OVERCOMING PARTICIPATION CONSTRAINTS

By

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Overcoming Participation Constraints*

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Abstract

In incomplete information environments with transferable utility, efficient outcomes are generally implementable unless interim or ex post participation constraints are imposed on the problem. In this paper we show that linking a sufficiently large number of independent but possibly unrelated social decisions, a slightly perturbed Groves mechanism can implement an efficient outcome with probability arbitrarily close to one, while respecting all participation, incentive and balanced budget constraints.

Keywords: Linking, Participation Constraints, Perturbed Groves Mechanism

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1 Introduction

A common source of inefficiencies in collective decisions problems with privately informed agents is the interaction between incentive constraints, participation constraints, and resource constraints. In the absence of participation constraints, ex post efficiency is attainable in models with transferable utility and private values, since transfers can be set up so that all agents internalize their externalities (Groves [13], d'Aspremont and Gerard-Varet [11] and Arrow [2]). In contrast, if participation and resource constraints are both added to the problem, it is in general impossible to achieve ex post efficient outcomes.[1]

In this paper we argue that a natural way to overcome these inefficiencies is to "link" *unrelated* problems. The key effect of considering two problems at the same time is that, for each type-vector realization of each agent, the mechanism designer needs only to be concerned about one participation constraint, rather than one for each problem. Linking must therefore weakly improve the allocation, because the set of feasible mechanisms is enlarged.

Our first result is that, in an economy with a sufficiently large number of unrelated decision problems (that satisfy stochastic independence and some regularity conditions), we can construct an ex ante budget-balancing Groves mechanism under which the failure of participation constraints occur with vanishingly small probability. This is both intuitive and straightforward to show. The key intuition is that, since Groves mechanisms are efficient, lump sum transfers can be constructed that budget balances in expectation and all agents have a strict incentive to participate, if making the decision behind a veil of ignorance. But, the average interim utility (where the averaging is over the decision problems) converges in probability to the average ex ante expected utility. It follows that violations of participation constraints become rare as the number of decision problems grow large.

While rare, violations of the participation constraints do in general occur with positive probability regardless of how many problems are linked. We therefore also ask whether a nearly efficient mechanism that always fulfill the participation constraints and budget balance can be constructed. This analysis turns out to be more subtle than the rather direct law of large numbers style reasoning used to argue that a Groves mechanism is "almost implementable". The main difficulty is how to rule out unravelling of the participation constraints. Specifically, to always satisfy the participation constraints we need to allow agents to opt out of the Groves mechanism. But then, the incentives

[1] Cramton et al [9], Mailath and Postlewaite [15], and Myerson and Satterthwaite [17] are some well-known examples.

to opt into the mechanism depends on which other types will do so. Hence, even if few types have an incentive to opt out when assuming that all other agents will opt in, the types that opt out may upset the participation constraints for other types. In general, it is possible that the types with the most to gain from opting out are the types that contribute the most to social surplus, implying that there could be negative selection of agents into the mechanism if allowing agents to veto implementation.

The main result of this paper shows that a mechanism can be constructed so as to rule out any unravelling of participation constraints. Specifically, we consider a slight *perturbation* of the Groves mechanism, which satisfies all constraints – including the participation constraints – for *all* type realizations. We show that this mechanism can implement the efficient outcome with a probability arbitrarily close to one, provided that the number of unrelated decisions is sufficiently large. The perturbed Groves mechanism has an sequential interpretation where in first stage, all agents decide whether to agree to play a (particular) Groves mechanism. If *some* agent decides to opt out of the mechanism then the status quo outcome that gives all agents their reservation utility is implemented (i.e., all agents have veto power); if all agents agree to participate in the Groves mechanism, truth-telling is a conditionally dominant strategy, thus the efficient outcome is implemented. Interestingly, the use of a Groves mechanism (as opposed to an ex post budget balanced mechanism) is crucial for our argument, despite the fact that our result "only" is a result in terms of Bayesian implementability. The reason is that selection becomes irrelevant for behavior in the "second stage", which in turn gives a tractable characterization of the "veto game".

Our paper is most closely related to a recent contribution by Jackson and Sonnenschein [14], who show that inefficiencies due to *incentive constraints* can be overcome by linking a large number of independent *replicas* of some underlying decision problems. In their mechanism, agents are required to announce preference profiles such that the frequency of any particular type in the problem that is replicated coincides with its underlying probability distribution. They show that if the social choice function the planner seeks to implement is efficient, then a rationing mechanism can be constructed under which agents try to be as truthful as they can, which in turn implies that with a large number of independent replicas of the same decision problem, the frequency distribution of announced preference types will be close to the true probability distribution. As a result, all equilibria result in approximately efficient allocations with high probability when there are many replicas.

There are some key differences between our paper and Jackson and Sonnenschein [14]. Most importantly, they focus on non-transferable utility environments, while transferable utility is crucial for our construction.[2] In other words, their linking mechanism is primarily designed to overcome the difficulties involved in creating incentives to reveal intensity of preferences, which is the main concern in non-transferable utility models.[3] In contrast, incentive constraints can be easily satisfied in our environment, and the role of linking is directly designed to overcome the participation constraints. Moreover, while Jackson and Sonnenschein [14] impose few restrictions on their underlying social decision problem, it is crucial for their construction that they can link a large number of *identical replicas*. In contrast, our results apply to arbitrary sequences of (transferable utility) problems that satisfy some mild regularity conditions. As a result of these differences, we view our contribution as complementary to Jackson and Sonnenschein [14].

There is also a connection between our paper and papers on bundling, as well as many other papers that have noted the advantages of linking across problems in more specific environments. Here it is very instructive to observe that our construction is a direct generalization of the construction in Armstrong [1], who used a two-part tariff to allow a monopolist to almost fully extract the surplus in a standard monopoly problem with a large number of commodities (similar constructions are used by Bakos and Brynjolfson [3] and Bergstrom and Bergstrom [4] in the context of excludable public goods). In contrast, the mechanism utilized by Jackson and Sonnenschein [14] is more akin to various rationing schemes, such as a vote storage mechanism proposed in Casella [7], a risk-sharing arrangement proposed by Townsend [19], which limits how often the risk averse agent can claim a loss, and a compromising scheme proposed by Borgers and Postl [6].

2 Notation and Preliminaries

Consider an environment with $n \geq 2$ agents and a set D of possible *social decisions.* Each agent $i \in I \equiv \{1, 2, ..., n\}$ is privately informed about a preference parameter $\theta^i \in \Theta^i$ and has a quasi-linear von Neumann-Morgenstern utility function given by

$$v^i \left(d, \theta^i\right) - t^i, \tag{1}$$

[2] An interpretation of Jackson and Sonnenscheins' rationing mechanism is that it provides an ingenious way of creating transferrable utility out of a non-transferrable utility environment by adding constraints on the available announcements agents can make.

[3] They do also discuss how to deal with participation constraints.

where t^i is interpreted as a transfer from agent i in terms of a numeraire good. Also assume that implementing decision $d \in D$ costs $C(d) \in \mathbb{R}$ units of the numeraire good. Moreover, let $\Theta = \times_{i=1}^{n} \Theta^i$ denote the set of all possible type profiles, and denote a generic element of Θ by θ.

By appeal to the revelation principle we restrict our attention to direct mechanisms. A *pure direct revelation mechanism* (henceforth called *mechanism*) is a pair $\langle x, t \rangle$, where $x : \Theta \to D$ is the *allocation rule* and $t : \Theta \to R^n$ is the *cost sharing rule,* where $t(\theta) = \left(t^1(\theta), t^2(\theta) ..., t^n(\theta)\right)$. We adopt the convention that a positive $t^i(\theta)$ is a transfer from agent i when θ is announced. We say that a mechanism is *incentive compatible* if truth-telling is a Bayesian Nash equilibrium in the incomplete information game induced by $\langle x, t \rangle$. Dominant strategy incentive compatibility will be explicitly referred to as such.

Since utility function (1) is quasi-linear and the valuation function v^i is one with private values (that is, v^i depends only on θ^i), many classical implementation results apply. In particular, as long as neither ex post budget balance or participation constraints and ex ante budget balancing constraints are simultaneously imposed, any efficient social decision rule can be implemented in dominant strategies by a Groves mechanism. For easy reference, it will be useful to define this class of mechanisms explicitly.

Definition 1 *A mechanism $\langle x, t \rangle$ is a Groves mechanism if for each θ*

$$x(\theta) \in \arg\max_{d \in D} \sum_{i=1}^{n} v^i\left(d, \theta^i\right) - C(d) \tag{2}$$

and, where for each $i \in I$,

$$t^i(\theta) = C(x(\theta)) - \sum_{j \neq i} v^j\left(x(\theta), \theta^j\right) + \tau^i\left(\theta^{-i}\right), \tag{3}$$

and $\tau^i : \Theta^{-i} \to \mathbb{R}$ may be arbitrarily chosen.

