to all of these models is the unchallenged assumption that choice under uncertainty derives from maximization of preferences. This is the rational agent paradigm that dominates economic theory — see Kreps (1990) and MasColell et al. (1995). An alternative paradigm for ADM has recently been proposed by Bracha (2005). In her model, the interaction between cognitive and affective neural processes is described as an intrapersonal potential game where observed behavior is a Nash equilibrium of the game resulting from simultaneous play of the cognitive and affective processes. Recall that a potential game is a strategic-form game where the payoff functions of the players can be represented by a single real-valued function of the outcomes in the payoff matrix, the potential — see Monderer and Shapley (1996).

In this paper, we consider the implications of her model for a hypothetical experiment in discrete choice under risk — the Casino Game. The experimental design was suggested in part by a recent paper of Ernst et al. (2004) where they consider a variant of the popular TV game show: "The Wheel of Fortune." We derive the testable implications of her model for this experiment from Sprumont's paper (2000) on collective choice theories and Ibragimov and Brown' (2005)

[1] In the video slot example, you are willing to pay 20 to 25 cents for the entertainment.

paper on randomized sign tests for dependent observations. That is, given a finite sequence of dependent observations on subjects choosing between decks of cards with random monetary payoffs paired with subjects choosing between decks of cards with the same random monetary payoffs, but containing affective payoffs, images, we test the null hypothesis that the conditional distributions of choices are the same in both groups against the alternative hypothesis that the conditional distributions of choices differ as predicted by Bracha's model – the Casino Affect. The proposed test has very good power, even in small samples.

The analysis in the present paper is based on the new sign tests developed in Ibragimov and Brown (2005) that provide a device for testing for conditionally symmetric martingale-difference assumptions as well as for testing that conditional distributions of two (arbitrary) adapted sequences are the same. The foundation for the results in Ibragimov and Brown (2005) is given by general estimates for the tail probabilities of sums of signs of random variables (r.v.'s) forming a conditionally symmetric martingale-difference sequence or signs of differences of the components of two adapted sequences of interest. The bounds give sharp (i.e., attainable either in finite samples or in the limit) estimates for these tail probabilities in terms of (generalized) moments of sums of i.i.d. Bernoulli r.v.'s (or corresponding moments of Binomial distributions) and standard normal r.v.'s (see Corollaries 2, 3, 6 and 7). Similar estimates hold as well for expectations of arbitrary functions of the signs that are convex in each of their arguments (Theorem 5 and Corollary 5). The bounds in Ibragimov and Brown (2005) are based on the results that demonstrate that randomization over zero values of three-valued r.v.'s in a conditionally symmetric martingale-difference sequence produces a stream of i.i.d. symmetric Bernoulli r.v.'s and thus reduces the problem of estimating the critical values of the tests to computing the quantiles or moments of Binomial or normal distributions (Theorem 4 and Corollary 1). The same is the case for randomization over ties in sign tests for equality of conditional distributions of two adapted sequences (see Theorem 6 and Corollary 4).

The analysis in Ibragimov and Brown (2005) and in this paper is based, in large part, on general characterization results for two-valued martingale difference sequences and multiplicative forms obtained recently in Sharakhmetov and Ibragimov (2002) (see also de la Peña and Ibragimov, 2003, and de la Peña, Ibragimov and Sharakhmetov, 2003). These characterization results reviewed in Section 2 demonstrate, in particular, that martingale-difference sequences consisting of r.v.'s each of which takes two values are, in fact, sequences of independent r.v.'s. The results allow one to reduce the study of many problems for three-valued martingales to the case of i.i.d. symmetric Bernoulli r.v.'s and provide the key to the development of sign tests for dependent observations.

There are many studies focusing on procedures for dealing with ties in independent observations (see Coakley and Heise, 1996, for a review and comparisons of sign tests in the presence of ties). Basing the conclusions on a size and power study, Coakley and Heise (1996) recommended using the asymptotic uniformly most powerful non-randomized (ANU) test due to Putter (1955) if ties occur in the sign test. The results obtained by Putter (1955) show that randomization over ties reduces the exact power of the sign test and the asymptotic efficiency of the sign test. It is known, however, that the exact version of the ANU test is conservative for small samples compared to both its randomized conditional version as well as to ANU (see Coakley and Heise, 1996; Wittkowski, 1998). The estimates obtained in Ibragimov and Brown (2005) shed new light on sign tests comparisons and suggest that randomization over ties leads, in general, to more conservative unconditional sign tests since it provides bounds for the tail probabilities of signs in terms of generalized moments of i.i.d. Bernoulli r.v.'s. According to the results in Ibragimov and Brown (2005), the advantage of randomization over ties or zero observations is that it allows one to use the sign tests in the presence of dependence while nonrandomized sign tests can only be used in the case of independence in data. In this regard, the results in that paper demonstrate that sign tests have the important property of robustness to dependence. Sign tests have other appealing properties.

First, a simple linear transformation of a test statistic based on signs leads to a Binomial distribution, and, thus, its distribution can be computed exactly. This is in contrast to other commonly used test statistics for which the exact distributions are frequently unknown. Even if known, the exact distributions of such test statistics are usually difficult to compute and have to be obtained by relying on computationally intensive algorithms or Monte-Carlo techniques.

The second important property of sign tests is that they can be applied in the case of a small number of observations. This is very important since large sample approximations, e.g., those based on the central limit theorems, require special regularity assumptions on the distribution of the observations such as existence of the second or higher moments or identical distribution.

The third property of sign tests is their robustness to distributional assumptions. Sign tests can be used for statistical inference in models driven by innovations with heavy-tailedness since the distributions of their test statistics are known under very mild assumptions such as symmetry. Robustness of sign tests is appealing since it has been shown in numerous studies that many time series encountered in economics and finance are heavy-tailed and involve r.v.'s X with the power tail decline $P(|X| > x) \sim x^{-\alpha}$ (see the discussion in Ibragimov, 2004, 2005, and references therein). One should also emphasize here that the limiting distributions for test statistics in setups based on heavy-tailedness assumptions are non-standard and usually involve functionals of stable processes; therefore, one has to rely on computationally intensive Monte-Carlo simulations to compute the critical values of the tests.

The paper is organized as follows. Section 2 reviews the characterization results for two-valued martingale difference sequences and multiplicative forms in Sharakhmetov and Ibragimov (2002) that provide the basis for the analysis in Ibragimov and Brown (2005) and in this paper. In Section

3, we review the main results of Ibragimov and Brown (2005) on the distributional properties of sign tests for martingale-difference sequences that are the key to the development of statistical procedures based on signs of dependent observations in subsequent sections. Section 4 describes the sign tests of Ibragimov and Brown (2005) derived from the results obtained in Section 3. These sign tests provide the statistical procedures for testing for conditionally symmetric martingale-difference assumptions as well as for testing that conditional distributions of two (arbitrary) adapted sequences are the same.[2] In Section 5, we explain in details the experimental design of the Casino Game and in Section 6 we apply the statistical analysis developed in Sections 3 and 4 to the Casino Game.

2 Probabilistic Foundations of the Analysis[3]

Let $(\Omega, \Im, P)$ be a probability space equipped with a filtration $\Im_0 = (\Omega, \emptyset) \subseteq \Im_1 \subseteq \cdots \subseteq \Im$. Further, let $(a_t)_{t=1}^{\infty}$ and $(b_t)_{t=1}^{\infty}$ be arbitrary sequences of real numbers such that $a_t \neq b_t$ for all t.

The key to the analysis in Ibragimov and Brown (2005) and in this paper is provided by the following theorems. These theorems are consequences of more general results obtained in Sharakhmetov and Ibragimov (2002) that show that r.v.'s taking $k+1$ values form a multiplicative system of order k if and only if they are jointly independent (see also de la Peña and Ibragimov, 2003; de la Peña, Ibragimov and Sharakhmetov, 2003). These results imply, in particular, that r.v.'s each taking two values form a martingale-difference sequence if and only if they are jointly independent.

To illustrate the main ideas of the proof, we first consider the case of r.v.'s taking values ± 1.

In what follows, $I(\cdot)$ stands for the indicator function.

Theorem 1 *If r.v.'s U_t, $t = 1, 2, ...$, form a martingale-difference sequence with respect to a filtration $(\Im_t)_t$ and are such that $P(U_t = 1) = P(U_t = -1) = 1/2$ for all t, then they are jointly independent.*

Proof It is easy to see that, under the assumptions of the theorem, one has that, for all $1 < \ell_1 < \ell_2 < \cdots < \ell_k$, $k = 2, 3, ...$,

$$EU_{\ell_1}...U_{\ell_{k-1}}U_{\ell_k} = E\left(U_{\ell_1}...U_{\ell_{k-1}}E(U_{\ell_k}|\Im_{\ell_{k-1}})\right) = E(U_{\ell_1}...U_{\ell_{k-1}} \times 0) = 0 \qquad (1)$$

[2]For completeness of the presentation, the complete proofs of the results in Sections 2-4 are provided.

