EX POST IMPLEMENTATION

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Abstract

We analyze the problem of fully implementing a social choice set in ex post equilibrium. We identify an *ex post monotonicity* condition that is necessary and - in economic environments - sufficient for full implementation in ex post equilibrium. We also identify an ex post monotonicity no veto condition that is sufficient.

Ex post monotonicity is satisfied in all single crossing environments with *strict* ex post incentive constraints. In many economically significant environments, ex post implementation can be achieved in the direct mechanism.

We show by means of two classic examples that ex post monotonicity does not imply nor is it implied by Maskin monotonicity (necessary and almost sufficient for complete information implementation). The single unit auction with interdependent valuations is shown to satisfy ex post monotonicity but not Maskin monotonicity. Ex post implementation in the direct mechanism is also possible in this case. We describe an example where the Pareto correspondence fails ex post monotonicity but Maskin monotonicity is satisfied.

KEYWORDS: Ex Post Equilibrium, Implementation, Single Crossing, Interdependent Values.
JEL CLASSIFICATION: C79, D82

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1 Introduction

Recent research in auction theory, and mechanism design theory more generally, has lead to a better understanding of models with interdependent rather than private values. Much of this work has used the solution concept of ex post rather than Bayesian equilibrium.[1] The analysis of ex post equilibrium is considerably more tractable, because incentive compatible transfers can frequently be derived with ease and single crossing conditions generating incentive compatibility are easy to identify. A conceptual advantage of ex post equilibrium is its robustness to the informational assumptions about the environment. In particular, it often seems unrealistic to allow the mechanism to depend on the designer's knowledge of the type space as Bayesian mechanisms do.[2]

Research on interdependent values has focussed almost exclusively on the incentive compatibility of the social choice problem in the direct mechanism. In contrast, this paper focuses on the problem of *full* rather than *partial* implementation. The task for the designer, who does not know the agents' types, is to choose a mechanism such that in *every* equilibrium of the mechanism, agents' play of the game results in the outcome specified by the social choice objective at every type profile. If the social choice problem is described by a social choice set, a set of social choice functions, then full implementation also requires that every selection from the set can be realized as an ex post equilibrium under the mechanism. This problem has been analyzed under the assumption of complete information, i.e., there is common knowledge among the agents of their types (see Maskin (1999)). It has also been analyzed under the assumption of incomplete information, on the assumption that there is common knowledge among the agents of the prior (or the priors) according to which agents form their beliefs (see Postlewaite & Schmeidler (1986), Palfrey & Srivastava (1989*a*) and Jackson (1991)). While complete information (or Nash) implementation and incomplete information (or Bayesian) implementation are well understood, the ex post implementation problem has not been analyzed. In this paper, we develop necessary and sufficient conditions for ex post implementation, both in general environments and also in settings of special interest for auction theory.

A strategy profile in an incomplete information game is an ex post equilibrium if each action profile is a Nash equilibrium at every type profile. Put differently, each player's incomplete information strategy mapping types to messages must remain a best response even if he knew the

[1] See Dasgupta & Maskin (2000), Jehiel & Moldovanu (2001), Perry & Reny (2002) and Bergemann & Valimaki (2002) among many others.

[2] In an earlier contribution, Bergemann & Morris (2004), we pursued this argument formally in the context of truthful (or partial) implementation. We showed that in many environments a social choice problem can be truthfully Bayesian implemented for all priors if and only if it can be truthfully ex post implemented. In this paper, we take the solution concept of ex post equilibrium as given.

types of his opponents. We introduce an ex post monotonicity condition that - along with ex post incentive compatibility - is necessary for ex post implementation. We show that a slight strengthening of ex post monotonicity - the ex post monotonicity no veto condition - is sufficient for implementation with at least three agents. The latter condition reduces to ex post monotonicity in economic environments. These results are the ex post analogues of the Bayesian implementation results of Jackson (1991), and we employ similar arguments to establish our results. But just as ex post incentive compatibility conditions are easier to verify and interpret, the ex post monotonicity condition is easier to verify and interpret than the Bayesian monotonicity condition, because it depends on complete information utilities and do not involve the prior or posterior distributions of the agents.

Because an ex post equilibrium is a Nash equilibrium at every type profile, there is a natural relationship between ex post and Nash implementation. When we compare the *complete* with the *incomplete* information settings, two important differences regarding the ability of the agents to sustain equilibrium behavior emerges. On the one hand, with complete information, the agents have the ability to coordinate their actions at every preference profile. This makes the designer's problem harder. On the other hand, with complete information the designer can detect individual deviations from the reports of the other agents. This makes the designer's problem easier. The ability of the agents to coordinate in complete information settings makes the task of implementing the social choice outcome more difficult for the designer, but it is made easier by the lack of individual incentive constraints. With incomplete information, the first problem becomes easier, but the second becomes harder. As these two effects are in conflict, we will show that ex post and Maskin monotonicity are not nested notions. In particular, either one of them can hold while the other one can fail. Interestingly, in the class of single crossing environments, ex post monotonicity is always guaranteed as is Maskin monotonicity. Even though ex post monotonicity has to include ex post incentive constraints absent in the complete information world, it turns out that the local property of single crossing indifference curves is sufficient to guarantee ex post monotonicity in the presence of strict rather than weak ex post incentive constraints.

The "augmented" mechanisms used to obtained our general positive ex post implementation results inherit some complex and unsatisfactory features from their complete information and Bayesian counterparts. The hope often expressed in the literature is that it should be possible to show in specific settings that less complex mechanisms are required. We are able to identify a number of important settings where ex post implementation is only possible when it is possible in the direct mechanism. This is true, for example, if the social choice function has a sufficiently wide range or if the environment is supermodular.

We also use the methods developed for the general case to show that the direct mechanism has a unique ex post equilibrium in the problem of efficiently allocating goods when bidders have independent values (see Dasgupta & Maskin (2000) and Perry & Reny (2002)). And in this context, the interdependent value model delivers new and positive results. With at least three symmetric bidders, we show that the generalized Vickrey-Clark-Groves (VCG) allocation can be ex post implemented in the direct mechanism, even though Maskin monotonicity fails. This result is in stark contrast with the impossibility to Nash implement the single unit auction with private values. The positive result relies on interdependence. The latter intuition is also confirmed by contrasting our positive results with a recent result by Birulin (2003). He shows that with two bidders there are multiple and inefficient ex post equilibria in the single unit auction. With two agents, bidder i can use a non-truthful bidding strategy to exactly offset a non-truthful bidding strategy of bidder j. With more than two agents, the strategy of bidder i cannot incorporate anymore the bidding behavior by j and k and truthtelling becomes the unique ex post equilibrium strategy.