3 Problems with Many Independent Social Decisions

In this section we describe an economy with many independent social decisions. In doing this, it is convenient to refer to the social decision in an economy with a single decision as an *issue* and use the term *social decision* only when considering the fully linked problem. Formally, let $e_k \equiv \left\{\left\langle \Theta_k^i, F_k^i, v_k^i \right\rangle_{i=1}^{n}, D_k, C_k\right\}$ denote an economy with a single issue k. Here, Θ_k^i denotes the issue-k type-space for agent i. For each issue k, the types $\theta_k^i \in \Theta_k^i$ are stochastically independent across agents, and F_k^i is consequently both the prior probability distribution over Θ_k^i and the percieved probability distribution over Θ_k^i for all other agents. Preferences over different ways to resolve issue k are described by the valution function $v_k^i : \Theta_k^i \times D_k \to \mathbb{R}$, where D_K is the set of possible alternatives for issue k. The von Neumann-Morgernstern utility for a type-θ_k^i agent i in economy e_k is given by

$$v_k^i\left(\theta_k^i, d_k\right) - t^i \tag{4}$$

for each $d_k \in D_k$, where t^i is a transfer of the numeraire good. Finally, $C_k : D_k \to \mathbb{R}$ describes the cost of implementing each alternative $d_k \in D_k$ in terms of the numeraire good.

Let $\mathcal{E}_K = \{e_k\}_{k=1}^K$ denote an economy with K social issues indexed by $k \in \{1, ..., K\}$. A social decision in economy $\mathcal{E}_K$ corresponds with a resolution for each issue and a transfer scheme. We let $\mathcal{D}_K = \times_{k=1}^K D_k$ denote the set of social decisions in the $K-$issue economy . For each $d \in \mathcal{D}_K$, the cost of implementing d is,

$$\mathcal{C}_K\left(d\right) \equiv \sum_{k=1}^K C_k\left(d_k\right). \tag{5}$$

The type-space for agent i in economy $\mathcal{E}_K$ is given by $\Theta^i\left(K\right) = \times_{k=1}^K \Theta_k^i$, and we write $\Theta\left(K\right) \equiv \times_{i=1}^n \Theta^i\left(K\right)$, and $\Theta_{-i}\left(K\right) \equiv \times_{j \neq i} \Theta^j\left(K\right)$. The von Neumann-Morgernstern utility function for a agent i of type $\theta^i \in \Theta^i\left(K\right)$ is of the form in (1), with v^i being replaced by $V_K^i : \Theta^i\left(K\right) \times \mathcal{D}_K \to \mathbb{R}$ given by

$$V_K^i\left(d, \theta^i\right) = \sum_{k=1}^K v_k^i\left(d_k, \theta_k^i\right). \tag{6}$$

Next, we assume that the issue-specific types of each agent are independently distributed across issues. That is, for each $\theta^i \in \Theta^i\left(K\right)$, the cumulative distribution function is given by

$$\mathcal{F}_K^i\left(\theta^i\right) = \times_{k=1}^K F_k^i\left(\theta_k^i\right) \tag{7}$$

To sum up we can describe the primitives of an economy $\mathcal{E}_K$ in terms of the components $\{e_k\}_{k=1}^K$ as follows:

$$\mathcal{E}_K = \left\{ \left\langle \underbrace{\times_{k=1}^K \Theta_k^i}_{\Theta^i(K)}, \underbrace{\sum_{k=1}^K v_k^i}_{V_K^i}, \underbrace{\times_{k=1}^K F_k^i}_{\mathcal{F}_K^i} \right\rangle_{i=1}^n, \underbrace{\times_{k=1}^K D_k}_{\mathcal{D}_K}, \underbrace{\sum_{k=1}^K C_k}_{\mathcal{C}_K} \right\}. \tag{8}$$

To deal with participation constraints at a reasonable level of generality, we postulate that for each issue k there is a "status quo outcome" $d_k^0 \in D_k$ such that $C_k\left(d_k^0\right) = 0$. We let $r_k^i\left(\theta_k^i\right) \equiv$

$v_k^i\left(d_k^0, \theta_k^i\right)$ denote agent $i's$ "reservation utility" on issue k and for each $\theta^i \in \Theta^i(K)$, let

$$R_K^i\left(\theta^i\right) = \sum_{k=1}^{K} r_k^i\left(\theta_k^i\right) \tag{9}$$

denote agent $i's$ reservation utility in a K-issue economy $\mathcal{E}_K$.

For notational convenience we let $\theta_k = \left(\theta_k^1, ..., \theta_k^n\right) \in \times_{i=1}^n \Theta_k^i$ denote an issue-k type profile. Let $x_k^*(\theta_k)$ denote an efficient social decision rule and $s_k(\theta_k)$ the associated maximized value of the social surplus in this economy with a single issue k, that is

$$\begin{aligned} x_k^*(\theta_k) &\in \arg\max_{d_k \in D_k} \sum_{i=1}^{n} v_k^i\left(\theta_k^i, d_k\right) - C_k(d_k) \\ s_k(\theta_k) &= \max_{d_k \in D_k} \sum_{i=1}^{n} v_k^i\left(\theta_k^i, d_k\right) - C_k(d_k), \end{aligned} \tag{10}$$

where we assume that appropriate continuity and compactness assumptions are satisfied to make the optimization problem in (10) well defined. It follows by definition that $s_k(\theta_k) \geq \sum_{i=1}^n r_k^i\left(\theta_k^i\right)$ for every $\theta_k \in \times_{i=1}^n \Theta_k^i$. However, to avoid trivialities we restrict attention to the case where the expected social surplus from each issue is uniformly bounded away from the reservation utilities. Moreover, we also require that the social surplus associated with each issue be bounded from above and below. Together, we refer to these restrictions as *regularity conditions.* Formally,

Definition 2 *We say that* $\{e_k\}_{k=1}^{\infty}$ *is a* regular sequence *of single-issue economies if*

R1 [Non-vanishing Gains from Trade]. *There exists some* $\delta > 0$ *such that*

$$\mathrm{E}\left[s_k(\theta_k)\right] \geq \mathrm{E}\left[\sum_{i=1}^{n} r_k^i\left(\theta_k^i\right)\right] + \delta, \text{ for all } k;$$

R2 [Uniform Bounds on Surplus]. *There exist some* $\underline{a} < \bar{a}$ *such that* $\underline{a} < s_k(\theta_k) < \bar{a}$ *for all* $\theta_k \in \times_{i=1}^n \Theta$ *and for all* k;[4]

R3 [Uniform Bounds on Reservation Utilities]. *There exist some* $\underline{b} < \bar{b}$ *such that* $\underline{b} < r_k^i\left(\theta_k^i\right) < \bar{b}$ *for all* i, *all* $\theta_k^i \in \Theta_k^i$, *and all* k.

[4] We can, without loss of generality, assume that $\bar{a} > 0$. If $\bar{a} < 0$ we can always add a constant larger than $|\bar{a}|/n$ to all utility functions. Since the reservation utilities are defined in terms of the default outcomes this does not change the problem.

3.1 Remarks about the Environment

1. We kept the set of agents the same and finite for all problems. Since we can allow agents with trivial roles for any issue k, the restriction can be rephrased as saying that the union of the set of agents over all issues is finite.

2. Throughout the model we made many strong separability assumptions. For example, payoffs and costs are assumed to be additively separable across different issues; and for each agent the list of issue-specific types are stochastically independent. Moreover, the type parameter θ_k^i is assumed to affect how agent i evaluates issue k and no other issues. These separable assumptions are weaker than it appears. If any of these assumptions fail for a pair of issues, we can always bunch these two issues into a single issue. We can do this because we do not impose any dimensionality restrictions on either the issue specific set of alternatives or type spaces. As it will become clear later, our results hinge on the existence of a sufficiently large number of truly independent and separate problems for our results, but do not require that all problems be independent and separate.

3. Our model also assumes that, for each issue, agents' types are stochastically independent. This assumption is maintained because, if types were correlated across agents, then it is usually possible to construct efficient mechanisms similar to that of Cremer and McLean [10] that respects both participation and resource constraints, without any role for linking the issues.

4. Our environment is general enough to incorporate both public goods and private goods problems. For example, Armstrong's [1] non-linear pricing problem with many products fits nicely in this framework. Consider each issue k as corresponding to a product. The set of alternatives for issue $D_k \in \mathbb{R}^n$ will then correspond to a quantity choice for product k for each consumer. Agent $i's$ valuation function v_k^i depends on the quantity choice for i, i.e. i-th component of d_k (see Section 5.1 for more discussion). Myerson and Satterswaite's [17] bilateral bargaining problem also fits in as an issue k in our setup. A bilateral bargaining problem can be represented by an action space $D_k = [0, 1]$, denoting the probability that agent 1 will obtain the good. Besides agents 1 and 2 – the two agents in the bargaining situation – all other agents' preferences on this issue are simply independent of d_k. Mailath and Postlewaite's [15] many-agent bargaining problem with public goods can similarly be incorporated as an issue in our setup.