[3]An excellent introduction to the theory of martingales, including discussions of filtrations, adapted stochastic processes and martingale-difference sequences may be found in Grimmett and Stirzaker (2001).

It is easy to see that, for $x_t \in \{-1, 1\}$, $I(X_t = x_t) = (1 + x_t U_t)/2$. Consequently, for all $1 \leq j_1 < j_2 < \cdots < j_m$, $m = 2, 3, ...$, and any $x_{j_k} \in \{-1, 1\}$, $k = 1, 2, ..., m$, we have

$$\begin{aligned}
P(U_{j_1} &= x_{j_1}, U_{j_2} = x_{j_2}, ..., U_{j_m} = x_{j_m}) = EI(U_{j_1} = x_{j_1})I(U_{j_2} = x_{j_2})...I(U_{j_m} = x_{j_m}) \\
&= \frac{1}{2^m} E(1 + x_{j_1}U_{j_1})(1 + x_{j_2}U_{j_2})...(1 + x_{j_m}U_{j_m}) \\
&= \frac{1}{2^m}\left(1 + \sum_{c=2}^{m} \sum_{i_1 < \cdots < i_c \in \{j_1, j_2, ..., j_m\}} EU_{i_1}...U_{i_c}\right) \\
&= \frac{1}{2^m} = P(U_{j_1} = x_{j_1})P(U_{j_2} = x_{j_2})...P(U_{j_m} = x_{j_m})
\end{aligned}$$

by (1). ■

The proof of the analogue of the result in the case of r.v.'s each of which takes arbitrary two values is completely similar and the following more general result holds.

Theorem 2 *If r.v.'s X_t, $t = 1, 2, ...$, form a martingale-difference sequence with respect to a filtration $(\Im_t)_t$ and each of them takes two (not necessarily the same for all t) values $\{a_t, b_t\}$, then they are jointly independent.*

Proof Let the random variable X_t take the values a_t and b_t, $a_t \neq b_t$, with probabilities $P(X_t = a_t) = p_t$ and $P(X_t = b_t) = q_t$, respectively. It is not difficult to check that, for $x_t \in \{a_t, b_t\}$,

$$\begin{aligned}
I(X_t = x_t) &= P(X_t = x_t)\left(1 + \frac{(X_t - a_t p_t - b_t q_t)(x_t - a_t p_t - b_t q_t)}{(a_t - b_t)^2 p_t q_t}\right) \\
&= P(X_t = x_t)\left(1 + \frac{(X_t - EX_t)(x_t - EX_t)}{(a_t - b_t)^2 p_t q_t}\right) \\
&= P(X_t = x_t)\left(1 + \frac{X_t x_t}{(a_t - b_t)^2 p_t q_t}\right) = P(X_t = x_t)\left(1 + \frac{X_t x_t}{\mathrm{Var}(X_t)}\right),
\end{aligned}$$

where $EX_t = a_t p_t + b_t q_t = 0$ and $\mathrm{Var}(X_t) = (b_t - a_t)^2 p_t q_t$ are the mean and the variance of X_t. Since the r.v.'s X_t satisfy property (1) with U_{ℓ_j} replaced by X_{ℓ_j}, $j = 1, ..., k$, similar to the proof of Theorem 1 we have that, for all $1 \leq j_1 < j_2 < \cdots < j_m$, $m = 2, 3, ...$, and any $x_{j_k} \in \{a_{j_k}, b_{j_k}\}$, $k = 1, 2, ..., m$,

$$\begin{aligned}
&P(X_{j_1} = x_{j_1}, X_{j_2} = x_{j_2}, ..., X_{j_m} = x_{j_m}) = EI(X_{j_1} = x_{j_1})I(X_{j_2} = x_{j_2})...I(X_{j_m} = x_{j_m}) \\
= &\prod_{s=1}^{m} P(X_{j_s} = x_{j_s}) E\left(1 + \frac{X_{j_1}x_{j_1}}{\mathrm{Var}(X_{j_1})}\right)...\left(1 + \frac{X_{j_m}x_{j_m}}{\mathrm{Var}(X_{j_m})}\right) \\
= &\prod_{s=1}^{m} P(X_{j_s} = x_{j_s})\left(1 + \sum_{c=2}^{m} \sum_{i_1 < \cdots < i_c \in \{j_1, j_2, ..., j_m\}} EX_{i_1}...X_{i_c}x_{i_1}...x_{i_c}/(\mathrm{Var}(X_{i_1})..., \mathrm{Var}(X_{i_c}))\right) \\
= &P(X_{j_1} = x_{j_1})P(X_{j_2} = x_{j_2})...P(X_{j_m} = x_{j_m}).
\end{aligned}$$

■

Let X_t, $t = 1, 2, ...$, be an $(\Im_t)$-martingale-difference sequence consisting of r.v.'s each of which takes three values $\{-a_t, 0, a_t\}$. Denote by ε_t, $t = 1, 2, ...$, a sequence of i.i.d. symmetric Bernoulli r.v.'s independent of $(X_t)_{t=1}^{\infty}$. The following theorem provides an upper bound for the expectation of arbitrary convex function of X_t in terms of the expectation of the same function of the r.v.'s ε_t.

Theorem 3 *If $f : R^n \rightarrow R$ is a function convex in each of its arguments, then the following inequality holds:*

$$Ef(X_1, ..., X_n) \leq Ef(a_1\varepsilon_1, ..., a_n\varepsilon_n). \tag{2}$$

Proof Let $\bar{\Im}_0 = \Im_n$. For $t = 1, 2, ..., n$, denote by $\bar{\Im}_t$ the σ-algebra spanned by the r.v.'s $X_1, X_2, ..., X_n$, $\varepsilon_1, ..., \varepsilon_t$. Further, let, for $t = 0, 1, ..., n$, E_t stand for the conditional expectation operator $E(\cdot|\bar{\Im}_t)$ and let η_t, $t = 1, ..., n$, denote the r.v.'s $\eta_t = X_t + \varepsilon_t I(X_t = 0)$.

Using conditional Jensen's inequality, we have

$$\begin{aligned} Ef(X_1, X_2, ..., X_n) &= Ef(X_1 + E_0[\varepsilon_1 I(X_1 = 0)], X_2, ..., X_n) \\ &\leq E[E_0 f(X_1 + \varepsilon_1 I(X_1 = 0), ..., X_2, ..., X_n)] = Ef(\eta_1, X_2, ..., X_n). \end{aligned} \tag{3}$$

Similarly, for $t = 2, ..., n$,

$$\begin{aligned} & Ef(\eta_1, \eta_2, ..., \eta_{t-1}, X_t, X_{t+1}, ..., X_n) \\ = \; & Ef(\eta_1, \eta_2, ..., \eta_{t-1}, X_t + E_{t-1}[\varepsilon_t I(X_t = 0)], X_{t+1}, ..., X_n) \\ \leq \; & E[E_{t-1} f(\eta_1, \eta_2, ..., \eta_{t-1}, X_t + \varepsilon_t I(X_t = 0), X_{t+1}, ..., X_n)] \\ = \; & Ef(\eta_1, \eta_2, ..., \eta_{t-1}, \eta_t, X_{t+1}, ..., X_n). \end{aligned} \tag{4}$$

From equations (3) and (4) by induction it follows that

$$Ef(X_1, X_2, ..., X_n) \leq Ef(\eta_1, \eta_2, ..., \eta_n). \tag{5}$$

It is easy to see that the r.v.'s η_t, $t = 1, 2, ..., n$, form a martingale-difference sequence with respect to the sequence of σ-algebras $\bar{\Im}_0 \subseteq \bar{\Im}_1 \subseteq \cdots \subseteq \bar{\Im}_t \subseteq \cdots$, and each of them takes two values $\{-a_t, a_t\}$. Therefore, from Theorems 1 and 2 we get that η_t, $t = 1, 2, ..., n$ are jointly independent, and therefore, the random vector $(\eta_1, \eta_2, ..., \eta_n)$ has the same distribution as $(a_1\varepsilon_1, a_2\varepsilon_2, ..., a_n\varepsilon_n)$. This and (5) imply estimate (2). ■

3 Distributions of Sign Test Statistics for Dependent Observations[4]

The present section of the paper reviews the results in Ibragimov and Brown (2005) on the distributional properties of the sign tests for martingale-difference sequences that provide the basis for the development of the statistical procedures in Section 4, based on signs of dependent observations.

Let X_n, $n = 1, 2, ...$, be an $(\Im_n)$-conditionally symmetric martingale-difference sequence (so that $P(X_n < x|\Im_{n-1}) = P(X_n < -x|\Im_{n-1})$, $n = 1, 2, ...$, for all $x > 0$) consisting of r.v.'s each of which takes three values $\{-a_n, 0, a_n\}$. Further, let, for $z \in \mathbb{R}$, $\text{sign}(z)$ denote the sign of z defined by $\text{sign}(z) = 1$, if $z > 0$, $\text{sign}(z) = -1$, if $z < 0$, and $\text{sign}(0) = 0$.