For twenty years from the mid-1970s to the mid-1990s, there was a large literature devoted to the problem of full implementation. While elegant characterizations of implementability were developed, the "augmented" mechanisms required to achieve positive results were complex and seemed particularly implausible. While the possibility of multiple equilibria does seem to be a relevant one in practical mechanism design problems, particularly in the form of collusion and shill bidding, the theoretical literature is not seen as having developed practical insights (with a few recent exceptions such as Ausubel & Milgrom (2005) and Yokoo, Sakurai & Matsubara (2004)). For some reason, the gap between pure implementation theory and practical market design has appeared especially stark when thinking about full implementation. Following Wilson (1987), we hope that by relaxing unrealistic implicit common knowledge assumptions, we will deliver predictions that are more robust and practical. While the complete information implementation literature makes the assumption of common knowledge of preferences, the Bayesian implementation literature makes the assumption that there is common knowledge of a prior on a fixed set of types; this both seems unlikely to practical market designers and is a substantive constraint when viewed as a restriction on all possible beliefs and higher order beliefs.[3] Our hope is partially vindicated by the results of this paper: it does turn out that in many environments of interest, augmented mechanisms cannot deliver ex post implementation when direct mechanisms cannot. Of course, direct mechanisms also have their own robustness critiques, but a number of important papers have shown that, in the type of interdependent environments discussed in this paper, direct mechanisms can be replicated by more plausible auction mechanisms.[4]

[3] The common knowledge of a common prior assumption has particularly strong consequences in mechanism design, as argued by Neeman (2004) and Bergemann & Morris (2004).

[4] See Dasgupta & Maskin (2000) and Perry & Reny (2002).

In this paper, we take the solution concept of ex post equilibrium as given. In a companion paper, Bergemann & Morris (2005), we address the problem of "robust implementation". There we analyze the problem of Bayesian implementation under the assumption that the designer knows nothing about what the agents know or believe about the other agents' types, their payoffs or their high-order beliefs. While the incentive compatibility constraints for this problem are the same as for the ex post implementation problem,[5] the resulting "robust monotonicity" condition (equivalent to Bayesian monotonicity on all type spaces) is strictly stronger then ex post monotonicity (and Maskin monotonicity). The resulting robust monotonicity notions provide the full implementation counterparts to the robust mechanism design (i.e. partial implementation) questions pursued in Bergemann & Morris (2004).

The paper is organized as follows. Section 2 describes the formal environment and solution concepts. Section 3 introduces the notion of ex post monotonicity and compares it to Maskin monotonicity in a simple public good example. Section 4 shows that ex post monotonicity is necessary and, in economic environments, also sufficient for ex post implementation. We also provide a sufficient condition - ex post monotonicity no veto - for non-economic environments. Section 5 considers an important class of single crossing environments; we show that ex post monotonicity is satisfied in all single crossing environments if the social allocation problem satisfies strict rather than weak ex post incentive constraints. Section 6 provides sufficient conditions under which ex post implementation is possible in the direct mechanism. Section 7 considers the single unit auction environment. It is an important example as it fails Maskin monotonicity and has weak ex post incentive constraints almost everywhere; yet it satisfies ex post monotonicity and ex post implementation is possible in the direct mechanism. Section 8 presents a Pareto social choice set with the converse implementation properties. It satisfies Maskin monotonicity but fails ex post monotonicity. Section 9 extends the analysis to mixed strategy implementation and the use of stochastic mechanisms. Section 10 concludes.

2 Model

We fix a finite set of agents, $1, 2, ..., I$. Agent i's *type* is $\theta_i \in \Theta_i$. We write $\theta \in \Theta = \Theta_1 \times ... \times \Theta_I$. There is a set of outcomes Y. Each agent has utility function $u_i : Y \times \Theta \to \mathbb{R}$. Thus we are in the world of interdependent values, where an agent's utility may depend on other agents' types.[6] A

[5] This follows from results in Bergemann & Morris (2004).

[6] We represent the preferences of the agents by utility functions rather than by preference relations as much of the mechanism design literature with interdependent values uses utility functions rather than preferences. However, all our results (with the exception of the mixed strategy implementation results in Section 9) only rely on ordinal properties of the preferences and all results could be restated in terms of preferences rather than utility functions.

social choice function is a function from states to allocations, or $f : \Theta \to Y$. The set of all social choice functions is $\mathcal{F} = \{f \,|f : \Theta \to Y\}$. A *social choice set* F is a subset of $\mathcal{F}$.[7]

In the tradition of the implementation literature, we describe the implementation problem here for deterministic mechanisms and pure strategies. In Section 9 we extend the analysis to implementation to stochastic mechanisms and mixed strategies. We postpone the relevant modifications to accommodate mixed strategies until then.

A planner must choose a *game form* or *mechanism* for the agents to play in order to determine the social outcome. Let m_i be a message of agent i, M_i be the set of messages available to i and a message profile is denoted by $m = (m_1, m_2, ..., m_I) \in M = \times_{i=1}^{I} M_i$. Let $g : M \to Y$ be the outcome function and $g(m) = y$ be a specific outcome if message profile m is chosen. Thus a mechanism is a collection:

$$\mathcal{M} = (M_1, ..., M_I, g(\cdot)) .$$

Until Section 9, we make no additional assumptions on the structure of the type space Θ, the outcome space Y, or the message space M.

For the given environment, we can combine the type space Θ with a mechanism $\mathcal{M}$ to get an incomplete information game $(\Theta, \mathcal{M})$. We wish to analyze the ex post equilibria of the incomplete information game $(\Theta, \mathcal{M})$. A pure strategy in this game is a function $s_i : \Theta_i \to M_i$.

Definition 1 (Ex post equilibrium)
A pure strategy profile $s^* = (s_1^*, ..., s_I^*)$ is an ex post equilibrium if

$$u_i (g (s^* (\theta)), \theta) \geq u_i \left(g \left(m_i, s_{-i}^* (\theta_{-i})\right), \theta\right),$$

for all i, θ and m_i.

An ex post equilibrium is a Nash equilibrium for every type profile θ. We observe that the notion of an ex post equilibrium does not refer to prior or posterior probability distributions of the types as the Bayes-Nash equilibrium does. The ex post equilibrium has an ex post no regret property in the incomplete information game, as no agent would like to change his message even if he were to know the true type profile of the remaining agents.[8]

This is made precise in Section 4.

[7] In the literature on complete information implementation, it is customary to use social choice correspondences (see Maskin (1999)) whereas in the literature on incomplete information implementation (see Postlewaite & Schmeidler (1986) and Jackson (1991)) it is customary to use social choice sets. We shall discuss some of the issues regarding ex post implementation of functions, sets and correspondences in Section 8 in conjunction with the Pareto correspondence.

[8] Ex post incentive compatibility was discussed as "uniform incentive compatibility" by Holmstrom and Myerson (1983). Ex post equilibrium is increasingly studied in game theory (see Kalai (2002)) and is often used in mechanism design as a more robust solution concept (see Cremer and McLean (1985) and the references in footnote 1).

In an environment with private values, the notion of ex post incentive compatibility is equivalent to the notion of dominant strategy incentive compatibility. If in addition one could guarantee *strict* dominant strategy incentive compatibility, then full implementation can be achieved by fiat. The importance of the distinction between weak and strict incentive compatibility for implementation will be discussed in detail in the context of the single unit auction in Section 7. Results about the private value special case and dominant strategy incentive compatibility are collected in the Appendix.

Definition 2 (Ex Post Implementation)
Social choice set F is ex post implementable (in pure strategies) if there exists a mechanism $\mathcal{M}$ such that:

1. *for every $f \in F$, there exists an ex post equilibrium s^* of the game $\mathcal{M}$ that satisfies:*

$$g\left(s^*\left(\theta\right)\right) = f\left(\theta\right), \forall \theta \in \Theta;$$

2. *for every ex post equilibrium s^* of the game there exists $f \in F$ such that:*

$$g\left(s^*\left(\theta\right)\right) = f\left(\theta\right), \ \forall \theta \in \Theta.$$

Implementation then requires that the equilibria of the mechanism exactly coincide with the given social choice set. The notion of implementation defined above is sometimes referred to as "full" implementation (see Dasgupta, Hammond & Maskin (1979), Maskin (1999) and Postlewaite & Schmeidler (1986)).