4 The Implementation Problem

The goal of the mechanism designer is to implement an efficient allocation, subject to incentive compatibility, individual rationality and resource constraints. We write $\langle x_K(\cdot), t_K(\cdot)\rangle$ for a mechanism in economy $\mathcal{E}_K$ where $x_K : \Theta(K) \to \mathcal{D}_K$ is the allocation rule and $t_K \equiv \left(t_K^1, ..., t_K^n\right) : \Theta(K) \to \mathbb{R}^n$ is the transfer rule. We can now express the incentive compatibility constraint as

$$\begin{aligned} \mathrm{E}_{-i}\left[V_K^i\left(x_K(\theta), \theta^i\right) - t_K^i(\theta)\right] &\geq \mathrm{E}_{-i}\left[V_K^i\left(x_K\left(\theta_i', \theta_{-i}\right), \theta^i\right) - t_K^i\left(\theta_i', \theta\right)\right] \\ \forall i &\in I, \theta_i, \theta_i' \in \Theta^i(K). \end{aligned} \tag{11}$$

We impose participation constraints in the interim state as

$$\mathrm{E}_{-i}\left[V_K^i\left(x_K(\theta), \theta^i\right) - t_K(\theta)\right] \geq R_K^i\left(\theta^i\right) \qquad \forall i \in I, \theta_i \in \Theta^i(K), \tag{12}$$

where $R_K^i\left(\theta^i\right)$ is defined in (9). Finally, the resource constraint is imposed in the ex ante form,

$$\mathrm{E}\left[\mathcal{C}_K\left(x_K(\theta)\right)\right] = \mathrm{E}\left[\sum_{i=1}^n t_K^i(\theta)\right]. \tag{13}$$

A seemingly more stringent way to impose the resource constraint would be to require that the resources balance for each realized type profile, that is

$$\mathcal{C}_K\left(x_K(\theta)\right) = \sum_{i=1}^n t_K^i(\theta) \qquad \forall \theta \in \Theta(K). \tag{14}$$

When types are correlated or when stronger notions of incentive compatibility are considered, results sometimes depend on the form of the budget balance constraint. However, for the setup we consider in this paper (13) and (14) are equivalent:

Proposition 1 (Borgers and Norman [5]) *Suppose that each agent has a utility function of the form $v^i(d, \theta) - t^i$ and that there exists a pair of agents (i, j) such that θ^i and θ^j are stochastically independent. Then, for every mechanism satisfying (13) there is a mechanism with the same allocation rule x that satisfies (14) and generates the same interim expected payments for all agents as the original mechanism.*

By Proposition 1 we note that if, as we assume, types are stochastically independent, then for any $\langle x, t\rangle$ that is (Bayesian) incentive compatible, individually rational and ex ante budget balanced we can find transfers such that the same social decision rule satisfies ex post budget balance and all other constraints.[5]

[5] Applied to a Groves mechanism, this is the well-known AGV implementation result due to d'Aspremont and Gerard-Varet [11].

5 The Groves Mechanism Almost Works when Many Problems Are Linked

We now consider a sequence of economies $\{\mathcal{E}_K\}_{K=1}^{\infty}$ and examine the results when K is sufficiently large. Let

$$x_K^*(\theta) \in \arg\max_{d \in \mathcal{D}_K} \sum_{i=1}^{n} V_K^i\left(d, \theta^i\right) - \mathcal{C}_K(d), \tag{15}$$

$$S_K(\theta) = \max_{d \in \mathcal{D}_K} \sum_{i=1}^{n} V_K^i\left(d, \theta^i\right) - \mathcal{C}_K(d) \tag{16}$$

where $V_K^i\left(d, \theta^i\right)$ is defined in (6) and $\mathcal{C}_K(d)$ in (5). Consider a Groves mechanism

$$\left\langle x_K^*, t_{G,K} \equiv \left(t_{G,K}^1, ..., t_{G,K}^n\right)\right\rangle, \tag{17}$$

where x_K^* is given by (15) and for each i the transfer $t_{G,K}^i$ is given by

$$t_{G,K}^i(\theta) = V_K^i\left(x_K^*(\theta), \theta^i\right) - S_K(\theta) \underbrace{+\frac{(n-1)}{n}\mathrm{E}\left[S_K(\theta)\right] - \mathrm{E}\left[R_K^i\left(\theta^i\right)\right] + \frac{1}{n}\mathrm{E}\left[\sum_{j=1}^{n} R_K^j\left(\theta^j\right)\right]}_{\text{lump sum transfer independent of } i\text{'s announcement}} \tag{18}$$

Clearly truth-telling is a dominant strategy, implying that incentive constraints (11) are satisfied. Moreover, routine calculations show that the ex ante balanced budget constraint (13) holds. The only issue is thus the participation constraints (12). We denote the interim expected payoff for agent i with type θ^i in economy $\mathcal{E}_K$ under the proposed Groves mechanism $\langle x_K^*, t_{G,K}\rangle$, when other agents report truthfully, by $U_K^i\left(\theta^i\right)$. It is given by

$$U_K^i\left(\theta^i\right) = \mathrm{E}_{-i}\left[S_K(\theta)\right] - \frac{n-1}{n}\mathrm{E}\left[S_K(\theta)\right] + \mathrm{E}\left[R_K^i\left(\theta^i\right)\right] - \frac{1}{n}\mathrm{E}\left[\sum_{j=1}^{n} R_K^j\left(\theta^j\right)\right]. \tag{19}$$

Since the maximum of a sum can be obtained by maximizing each component, we note that

$$S_K(\theta) = \sum_{k=1}^{K} s_k(\theta_k), \tag{20}$$

where $s_k(\theta_k)$ is defined in (10). The additive form of (20) and stochastic independence across issues suggests that $S_K(\theta)/K$ should converge in probability to its expectation as K goes to infinity, implying that $\mathrm{E}_{-i}\left[S_K(\theta)\right]/K$ should be close to the unconditional expectation. Similarly, $R_K^i\left(\theta^i\right)$

is a sum of single-issue reservation utilities, so $R_K^i(\theta^i)/K$ should converge in probability to its expectation. But, inspecting the expressions in (19), this suggests that the interim expected payoff should be close to its expectation, which is the difference between social surplus and the reservation utilities summed over all agents. Hence, it seems quite plausible that almost all participation constraints will be satisfied under some appropriate regularity conditions. Indeed, the proposition below establishes that this is the case for a sequence of economies consisting of regular issues:

Proposition 2 *Suppose that $\{\mathcal{E}_K\}_{K=1}^{\infty}$ is sequence of economies consisting of stochastically independent regular issues (in the sense of Definition 2). Moreover, for each K, let $(x_K^*, t_{G,K})$ be the Groves mechanism as specified in (17). Then, for every $\varepsilon > 0$ there exists some finite $K^*(\varepsilon)$ such that*

$$\Pr\left[U_K^i(\theta^i) - R_K^i(\theta^i) \geq 0\right] \geq 1 - \varepsilon \tag{21}$$

for every i and every $K \geq K^*(\varepsilon)$.

The interpretation of the result is that the probability that all participation constraints in (12) are satisfied (i.e., $(1-\varepsilon)^n$) can be made arbitrarily close to one when we link a sufficiently large number of independent social decisions using a standard Groves mechanism with appropriately chosen lump sum transfers.

Proof. Since $S_K(\theta) = \sum_{k=1}^K s_k(\theta_k)$ and $R_K^i(\theta^i) = \sum_{k=1}^K r_k^i(\theta_k^i)$, we may express (19) as

$$U_K^i(\theta^i) = \sum_{k=1}^K \left\{ \mathrm{E}_{-i}\left[s_k(\theta_k)\right] - \frac{n-1}{n}\mathrm{E}\left[s_k(\theta_k)\right] + \mathrm{E}\left[r_k^i(\theta_k^i)\right] - \frac{1}{n}\mathrm{E}\left[\sum_{j=1}^n r_k^j\left(\theta_k^j\right)\right]\right\} \tag{22}$$

For notational brevity, define $\phi_k^i(\theta_k^i) \equiv \mathrm{E}_{-i}\left[s_k(\theta_k)\right] - r_k^i(\theta_k^i)$. This allows us to express $U_K^i(\theta^i) - R_K^i(\theta^i)$ as

$$U_K^i(\theta^i) - R_K^i(\theta^i) = \sum_{k=1}^K \left\{ \phi_k^i(\theta_k^i) - \mathrm{E}\left[\phi_k^i(\theta_k^i)\right] + \frac{1}{n}\left(\mathrm{E}\left[s_k(\theta_k)\right] - \mathrm{E}\left[\sum_{j=1}^n r_k^j\left(\theta_k^j\right)\right]\right)\right\}. \tag{23}$$

By assumption all issues in economy $\mathcal{E}_K$ are regular, thus by (R1), we have:

$$\mathrm{E}_i\left[U_K^i(\theta^i) - R_K^i(\theta^i)\right] = \frac{1}{n}\sum_{k=1}^K \left\{ \mathrm{E}\left[s_k(\theta_k)\right] - \mathrm{E}\left[\sum_{j=1}^n r_k^j(\theta^j)\right]\right\} \geq \frac{K}{n}\delta. \tag{24}$$