As before, throughout Sections 3 and 4, ε_n, $n = 1, 2, ...$, stands for a sequence of i.i.d. symmetric Bernoulli r.v.'s independent of X_n, $n = 1, 2, ...$; in addition to that, in what follows, we denote by Z the standard normal r.v. if not stated otherwise.

Theorem 4 *The r.v.'s* $\eta_t = \text{sign}(X_t) + \varepsilon_t I(X_t = 0)$ *are i.i.d. symmetric Bernoulli r.v.'s.*

Proof The theorem follows from Theorem 1 since, as it is easy to see, the r.v.'s (η_t) form an $(\Im_t)$-martingale-difference sequence and each of them takes two values -1 and 1. ■

Corollary 1 *The statistic* $S_n = (\sum_{t=1}^n \text{sign}(X_t) + \varepsilon_t I(X_t = 0) + n)/2$ *has Binomial distribution* $Bin(n, 1/2)$ *with parameters* n *and* $p = 1/2$.

Proof The corollary is an immediate consequence of Theorem 4.

Theorem 5 *For any function* $f : R^n \to R$ *convex in each of its arguments,*

$$Ef(\text{sign}(X_1), \text{sign}(X_2), ..., \text{sign}(X_n)) \leq Ef(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n).$$

Proof The theorem follows from Theorem 3 applied to the martingale-difference sequence $Y_n = \text{sign}(X_n)$, $n = 1, 2, ...$, consisting of r.v.'s each of which takes three values $\{-1, 0, 1\}$. ■

Corollary 2 *For any* $x > 0$,

$$P\left(\sum_{t=1}^{n} \text{sign}(X_t) > x\right) \leq \inf_{0<c<x} \frac{E\max\left(\sum_{t=1}^n \varepsilon_t - c, 0\right)}{(x-c)}. \quad (6)$$

[4] The classical reference for sign tests with independent observations is Lehmann (1986).

Proof The corollary is an immediate consequence of Markov's inequality and Theorem 5 applied to the functions $f_c(x_1, x_2, ..., x_n) = \max(\sum_{t=1}^n x_t - c, 0)$, $0 < c < x$. ■

Remark 1 *For a fixed $x > 0$, consider the class of functions ϕ satisfying $\phi(y) = \int_0^y \max(y - u, 0)dF(u)$, $y \geq 0$, $\phi(y) = 0$, $y < 0$, and $\phi(x) = \int_0^x \max(x - u, 0)dF(u) = 1$, for a nonnegative bounded nondecreasing function $F(x)$ on $[0, +\infty)$ with $F(0) = 0$. Similar to the proof of Corollary 2 we obtain*

$$P\left(\sum_{t-1}^{n} \text{sign}(X_t) > x\right) \leq E\phi\left(\sum_{t=1}^{n} \varepsilon_t\right) \tag{7}$$

for all ϕ. It is not difficult to show, similar to Proposition 4 in Eaton (1974) (see also the discussion following Theorem 5 in de la Peña, Ibragimov and Jordan, 2004, for related optimality results for bounds on the expected payoffs of contingent claims in the binomial model) that bound (6) is the best among all estimates (7), that is

$$\inf_{\phi} E\phi\left(\sum_{t=1}^{n} \varepsilon_t\right) = \inf_{0<c<x} \frac{E\max(\sum_{t=1}^n \varepsilon_t - c, 0)}{x - c}.$$

The following result gives sharp bounds for the tail probabilities of the normalized sum of sign of the r.v.'s X_t in terms of (generalized) moments of the standard normal r.v.

Corollary 3 *For any $x > 0$,*

$$P\left(\frac{\sum_{t=1}^n \text{sign}(X_t)}{\sqrt{n}} > x\right) \leq \inf_{0<c<x} \frac{E\max\left(\frac{\sum_{t=1}^n \varepsilon_t}{\sqrt{n}} - c, 0\right)}{x - c} \leq \inf_{0<c<x} \frac{(E[\max(Z - c, 0)]^3)^{1/3}}{x - c}. \tag{8}$$

Proof Using Markov's inequality and Theorem 5 applied to the functions

$$f_c(x_1, x_2, ..., x_n) = \max\left(\frac{\sum_{t=1}^n x_t}{\sqrt{n}} - c, 0\right),$$

$0 < c < x$, we get the first estimate in (8). From Jensen's inequality we obtain

$$E\max\left(\frac{\sum_{t=1}^n \varepsilon_t}{\sqrt{n}} - c, 0\right) \leq \left\{E\left[\max\left(\frac{\sum_{t=1}^n \varepsilon_t}{\sqrt{n}} - c, 0\right)\right]^3\right\}^{1/3}, \tag{9}$$

$0 < c < x$. The second bound in (8) is a consequence of estimate (9) and the inequality

$$E\left[\max\left(\frac{\sum_{t=1}^n \varepsilon_t}{\sqrt{n}} - c, 0\right)\right]^3 \leq E[\max(Z - c, 0)]^3 \tag{10}$$

for all $c > 0$ implied by the results in Eaton (1974). ■

Let (X_t), $n = 1, 2, ...$, and (Y_n), $n = 1, 2, ...$, be two $(\Im_n)$-adapted sequences.

The following results provide analogues of Theorem 4 and Corollaries 1–3 that concern the distributional properties of sign tests for equality of conditional distributions of two adapted sequences (X_n) and (Y_n). They follow from Theorem 4 and Corollaries 1–3 applied to the r.v.'s $Z_n = X_n - Y_n$ that form a conditionally symmetric martingale-difference sequence under the assumption that the conditional distributions of (X_n) and (Y_n) are the same.

Theorem 6 *If the conditional (on $\Im_{n-1}$) distributions of (X_n) and (Y_n) are the same: $L(X_n|\Im_{n-1}) = L(Y_n|\Im_{n-1})$, then the r.v.'s $\tilde{\eta}_n = \text{sign}(X_n - Y_n) + \varepsilon_n I(X_n = Y_n)$ are i.i.d. symmetric Bernoulli r.v.'s.*

Corollary 4 *If the conditional (on $\Im_{n-1}$) distributions of (X_n) and (Y_n) are the same: $L(X_n|\Im_{n-1}) = L(Y_n|\Im_{n-1})$, then the statistic $\tilde{S}_n = (\sum_{t=1}^n \text{sign}(X_t - Y_t) + \varepsilon_t I(X_t - Y_t) + n)/2$ has Binomial distribution $\text{Bin}(n, 1/2)$ with parameters n and $p = 1/2$.*

Corollary 5 *If the conditional (on $\Im_{n-1}$) distributions of (X_n) and (Y_n) are the same: $L(X_n|\Im_{n-1}) = L(Y_n|\Im_{n-1})$, then, for any function $f : R^n \to R$ convex in each of its arguments,*

$$Ef(\text{sign}(X_1 - Y_1), \text{sign}(X_2 - Y_2), ..., \text{sign}(X_n - Y_n)) \leq Ef(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n).$$

Corollary 6 *If the conditional (on $\Im_{n-1}$) distributions of (X_n) and (Y_n) are the same: $L(X_n|\Im_{n-1}) = L(Y_n|\Im_{n-1})$, then, for any $x > 0$,*

$$P\left(\sum_{i=1}^n \text{sign}(X_t - Y_t) > x\right) \leq \inf_{0<c<x} \frac{E\max(\sum_{t=1}^n \varepsilon_t - c, 0)}{x - c}.$$

Corollary 7 *If the conditional (on $\Im_{n-1}$) distributions of (X_n) and (Y_n) are the same: $L(X_n|\Im_{n-1}) = L(Y_n|\Im_{n-1})$, then, for any $x > 0$,*

$$P\left(\frac{\sum_{t=1}^n \text{sign}(X_t - Y_t)}{\sqrt{n}} > x\right) \leq \inf_{0<c<x} \frac{E\max\left(\frac{\sum_{t=1}^n \varepsilon_t}{\sqrt{n}} - c, 0\right)}{x - c} \leq \inf_{0<c<x} \frac{(E[\max(Z - c, 0)]^3)^{1/3}}{x - c}.$$

Remark 2 *Bounds for the tail probabilities of sums of bounded r.v.'s forming a conditionally symmetric martingale-difference sequence implied by the results in the present section provide better estimates than many inequalities implied, in the trinomial setting, by well-known estimates in martingale theory. In particular, from Markov's inequality and Theorem 5 applied to the function $f(x_1, x_2, ..., x_n) = \exp(h\sum_{t=1}^n u_t x_t)$, $h > 0$, it follows that the tail probability $P(\sum_{t=1}^n X_t > x)$, $x > 0$, of the sum of r.v.'s X_t that take three values $\{-u_t, 0, u_t\}$ is bounded from above by $\exp(-hx)E\exp(h\sum_{t=1}^n u_t\varepsilon_t)$, $h > 0$:*

$$P\left(\sum_{t=1}^n X_t > x\right) \leq \inf_{h>0} \exp(-hx)E\exp\left(h\sum_{t=1}^n u_t\varepsilon_t\right). \tag{11}$$