3 Monotonicity

3.1 Ex Post Monotonicity

Implementation is meant to address the problem that privately informed agents may consistently misrepresent their information and jointly establish equilibrium behavior which fails to realize the social choice objective of the planner. The notion of ex post monotonicity is easiest to grasp by considering the direct revelation game. If we were just interested in partially implementing F - i.e., constructing a mechanism with an ex post equilibrium achieving a selection $f \in F$ - then by the revelation principle we could restrict attention to the direct mechanism and a necessary and sufficient condition is the following ex post incentive compatibility condition.

Definition 3 (Ex Post Incentive Compatibility)
F is ex post incentive compatible $(EPIC)$ if for every $f \in F$:

$$u_i\left(f\left(\theta\right),\theta\right) \geq u_i\left(f\left(\theta_i',\theta_{-i}\right),\theta\right),$$

for all i, θ and θ_i'.

In the direct mechanism, a misrepresentation by an agent is a non-truthtelling strategy. As such it is an attempt by the agent to deceive the designer and we refer to such a misrepresentation as a deception α_i by agent i:

$$\alpha_i : \Theta_i \to \Theta_i.$$

The deception α_i represents i's reported type as a function of his true type. The entire profile of deceptions is denoted by:

$$\alpha\left(\theta\right) = \left(\alpha_1\left(\theta_1\right), ..., \alpha_I\left(\theta_I\right)\right).$$

In the direct mechanism, if agents report the deception $\alpha\left(\theta\right)$ rather than truthfully report θ, then the resulting social outcome is given by $f\left(\alpha\left(\theta\right)\right)$ rather than $f\left(\theta\right)$. We write $f \circ \alpha\left(\theta\right) \triangleq f\left(\alpha\left(\theta\right)\right)$. The notion of ex post monotonicity guarantees that there exists a whistle-blower (among the agents) who (i) will alert the designer of deceptive behavior α by receiving a reward for his alert; and (ii) will not falsely report a deception in a truth-telling equilibrium.

Definition 4 (Ex Post Monotonicity)
Social choice set F satisfies ex post monotonicity (EM) if for every $f \in F$ and deception α with $f \circ \alpha \notin F$, there exists i, θ and $y \in Y$ such that

$$u_i\left(y,\theta\right) > u_i\left(f\left(\alpha\left(\theta\right)\right),\theta\right), \tag{1}$$

while

$$u_i\left(f\left(\theta_i',\alpha_{-i}\left(\theta_{-i}\right)\right),\left(\theta_i',\alpha_{-i}\left(\theta_{-i}\right)\right)\right) \geq u_i\left(y,\left(\theta_i',\alpha_{-i}\left(\theta_{-i}\right)\right)\right),\ \forall \theta_i' \in \Theta_i. \tag{2}$$

It is convenient to denote the set of allocations that make agent i worse off (relative to the social choice function f) at all of his types, $\theta_i' \in \Theta_i$, and a given type profile $\theta_{-i} \in \Theta_{-i}$ of the other agents by $Y_i^f\left(\theta_{-i}\right)$:

$$Y_i^f\left(\theta_{-i}\right) \triangleq \left\{y : u_i\left(f\left(\theta_i',\theta_{-i}\right),\left(\theta_i',\theta_{-i}\right)\right) \geq u_i\left(y,\left(\theta_i',\theta_{-i}\right)\right),\ \forall \theta_i' \in \Theta_i.\right\}. \tag{3}$$

Thus (2) can be replaced with the requirement that $y \in Y_i^f\left(\alpha_{-i}\left(\theta_{-i}\right)\right)$. The set $Y_i^f\left(\theta_{-i}\right)$ depends on the selection $f \in F$ and is referred to as the *reward set*. It is the set of allocations which can be used to reward the whistle-blower without upsetting the truthtelling equilibrium realizing the social choice function f. If the social choice objective is a function rather than a set than we can omit the superscript on the reward set for notational ease. We refer to the subset of the reward set which also satisfies the reward inequality (1) as the *successful reward set* and denote it by $Y_i^*(\theta_{-i})$.

The definition of ex post monotonicity suggests a rather intuitive description as to why monotonicity is a necessary condition for implementation. Suppose that some selection $f \in F$ is ex post implementable. Then if the agents were to deceive the designer by misreporting $\alpha(\theta)$ rather than reporting truthfully θ and if the deception $\alpha(\cdot)$ would lead to an allocation outside of the social choice set, i.e. $f \circ \alpha \notin F$, then the designer should be able to fend off the deception. This requires that there is some agent i and profile θ such that the designer can offer agent i a reward y for denouncing the deception $\alpha(\theta)$ if the true type profile is θ. Yet, the designer has to be aware that the reward could be used in the wrong circumstances, namely when the true type profile of the remaining agents is $\alpha_{-i}(\theta_{-i})$ *and* truthfully reported to be $\alpha_{-i}(\theta_{-i})$. The strict inequality (1) then guarantees the existence of a whistle-blower, whereas the weak inequalities (2) guarantee ex post incentive compatible behavior by the whistle-blower.

3.2 Maskin Monotonicity

Maskin (1999) introduced a celebrated monotonicity notion which is a necessary and almost sufficient condition for complete information implementation. In the complete information environment, each agent i is assumed to know the entire type profile θ rather than just his private type θ_i. In consequence, report and deception of each individual agent pertain to the entire type profile $\theta \in \Theta$, or:

$$\alpha_i : \Theta \to \Theta.$$

With complete information, it is easy to detect *individual* deceptions and hence it suffices to consider collective and coordinated deceptions in which all agents pursue a common deception strategy, $\alpha_i = \alpha$, for all i.

Definition 5 (Maskin Monotonicity)
Social choice set F satisfies Maskin monotonicity (MM) if for every $f \in F$, α and θ with $f \circ \alpha(\theta) \notin \hat{f}(\theta)$ for all $\hat{f} \in F$, there exists i and $y \in Y$ such that

$$u_i(y, \theta) > u_i(f(\alpha(\theta)), \theta), \tag{4}$$

while

$$u_i(f(\alpha(\theta)), \alpha(\theta)) \geq u_i(y, \alpha(\theta)). \tag{5}$$

We state the notion of Maskin monotonicity in such a way as to facilitate a simple comparison with ex post monotonicity. Typically, Maskin monotonicity is defined for social choice correspondences rather than social choice sets. If we start with a social choice correspondence $\phi : \Theta \to 2^Y / \varnothing$, then we can define an associated social choice set $F = \{f \,|f : \Theta \to Y\}$ by including all social choice functions f which select at all profiles allocations in the image of the correspondence:

$$F = \{f \,|f (\theta) \in \phi (\theta) , \ \forall \theta \in \Theta\} .$$

We compare the notions of social choice set and correspondence in more detail in Section 8.

Comparing ex post and Maskin monotonicity, it may initially appear that ex post monotonicity is a stronger requirement: the truthtelling constraint is required to hold at $\left(\theta_i', \alpha_{-i} (\theta_{-i})\right)$ for all $\theta_i' \in \Theta_i$ rather than just at $\alpha (\theta)$. Thus (2) is stronger requirement than (5) because the incomplete information reward set $Y_i^f (\theta_{-i})$ is (weakly) contained in the complete information counterpart

$$Y_i^f (\theta) = \{y : u_i (f (\theta) , \theta) \geq u_i (y, \theta)\} .$$

The complete information reward set depends on the entire profile θ rather than the profile θ_{-i} of all agents but i. This difference in the reward sets arises from the informational assumption. With complete information, all individual deceptions can easily be detected and the designer only needs to worry about coordinated misrepresentations by all the agents. In the incomplete information environment, agent i has private information about θ_i' and hence incentive compatibility is required to hold for all types $\theta_i' \in \Theta_i$.