Moreover, since the issues are stochastically independent, $\left\{\theta_k^i\right\}_{k=1}^K$ is a sequence of stochastically independent variables, thus $\left\{\phi_k^i(\theta_k^i)\right\}_{k=1}^K$ is a sequence of independent variables. Finally, since all issues are regular, $\phi_k^i\left(\theta_k^i\right)$ is bounded from above and below by (R2) and (R3), thus there exists some σ^2 such that $\operatorname{Var}\left[\phi_k^i\left(\theta_k^i\right)\right] \leq \sigma^2$ for every i and every k. From the above observations, we have

$$\begin{aligned}
\Pr\left[U_K^i\left(\theta^i\right) - R_K^i\left(\theta^i\right) < 0\right] & \overset{(23)}{=} \Pr\left[\sum_{k=1}^K\left(\phi_k^i\left(\theta_k^i\right) - \mathrm{E}\left[\phi_k^i\left(\theta_k^i\right)\right]\right) < -\frac{1}{n}\sum_{k=1}^K\left(\mathrm{E}\left[s_k\left(\theta_k\right)\right] - \mathrm{E}\left[\sum_{j=1}^n r_k^j\left(\theta_k^j\right)\right]\right)\right] \\
\text{/By inequality in (24)/} \quad & \leq \Pr\left[\sum_{k=1}^K\left(\phi_k^i\left(\theta_k^i\right) - \mathrm{E}\left[\phi_k^i\left(\theta_k^i\right)\right]\right) < -\frac{\delta K}{n}\right] \\
& \leq \Pr\left[\left|\sum_{k=1}^K\left(\phi_k^i\left(\theta_k^i\right) - \mathrm{E}\left[\phi_k^i\left(\theta_k^i\right)\right]\right)\right| > \frac{\delta K}{n}\right] \\
\text{/Chebyshev's inequality/} \quad & \leq \left(\frac{1}{\frac{\delta K}{n}}\right)^2 \operatorname{Var}\left[\sum_{k=1}^K \phi_k^i\left(\theta_k^i\right)\right] \leq \frac{n^2\sigma^2}{\delta^2 K} \qquad (25)
\end{aligned}$$

Since $\frac{n^2\sigma^2}{\delta^2 K} \to 0$ as $K \to \infty$, we conclude that for every $\varepsilon > 0$, there exists some finite integer $K_i^*\left(\varepsilon\right)$ such that (21) is satisfied for agent i. Since n is finite we then just let $K^*\left(\varepsilon\right) = \max_{i \in I} K_i^*\left(\varepsilon\right)$ and the result follows. ■

5.1 Application: Armstrong's [1] Multiproduct Monopolist Problem

While Proposition 2 is about "almost implementing" efficient allocations, the logic in the proof of Proposition 2 can be used to understand Armstrong's [1] analysis of a multiproduct monopolist with K private goods. Armstrong assumes that each good k is produced at constant unit cost c_k. Efficiency is trivially implementable in this environment by marginal cost pricing, but profit maximization will lead to inefficiencies due to informational rents for the consumers. His main result is that when K is sufficiently large, a cost-based two-part tariff can be almost profit maximizing in the sense that the monopolist extracts almost all consumer surplus.

Due to the constant unit cost assumption, each consumer can be treated separately, so there is no loss in assuming that there is a single consumer. We therefore drop the index i and write

$$V_K(d,\theta) = \sum_{k=1}^K v_k(d_k, \theta_k)$$

for the utility function of the consumer, where d_k is now is to be interpreted as the quantity of good k consumed. It can be seen from (15) and (18) that the Groves mechanism in this environment reduces to marginal cost pricing together with a lump sum transfer where

$$x_K^*(\theta) = \arg\max_{d \in R_+^K} \left[\sum_{k=1}^{K} v_k(d_k, \theta_k) - \sum_{k=1}^{K} c_k d_k \right] \quad (26)$$

$$t_{G,K}(\theta) = \sum_{k=1}^{K} c_k x_k^*(\theta) + T, \quad (27)$$

where T is a lump sum transfer that does not depend on θ. Thus mechanism (17) reduces to a two-part tariff in this case, where the consumer pays a fixed fee of T for the right to purchase goods at marginal costs. While the budget-balanced Groves mechanism will imply $T = 0$ (obvious from (15) by setting $n = 1$), the approximately profit maximizing two-part tariff is to set $T = (1 - \varepsilon) \mathrm{E}[S_K(\theta)]$ where $S_K(\theta) = \left[\sum_{k=1}^{K} v_k(x_k^*(\theta), \theta_k) - \sum_{k=1}^{K} c_k x_k^*(\theta) \right]$ and $\varepsilon > 0$ can be made arbitrarily small. A straightforward application of law of large numbers imply that such a two-part tariff satisfies the consumer's participation constraint almost always, and, since the problem can be solved separately for each consumer the mechanism becomes exactly incentive feasible if types that are not willing to pay T are allowed to opt out. This is exactly the construction Armstrong [1] uses to establish that the monopolist can extract almost the full surplus if there are many goods.

6 An Asymptotically Efficient Perturbation of the Groves Mechanism

While Proposition 2 is suggestive, it does not directly provide a solution to the implementation problem stated in Section 4. The reason is as follows. Even though the probability of the violation of participation constraints can be made arbitrary small by linking sufficiently many problems, for any finite K the probability of a violation will in general be strictly positive.

An obvious remedy for this is to revert to the status quo outcome whenever a participation constraint is violated, which, since the failures are rare when K is large, would create a mechanism that is almost efficient. The problem with this idea, however, is that the agents that opt out change the interim expected payoff from participation for the agents that opt in.[6] It is therefore not obvious

[6] This issue is not relevant for private goods problems such as Armstrong [1] where the efficient outcome for agent i does not depend on others' types. Similarly, in the analysis of excludable public goods (Bakos and Brynjofson [3], Bergstrom and Bergstrom [4] and Fang and Norman [12]) the reservation utility for a consumer can be attained by simply excluding the consumer from usage. As a result, approximate efficiency can be attained by a slight modification of the Groves mechanism in Section 5.

whether such a mechanism can be constructed in a way that guarantees that the participation constraints hold for all types, and, at the same time, the Groves mechanism is implemented for almost all type profiles. More precisely, we have to deal with the following issue. When some agents opt out of the mechanism, this may diminish the value of playing the Groves mechanism for the remaining agents. In fact, the types that opt out may actually be the types that contribute the most to the social surplus, if these types have reservation utilities that are high enough. Hence, if types with interim payoffs in the Groves mechanism below the reservation utilities opt out, the best response for the remaining types may be that some additional types should opt out. The set of types that agree to play the Groves mechanism must therefore somehow be determined through a "fixed point" argument, and a priori it is unclear how to rule out that the only "fixed point" implements the status quo for sure, even when the probability that the original Groves mechanism violates a participation constraint is small.

6.1 A Perturbed Groves Mechanism

Constructing a sequence of almost efficient mechanisms satisfying the constraints (11), (12), and (13) for all type realizations is complicated by the fixed point problem discussed above. We choose to deal with this by constructing a mechanism where the message space is enlarged to include a choice on whether or not to play the Groves mechanism. This construction has the advantage that the mechanism is easy to describe, and that we can "solve" the fixed point problem by appealing to an off-the-shelf existence result from Milgrom and Weber [16].[7]

Formally, we add a message $m^i \in \{0,1\}$ to the message space, so that each agent reports a pair $\left(m^i, \theta^i\right)$. The extra message m^i is interpreted as the decision on whether or not to accept to play the Groves mechanism. Specifically, we assume that if each agents report $m^i = 1$, then the Groves mechanism is implemented, whereas it *any* agent i chooses $m^i = 0$, the default outcome d^0 is implemented and all transfers are zero. That is, we consider a sequence of mechanisms $\left\{\widehat{x}_K\left(\cdot\right), \widehat{t_K}\left(\cdot\right)\right\}_{K=1}^{\infty}$, where for each K

$$\begin{aligned}
\widehat{x}\left(\theta, m\right) &= \begin{cases} x_K^*\left(\theta\right) & \text{if } m = (1,1,...,1) \\ d^0 = \left(d_1^0, ..., d_K^0\right) & \text{otherwise} \end{cases} \qquad (28) \\
\widehat{t}_K^i\left(\theta, m\right) &= \begin{cases} t_{G,K}^i\left(\theta\right) + \frac{\delta K}{4n} & \text{if } m = (1,1,...,1) \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

[7] In contrast, constructing sequences of direct revelation mechanisms generates a fixed point problem in the set of "participating types", which is harder to deal with and makes the mechanism awkward to describe.

where $t^i_{G,K}(\theta)$ is given by the Groves transfer scheme in (18), and δ is the lower bound on the per issue gains from trade defined in (R1). One may think of (28) as a sequential mechanism, where in stage 1 each agent announces whether or not to opt in. If any agents opt out, then the game is over and the status quo outcome is implemented. If, instead, all agents opt in, then the game proceeds into the second stage, where Groves mechanism with almost the same transfers as in Section 5 is implemented.

The reason for the added term $\frac{\delta K}{4n}$ in the transfer scheme is to ensure that

$$\mathrm{E}\left[\mathcal{C}_K\left(\widehat{x}\left(\theta,m\right)\right)\right] < \mathrm{E}\left[\sum_{i=1}^{n} \widehat{t}^i_K\left(\theta,m\right)\right], \tag{29}$$

when K is sufficiently large, which ensures existence of a transfer that balances the budget exactly (see the discussion following Proposition 3).