From estimate (11) it follows that Hoeffding–Azuma inequality for martingale-differences in the above setting

$$P\left(\sum_{t=1}^{n} X_t > 0\right) \le \exp\left(-\frac{x^2}{2\sum_{t=1}^{n} u_t^2}\right) \tag{12}$$

is implied by the corresponding bounds on the expectation of exponents of weighted i.i.d. Bernoulli r.v.'s $E\exp(h\sum_{t=1}^{n} u_t\varepsilon_t)$ *(see Hoeffding, 1963; Azuma, 1967). More generally, Markov's inequality and Theorem 5 imply the following bound for the tail probabilities of three-valued r.v.'s forming a conditionally symmetric martingale-difference sequence with the support on* $\{-u_t, 0, u_t\}$*:*

$$P\left(\sum_{t=1}^{n} X_t > x\right) \le \inf_{\phi} \frac{\phi\left(\sum_{t=1}^{n} u_t\varepsilon_t\right)}{\phi(x)}, \tag{13}$$

where the infimum is taken over convex increasing functions $\phi : \mathbb{R} \to \mathbb{R}_+$*. It is easy to see that estimate (13) is better than Hoeffding–Azuma inequality (12) since the latter follows from choosing a particular (close to optimal)* h *in estimates for the right-hand side of (11) which is a particular case of (13) (see Hoeffding, 1963).*

4 Sign Tests under Dependence

As discussed in Ibragimov and Brown (2005), from the results in the previous section it follows that sign tests for testing the null hypothesis that (X_n) is an $(\Im_n)$-conditionally symmetric martingale-difference sequence with $P(X_n > x|\Im_{n-1}) = P(X_n < -x|\Im_{n-1})$, $x > 0$, or that the conditional distributions of two adapted sequences (X_n) and (Y_n) are the same ($\mathcal{L}(X_n|\Im_{n-1}) = \mathcal{L}(Y_n|\Im_{n-1})$ for all n) can be based on the procedures described below. As most of the testing procedures in statistics and econometrics, they can be classified as falling into one of the following classes: exact tests, conservative tests and testing procedures based on asymptotic approximations. The exact tests are based on the fact that, according to Corollaries 1 and 4, the distribution of the transformation of signs in the model is known precisely to be Binomial and thus the statistical inference can be based on critical values for the sum of i.i.d. Bernoulli r.v.'s (the case of exact randomized ER tests below). The asymptotic tests use approximations for the quantiles of the Binomial distribution in terms of the limiting normal distribution (the case of asymptotic randomized AR tests). The conservative testing procedures in the present section are based on sharp estimates for the tail probabilities of sums of dependent signs in the model in terms of sums of i.i.d. Bernoulli or normal r.v.'s implied by Corollaries 2, 3, 6 and 7 and corresponding estimates for the critical values of the sign tests for dependent observations in terms of quantiles of the Binomial or Gaussian distributions (Binomial conservative non-randomized BCN and normal conservative non-randomized NCN testing procedures). The classification of the sign tests in the present section as non-randomized or randomized refers, respectively, to whether the inference is based on the original (three-valued) signs $\text{sign}(X_t)$ (resp., $\text{sign}(X_t - Y_t)$) in the model with dependent observations or the r.v.'s $\text{sign}(X_t - Y_t) + \varepsilon_t I(X_t = Y_t))$ that form, according to the results in the previous section, a sequence of symmetric i.i.d. Bernoulli r.v.'s.

1. The exact randomized (ER) sign test with the test statistic $S_n^{(1)} = (\sum_{t=1}^n \text{sign}(X_t) + \varepsilon_t I(X_t = 0) + n)/2$ rejects the null hypothesis $P(X_n > x|\Im_{n-1}) = P(X_n < -x|\Im_{n-1})$, $x > 0$, for all n in favor of $P(X_n > x|\Im_{n-1}) > P(X_n < -x|\Im_{n-1})$, $x > 0$, for all n at the significance level $\alpha \in (0, 1/2)$, if $S_n > B_\alpha$, where B_α is the $(1-\alpha)$-quantile of the Binomial distribution $\text{Bin}(n, 1/2)$.

Using the central limit theorem for the statistic $(\sum_{t=1}^n \text{sign}(X_t)+\varepsilon_t I(X_t = 0))/\sqrt{n}$, in the case of large sample sizes n one can also use the following asymptotic version of the previous testing procedure.

2. The asymptotic randomized (AR) sign test with the test statistic $S_n^{(2)} = (\sum_{t=1}^n \text{sign}(X_t) + \varepsilon_t I(X_t = 0) + n)/\sqrt{n}$ rejects the null hypothesis $P(X_n > x|\Im_{n-1}) = P(X_n < -x|\Im_{n-1})$, $x > 0$, for all n in favor of $P(X_n > x|\Im_{n-1}) > P(X_n < -x|\Im_{n-1})$, $x > 0$, for all n at the significance level $\alpha \in (0, 1/2)$, if $S_n > z_\alpha$, where z_α is the $(1-\alpha)$-quantile of the standard normal distribution $\mathcal{N}(0, 1)$.

3. The binomial conservative non-randomized (BCN) sign test with the test statistic $S_n^{(3)} = \sum_{t=1}^n \text{sign}(X_t)$ rejects the null hypothesis $P(X_n > x|\Im_{n-1}) = P(X_n < -x|\Im_{n-1})$, $x > 0$, for all n in favor of $P(X_n > x|\Im_{n-1}) > P(X_n < -x|\Im_{n-1})$, $x > 0$, for all n at the significance level $\alpha \in (0, 1/2)$, if $S_n > B_\alpha$, where B_α is such that

$$\inf_{0<c<z_\alpha} \frac{E[\max(\sum_{t=1}^n \varepsilon_t - c, 0)]}{B_\alpha - c} < \alpha.$$

4. The normal conservative non-randomized (NCN) sign test with the test statistic $S_n^{(4)} = \sum_{t=1}^n \text{sign}(X_t)$ rejects the null hypothesis $P(X_n > x|\Im_{n-1}) = P(X_n < -x|\Im_{n-1})$, $x > 0$, for all n in favor of $P(X_n > x|\Im_{n-1}) > P(X_n < -x|\Im_{n-1})$, $x > 0$, for all n at the significance level $\alpha \in (0, 1/2)$, if $S_n > z_\alpha$, where z_α is such that

$$\inf_{0<c<z_\alpha} \frac{(E[\max(Z - c, 0]^3)^{1/3}}{z_\alpha - c} < \alpha.$$

The analogues of the above tests in the case of the two-sided alternative $P(X_n > x|\Im_{n-1}) \neq P(X_n < -x|\Im_{n-1})$ are completely similar.

The tests also have the following analogues for testing the null hypothesis that conditional distributions of components of two adapted sequences are the same: $\mathcal{L}(X_n|\Im_{n-1}) = \mathcal{L}(Y_n|\Im_{n-1})$.[5]

1. The exact randomized (ER) sign test with the test statistic $\tilde{S}_n^{(1)} = (\sum_{t=1}^n \text{sign}(X_t) + \varepsilon_t I(X_t = 0) + n)/2$ rejects the null hypothesis $\mathcal{L}(X_n|\Im_{n-1}) = \mathcal{L}(Y_n|\Im_{n-1})$ for all n in favor of the

[5] We describe the tests for the two-sided alternative since this is usually the case of interest in most of the applications.

(two-sided) hypothesis $\mathcal{L}(X_n|\Im_{n-1}) \neq \mathcal{L}(Y_n|\Im_{n-1})$ for all n at the significance level $\alpha \in (0, 1/2)$, if $\tilde{S}_n^{(1)} < B_{\alpha/2}^{(1)}$ or $\tilde{S}_n^{(1)} > B_{\alpha/2}^{(2)}$ where $B_{\alpha/2}^{(1)}$ and $B_{\alpha/2}^{(2)}$ are, respectively, the $(\alpha/2)$- and $(1-\alpha/2)$-quantiles of the Binomial distribution $\text{Bin}(n, 1/2)$.

2. The asymptotic randomized (AR) sign test with the test statistic $\tilde{S}_n^{(2)} = (\sum_{t=1}^n \text{sign}(X_t) + \varepsilon_t I(X_t = Y_t))/\sqrt{n}$ rejects the null hypothesis $\mathcal{L}(X_n|\Im_{n-1}) = \mathcal{L}(Y_n|\Im_{n-1})$ for all n in favor of $\mathcal{L}(X_n|\Im_{n-1}) \neq \mathcal{L}(Y_n|\Im_{n-1})$ for all n at the significance level $\alpha \in (0, 1/2)$, if $\left|\tilde{S}_n^{(2)}\right| > z_{\alpha/2}$, where $z_{\alpha/2}$ is the $(1-\alpha/2)$-quantiles of the standard normal distribution $\mathcal{N}(0, 1)$.