But for either ex post or Maskin monotonicity, we need a preference reversal relative to the allocation $f (\alpha (\theta))$. If the behavior of the incomplete information reward set $Y_i (\alpha_{-i} (\theta_{-i}))$ is locally similar to the complete information set $Y_i (\alpha (\theta))$, then the difference between them may not matter for implementation purposes. Indeed, we will show that in the important class of single crossing environments, ex post and Maskin monotonicity coincide. We will illustrate this coincidence in the public good example that follows. But this gap in reward sets may have important implications. While the Pareto correspondence is always Maskin monotonic, we will later give an example showing that it is not always ex post monotonic.

But outside the single crossing environment, ex post monotonicity is not necessarily a stronger notion than Maskin monotonicity. In the complete information environment, the agents are (implicitly) allowed to perfectly coordinate their misrepresentation for every societal type profile θ. In contrast, in the incomplete information world, agent i has to deceive, i.e. determine $\alpha_i : \Theta_i \to \Theta_i$, independently of the type profile of the other agents. For this reason, it is strictly more difficult to find a reward y for Maskin monotonicity than for ex post monotonicity. In other words, the independent choice of deception α_i leads to a strictly smaller number of feasible deceptions α in the incomplete information context. In the context of a single unit auction with interdependent valuations, this second difference will enable us to show that the single unit auction can be implemented in ex post equilibrium, yet fails to be implementable in complete information, and hence fails Maskin monotonicity.

3.3 Public Good Example

We will illustrate ex post monotonicity, and the relation to Maskin monotonicity, with the following public good example. The utility of each agent is given by:

$$u_i(\theta, x, t) = \left(\theta_i + \gamma \sum_{j \neq i} \theta_j\right) x + t_i, \tag{6}$$

where x is the level of public good provided and t_i is the monetary transfer to agent i. The utility of agent i depends on his own type $\theta_i \in [0,1]$ and the type profile of other agents, with $\gamma \geq 0$. The cost of establishing the public good is given by $c(x) = \frac{1}{2}x^2$. The planner must choose $(x, t_1, ..., t_I) \in \mathbb{R}_+ \times \mathbb{R}^I$ to maximize social welfare, i.e., the sum of gross utilities minus the cost of the public good:

$$\left((1 + \gamma(I-1)) \sum_{i=1}^{I} \theta_i\right) x - \frac{1}{2}x^2.$$

The socially optimal level of the public good is therefore equal to

$$x(\theta) = (1 + \gamma(I-1)) \sum_{i=1}^{I} \theta_i.$$

The social choice set F can then be described by:

$$F = \left\{ (x, t_1, ..., t_I) : \Theta \to \mathbb{R}^{I+1} \middle| x(\theta) = (1 + \gamma(I-1)) \sum_{i=1}^{I} \theta_i \right\},$$

where the level of the public good is determined uniquely, but the designer is unrestricted in his choice of transfers. By standard arguments, ex post incentive compatibility pins down the levels of transfers[9]:

$$t_i(\theta) = h_i(\theta_{-i}) - (1 + \gamma(I-1)) \left(\gamma \left(\sum_{j \neq i} \theta_j\right) \theta_i + \frac{1}{2}\theta_i^2\right). \tag{7}$$

[9] In this example, and in interdependent public good problems with more general function forms, it is possible to find ex post incentive compatible transfers for all values of $\gamma \geq 0$. This can be established using conditions in Bergemann & Valimaki (2002). This contrasts with the case of allocating a private good with interdependent values, where ex post incentive compatibility puts an upper bound on the amount of interdependence (Dasgupta & Maskin (2000)). However, Fieseler, Kittsteiner & Moldovanu (2003) point out that negative interdependence, or $\gamma < 0$, relaxes the ex post incentive constraints in the private good problem.

The complete information reward set $Y_i(\theta)$ is now characterized by an indifference curve in the (x, t_i) space. With the linear preferences here it is simply a straight line. The reward set is given by the set of allocations below the indifference curve. In contrast, the incomplete information reward set $Y_i(\theta_{-i})$ is characterized by the intersection of the reward sets for all $\theta_i' \in \Theta_i$, or:

$$Y_i(\theta_{-i}) = \bigcap_{\theta_i' \in \Theta_i} Y_i(\theta_i', \theta_{-i}).$$

The boundary of the set $Y_i(\theta_{-i})$ is the set of all truthtelling allocations $\{x(\theta_i', \theta_{-i}), t_i(\theta_i', \theta_{-i})\}_{\theta_i' \in \Theta_i}$. The respective reward sets are depicted for $I = 3, \gamma = \frac{1}{4}$, and $\theta_i = \frac{1}{4}$ for all i in Figure 1 (setting $h_i(\theta_{-i}) = t_i(0, \theta_{-i}) = 0$ for all θ_{-i}).

INSERT FIGURE 1: REWARD SETS $Y_i(\theta)$ AND $Y_i(\theta_{-i})$.

The crucial observation is now that the slope of the boundary of the set $Y_i(\theta_{-i})$ at $\theta = (\theta_i, \theta_{-i})$ is identical to the slope of the boundary of the set $Y_i(\theta)$. In other words, locally, the slope of the boundary of $Y_i(\theta_{-i})$ is determined by the preferences of type θ. It then follows that if we can guarantee preference reversal at the allocation $f(\alpha(\theta))$, essentially the crossing of the indifference curves for θ and $\alpha(\theta)$, then the indifference curve of type θ will also cross with the boundary of the set $Y_i(\theta_{-i})$. This is illustrated with $\theta_i = \frac{3}{4}$ and $\alpha_i(\theta_i) = \frac{1}{4}$ for all i in Figure 2.

INSERT FIGURE 2: SUCCESSFUL REWARD SETS.

This basic insight allows us later to conclude that despite the additional incentive constraints imposed by the ex post monotonicity condition, the single crossing environment by itself is strong enough to guarantee the ex post monotonicity condition. The only modification we need is to strengthen the necessary condition from weak to strict ex post incentive compatibility.

4 Ex Post Implementation

We present necessary and sufficient conditions for a social choice set F to be ex-post implementable. Our results extend the work of Maskin (1999) for complete information implementation and of Postlewaite & Schmeidler (1986), Palfrey & Srivastava (1989*a*) and Jackson (1991) on Bayesian implementation to the notion of ex post equilibrium.

4.1 Necessary Conditions

Ex post incentive and monotonicity conditions are necessary conditions for ex post implementation.

Theorem 1 (Necessity)
If F is ex post implementable, then it satisfies $(EPIC)$ and (EM).