6.2 The Approximate Efficiency Result

Our main result of the paper establishes that, if the underlying sequence of single-issue economies is regular, then mechanism (28) has equilibria that implement the efficient outcome with probability arbitrarily close to one, provided that sufficiently many issues are linked. Formally,

Proposition 3 *Suppose that $\{\mathcal{E}_K\}_{K=1}^{\infty}$ is sequence of economies consisting of stochastically independent regular issues (in the sense of Definition 2). Then, for every $\varepsilon > 0$ there exists some finite $K^*(\varepsilon)$ such that, for every economy $\mathcal{E}_K$ with $K \geq K^*(\varepsilon)$, there exists an equilibrium in the game induced by mechanism (28) where the efficient outcome is implemented with probability at least $1-\varepsilon$, and where (29) is satisfied.*

Notice that the participation constraints are trivially satisfied by mechanism (28) since every agent can guarantee her reservation utility by announcing $m^i = 0$. Since any (expected) budget surplus can be repaid to the agents in a lump-sum fashion without upsetting any incentive or participation constraints, this ensures existence of a mechanism that satisfies the ex ante budget balance constraint (13), and, by appeal to Proposition 1 this in turns implies the existence of a mechanism that satisfies the ex post budget balance constraint (14). Hence, all the constraints discussed in Section 4 can be satisfied by an approximately optimal mechanism if K is large enough.

Intuitively, the explanation for Proposition 3 is that truth-telling is a dominant strategy, *conditional on opting in.* Moreover, when opting out, the reported type is irrelevant. Together, this implies that assuming truth-telling of the type message is without loss of generality, and the only issue is whether to opt in or out. Assuming that almost all types of the other agents decide to opt in, the situation is not all that different from the regular Groves mechanism. Indeed, the formal proof shows that, if the number of issues is large enough, then there is a set of types with probability measure close to unity, such that all these types want to opt in if they believe that other agents of these types also opt in. This argument is for arbitrary behavioral strategies for types outside the set. Hence, it only remains to show that there are *some* equilibrium strategies for the remaining types, which can be done by appealing to a standard existence argument.

6.3 The Proof

Truth-telling is dominant "in the second stage" if all players opt in, and there is nothing to loose from truth-telling if opting out. This is true regardless of whether or not other players report truthfully. However, to reduce the need for additional notation we will only show this assuming truth-telling by all other agents.[8]

Allowing randomizations over m^i we let $\psi_K^i : \Theta^i(K) \to [0,1]$, where $\psi_K^i(\theta^i)$ is the probability that type θ^i chooses $m^i = 1$. Following standard conventions we let $\psi_K = (\psi_K^1, ..., \psi_K^n)$ and $\psi_K^{-i} = (\psi_K^1, .., \psi_K^{i-1}, \psi_K^{i+1}, ..., \psi_K^n)$. We let $\widehat{U}^i(m_i, \psi_K^{-i}, \theta^i)$ denote the interim expected payoff for type θ^i from announcement (m^i, θ^i) under the assumption that all other agents report type truthfully and choose to opt in according to an arbitrary $\psi_K^{-i} : \Theta^{-i}(K) \to [0,1]^{n-1}$. For notational brevity, let

$$\begin{aligned} \Psi_K^{-i}(\theta^{-i}) &\equiv \prod_{j\neq i} \psi_K^j(\theta^j) \\ F_K^{-i}(\theta^{-i}) &= \prod_{j\neq i} F_K^j(\theta^j) \end{aligned} \quad (30)$$

and express the interim expected payoff (conditional on truth-telling) for player i as

$$\widehat{U}^i\left(m_i, \psi_K^{-i}, \theta^i\right) = \begin{cases} \int \left(\left[S_K(\theta) + T_K^i\right] \Psi_K^{-i}(\theta^{-i}) + R_K^i(\theta^i)\left[1 - \Psi_K^{-i}(\theta^{-i})\right]\right) dF_K^{-i}(\theta^{-i}) & \text{if } m_i = 1 \\ R_K^i(\theta^i) & \text{if } m_i = 0 \end{cases}, \quad (31)$$

where $S_K(\theta)$ is the surplus and

$$T_K^i \equiv -\frac{n-1}{n}\mathrm{E}\left[S_K(\theta)\right] + \mathrm{E}\left[R_K^i(\theta^i)\right] - \frac{1}{n}\mathrm{E}\left[\sum_{j=1}^n R_K^j(\theta^j)\right] - \frac{\delta K}{4n} \quad (32)$$

[8] In the end, the extra "bite" that comes from the dominance of truth-telling will not be of any help, since agents need to make probabilistic assessments when deciding on the choice of m^i.

is the (negative of the) "conditional lump-sum" part of the transfer in (28). A best response where agent i always tells the truth can then be characterized in terms of the interim expected payoff function as follows:

Lemma 1 *Suppose that each agent $j \neq i$ reports type truthfully and chooses m^j in accordance to rule $\psi_K^j : \Theta^j(K) \to [0,1]$. Then, it is a best response for agent i to always report truthfully and to follow the rule $\widetilde{\psi}_K^i$, where;*

1. $\widetilde{\psi}_K^i\left(\theta^i\right) = 1$ *whenever* $\widehat{U}^i\left(m_i, \psi_K^{-i}, \theta^i\right) > R_K^i\left(\theta^i\right)$
2. $\widetilde{\psi}_K^i\left(\theta^i\right) = 0$ *whenever* $\widehat{U}^i\left(m_i, \psi_K^{-i}, \theta^i\right) < R_K^i\left(\theta^i\right)$

Lemma 1 is a more or less direct consequence of truth-telling being dominant in the Groves mechanism, but a proof is in the appendix for completeness.

Define the set,

$$\overline{\Theta}^i(K) = \left\{\theta^i \in \Theta^i(K) \,|U_K^i\left(\theta^i\right) - R_K^i\left(\theta^i\right) \geq \frac{\delta K}{2n}\right\}, \tag{33}$$

where the interim utility function $U_K^i\left(\theta^i\right)$ is defined in (19), using the Groves mechanism in the previous section. To interpret $\overline{\Theta}^i(K)$, recall that $\delta K/n$ is a lower bound for the expected value for $U_K^i\left(\theta^i\right) - R_K^i\left(\theta^i\right)$, so $\overline{\Theta}^i(K)$ is a set of agent i's types for which her interim expected surplus is at least 1/2 of this lower bound. The reason why this set is useful is that, just like the set of agents for which the participation constraints are fulfilled under the Groves mechanism, the probability that θ^i belongs to (33) can be made arbitrarily close to one by considering a sufficiently large K. However, since the types in the set (33) have a strict incentive to participate in the Groves mechanism, there is now some room to consider small deviations from the Groves mechanism without upsetting their participation constraints.

It will be convenient to introduce some notation for sets of type profiles where all agents have types in this set, so we let

$$\overline{\Theta}(K) = \left\{\theta \in \Theta(K) \,|\theta^i \in \overline{\Theta}^i(K) \text{ for each } i\right\}, \tag{34}$$

and

$$\overline{\Theta}^{-i}(K) = \left\{\theta^{-i} \in \Theta^{-i}(K) \,|\theta^j \in \overline{\Theta}^j(K) \text{ for each } j \neq i\right\}. \tag{35}$$

Using an argument which is very similar to the proof of Proposition 2 we first establish that for large K, the probability that θ^{-i} belongs to $\overline{\Theta}^{-i}(K)$ is near one:

Lemma 2 *Suppose that $\{\mathcal{E}_K\}_{K=1}^{\infty}$ is sequence of economies consisting of stochastically independent regular issues. Then, for every $\varepsilon > 0$ there exists finite $K'(\varepsilon)$ such that* $\Pr\left[\overline{\Theta}^{-i}(K)\right] \geq 1 - \varepsilon$ *for all $K \geq K'(\varepsilon)$.*

Define $\mathrm{E}_{-i}\left[S_K(\theta)\,|\Theta^{-i}(K)\right]$ as the interim expectation of the social surplus under an efficient social decision rule for agent i conditional on his own type θ^i *and conditional on the types of the other agents being in the set* $\Theta^{-i}(K)$. Intuitively, this conditional expectation $\mathrm{E}_{-i}\left[S_K(\theta)\,|\Theta^{-i}(K)\right]$ should be near the unconditional expectation of the social surplus $\mathrm{E}_{-i}[S_K(\theta)]$ in the Groves mechanism when K is large. That is, since the probability that θ^{-i} lies in $\Theta^{-i}(K)$ can be made arbitrarily close to one by linking sufficiently many problems, the regular conditional expectation is almost exclusively taken over $\Theta^{-i}(K)$. Using the regularity assumptions (R2) and (R3) we show that:

Lemma 3 *Suppose that $\{\mathcal{E}_K\}_{K=1}^{\infty}$ is sequence of economies consisting of stochastically independent regular issues. Then, for every $\varepsilon > 0$ there exists finite $K''(\varepsilon)$ such that for all $K \geq K''(\varepsilon)$,*

$$\frac{\mathrm{E}_{-i}\left[S_K(\theta)\,|\Theta^{-i}(K)\right]}{K} \geq \frac{\mathrm{E}_{-i}\left[S_K(\theta)\right]}{K} - \varepsilon, \textit{ for all } \theta^i \in \Theta^i(K).$$

In the next step, we combine Lemma 2 with Lemma 3, and use the uniform bounds on the per-issue surplus and reservation utilities (R2 and R3) to show that, when K is sufficiently large, it is a best response for agent i with types in $\overline{\Theta}^i(K)$ defined in (33) to play the Groves mechanism if all other agents with types in the set $\overline{\Theta}^{-i}(K)$ defined in (35) decide to do so. In essence, the logic is that since all agents opt in with a probability close to one, it doesn't matter what the remaining agents do.