3. The binomial conservative non-randomized (BCN) sign test with the test statistics $\tilde{S}_n^{(3)} = \sum_{t=1}^n \text{sign}(X_t - Y_t)$ rejects the null hypothesis $\mathcal{L}(X_n|\Im_{n-1}) = \mathcal{L}(Y_n|\Im_{n-1})$ for all n in favor of $\mathcal{L}(X_n|\Im_{n-1}) \neq \mathcal{L}(Y_n|\Im_{n-1})$ for all n at the significance level $\alpha \in (0, 1/2)$, if $\left|\tilde{S}_n^{(3)}\right| > B_{\alpha/2}$, where $B_{\alpha/2}$ is such that

$$\inf_{0<c<B_{\alpha/2}} \frac{E \max(\sum_{t=1}^n \varepsilon_t - c, 0)}{B_{\alpha/2} - c} < \alpha/2.$$

4. The normal conservative non-randomized (NCN) sign test with the test statistic $\tilde{S}_n^{(4)} = \sum_{t=1}^n \text{sign}(X_t - Y_t)$ rejects the null hypothesis $\mathcal{L}(X_n|\Im_{n-1}) = \mathcal{L}(Y_n|\Im_{n-1})$ for all n in favor of $\mathcal{L}(X_n|\Im_{n-1}) \neq \mathcal{L}(Y_n|\Im_{n-1})$ for all n at the significance level $\alpha \in (0, 1/2)$, $\left|\tilde{S}_n^{(4)}\right| > z_{\alpha/2}$, where $z_{\alpha/2}$ is such that

$$\inf_{0<c<B_{\alpha/2}} \frac{(E[\max(Z - c, 0)]^3)^{1/3}}{z_{\alpha/2} - c} < \alpha/2.$$

For illustration, in Table 1 in the Appendix, we provide the results on calculations of the power of the AR sign test for testing the null hypothesis $H_0 : P(X_n > x|\Im_{n-1}) = P(X_n < -x|\Im_{n-1})$, $x > 0$, for all n against particular cases of the alternative hypothesis, namely, against the alternatives $P(X_n > x|\Im_{n-1}) > P(X_n < -x|\Im_{n-1})$, $x > 0$, such that $P(X_n > 0|\Im_{n-1}) = p > 1 - p = P(X_n < 0|\Im_{n-1})$, where $p \in (1/2, 1]$ (the power of other tests discussed in the present section against this particular alternative may be calculated in complete similarity). One should note that, as it is not difficult to see, the power calculations are the same for the AR test for testing H_0 against the alternatives $P(X_n > x|\Im_{n-1}) > P(X_n < -x|\Im_{n-1})$, $x > 0$, such that $P(X_n > 0|\Im_{n-1}) = p_1 > q_1 = P(X_n < 0|\Im_{n-1})$, where $p_1, q_1 \in [0, 1]$ and $1/2 + (p_1 - q_1)/2 = p$. They are also the same for the AR sign test for testing the null hypothesis of equality of conditional distributions of two $(\Im_t)$-adapted sequences X_t and Y_t against the alternative that $P(X_t > Y_t|\Im_{t-1}) = p_2 > 1/2 > q_2 = P(Y_t > Y_t|\Im_{t-1})$, where $p_2, q_2 \in [0, 1]$ are such that $1/2 + (p_2 - q_2)/6 = p$. According to the table, the test has very good power properties, even in the case of small samples.

5 The Casino Affect

There are two major challenges to testing a theory of affective decision making predicated on equilibrium outcomes in an intrapersonal game. First, how do you elicit behavior consistent with the existence of Nash equilibrium in a strategic form game? Second, given a small sample of dependent observations of this behavior — as is often the case in experimental economics when there is learning, see Camerer (1995) — how do you carry out the statistical analysis?

To answer the first question, we turn to Sprumont's (2000) paper on collective choice theories. For our purposes it suffices to consider a 2×2 strategic form game — see Figure 1.

	c_1	c_2
r_1	O_{11}	O_{12}
r_2	O_{21}	O_{22}

Figure 1

Sprumont proposes that we consider the game and all of its subgames, i.e., both columns and both rows. Suppose the outcomes in these games are x_1, x_2, x_3, x_4, and x_5. That is, x_1 is the outcome in the subgame defined by c_1; x_2 is the outcome in the subgame defined by c_2; x_3 is the outcome in the subgame defined by r_1; x_4 is the outcome in the subgame defined by r_2; and x_5 is the outcome in the game. The following proposition is Theorem 1 in Sprumont (2000, p. 211).

Proposition *x_5 is a Nash equilibrium of the game iff it is the chosen outcome in its row and in its column.*

Moreover, Sprumont observes in his Theorem $1'$, on page 212, that the game is a potential game relative to some ranking of the outcomes iff it has a Nash equilibrium — his term is team-rationalizable rather than a potential game.

Following Sprumont, we construct a 2×2 intrapersonal, strategic form game as a model of affective decision making, where we elicit choices for the game and all of its subgames.

Suppose two imaginary casinos: Winners and Losers. Subjects are divided into two equal groups. Each "winner" is paired anonomously with a "loser." There are four decks of cards in each casino, say A, B, C, and D. The decks have different random payoffs. In the Winners' casino, the only payoffs are monetary, but in the Losers' casino, the cards have both monetary and affective payoffs. In each casino, subjects are told the composition of the various decks, i.e., the payoffs and the probability of the payoffs in their casino.

A game consists of five rounds, where in each of the first four rounds the subjects are asked to choose from the columns and rows of the following matrix in Figure 2.[6]

[6] By round we mean subgame, i.e., column, row or all four elements of the matrix

A	B
C	D

Figure 2

More precisely, in the first subgame of each game, each of the two subjects is asked to choose between decks (A, C); in the second subgame, the subjects choose between (B, D); in the third round, they choose between (A, B); and in the forth round of each game, the choice is between (C, D).

In the fifth round, the subjects are asked to choose a deck from all four decks.

These are computerized games and subjects learn their outcomes after each round. Moreover, after each round they also learn the monetary payoffs of their "partner" in the other casino, but the "winners" don't observe the affective payoffs of the "losers." That is, winners and losers have different information sets in the sense that the winners only observe part of what the losers observe in each round. However, the information sets generated by the observed monetary payoffs are the same for both the paired players (see below).

The decks are constructed such that Bracha's model predicts the choices of the "winners" will be Nash equilibria on the diagonal $\{A, D\}$ and the choices of the "losers" will be Nash equilibria on the diagonal $\{B, C\}$. That is, in the casino for losers, the decks B and C have "positive" images, e.g., flowers or smiling faces, and decks A and D have "negative" images, e.g., snakes or angry faces.

Several testable hypotheses are of interest for the model.

First, it is of interest to test the strongest (among all the testable conclusions for the model described in this section) behavioral hypothesis whether seeing "positive" or "negative" images affects the players' choices between decks in the casino for losers.

It is natural to assume that players' decisions in each subgame in both casinos depend only on the outcomes of previous rounds observed by them. Thus, if the opposite of the above hypothesis holds and seeing the images has no effect on the losers' decisions, then, provided that each subject observes the outcomes played by her and her partner, the following statement (the null) holds:

$H_0^{(1)}$: Conditional on the outcomes of previous rounds played by a subject and her partner in the other casino, the current choices for both paired subjects are the same.

The following "two-sided" and "one-sided" alternatives can be considered for the null hypothesis $H_0^{(1)}$.

$H_{a1}^{(1)}$ (two-sided alternative): Conditional on the outcomes of previous rounds played by a subject and her partner in the other casino, the current choices for both paired subjects in the first four subgames are (always) different from each other (that is, the choices of the paired players are the opposites of each other, so that if a winner chooses A in the first subgame then her loser partner chooses B in that subgame and vice versa);

$H_{a2}^{(1)}$ (one-sided alternative): Conditional on the outcomes of previous rounds played by a subject and her partner in the other casino, the winner chooses decks A or D in each of the five subgames and the loser chooses B or C in each of the subgames (in other words, the winners always choose a deck with a better expected monetary payoffs; the losers, on the other hand, always choose a deck which is worse in terms of the monetary payoffs but is more preferable in terms of its images).

Of course, one can consider the analogues of the above hypotheses restricted to the first four subgames in each game or to the fifth round in each game only.