Proof. Let (M,g) implement F. Fix any $f \in F$. By the implementation hypothesis, there must exist an equilibrium $s = (s_1, ...s_I)$, each $s_i : \Theta_i \to M_i$ such that $f = g \circ s$. Consider any $i, \theta_i' \in \Theta_i$. Since s is an equilibrium,

$$u_i \left(g\left(s\left(\theta\right)\right), \theta\right) \geq u_i \left(g\left(s_i\left(\theta_i'\right), s_{-i}\left(\theta_{-i}\right)\right), \theta\right),$$

for all $\theta_i' \in \Theta_i$ and all $\theta \in \Theta$. Noting that $g\left(s\left(\theta\right)\right) = f\left(\theta\right)$ and $g\left(s_i\left(\theta_i'\right), s_{-i}\left(\theta_{-i}\right)\right) = f\left(\theta_i', \theta_{-i}\right)$ establishes $(EPIC)$.
Now fix any deception α with $f \circ \alpha \notin F$. It must be that $s \circ \alpha$ is not an equilibrium at some $\theta \in \Theta$. Therefore there exists i and $m_i \in M_i$ such that we have

$$u_i \left(g\left(m_i, s_{-i}\left(\alpha_{-i}\left(\theta_{-i}\right)\right)\right), \theta\right) > u_i \left(g\left(s\left(\alpha\left(\theta\right)\right)\right), \theta\right).$$

Let $y \triangleq g\left(m_i, s_{-i}\left(\alpha_{-i}\left(\theta_{-i}\right)\right)\right)$. Then, from above,

$$u_i\left(y, \theta\right) > u_i\left(f\left(\alpha\left(\theta\right)\right), \theta\right).$$

But since s is an equilibrium and $f = g \circ s$, it follows that

$$\begin{aligned} u_i\left(f\left(\theta_i', \alpha_{-i}\left(\theta_{-i}\right)\right), \left(\theta_i', \alpha_{-i}\left(\theta_{-i}\right)\right)\right) &= u_i\left(g\left(s\left(\theta_i', \alpha_{-i}\left(\theta_{-i}\right)\right)\right), \left(\theta_i', \alpha_{-i}\left(\theta_{-i}\right)\right)\right) \\ &\geq u_i\left(g\left(m_i, s_{-i}\left(\alpha_{-i}\left(\theta_{-i}\right)\right)\right), \left(\theta_i', \alpha_{-i}\left(\theta_{-i}\right)\right)\right) \\ &= u_i\left(y, \left(\theta_i', \alpha_{-i}\left(\theta_{-i}\right)\right)\right), \forall \theta_i' \in \Theta_i. \end{aligned}$$

This establishes the incentive compatibility of the whistle-blower, or $y \in Y_i^f\left(\alpha_{-i}\left(\theta_{-i}\right)\right)$. ■

We defined ex post monotonicity in terms of the type profiles and associated utility functions. As ex post monotonicity is the central condition in the subsequent analysis, we want to verify, as promised earlier, that ex post monotonicity is nonetheless an ordinal rather than a cardinal concept.

Definition 6 (Ordinality)
The types θ_i and θ_i' are ordinally equivalent $(\theta_i \sim \theta_i')$ if for all j, $\theta_{-i} \in \Theta_{-i}$, y and y',

$$u_j\left(y, \left(\theta_i, \theta_{-i}\right)\right) \geq u_j\left(y', \left(\theta_i, \theta_{-i}\right)\right) \Leftrightarrow u_j\left(y, \left(\theta_i', \theta_{-i}\right)\right) \geq u_j\left(y', \left(\theta_i', \theta_{-i}\right)\right).$$

In other words, any two types of agent i, θ_i and θ_i', are ordinally equivalent if the ranking of any pair of alternatives by any agent and for any profile of agents other than i remain unchanged. With interdependent values, it is important that the ranking remains unchanged not only for agent i but for all other agents as well.

Definition 7 (Ordinal Social Choice Set)
Social choice set F is ordinal if $f \in F$ and $\alpha_j(\theta_j) \sim \theta_j$ for all j and θ_j imply $f \circ \alpha \in F$.

A social choice set is hence called ordinal if for any two profiles, θ and $\alpha(\theta)$, which only differ by ordinally equivalent types, the corresponding allocations remain in the social choice set.

Lemma 1
If F satisfies ex post monotonicity, then F is ordinal.

Proof. Suppose $f \in F$ and $\alpha_j(\theta_j) \sim \theta_j$ for all j and θ_j and that

$$u_i(f(\alpha(\theta)), \alpha(\theta)) \geq u_i(y, \alpha(\theta)).$$

By ordinality we have

$$u_i(f(\alpha(\theta)), \theta) \geq u_i(y, \theta).$$

But if ex post monotonicity holds, we must have $f \circ \alpha \in F$. ■

We proceed by first showing that in a wide class of environments, to be referred to as economic environments, ex post incentive and monotonicity conditions are also sufficient conditions for ex post implementation. We then present weaker sufficiency conditions, in the spirit of the conditions used in Bayesian implementation, to obtain positive results outside of economic environments.

4.2 Sufficient Conditions in Economic Environments

The sufficiency arguments - for both the economic and the non-economic environment - will rely on the use of an augmented mechanism. The mechanism suggested here is similar to the one used to establish sufficiency in the complete information implementation literature (e.g., Maskin (1999)). Each agent sends a message of the form $m_i = (\theta_i, f_i, z_i, y_i)$, where θ_i is the reported type, f_i is the social choice function suggested by i, z_i is a positive integer from the set $\mathcal{I} = \{1, 2, ..., I\}$ and y_i is the reward claimed by i. The set of feasible messages for agent i is given by:

$$M_i = \Theta_i \times F \times \mathcal{I} \times Y.$$

The mechanism is described by three rules:

Rule 1. If $f_i = f$ for all i, then $g(m) = f(\theta)$.

Rule 2. If there exists j and f such that $f_i = f$ for all $i \neq j$ while $f_j \neq f$, then outcome y_j is chosen if $y_j \in Y_j^f(\theta_{-j})$; otherwise outcome $f(\theta)$ is chosen.

Rule 3. In all other cases, $y_{j(z)}$ is chosen, where $j(z)$ is the agent determined by the modulo game:

$$j(z) = \sum_{i=1}^{I} z_i \quad (\text{mod}\, I).$$

We refer to the mechanism described by Rule 1-3 as the *augmented mechanism.* A strategy profile in this game is a collection $s = (s_1, ..., s_I)$, with $s_i : \Theta_i \to M_i$ and we write:

$$s_i(\theta_i) = \left(s_i^1(\theta_i), s_i^2(\theta_i), s_i^3(\theta_i), s_i^4(\theta_i)\right) \in \Theta_i \times \mathcal{F} \times \mathcal{I} \times Y;$$

and $s^k(\theta) = \left(s_i^k(\theta)\right)_{i=1}^{I}$. We observe that if Y and Θ are finite, then the above mechanism is finite.

Next we define the notion of an economic environment.

Definition 8 (Economic Environment)
An environment is economic in state $\theta \in \Theta$ if, for every allocation $y \in Y$, there exist $i \neq j$ and allocations y_i and y_j, such that

$$u_i(y_i, \theta) > u_i(y, \theta),$$

and

$$u_j(y_j, \theta) > u_j(y, \theta).$$

An environment is economic if it is economic in every state $\theta \in \Theta$.

Theorem 2 (Economic Environment)
If $I \geq 3$, the environment is economic, and F satisfies (EPIC) and (EM), then F is ex post implementable.

Proof. The proposition is proved in three steps, using the augmented mechanism.

Claim 1. Fix any $f \in F$. There is an ex post equilibrium s with $g(s(\theta)) = f(\theta)$ for all θ.