Lemma 4 *Suppose that $\{\mathcal{E}_K\}_{K=1}^{\infty}$ is sequence of economies consisting of stochastically independent regular issues. Then, there exists finite K^*such that, for all $K \geq K^*$, if $\psi_K^j\left(\theta^j\right) = 1$ for each type $\theta^j \in \overline{\Theta}^j(K)$, $j \neq i$, then it is a best response for agent i to set $\psi_K^i\left(\theta^i\right) = 1$ for each $\theta^i \in \overline{\Theta}^i(K)$.*

Lemma 4 is the crucial step in the argument since it establishes that for K large enough each agent i and every type $\theta^i \in \overline{\Theta}^i(K)$ has an incentive to opt in to play the Groves mechanism, provided they expect every type $\theta^j \in \overline{\Theta}^j(K)$ to do so for every other agent j.

Lemma 5 *Suppose that $\{\mathcal{E}_K\}_{K=1}^{\infty}$ is sequence of economies consisting of stochastically independent regular issues and that, for every K, $\psi_K^i\left(\theta^i\right) = 1$ for each $\theta^i \in \overline{\Theta}^i(K)$. Then, there exists finite K^{**}such that (29) is satisfied for all $K \geq K^{**}$.*

This is more or less obvious since $\Pr\left[\overline{\Theta}(K)\right]$ converges in probability to unity and since the mechanism would run a budget surplus of $\frac{\delta K}{4}$ if all agents would agree to play the Groves mechanism for sure. A proof is in the appendix for completeness.

We have already established that the probability that θ^i belongs to $\overline{\Theta}^i(K)$ approaches one as K goes about of bounds, so the only remaining step is to argue that we can find equilibrium strategies for the remaining types:

Proof of Proposition 3. Consider a (fictitious) game where $m^i = 1$ is the only available action for types in $\overline{\Theta}^i(K)$ and where types in $\Theta^i(K) \backslash \overline{\Theta}^i(K)$ may choose $m^i \in \{0,1\}$, and the interim expected payoffs are given by (31). The action space is finite for each player, so payoffs are equicontinuous in the sense of Milgrom and Weber [16] (see Proposition 1 in Milgrom and Weber [16]). Moreover, stochastic independence implies that the information structure is absolutely continuous (Proposition 3 in Milgrom and Weber [16]). Applying Theorem 1 in Milgrom and Weber the game has an equilibrium in distributional strategies. Since the action space is finite, we can represent this equilibrium as a behavioral strategy $\psi_K^* : \Theta(K) \to [0,1]^n$, where by construction $\psi_K^{i*}\left(\theta^i\right) = 1$ whenever $\theta^i \in \overline{\Theta}^i(K)$. But, applying Lemma 4 it follows that that there exists $K^* < \infty$ such that ψ_K^* is an equilibrium also when types in $\overline{\Theta}^i(K)$ have the option to freely pick $m^i \in \{0,1\}$, which by use of Lemma 5 implies that (29) is satisfied for $K \geq \max\{K^*, K^{**}\}$. Moreover, for any $\varepsilon > 0$ Lemma 2 assures that we may pick $K'(\varepsilon)$ such that such that $\Pr\left[\overline{\Theta}^i(K)\right] \geq 1 - \varepsilon$ for each $K > K'(\varepsilon)$. Finally, Lemma 1 guarantees that truth-telling is optimal provided that each agent announce m^i in accordance with ψ_K^{i*}. The result follows by letting $K^*(\varepsilon) = \max\{K^*, K^{**}, K'(\varepsilon)\}$.
■

7 Discussion

7.1 Weak versus Strong Implementation

The approximate efficiency result in Proposition 3 is in terms of Bayesian implementation, whereas the "almost incentive feasible" Groves mechanism (17) obviously implements the efficient outcome in dominant strategies. Also notice that there are always an equilibria of the perturbed Groves mechanism (28) where the status quo outcome is implemented for sure. If one agent opts out regardless of realized type it is a best response for all other agents to do so as well. We don't think that there is any way to resolve this indeterminacy through additional non-type messages.

7.2 Governments as Linking Mechanisms

Real world government institutions usually fulfil many and arguably quite unrelated functions, and the results in this paper can be seen as a simple theory explaining this, as far as we know, previously unexplained fact. The existing theory of public finance has identified many potential sources of market failure, and interpreted these as a rationale for "government intervention." However, the existing theory only explains that it may be beneficial to set up some institution to deal with each particular market failure, but does not provide a foundation for why a *single* government institution should be responsible for dealing with all these problem. For example, we understand that there are externalities involved in garbage collection, that public parks would be under-provided by voluntary provisions, and that for-profit policing may be a bad idea, but it seems hard to argue that there are any technological reasons for why these services should be provided jointly as part of a local government bundle, as they tend to be. The results in this paper suggest a possible explanation that linking all these seemingly unrelated social decisions via a single government institution helps achieve efficiency by alleviating citizens' participation constraints.

7.3 Vetoing versus Exclusions

Applied to an economy with K public goods, mechanism (28) does not distinguish between excludable and non-excludable public goods. Either all agents opt in and consume all the public goods that are produced, or a veto is cast, in which case none of the public goods is provided. Since the probability of a veto goes to zero as K goes out of bounds, use exclusions are not needed for asymptotic efficiency.

This "asymptotic irrelevance of exclusions" depends crucially on the fact that we keep the number of agents n fixed. In contrast, Norman [18] and Fang and Norman [12] demonstrate that the exclusion instrument is a crucial feature of the constrained optimal mechanism when K is fixed (to 1 and 2 respectively) and n is large.

It should be clear that our proofs don't apply to sequences where both K and n tend to infinity. We don't have a proof, but we conjecture that in the case of non-excludable public goods, asymptotic efficiency is impossible if K and n go out of bounds at the same rate. On the other hand, if goods are excludable, there is no need to equip agents with veto power; reservation utilities can be attained by the "milder" exclusion instrument. As a result, asymptotic efficiency is attainable in this case (regardless of the asymptotic behavior of K/n). Indeed, since the efficient provision for good k converges either to "always provide" or "never provide" as n tends to infinity approximate efficiency can be implemented with a mechanism where the provision decisions are done ex ante and a fixed price is charged for access to all public goods, much in the same spirit as Armstrong's [1] two-part tariff scheme.

References

[1] Armstrong, Mark, "Price Discrimination by a Many-Product Firm," *Review of Economic Studies*, 90(1), January 1999, 151-168.

[2] Arrow, Kenneth, "The Property Rights Doctrine and Demand Revelation Under Incomplete Information," in *Economics and Human Welfare*, Academic Press, 1979.

[3] Bakos, Yannis and Erik Brynjolfsson, "Bundling Information Goods: Pricing, Profits and Efficiency," *Management Science*, 45(12), December 1999, 1613-1630.

[4] Bergstrom, Carl and Ted Bergstrom, "The Costs and Benefits of Library Site Licenses to Academic Journals," *Proceedings of the National Academy of Sciences*, 101(3), January 2004, 897-902.

[5] Borgers, Tilman and Peter Norman, "Ex-Ante and Ex-Post Budget-Balanced Mechanisms When Signals are Stochastically Independent," *unpublished manuscript*, University College London and University of British Columbia, February 2005.

[6] Borgers, Tilman and Peter Postl, "Efficient Compromising,", *unpublished manuscript,* University College London and Birmingham, June 2004.

[7] Casella, Alessandra, "Storable Votes," *unpublished manuscript,* Columbia University, October 2003.

[8] Clarke, Edward H., "Multipart Pricing of Public Goods", *Public Choice* 8, 1971, 19-33.

[9] Cramton, Peter C., Robert H. Gibbons and Paul Klemperer, "Dissolving a Partnership Efficiently", *Econometrica* 55, 1987, 615-632.

[10] Cremer, Jacques and Richard P. McLean, "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions", *Econometrica,* 56, 1988, 1247-57.

[11] D'Aspremont, Claude and Louis-Andre' Gérard-Varet, "Incentives and Incomplete Information", *Journal of Public Economics* 11, 1979, 25-45.

[12] Fang, Hanming and Peter Norman, "An Efficiency Rationale for Bundling of Public Goods" , *Cowles Foundation Discussion Papers*, 1441, November 2004.

[13] Groves, Theodore, "Incentives in Teams," *Econometrica*, 41, 1973, 617-631.

[14] Jackson, Matt and Hugo Sonnenschein, "Overcoming Incentive Constraints by Linking Decisions," forthcoming, 2005, *Econometrica.*

[15] Mailath, George and Andrew Postlewaite, "Asymmetric Information Bargaining Problems with Many Agents", *Review of Economic Studies* 57, 1990, 351-368.

[16] Milgrom, Paul and Robert Weber, "Distributional Strategies for Games with Incomplete Information," *Mathematics of Operations Research,* 10, 1985.

[17] Myerson, Roger and Mark Satterthwaite, "Efficient Mechanisms for Bilateral Trading", *Journal of Economic Theory* 29, 1983, 265-281.

[18] Norman, Peter, "Efficient Mechanisms for Public Goods with Use Exclusion." *Review of Economic Studies*, 71, 2004, 1163-1188.

[19] Townsend, Robert, "Optimal Multi-Period Contracts and the Gain From Enduring Relationships Under Private Information," *Journal of Political Economy* 90, 1166-1186.