In order to be able to conduct tests of the above hypotheses, it is important that the players' choices are adapted to the same filtration generated by the monetary outcomes they both observe. In other words, if seeing the images has no effect on the losers' decisions, then the players' decisions are determined only by the outcomes in the previous rounds in both of the casinos that both the subject and her partner observe. Indeed, if this is the case and the above null hypothesis $H_0^{(1)}$ holds then the players' payoffs in each subgame have the same distributions conditional on the information set of the previous outcomes they both have the access to. Thus, the machinery for testing equality of conditional distributions of two adapted sequences (of payoffs) described in Sections 3 and 4 can be used to make inferences on the hypotheses of interest. The importance of the assumption that the random sequences of interest in Sections 3 and 4 are adapted to *the same* filtration is the reason why the subjects are paired in the experiment and observe not only their own payoffs but also the payoffs of their partners.

Formally, if the behavioral null hypothesis $H_0^{(1)}$ is true then the following distributional conclusion also holds.

$H_0^{(2)}$: Conditional on the outcomes of previous rounds played by a subject and her partner in the other casino, the distribution of current payoffs for both paired subjects is the same.

The two sided alternative for the null hypothesis is, naturally, the following:

$H_{a1}^{(2)}$ (two-sided alternative): Conditional on the outcomes of previous rounds played by a subject and her partner in the other casino, the distribution of current payoffs for both paired subjects are different from each other.

The one-sided distributional alternative to $H_0^{(2)}$ can be formulated as follows.

$H_{a2}^{(2)}$ (one-sided alternative): Conditional on the outcomes of previous rounds played by a subject and her partner in the other casino, the winners' monetary payoffs are likely to be greater than those of losers.

Note that, since the behavioral null hypothesis $H_0^{(1)}$ is evidently stronger than the distributional null hypothesis $H_0^{(2)}$, then rejection of $H_0^{(2)}$ also implies rejection of $H_0^{(1)}$.

Due to the importance of the diagonals in the model, it is also of interest to consider the following distributional hypothesis.

$H_0^{(3)}$: Conditional on the outcomes of previous rounds played by a subject and her partner in the other casino, the distribution of current payoffs from the diagonal AD for both paired subjects is the same.

The alternative that can be regarded as an analogue of the above alternative hypothesis $H_{a2}^{(1)}$ (in the case of the null $H_0^{(3)}$) is the following.

$H_a^{(3)}$: Conditional on the outcomes of previous rounds played by a subject and her partner in the other casino, the winners' monetary payoffs from the diagonal AD are likely to be greater than those of losers from the same diagonal.

The terms "likely to be greater than" in the alternative hypotheses $H_{a2}^{(2)}$ and $H_a^{(3)}$ above can be made operational and formalized as follows. Let $\pi_t^{(1)}$ and $\pi_t^{(2)}$ be the sequences of monetary payoffs of interest (either from each subgame or those from the diagonal AD) for winners and losers adapted to the same filtration $\Im_t$ (see the next section for details). If the winners' monetary payoffs are likely to dominate those of losers conditionally on the information sets $\Im_{t-1}$ of previous outcomes that they both observe, then $P(\pi_t^{(1)} > \pi_t^{(2)}|\Im_{t-1}) > P(\pi_t^{(1)} < \pi_t^{(2)}|\Im_{t-1})$ for all t. This a precise formulation of the alternatives to the null hypotheses of equality of conditional distributions $\mathcal{L}(\pi_t^{(1)}|\Im_{t-1}) = \mathcal{L}(\pi_t^{(2)}|\Im_{t-1})$ for all t in $H_0^{(2)}$ and $H_0^{(3)}$ that imply, in particular, that $P(\pi_t^{(1)} > \pi_t^{(2)}|\Im_{t-1}) = P(\pi_t^{(1)} < \pi_t^{(2)}|\Im_{t-1})$ (see also the notes on calculations of power of the sign tests at the end of Section 4).

It is important to note that, according to the results reviewed in Section 2, the above formulations of the null hypotheses and the one-sided alternatives are, in fact, the statements concerning martingale vs. submartingale behavior of the sums of signs of differences of the subjects' payoffs in each of the casinos. For instance, from the results in Section 2 it follows that the null hypothesis $H_0^{(2)}$ is equivalent to the statement that the sum of three-valued signs $\sum sign(\pi_t^{(1)} - \pi_t^{(2)})$ is a conditionally symmetric martingale with respect to the filtration that both players observe. The one-sided hypothesis $H_{a1}^{(2)}$, on the other hand, is equivalent to the statement that this sum of signs is a submartingale with respect to the same filtration.

The next section proposes a statistical model for conducting the tests. This will address the second challenge, raised above.

6 Sign Tests for the Casino Affect

Suppose that each of the subjects plays the game described in the previous section N times. In order to maintain incentive to pay attention to the outcomes of their partners' games, at the end of the last game, the subjects are allowed to choose between the two casinos and to play in the casino of their choice N more times.

Let X_A, X_B, X_C, and X_D stand for random variables representing monetary payoffs from decks A, B, C, and D, respectively. Further, let Y_{tj} and Z_{tj}, $t = 1, 2, ..., 5N$, $j \in \{A, B, C, D\}$, stand for the subject's decision variables for choices in the tth subgame, so that $Y_{tj} = 1$, if the first subject chooses deck j in the subgame and $Y_{tj} = 0$ otherwise; $Z_{tj} = 1$, if the second subject chooses deck j in the subgame and $Z_{tj} = 0$ otherwise (note that $Y_{tB} = Y_{tD} = Z_{tB} = Z_{tD} = 0$ if $t = 5m - 4$; $Y_{tA} = Y_{tC} = Z_{tA} = Z_{tC} = 0$ if $t = 5m - 3$; $Y_{tC} = Y_{tD} = Z_{tC} = Z_{tD} = 0$ if $t = 5m - 2$; $Y_{tA} = Y_{tB} = Z_{tA} = Z_{tB} = 0$ if $t = 5m - 1$, where m is the number of the game, $m = 1, ..., N$).

Let $(X_{tA}^{(i)}, X_{tB}^{(i)}, X_{tC}^{(i)}, X_{tD}^{(i)})$, $i = 1, 2$, denote, respectively, the $i-$th subjects' payoffs from decks A, B, C and D in subgame t. Evidently, $(X_{tA}^{(i)}, X_{tB}^{(i)}, X_{tC}^{(i)}, X_{tD}^{(i)})$, $i = 1, 2$, $t = 1, ..., N$, are independent copies of $(X_{tA}, X_{tB}, X_{tC}, X_{tD})$.

The players' payoffs in subgame t are given by

$$\pi_t^{(1)} = X_{tA}^{(1)} Y_{tA} + X_{tB}^{(1)} Y_{tB} + X_{tC}^{(1)} Y_{tC} + X_{tD}^{(1)} Y_{tD}$$

and

$$\pi_t^{(2)} = X_{tA}^{(2)} Z_{tA} + X_{tB}^{(2)} Z_{tB} + X_{tC}^{(2)} Z_{tC} + X_{tD}^{(2)} Z_{tD}.$$

On the other hand, the subjects' payoffs from the diagonal $\{A, D\}$ in the last subgame $t = 5m$ of game m (where they choose a deck from all four decks) are

$$\tilde{\pi}_t^{(1)} = X_{tA}^{(1)} Y_{tA} + X_{tD}^{(1)} Y_{tD}$$

and

$$\tilde{\pi}_t^{(2)} = X_{tA}^{(2)} Z_{tA} + X_{tD}^{(2)} Z_{tD}.$$

Let $\Pi_m^{(1)} = \sum_{j=1}^{4} \pi_{5m-j}^{(1)}$ and $\Pi_m^{(2)} = \sum_{j=1}^{4} \pi_{5m-j}^{(2)}$, $m = 1, 2, ..., N$, denote the players' total payoffs in the first 4 subgames of game m.

Let $\Im_t$, $t = 1, 2, ..., 5N$, denote the $\sigma-$algebra spanned by the random variables

$$X_{kA}^{(1)} Y_{kA}, X_{kB}^{(1)} Y_{kB}, X_{kC}^{(1)} Y_{kC}, X_{kD}^{(1)} Y_{kD}, X_{kA}^{(2)} Z_{kA}, X_{kB}^{(2)} Z_{kB}, X_{kC}^{(2)} Z_{kC}, X_{kD}^{(2)} Z_{kD},$$

$k = 1, 2, ..., t$. The players' decision variables Y_{tj}, and Z_{tj}, $j \in \{A, B, C, D\}$ are $\Im_{t-1}$-measurable.

The hypotheses described in the previous section have precise formalization in terms of the variables defined above.

For instance, the behavioral null hypothesis $H_0^{(1)}$ is equivalent to the hypothesis that, conditionally on $\Im_{t-1}$, $Y_{tj} = Z_{tj}$, $j \in \{A, B, C, D\}$ (a.s.).