Any strategy profile s of the following form is an ex post equilibrium:

$$s_i(\theta_i) = (\theta_i, f, \cdot, \cdot).$$

Suppose agent i thinks that his opponents are types θ_{-i} and deviates to a message of the form

$$s_i(\theta_i) = \left(\theta_i', f_i, \cdot, \cdot\right);$$

if $y_i \notin Y_i^f(\theta_{-i})$, then the payoff gain is

$$u_i\left(f\left(\theta_i', \theta_{-i}\right), f(\theta_i, \theta_{-i})\right) - u_i\left(f(\theta_i, \theta_{-i}), f(\theta_i, \theta_{-i})\right),$$

which is non-positive by (EPIC); if $f_i \neq f$ and $y_i \in Y_i^f(\theta_{-i})$, then the payoff gain is

$$u_i(y_i, (\theta_i, \theta_{-i})) - u_i(f(\theta_i, \theta_{-i}), f(\theta_i, \theta_{-i})),$$

which is non-positive by the definition of $Y_i^f(\theta_{-i})$.

Claim 2. In any ex post equilibrium, there exists $f \in F$ such that $s_i^2(\theta_i) = f$ for all i and θ_i.

Suppose that for all $f \in F$, there exists i and θ_i such that $s_i^2(\theta_i) \neq f$. Then there exists θ such that rule 1 does not apply in equilibrium.

First suppose that rule 2 applies at θ, so that there exists j and f such that $f_i = f$ for all $i \neq j$. Then any agent $i \neq j$ of type θ_i who thought his opponents were types θ_{-i}, could send a message of the form $m_i = (\cdot, f_i, z_i, y_i)$, with $f_i \neq f$ and $i = \sum_{k=1}^{I} z_k$ and obtain utility $u_i(y_i, \theta)$. Thus we must have $u_i(g(s(\theta)), \theta) \geq u_i(y, \theta)$ for all y and all $i \neq j$. This contradicts the economic environment assumption.

Now suppose that rule 3 applies at θ. Then every agent i of type θ_i who thought his opponents were types θ_{-i}, could send a message of the form $m_i = (\cdot, f_i, z_i, y_i)$, with $i = \sum_{k=1}^{I} z_k$ and obtain utility $u_i(y_i, \theta)$. Thus we must have $u_i(g(s(\theta)), \theta) \geq u_i(y, \theta)$ for all y and i. This again contradicts the economic environment assumption.

Claim 3. For any $f \in F$ and in any ex post equilibrium with $s_i^2(\theta_i) = f$ for all i and θ_i, $f \circ s^1 \in F$.

Suppose that $f \circ s^1 \notin F$. By (EM), there exists i, θ and $y \in Y_i^f\left(s_{-i}^1(\theta_{-i})\right)$ such that

$$u_i(y, \theta) > u_i\left(f\left(s^1(\theta)\right), \theta\right).$$

Now suppose that type θ_i of agent i believes that his opponents are of type θ_{-i} and sends message $m_i = (\cdot, f_i, \cdot, y)$, with $f_i \neq f$, while other agents send their equilibrium messages, then from the definition of $g(\cdot)$:

$$g(m_i, s_{-i}(\theta_{-i})) = y,$$

so that:

$$u_i(g(m_i, s_{-i}(\theta_{-i})), \theta) = u_i(y, \theta) > u_i\left(f\left(s^1(\theta)\right), \theta\right) = u_i(g(s(\theta)), \theta),$$

and this completes the proof of sufficiency. ■

The economic environment condition was used to show that in equilibrium, the suggested social choice functions all have to agree: $f_i = f$ for all i. If not, then some agent j could profitably change his suggestion to $f_j \neq f$ and obtain a more desirable allocation than f. The economic environment assumption guaranteed the existence of an agent j with a preferred allocation.

4.3 Sufficiency Conditions in Non-Economic Environments

We now establish sufficient conditions for ex post implementation outside of economic environments. For simplicity, we focus on the implementation of social choice functions, rather than social choice sets, in this section.

The ex post sufficient conditions are the natural complements of the conditions obtained earlier for Bayesian implementation. But because only ex post utilities matter, they are more easily verified than their Bayesian analogues. We show that a joint strengthening of Maskin monotonicity and ex post monotonicity, together with a no veto condition, is sufficient for ex post implementation. In Section 9 we permit random mechanisms, which will allow us to strengthen the sufficient conditions presented here substantially.

Within the augmented mechanism there are essentially two ways in which the play of agents can lead to equilibrium behavior outside of the social choice objective. At any profile $\theta \in \Theta$, the agents can either misrepresent their true type and fail to alert the designer of the misrepresentation. Or, some agents alert the designer and thus lead him to choose an allocation different from $f(\theta)$. In the former case, Rule 1 of the augmented mechanism applies whereas in latter case, either Rule 2 or Rule 3 applies. With an economic environment, it was impossible that in any equilibrium Rule 2 or Rule 3 would apply. It followed that in every equilibrium Rule 1 would apply at all profiles $\theta \in \Theta$. The ex post monotonicity condition then guaranteed that the equilibrium conformed with the social choice set. As we abandon the assumption of an economic environment, we cannot anymore exclude the possibility that in equilibrium either Rule 2 or Rule 3 might apply. In consequence, the sufficient conditions have to account for these complications. There are now basically two ways to achieve this goal. Either behavior under Rule 2 or Rule 3 can be made to conform with the social choice, or a reward can be offered in the *subset* of profiles where Rule 1 applies. The sufficient condition will contain both elements: either an application of a no veto condition will make behavior under Rule 2 or Rule 3 consistent with the social choice objective or an ex post monotonicity condition on subsets of Θ guarantees that a reward can be offered.

The relevant no veto condition is simply the "no veto power" property of Maskin (1999).

Definition 9 (No Veto Power)
Social choice function f satisfies no veto power at θ if, for any j, if $u_i(b,\theta) \geq u_i(y,\theta)$ for all $y \in Y$ and $i \neq j$, then $f(\theta) = b$.

Note that no veto power is vacuously true at θ if the environment is economic at θ, since the latter implies that the premise in the definition is never satisfied.

Under either Rule 2 or Rule 3, (almost) every agent can change the outcome to his most preferred outcome. If the candidate allocation under either rule is part of an equilibrium, it follows that at least $I-1$ (if Rule 2 applies and I if Rule 3 applies) agents rank the candidate allocation y higher than any other allocation. The no veto power property guarantees that allocation y coincides with the social choice set. In other words, the possibility of *undesirable* equilibrium behavior is eliminated by no veto power. However, if the no veto power property fails, it might still be possible to generate a reward on the set of profiles where Rule 1 applies. For any strategy profile of the agents, the set of profiles at which Rule 1 applies always satisfies a product structure. Given a strategy profile of the agents, a subset Φ_i identifies the types of agent i at which Rule 1 applies. The product set Φ:

$$\Phi \triangleq \times_{i=1}^{I} \Phi_i,$$

is the set of profiles at which Rule 1 applies, and the complementary set $\Theta - \Phi$ at which either Rule 2 or Rule 3 applies. We state the sufficient condition combining ex post monotonicity on subsets and the no veto power property. We state the conditions for the case of a social choice function and the straightforward extension to general social choice sets is provided in the Appendix.