A Appendix

A.1 Proof of Lemma 1

Proof. Assuming that all other agents announce truthfully and that $m = (1, ..., 1) = \mathbf{1}$ is announced, the ex post payoff for agent i of type θ^i from announcement $\widehat{\theta}^i$ is

$$u^i\left(\widehat{\theta}^i, \mathbf{1}, \theta\right) = V_K^i\left(x_K^*\left(\widehat{\theta}^i, \theta^{-i}\right), \theta^i\right) + \sum_{j \neq i} V_K^j\left(x_K^*\left(\widehat{\theta}^i, \theta^{-i}\right), \theta^j\right) + T_K^i,$$

where T_K^i is defined in (32). By construction, $x_K^*(\cdot)$ is maximized at $\widehat{\theta}^i = \theta^i$, resulting in ex post payoff $u^i\left(\theta^i, \mathbf{1}, \theta\right) = S_K(\theta) - T_K^i$. For $m \neq \mathbf{1}$ we have that $u^i\left(\widehat{\theta}^i, m, \theta\right) = R_K^i\left(\theta^i\right)$ for any $\widehat{\theta}^i$. Taking expectations over θ^{-i} gives the result. ∎

A.2 Proof of Lemma 2

Proof. Using the expression for $U_K^j\left(\theta^j\right) - R_K^j\left(\theta^j\right)$ in equation (23), we can proceed just like in that proof, with the only difference being that now the calculation is for a bound on the probability that the payoff is less than half of the lower bound on the expected value of the payoff. That is,

$$\begin{aligned}
1 - \Pr\left[\overline{\Theta}^j(K)\right] &= \Pr\left[U_K^j\left(\theta^j\right) - R_K^j\left(\theta^j\right) < \frac{\delta K}{2n}\right] \\
&= \Pr\left[\sum_{k=1}^{K}\left(\phi_k^j\left(\theta_k^j\right) - \mathrm{E}\left[\phi_k^j\left(\theta_k^j\right)\right] + \frac{\mathrm{E}\left[s_k\left(\theta_k\right)\right] - \mathrm{E}\left[\sum_{j=1}^{n} r_k^j\left(\theta^j\right)\right]}{n}\right) < \frac{\delta K}{2n}\right] \\
\text{/By inequality in (24)/} \quad &\leq \Pr\left[\sum_{k=1}^{K}\left(\phi_k^j\left(\theta_k^j\right) - \mathrm{E}\left[\phi_k^j\left(\theta_k^j\right)\right]\right) < -\frac{\delta K}{2n}\right] \\
&\leq \Pr\left[\left|\sum_{k=1}^{K}\left(\phi_k^j\left(\theta_k^j\right) - \mathrm{E}\left[\phi_k^j\left(\theta_k^j\right)\right]\right)\right| > \frac{\delta K}{2n}\right] \\
\text{/Chebyshev's inequality/} \quad &\leq \left(\frac{1}{\frac{\delta K}{2n}}\right)^2 \mathrm{Var}\left[\sum_{k=1}^{K}\phi_k^j\left(\theta_k^j\right)\right] \leq \frac{4n^2\sigma^2}{\delta^2 K}
\end{aligned}$$

Since $\frac{n^2\sigma^2}{\delta^2 K} \to 0$ as $K \to \infty$, we conclude that, for every $\varepsilon > 0$, there exists some finite integer $K_j'(\varepsilon)$ such that for every $K \geq K_j'(\varepsilon)$,

$$1 - \Pr\left[\overline{\Theta}^j(K,\delta)\right] \leq 1 - (1-\varepsilon)^{\frac{1}{n-1}}.$$

Hence, by letting $K'(\varepsilon) = \max_{j\neq i} K_j'(\varepsilon)$ and using stochastic independence we have that

$$\Pr\left[\overline{\Theta}^{-i}(K)\right] = \times_{j\neq i}\Pr\left[\overline{\Theta}^j(K,\delta)\right] \geq 1-\varepsilon$$

for every $K \geq K'(\varepsilon)$. ■

A.3 Proof of Lemma 3

Proof. By regularity assumption (R2), there is a uniform bound $\bar{a} > 0$ so that $s_k(\theta_k) \leq \bar{a}$ for every θ_k, which in turn implies that $S_K(\theta)/K \leq \bar{a}$ for every K and $\theta \in \Theta(K)$. Thus

$$\begin{aligned}
\frac{\mathrm{E}_{-i}\left[S_K(\theta)\right]}{K} &= \Pr\left[\overline{\Theta}^{-i}(K)\right]\frac{\mathrm{E}_{-i}\left[S_K(\theta)\,|\overline{\Theta}^{-i}(K)\right]}{K} \\
&\quad + \left(1 - \Pr\left[\overline{\Theta}^{-i}(K)\right]\right)\frac{\mathrm{E}_{-i}\left[S_K(\theta)\,|\theta^{-i} \notin \overline{\Theta}^{-i}(K)\right]}{K} \\
&\leq \frac{\mathrm{E}^{-i}\left[S_K(\theta)\,|\overline{\Theta}^{-i}(K)\right]}{K} + \left(1 - \Pr\left[\overline{\Theta}^{-i}(K)\right]\right)\bar{a}
\end{aligned}$$

Fix $\varepsilon > 0$. By Lemma 2 there exists $K'\left(\frac{\varepsilon}{\bar{a}}\right)$ such that $\Pr\left[\overline{\Theta}^{-i}(K)\right] \geq 1 - \frac{\varepsilon}{\bar{a}}$ for every $K \geq K'\left(\frac{\varepsilon}{\bar{a}}\right)$, implying that

$$\frac{\mathrm{E}_{-i}\left[S_K(\theta)\,|\overline{\Theta}^{-i}(K)\right]}{K} \geq \frac{\mathrm{E}^{-i}\left[S_K(\theta)\right]}{K} - \left[1 - \left(1 - \frac{\varepsilon}{\bar{a}}\right)\right]\bar{a} = \frac{\mathrm{E}_{-i}\left[S_K(\theta)\right]}{K} - \varepsilon,$$

which gives the result for $K''(\varepsilon) = K'\left(\frac{\varepsilon}{\bar{a}}\right)$. ∎

A.4 Proof of Lemma 4

Proof. Assuming that $\psi_K^j\left(\theta^j\right) = 1$ for all $j \neq i$ and all $\theta^j \in \overline{\Theta}^j(K)$, the interim expected payoff for agent i from setting $m^i = 1$ as defined in (31) can be written as

$$\begin{aligned}
\widehat{U}^i\left(1, \psi_K^{-i}, \theta^i\right) &= \int_{\theta^{-i}} \left(\left[S_K(\theta) + T_K^i\right]\Psi_K^{-i}\left(\theta^{-i}\right) + R_K^i\left(\theta^i\right)\left[1 - \Psi_K^{-i}\left(\theta^{-i}\right)\right]\right) dF_K^{-i}\left(\theta^{-i}\right) \\
&= \Pr\left[\overline{\Theta}^{-i}(K)\right]\left(\mathrm{E}_{-i}\left[S_K(\theta)\,|\overline{\Theta}^{-i}(K)\right] + T_K^i\right) \qquad \text{(A1)} \\
&\quad + \int_{\theta^{-i} \in \Theta^{-i}(K) \setminus \overline{\Theta}^{-i}(K)} \left(\left[S_K(\theta) + T_K^i\right]\Psi_K^{-i}\left(\theta^{-i}\right) + R_K^i\left(\theta^i\right)\left[1 - \Psi_K^{-i}\left(\theta^{-i}\right)\right]\right) dF_K^{-i}\left(\theta^{-i}\right).
\end{aligned}$$

Since $s_k(\theta_k)$ is uniformly bounded above by $\bar{a}$ (by R2) and $r_k^i\left(\theta_k^i\right)$ is uniformly bounded below by $\underline{b}$ (by R3), we can bound T_K^i as defined in (32) as follows:

$$\begin{aligned}
T_K^i &\equiv -\frac{n-1}{n}\mathrm{E}\left[S_K(\theta)\right] + \mathrm{E}\left[R_K^i\left(\theta^i\right)\right] - \frac{1}{n}\mathrm{E}\left[\sum_{j=1}^n R_K^j\left(\theta^j\right)\right] - \frac{\delta K}{4n} \\
\Big/ S_K(\theta) \geq \sum_{j=1}^n R_K^j\left(\theta^j\right) \Big/ &\geq -\mathrm{E}\left[S_K(\theta)\right] + \mathrm{E}\left[R_K^i\left(\theta^i\right)\right] - \frac{\delta K}{4n} \geq \left(\underline{b} - \bar{a} - \frac{\delta}{4n}\right)K.
\end{aligned}$$