The distributional null hypothesis $H_0^{(2)}$ restricted to the first four subgames can be formally written in terms of the equality of conditional distributions for the payoffs in each of those subgames of game m: $\mathcal{L}(\pi_t^{(1)} | \Im_{t-1}) = \mathcal{L}(\pi_t^{(2)} | \Im_{t-1})$, $t = 5m - j$, $j = 1, 2, 3, 4$.

$H_0^{(3)}$ restricted to the last subgames of each game can be formally written in terms of the equality of conditional distributions of the payoffs from the diagonal AD in the fifth subgame of game m : $\mathcal{L}(\tilde{\pi}_t^{(1)}|\Im_{t-1}) = \mathcal{L}(\tilde{\pi}_t^{(2)}|\Im_{t-1})$, $t = 5m$.

As discussed in the previous section, the natural alternatives to the above null hypotheses are

$H_{a1}^{(2)}$ (two-sided distributional alternative): $\mathcal{L}(\pi_t^{(1)}|\Im_{t-1}) \neq \mathcal{L}(\pi_t^{(2)}|\Im_{t-1})$, $t = 5m - j$, $j = 1, 2, 3, 4$.

$H_{a2}^{(2)}$ (one-sided distributional alternative for payoffs in the first four subgames of each game): $P(\pi_t^{(1)} > \pi_t^{(2)}|\Im_{t-1}) > P(\pi_t^{(1)} < \pi_t^{(2)}|\Im_{t-1})$, $t = 5m - j$, $j = 1, 2, 3, 4$.

$H_{a3}^{(2)}$ (one-sided distributional alternative for payoffs from the diagonal AD in the last subgame of each game): $P(\tilde{\pi}_t^{(1)} > \tilde{\pi}_t^{(2)}|\Im_{t-1}) > P(\tilde{\pi}_t^{(1)} < \tilde{\pi}_t^{(2)}|\Im_{t-1})$, $t = 5m$.

Various joint analogues of the above hypotheses involving both the payoffs from each of the four subgames as well as those from the diagonal AD in the last subgame or the total payoffs in each game can be considered as well.

As follows from the results discussed in Sections 2-4 of the present paper, exact and conservative sign tests for different hypotheses on behavioral differences in the two games can be based on the following statistics.

$$
\begin{aligned}
S_{4N}^{(1)} &= \sum_{m=1}^{N}\sum_{j=1}^{4} s_{5m-j}^{(1)} = \sum_{m=1}^{N}\sum_{j=1}^{4} \mathrm{sign}\left((\pi_{5m-j}^{(1)} - \pi_{5m-j}^{(2)}) + \varepsilon_{5m-j} I(\pi_{5m-j}^{(1)} = \pi_{5m-j}^{(2)})\right),\\
S_{N}^{(2)} &= \sum_{m=1}^{N} s_{5m}^{(2)} = \sum_{m=1}^{N} \mathrm{sign}\left((\tilde{\pi}_{5m}^{(1)} - \tilde{\pi}_{5m}^{(2)}) + \varepsilon_{5m} I(\tilde{\pi}_{5m}^{(1)} = \tilde{\pi}_{5m}^{(2)})\right),\\
S_{N}^{(3)} &= \sum_{m=1}^{N} s_{m}^{(3)} = \sum_{m=1}^{N} \mathrm{sign}\left((\Pi_{m}^{(1)} - \Pi_{m}^{(2)}) + \varepsilon_{m} I(\Pi_{m}^{(1)} = \Pi_{m}^{(2)})\right),\\
S_{N}^{(4)} &= \sum_{m=1}^{N} s_{t}^{(4)} = \sum_{m=1}^{N} \mathrm{sign}\left((\pi_{5m}^{(1)} - \pi_{5m}^{(2)}) + \varepsilon_{5m} I(\pi_{5m}^{(1)} = \pi_{5m}^{(2)})\right),\\
S_{N}^{(5)} &= \sum_{m=1}^{N} s_{m}^{(5)} = \sum_{m=1}^{N} (\mathrm{sign}(\Pi_{m}^{(1)} - \Pi_{m}^{(2)}) + \varepsilon_{m} I(\Pi_{m}^{(1)} = \Pi_{m}^{(2)})) + \\
&\quad \mathrm{sign}(\pi_{m}^{(1)} - \pi_{m}^{(2)}) + \tilde{\varepsilon}_{m} I(\pi_{m}^{(1)} = \pi_{m}^{(2)})),\\
S_{5N}^{(6)} &= S_{N}^{(1)} + S_{N}^{(4)} = \sum_{t=1}^{5N} s_{t}^{(6)} = \sum_{t=1}^{5N} \mathrm{sign}\left((\pi_{t}^{(1)} - \pi_{t}^{(2)}) + \varepsilon_{t} I(\pi_{t}^{(1)} = \pi_{t}^{(2)})\right).
\end{aligned}
$$

where ε_t, $\tilde{\varepsilon}_t$, $t = 1, 2, ..., 5N$, denote independent symmetric Bernoulli random variables independent of $X_{tA}^{(1)}Y_{tA}$, $X_{tB}^{(1)}Y_{tB}$, $X_{tC}^{(1)}Y_{tC}$, $X_{tD}^{(1)}Y_{tD}$, $X_{tA}^{(2)}Z_{tA}$, $X_{tB}^{(2)}Z_{tB}$, $X_{tC}^{(2)}Z_{tC}$, $X_{tD}^{(2)}Z_{tD}$, $t = 1, 2, ..., 5N$ (and, thus, independent of $\Im_{5N}$).

The test statistics $S_{4N}^{(1)}$ provides a device for testing the null hypothesis $H_0^{(2)}$ (and of $H_0^{(1)}$) against the alternative $H_{a1}^{(2)}$ or $H_{a2}^{(2)}$. The statistic $S_N^{(2)}$ can be used to test $H_0^{(3)}$ against $H_{a1}^{(3)}$ or its two-sided analogue. The interpretation of other statistics is similar, with, for instance, $S_N^{(3)}$ being a test statistics for the analogue of the distributional null hypothesis $H_0^{(2)}$ for the total payoffs from the first four subgames.

For instance, under the null hypothesis $H_0^{(1)}$ that, conditionally on $\Im_{t-1}$, $Y_{tj} = Z_{tj}$, $j \in \{A, B, C, D\}$ (a.s.) (and, thus, $H_0^{(2)}$ also holds), the distribution of the statistic $S_{4N}^{(1)}/2 + 2N$ is binomial with parameters $4N$ and $p = 0.5$: $S_{4N}^{(1)}/2 + 2N \sim \text{Bin}(4N, 0.5)$; $P(S_{4N}^{(1)}/2 + 2N = k) = C_{4N}^k/2^{4N} = p_k$; $k = 0, 1, ..., 4N$, where $C_{4N}^k = (4N)!/(k!(4N-k)!)$.

Indeed, under the null, the random variable

$$\begin{aligned}
&\pi_t^{(1)} - \pi_t^{(2)} = X_{tA}^{(1)} Y_{tA} + X_{tB}^{(1)} Y_{tB} + X_{tC}^{(1)} Y_{tC} + X_{tD}^{(1)} Y_{tD} - \\
&(X_{tA}^{(2)} Z_{tA} + X_{tB}^{(2)} Z_{tB} + X_{tC}^{(2)} Z_{tC} + X_{tD}^{(2)} Z_{tD}) = \\
&(X_{tA}^{(1)} - X_{tA}^{(2)}) Y_{tA} + (X_{tB}^{(1)} - X_{tB}^{(2)}) Y_{tB} + (X_{tC}^{(1)} - X_{tC}^{(2)}) Y_{tC} + (X_{tD}^{(1)} - X_{tD}^{(2)}) Y_{tD}
\end{aligned}$$

is symmetric conditional on $\Im_{t-1}$. Therefore, $P(s_t^{(1)} = 1|\Im_{t-1}) = P(s_t^{(1)} = -1|\Im_{t-1}) = 1/2$, and $s_t^{(1)}$ is a martingale difference with respect to $(\Im_t)_t$.

(For instance, suppose that $t = 5m - 4$ and the realizations of the random variables $X_{kA}^{(1)} Y_{kA}$, $X_{kB}^{(1)} Y_{kB}$, $X_{kC}^{(1)} Y_{kC}$, $X_{kD}^{(1)} Y_{kD}$, $X_{kA}^{(2)} Y_{kA}$, $X_{kB}^{(2)} Y_{kB}$, $X_{kC}^{(2)} Y_{kC}$, $X_{kD}^{(2)} Y_{kD}$, $k = 1, 2, ..., t-1$, are such that $Y_{tA} = 1$ (so that, $Y_{tB} = Y_{tC} = Y_{tD} = 0$). Then, conditionally on $\Im_{t-1}$, we have that

$$\begin{aligned}
& P(s_t^{(1)} = 1) = P(X_{tA}^{(1)} > X_{tA}^{(2)}) + \frac{1}{2} P(X_{tA}^{(1)} = X_{tA}^2) \\
= \;& P(X_{tA}^{(1)} > X_{tA}^{(2)}) + \frac{1}{2}\left(1 - P(X_{tA}^{(1)} > X_{tA}^{(2)}) - P(X_{tA}^{(1)} < X_{tA}^{(2)})\right) \\
= \;& \frac{1}{2} + \frac{1}{2}\left(P(X_{tA}^{(1)} > X_{tA}^{(2)}) - P(X_{tA}^{(1)} < X_{tA}^{(2)})\right) = \frac{1}{2} = P(s_t^{(1)} = -1)
\end{aligned}$$

since $X_{tA}^{(1)}$ and $X_{tA}^{(2)}$ are independent copies of each other. In complete similarity we have that $P(s_t^{(1)} = 1) = P(s_t^{(1)} = -1) = \frac{1}{2}$ in the case when $Y_{tC} = 1$ and, thus, $Y_{tA} = Y_{tC} = Y_{tD} = 0$).