Definition 10 (Ex Post Monotonicity No Veto (EMNV))
Social choice function f satisfies ex post monotonicity no veto if, for any deception α and any product set $\Phi \subset \Theta$, the following holds: If the environment is non-economic at each $\theta \in \Theta - \Phi$, then

1. *f satisfies no veto power on $\Theta - \Phi$;*
2. *if $f(\alpha(\theta)) \neq f(\theta)$ for some $\theta \in \Phi$ then there exists i, $\theta \in \Phi$ and y such that*

$$u_i(y, \theta) > u_i(f(\alpha(\theta)), \theta),$$

while

$$u_i\left(f\left(\theta_i', \alpha_{-i}(\theta_{-i})\right), \left(\theta_i', \alpha_{-i}(\theta_{-i})\right)\right) \geq u_i\left(y, \left(\theta_i', \alpha_{-i}(\theta_{-i})\right)\right), \ \forall \theta_i' \in \Theta_i.$$

The strategy profile of the agents could involve truthtelling on a subset Φ (and appealing to Rule 1) or involve whistle-blowing and misrepresentation on the complementary set $\Theta - \Phi$ (and appeal to Rule 2 or Rule 3). The sufficient condition guarantees that for all product sets Φ an appropriate reward can be found. The condition is weakened by the fact that we only need to consider those subsets Φ which guarantee that on the complementary subset $\Theta - \Phi$ *all* profiles are non-economic. It follows that the smaller the set Φ becomes (and hence restricting the ability of the designer to offer rewards), the more demanding is the requirement that *all* profiles are non-economic. This then conceivably puts a bound on the number of sets for which the ex post monotonicity part of the condition has to be verified.

Theorem 3 (Sufficiency)
For $I \geq 3$, if f satisfies $(EPIC)$ and $(EMNV)$, then f is ex post implementable.

Proof. We use the same mechanism as before. The argument that there exists an ex post equilibrium s with $g(s(\theta)) = f(\theta)$ is the same as before. Now we establish three claims that hold for all equilibria. Let

$$\Phi_i = \{\theta_i : s_i(\theta_i) = (\cdot, f, \cdot, \cdot)\}.$$

Claim 1. In any ex post equilibrium, for each $\theta \notin \Phi$, (a) there exists i such that $u_j(g(s(\theta)), \theta) \geq u_j(y, \theta)$ for all y and $j \neq i$; and thus (b) the environment is non-economic at θ.

First, observe that for each $\theta \notin \Phi$, there exists i such that $s_i^2(\theta_i) \neq f$. Given the strategies of the other agents, any agent $j \neq i$ who thought his opponents were types θ_{-j} could send any message of the form

$$(\cdot, f_j, z_j, y_j),$$

and obtain utility $u_j(y_j, \theta)$. Thus we must have $u_j(g(s(\theta)), \theta) \geq u_j(y, \theta)$ for all y and $j \neq i$; thus the environment is non-economic for all $\theta \notin \Phi$.

Claim 2. In any ex post equilibrium, for all $\theta \in \Phi$,

$$u_i\left(f\left(s^1(\theta)\right), \theta\right) \geq u_i(y, \theta)$$

for all $y \in Y_i^f\left(s_{-i}^1(\theta_{-i})\right)$. Suppose that $y \in Y_i^f\left(s_{-i}^1(\theta_{-i})\right)$ and that type θ_i of agent i believes that his opponents are of type θ_{-i} and sends message $m_i = (\cdot, f_i, z_i, y)$, while other agents send their equilibrium messages. Now

$$g(m_i, s_{-i}(\theta_{-i})) = y;$$

so ex post equilibrium requires that

$$\begin{aligned} u_i(g(s(\theta)), \theta) &= u_i\left(f\left(s^1(\theta)\right), \theta\right) \\ &\geq u_i(g(m_i, s_{-i}(\theta_{-i})), \theta) \\ &= u_i(y, \theta). \end{aligned}$$

Claim 3. If EMNV is satisfied, then Claim 1 and 2 imply that $g(s(\theta)) = f(\theta)$ for all θ.

Fix any equilibrium. Claim 1(b) establishes that the environment is non-economic at all $\theta \notin \Phi$. Suppose $g(s(\theta)) \neq f(\theta)$ for some $\theta \in \Phi$. Now EMNV implies that there exists i, $\theta \in \Phi$ and $y \in Y_i^f\left(s_{-i}^1(\theta_{-i})\right)$ such that $u_i(y, \theta) > u_i\left(f\left(s^1(\theta)\right), \theta\right)$, contradicting Claim 2. Suppose $g(s(\theta)) \neq f(\theta)$ for some $\theta \notin \Phi$. By claim 1(a), there exists i such that $u_j(g(s(\theta)), \theta) \geq u_j(y, \theta)$ for all y and $j \neq i$. This establishes that no veto power applies at θ. So again EMNV implies that $g(s(\theta)) = f(\theta)$. ■

EMNV is almost equivalent to requiring ex post monotonicity and no veto power everywhere. More precisely, we have:

1. If ex post monotonicity holds and no veto power holds at every type profile θ, then EMNV holds.

2. If EMNV holds, then (1) ex post monotonicity holds and (2) if the environment is non-economic whenever $\theta_i = \theta_i^*$, then no veto power holds whenever $\theta_i = \theta_i^*$. To see (1), set $\Phi_i = \Theta_i$ for all i; to see (2), set α to be the truth-telling deception and, for some i, $\Phi_i = \Theta_i \backslash \{\theta_i^*\}$ and $\Phi_j = \Theta_j$ for all $j \neq i$.

In an economic environment, we only have to verify $\Phi = \Theta$. EMNV is then equivalent to ex post monotonicity as the no veto condition is vacuously satisfied. On the other hand, if the environment is non-economic at every profile $\theta \in \Theta$, then the EMNV condition simplifies considerably as it suffices to evaluate the hypothesis at the most restrictive sets, or $\Phi = \{\theta\}$ for every $\theta \in \Phi$. In particular, we can then state ex post monotonicity and no veto conditions separately.

Definition 11 (Local Ex Post Monotonicity (LEM))
f satisfies local ex post monotonicity if for all θ and all α such that $f(\alpha(\theta)) \neq f(\theta)$, there exists i and y with:

$$u_i(y, \theta) > u_i(f(\alpha(\theta)), \theta), \tag{8}$$

while

$$u_i\left(f\left(\theta_i', \alpha_{-i}(\theta_{-i})\right), \left(\theta_i', \alpha_{-i}(\theta_{-i})\right)\right) \geq u_i\left(y, \left(\theta_i', \alpha_{-i}(\theta_{-i})\right)\right), \forall \theta_i' \in \Theta_i. \tag{9}$$

With LEM, the designer can offer a reward y at *every* type profile θ at which α leads to a different allocation, or $f(\alpha(\theta)) \neq f(\theta)$. In contrast, with ex post monotonicity it suffices to find *some* θ at which a reward y can be offered. The local version of ex post monotonicity is in fact identical to Maskin monotonicity with the additional ex post incentive constraints (see (9)).

Corollary 1 (Sufficiency)
For $I \geq 3$, if f satisfies $(EPIC)$, (LEM) and (NVP), then f is ex post implementable.

For a non-economic environment, the separate conditions of LEM and NVP are exactly identical to EM. If the environment is economic in some profiles but not all profiles, the joint conditions are more restrictive than $EMNV$.

5 Single Crossing Environment

In this section we consider ex post implementation in single crossing environments. To make full use of the crossing conditions, we restrict attention to social choice problems which recommend allocations in the interior of the feasible set of Y. We show that under this mild restriction, single crossing preferences are essentially sufficient to guarantee ex post monotonicity.