Moreover $s_k(\theta_k)$ is uniformly bounded below by $\underline{a}$ (by R2), so $S_K(\theta) \geq \underline{a}K$. We conclude that

$$\begin{aligned}
S_K(\theta) + T_K^i &\geq \left(\underline{a} + \underline{b} - \bar{a} - \frac{\delta}{4n}\right)K \\
R_K^i\left(\theta^i\right) &\geq \underline{b}K > \left(\underline{a} + \underline{b} - \bar{a} - \frac{\delta}{4n}\right)K,
\end{aligned}$$

where the strict inequality follows since $\underline{a} < \bar{a}$. Hence,

$$\begin{aligned}
&\int_{\theta^{-i} \in \Theta^{-i}(K) \setminus \overline{\Theta}^{-i}(K)} \left(\left[S_K(\theta) + T_K^i\right]\Psi_K^{-i}\left(\theta^{-i}\right) + R_K^i\left(\theta^i\right)\left[1 - \Psi_K^{-i}\left(\theta^{-i}\right)\right]\right) dF_K^{-i}\left(\theta^{-i}\right) \\
\geq\ & K\left(\underline{a} + \underline{b} - \bar{a} - \frac{\delta}{4n}\right)\left(1 - \Pr\left[\overline{\Theta}^{-i}(K)\right]\right). \qquad \text{(A2)}
\end{aligned}$$

Combining (A1) and (A2) we obtain

$$\begin{aligned}
\widehat{U}^i\left(1, \psi^{-i}, \theta^i\right) &\geq \Pr\left[\overline{\Theta}^{-i}(K)\right]\left(\mathrm{E}_{-i}\left[S_K(\theta)\,|\overline{\Theta}^{-i}(K)\right] + T_K^i\right) \\
&\quad + \left(1 - \Pr\left[\overline{\Theta}^{-i}(K)\right]\right)K\left(\underline{a} + \underline{b} - \bar{a} - \frac{\delta}{4n}\right).
\end{aligned}$$

By Lemma 3, there exists K_1 so that for each $K \geq K_1$,

$$\frac{\mathrm{E}_{-i}\left[S_K(\theta)\,|\overline{\Theta}^{-i}(K)\right]}{K} \geq \frac{\mathrm{E}_{-i}\left[S_K(\theta)\right]}{K} - \frac{\delta}{8n},$$

where $\delta > 0$ is the uniform bound of the difference between the maximized expected surplus and the sum of the participation utilities (see Definition 2). Hence,

$$\begin{aligned}\frac{\widehat{U}^i\left(1,\psi^{-i},\theta^i\right)}{K} \quad &\geq \quad \Pr\left[\overline{\Theta}^{-i}(K)\right]\left(\frac{\mathrm{E}_{-i}S_K(\theta)+T_K^i}{K} - \frac{\delta}{8n}\right) \\ &\quad + \left(1-\Pr\left[\overline{\Theta}^{-i}(K)\right]\right)\left(\underline{a}+\underline{b}-\bar{a}-\frac{\delta}{4n}\right).\end{aligned}$$

for every $K \geq K_1$. By (19), we know that $\mathrm{E}_{-i}\left[S_K(\theta)\right]+T_K^i+\frac{\delta}{4n}$ is the interim expected payoff for type θ^i in the Groves mechanism. Since $\theta^i \in \overline{\Theta}^i(K)$ it follows from definition (33) that $\mathrm{E}_{-i}\left[S_K(\theta)\right]+T_K^i+\frac{\delta}{4n} \geq R_K^i\left(\theta^i\right)+\frac{\delta K}{2n}$. Hence

$$\begin{aligned}\frac{\widehat{U}^i\left(1,\psi^{-i},\theta^i\right)}{K} \quad &\geq \quad \Pr\left[\overline{\Theta}^{-i}(K)\right]\left(\frac{R_K^i\left(\theta^i\right)+\frac{\delta K}{4n}}{K} - \frac{\delta}{8n}\right) \\ &\quad + \left(1-\Pr\left[\overline{\Theta}^{-i}(K)\right]\right)\left(\underline{a}+\underline{b}-\bar{a}\right),\end{aligned}$$

or

$$\frac{\widehat{U}^i(1,\psi^{-i},\theta^i)-R_K^i\left(\theta^i\right)}{K} \geq \Pr\left[\overline{\Theta}^{-i}(K)\right]\left(\frac{\delta}{8n}\right)+\left(1-\Pr\left[\overline{\Theta}^{-i}(K)\right]\right)\left(\underline{a}+\underline{b}-\bar{a}\right),$$

The right hand side converges to $\delta/(8n)$ as $\Pr\left[\overline{\Theta}^{-i}(K)\right]$ approaches one, so there exists $\varepsilon_2 > 0$ such that $\widehat{U}^i(1,\psi^{-i},\theta^i)-R_K^i\left(\theta^i\right) \geq 0$ if $\Pr\left[\overline{\Theta}^{-i}(K)\right] \geq 1-\varepsilon_2$. Lemma 2 assures that there exists K_2 such that $\Pr\left[\overline{\Theta}^{-i}(K)\right] \geq 1-\varepsilon_2$ for every $K \geq K_2$. For $K^* = \max\{K_1,K_2\}$ it therefore follows that $\widehat{U}^i(1,\psi^{-i},\theta^i)-R_K^i\left(\theta^i\right) \geq 0$. ∎

A.5 Proof of Lemma 5

Let $\Psi_K(\theta) \equiv \prod_{i=1}^n \psi_K^i\left(\theta^i\right)$ denote the probability that $m = (1,....,1)$ given type profile $\theta \in \Theta(K)$. The expected budget tax revenues can then be expressed as

$$\begin{aligned}&\mathrm{E}\Psi_K(\theta)\left[\sum_{i=1}^n \widehat{t}_K^i(\theta,\mathbf{1})\right] \\ =\; &\mathrm{E}\Psi_K(\theta)\left[\sum_{i=1}^n V_K^i\left(x_K^*(\theta),\theta^i\right)-S_K(\theta)+\frac{n-1}{n}\mathrm{E}\left[S_K(\theta)\right]-\mathrm{E}\left[R_K^i\left(\theta^i\right)\right]+\frac{1}{n}\mathrm{E}\left[\sum_{j=1}^n R_K^j\left(\theta^i\right)\right]+\frac{\delta K}{4n}\right] \\ =\; &\mathrm{E}\Psi_K(\theta)\left[(n-1)\left(\mathrm{E}\left[S_K(\theta)\right]-S_K(\theta)\right)+\frac{\delta K}{4}+\mathcal{C}_K\left(\widehat{x}(\theta,m)\right)\right]\end{aligned}$$

By assumption, $\Psi_K(\theta) = 1$ when $\theta \in \overline{\Theta}(K)$, so the budget surplus/deficit satisfies

$$
\begin{aligned}
& \mathrm{E}\Psi_K(\theta)\left[\sum_{i=1}^{n} \widehat{t}_K^i(\theta, \mathbf{1}) - \mathcal{C}_K(\widehat{x}(\theta, m))\right] \\
= \quad & (n-1)\int_{\theta \in \overline{\Theta}(K)} \left[(\mathrm{E}[S_K(\theta)] - S_K(\theta)) + \frac{\delta K}{4}\right] dF_K(\theta) \\
& + (n-1)\int_{\theta \notin \overline{\Theta}(K)} \Psi_K(\theta)\left[(\mathrm{E}[S_K(\theta)] - S_K(\theta)) + \frac{\delta K}{4}\right] dF_K(\theta),
\end{aligned}
$$

But, $\mathrm{E}[S_K(\theta)] = \int_{\theta \in \overline{\Theta}(K)} S_K(\theta)\, dF(\theta) + \int_{\theta \notin \overline{\Theta}(K)} S_K(\theta)\, dF(\theta)$, so we may rearrange the expression above as

$$
\begin{aligned}
& \frac{\mathrm{E}\Psi_K(\theta)\left[\sum_{i=1}^{n} \widehat{t}_K^i(\theta, \mathbf{1}) - \mathcal{C}_K(\widehat{x}(\theta, m))\right]}{n-1} \\
= \quad & \Pr\left[\overline{\Theta}(K)\right]\left[\mathrm{E}[S_K(\theta)] + \frac{\delta K}{4}\right] - \mathrm{E}[S_K(\theta)] + \int_{\theta \notin \overline{\Theta}(K)} S_K(\theta)\, dF_K(\theta) \\
& + \int_{\theta \notin \overline{\Theta}(K)} \Psi_K(\theta)\left[(\mathrm{E}[S_K(\theta)] - S_K(\theta)) + \frac{\delta K}{4}\right] dF_K(\theta) \\
= \quad & \Pr\left[\overline{\Theta}(K)\right]\frac{\delta K}{4n} - \left[1 - \Pr\left[\overline{\Theta}(K)\right]\right]\mathrm{E}[S_K(\theta)] \\
& + \int_{\theta \notin \overline{\Theta}(K)} \left[\Psi_K(\theta)\left(\mathrm{E}[S_K(\theta)] + \frac{\delta K}{4n}\right) + (1 - \Psi_K(\theta))\, S_K(\theta)\right] dF_K(\theta) \\
> \quad & \Pr\left[\overline{\Theta}(K)\right]\frac{\delta K}{4n} + \left(1 - \Pr\left[\overline{\Theta}(K)\right]\right)(\underline{a} - \overline{a})\, K,
\end{aligned}
$$

where the inequality follows since $\mathrm{E}[S_K(\theta)] + \frac{\delta K}{4} > \underline{a}K$, $S_K(\theta) > \underline{a}K$, and $\mathrm{E}[S_K(\theta)] < \overline{a}K$ by assumption [R2]. Since $\Pr\left[\overline{\Theta}(K)\right] \to 1$ as $K \to \infty$ it follows that there exists some finite K^{**} such that the expected surplus is positive for any $K \geq K^{**}$. ■