By Theorem 6 and Corollary 4 we have that the random variables $s_t^{(1)}$, $t = 1, 2, ..., 4N$, are jointly independent symmetric Bernoulli random variables and, therefore, $S_{4N}/2 + 2N \sim \text{Bin}(4N, 0.5)$.

Similar conclusions concerning independence also hold for the summands in all the other statistics in the present section and, thus, the wide range of the sign tests described in Section 4 is applicable for hypotheses testing.

For instance, the exact randomized ER test (see Section 4) rejects the null hypothesis $H_0^{(1)}$ that, conditionally on $\Im_{t-1}$, $Y_{tj} = Z_{tj}$, $j \in \{A, B, C, D\}$ (a.s.), for the two-sided alternative $H_{a1}^{(2)}$: $Y_{tj} \neq Z_{tj}$ (and also rejects the distributional null hypothesis $H_0^{(2)} : \mathcal{L}(\pi_t^{(1)}|\Im_{t-1}) = \mathcal{L}(\pi_t^{(2)}|\Im_{t-1})$,

$t = 5m - j$, $j = 1, 2, 3, 4$, for the subjects' payoffs from the first four subgames in favor of the two-sided alternative $H_{a1}^{(2)} : \mathcal{L}(\pi_t^{(1)}|\Im_{t-1}) \neq \mathcal{L}(\pi_t^{(2)}|\Im_{t-1}))$ at the significance level α if $S_{4N}^{(1)}/2 + 2N < K_\alpha$ or $S_{4N}^{(1)}/2 + 2N > 4N - K_\alpha$, where K_α is such that $\sum_{k<K_\alpha} p_k + \sum_{k>K_\alpha} p_k < \alpha$.

The sign tests based on the above statistics can be conducted, in complete similarity, for all other null hypotheses and the alternatives discussed in the previous and in this section.

References

Azuma, K. (1967), 'Weighted sums of certain dependent variables', *Tôhoku Mathematical Journal* 4, 357–367.

Bracha, A. (2005), 'Multiple selves and endogenous beliefs,' Ph.D. dissertation, Yale University.

Breiter, H. C., I. Aharon, D. Kahneman, A. Dale and P. Shizgal (2001), 'Functional imaging of neural responses of expectancy and experience of monetary gains and losses,' *Neuron*, 30, 619–639.

Camerer, C. (1995), 'Individual decision making,' in J. H. Hagel and A. E. Roth, eds., *Handbook of Experimental Economics.* Princeton: Princeton University Press.

Coakley, C. W. and Heise, M. A. (1996), 'Versions of the sign test in the presence of ties', *Biometrics* 52(4), 1242–1251.

Damasio, A. R. (1994), *Descartes' Error: Emotion, Reason and the Human Brain.* New York: G. P. Putnam.

de la Peña, V. H. and Ibragimov, R. (2003), 'Sharp probability inequalities and conservative testing procedures for studentized processes and moving averages with applications to econometric models and heavy tails', Working paper, Yale University. Available at http://pantheon.yale.edu/ ri23/Papers/Selfnorm.pdf.

de la Peña, V. H., Ibragimov, R. and Jordan, S. (2004), 'Option bounds', *Journal of Applied Probability* 41A, 145–156.

de la Peña, V. H., Ibragimov, R. and Sharakhmetov, S. (2003), 'Characterizations of joint distributions, copulas, information, dependence and decoupling, with applications to time series', Working paper, Yale University . Available at http://pantheon.yale.edu/˜ri23/Papers/ Copulas.pdf.

Eaton, M. L. (1974), 'A probability inequality for linear combinations of bounded random variables', *Annals of Statistics* 2, 609–614.

Ernst, M., E. E. Nelson, E. B. McClure, C. S. Monk, S. Munson, N. Eshel, E. Zarahan, E. Leibenluft, A. Zametkin, K. Towbin, J. Blair, D. Charney and D. S. Pine (2004), 'Choice selection and reward anticipation: An FMRII study,' *Neuropsychologia*, 42, 1585–1597.

Grimmett, G.. and D. Stirzaker (2001), *Probability and Random Processes*, 3rd ed. Oxford.

Hoeffding, W. (1963), 'Probability inequalities for sums of bounded random variables', *Journal of the American Statistical Association* **58**, 13–30.

Ibragimov, R. (2004), 'On the robustness of economic models to heavy-tailedness assumptions', Mimeo, Yale University . Available at http://pantheon.yale.edu/~ri23/Papers/HeavyTails.pdf.

Ibragimov, R. (2005), *New majorization theory in economics and martingale convergence results in econometrics*. Ph.D. dissertation, Yale University.

Ibragimov, R. and D. Brown (2005), 'Sign tests for dependent observations and bounds for path-dependent options,' CFDP 1518.

Kahneman, D. and Tversky (1979), 'Prospect theory: An analysis of decision under risk,' *Econometrica*, 47, 263–291.

Krieger, L. A. Reber (2005), 'The reel truth,' Strictly Slots, Internet.

Kreps, D. M. (1990), *A Course in Microeconomic Theory.* Princeton: Princeton University Press.

Ledoux, J. E. (1996), *The Emotional Brain.* New York: Simon & Schuster.

Lehmann, E. C. (1986), *Testing Statistical Hypotheses*, 2nd ed. New York: Wiley.

MasColell, A., M. D. Whinsten and J. R. Green (1995), *Microeconomic Theory.* New York: Oxford University Press.

Mellers, B. andA. Schwartz (1999), 'Emotion-based choice,' *Journal of Experimental Psychology: General*, 128, 332–345.

Monderer, D. and L. S. Shapley (1996), 'Potential games,' *Games and Economic Behavior*, 14, 124–143.

Putter, J. (1955), 'The treatment of ties in some nonparametric tests,' *Annals of Mathematical Statistics*, 26, 368–386.

Rolls, E. T. (1999), *The Brain and Emotion.* Oxford: Oxford University Press.

Royer, V. H. (2003), *Powerful Profits from Slots.* New York: Kensington Publishing Corporation.

Sharakhmetov, S. and Ibragimov, R. (2002), 'A characterization of joint distribution of two-valued random variables and its applications', *Journal of Multivariate Analysis* 83, 389–408.

Sprumont, Y. (2000), 'On the testable implications of collective choice theories,' *Journal of Economic Theory*, 93, 205–232.

Starmer, C. (2004), 'Developments in nonexpected utility theory: The hunt for a descriptive theory of choice under risk,' in C. F. Camerer, G. Lowenstein and M. Rabin, eds., *Advances in Behavioral Economics.* Princeton: Princeton University Press.

Wittkowski, K. M. (1998), 'Versions of the sign test in the presence of ties', *Biometrics* 54, 789–791.

Appendix

Table 1. Power of the AR sign test for testing

$P(X_n > x \mid \Im_{n-1}) = P(X_n < x \mid \Im_{n-1})$ against the alternative

$P(X_n > x \mid \Im_{n-1}) > P(X_n < -x \mid \Im_{n-1}),\ x > 0,$

with $P(X_n > 0 \mid \Im_{n-1}) = p > 0.5 > 1 - p = P(X_n < 0 \mid \Im_{n-1})$

(N denotes the number of observations)

p \ N	10	20	30	40	50	60	70	80	90	100	150	200	250	300	500	1000
0.55	0.10	0.13	0.14	0.21	0.20	0.26	0.32	0.29	0.34	0.38	0.44	0.53	0.60	0.66	0.83	0.97
0.60	0.17	0.25	0.29	0.44	0.45	0.56	0.65	0.64	0.71	0.76	0.86	0.93	0.97	0.98	1.00	1.00
0.65	0.26	0.42	0.51	0.69	0.73	0.83	0.89	0.90	0.94	0.96	0.99	1.00	1.00	1.00	1.00	1.00
0.70	0.38	0.61	0.73	0.88	0.92	0.96	0.98	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.75	0.53	0.79	0.89	0.97	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.80	0.68	0.91	0.97	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.85	0.82	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.90	0.93	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.95	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00