Definition 12 (Interior Social Choice Set)
Social choice set F is interior if for all $f \in F$ and for all $\theta \in \Theta, f(\theta) \in \text{int}\, Y$ for all $\theta \in \Theta$.

The interior condition is essential to use the full strength of the single crossing environment. In this section we further assume that Y is a convex set and that $u_i(y, \theta)$ is continuous in y at all i and θ. The convexity and continuity assumptions appear in establishing that locally, around $f(\theta)$, it is only the single crossing condition with respect to the type profile θ, that matter for the monotonicity inequalities. We first give a general definition of preference reversal.

Definition 13 (Preference Reversal)
The environment is an environment with preference reversal if for all $z \in \text{int}\, Y$ and all θ, θ', every open set $\mathcal{O}$ including z contains allocations y, y' such that for some i:

$$u_i(y, \theta) > u_i(z, \theta) > u_i(y', \theta), \quad \textit{while} \quad u_i(y', \theta') > u_i(z, \theta') > u_i(y, \theta').$$

The above definition is weak in the sense that the preference reversal is required to occur only for one rather than all agents. This weaker version is helpful as the type profiles of the agents may sometimes interact so as to precisely offset each other in their effect on the preferences of the agents. We simple require that at two distinct profiles of society, θ and θ', there is at least one agent with a preference reversal.[10,11]

[10] With interdependent values the change in the type profile from θ to θ' may offset each other as can be easily verified within the earlier public good example (see Section 3.3). For example we can keep the utility of agent i constant as we move from θ to θ', where θ' is given by

$$\theta'_j = \begin{cases} \theta_j - \gamma\varepsilon, \text{ if } j = i \\ \theta_k + \varepsilon, \text{ if } j = k \\ \theta_j, \text{ if } j \neq i, k \end{cases}.$$

Now i has identical preferences at θ and θ', but it is easily verified that for all $j \neq i$, there is preference reversal.

[11] The condition can be weakened further: by requiring that each $f \in F$ satisfies $f(\theta) = f(\theta')$ if $u_i(\cdot, \theta) = u_i(\cdot, \theta')$ for all i, we can weaken the preference reversal condition to hold only at profile pairs, θ and θ', at which there exists i such that $u_i(\cdot, \theta) \neq u_i(\cdot, \theta')$.

As the above definition of single crossing applies to general allocation spaces, it is phrased as a preference reversal condition. In many applications of mechanism design, the allocation space for each agent is two-dimensional, say the level of private or public good and a monetary transfer. In this case, a sufficient condition for preference reversal is the well known intersection or single crossing condition.

Definition 14 (Single Crossing)
The environment is a single crossing environment if for all $z \in \operatorname{int} Y$, with $Y \subset \mathbb{R}^2$, the indifference curves for any two profiles θ and θ' generated by $u_i(z, \theta)$ and $u_i(z, \theta')$ intersect at z for some i.

We establish the positive implementation result for the environment given in Definition 13. We should point out that the important feature of the single crossing environment with respect to the monotonicity condition is the "crossing" or "reversal" aspect rather than the "single crossing" aspect. More precisely, in order to show that the monotonicity conditions are satisfied, we only need to require that the indifference curves intersect at z but we do not need them to have a unique intersection. By contrast, the uniqueness is of course helpful to establish the validity of the incentive compatibility conditions.

With strictness and interiority of F, a local argument allows us to show that, even though relative to Maskin monotonicity, ex post monotonicity imposes additional ex post incentive constraints, these additional constraints do not bind. In consequence, the set of dominated allocations y is locally identical to the complete information set. For the local argument to go through, we need to strengthen the ex post incentive constraints in the direct mechanism to strict rather than weak inequalities.

Definition 15 (Strict Ex Post Incentive Compatibility)
Social choice set F is strictly ex post incentive compatible if for all $f \in F$:

$$u_i(f(\theta), \theta) > u_i(f(\theta_i', \theta_{-i}), \theta)$$

for all i, θ and $\theta_i' \neq \theta_i$.

The public good example in Section 3.3 is an example where the (singleton) social choice set satisfies single crossing, strict EPIC and interiority.

Theorem 4 (Single Crossing)
In an environment with preference reversal, every strict ex post incentive compatible and interior F satisfies ex post monotonicity.

Proof. We start with the contrapositive version of ex post monotonicity, which can be stated as follows. Fix a deception α. If, for all i and all $\widehat{\theta} \in \Theta$, we have that

$$u_i\left(f\left(\theta_i', \alpha_{-i}\left(\widehat{\theta}_{-i}\right)\right), \left(\theta_i', \alpha_{-i}\left(\widehat{\theta}_{-i}\right)\right)\right) \geq u_i\left(y, \left(\theta_i', \alpha_{-i}\left(\widehat{\theta}_{-i}\right)\right)\right) \text{ for all } \theta_i' \in \Theta_i \text{ and } y \in Y,$$

implies that

$$u_i\left(f\left(\alpha\left(\widehat{\theta}\right)\right), \widehat{\theta}\right) \geq u_i\left(y, \widehat{\theta}\right),$$

then $f \circ \alpha \in F$. For a given $\widehat{\theta} \in \Theta$ with $\alpha\left(\widehat{\theta}\right) \neq \widehat{\theta}$, let us define for notational ease $\theta \triangleq \alpha\left(\widehat{\theta}\right)$. Now consider the indifference curve for θ and $\widehat{\theta}$ at $f(\theta)$. Since the preferences are single crossing, there is a sequence of allocations $\{y_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} y_n = f(\theta)$ such that for all y_n:

$$u_i(f(\theta), \theta) > u_i(y_n, \theta)$$

and

$$u_i\left(f(\theta), \widehat{\theta}\right) < u_i\left(y_n, \widehat{\theta}\right).$$

We shall now argue that for every $\theta_i' \in \Theta_i$ there exists sequence $\{y_n\}_{n=1}^{\infty}$ such that:

$$u_i\left(f\left(\theta_i', \theta_{-i}\right), \left(\theta_i', \theta_{-i}\right)\right) \geq u_i\left(y_n, \left(\theta_i', \theta_{-i}\right)\right). \tag{10}$$

The proof is by contrapositive. Suppose now that (10) were not to hold, and that there exists θ_i' such that for all y_n :

$$u_i\left(f\left(\theta_i', \theta_{-i}\right), \left(\theta_i', \theta_{-i}\right)\right) < u_i\left(y_n, \left(\theta_i', \theta_{-i}\right)\right),$$

then it would follow from continuity of the utility function that:

$$u_i\left(f\left(\theta_i', \theta_{-i}\right), \left(\theta_i', \theta_{-i}\right)\right) \leq u_i\left(f\left(\theta_i, \theta_{-i}\right), \left(\theta_i', \theta_{-i}\right)\right).$$

But this violates the hypothesis of strict ex post incentive compatibility. We have thus shown that the hypothesis in the definition of ex post monotonicity is never satisfied and hence the implication is never required. It follows that ex post monotonicity is vacuously satisfied in the single crossing environment. ■

The public good example of Section 3 is an example where the (singleton) social choice set satisfies single crossing, strict EPIC and interiority. The idea of the proof is that with strict ex post incentive compatibility, the set of allocations which are dominated by the social choice function is locally (around $f(\theta)$) determined by the preferences of the agents with type profile θ. The situation is represented in Figure 3.

INSERT FIGURE 3: STRICT EX POST INCENTIVE COMPATIBILITY

If the ex post incentive constraint only holds as an equality for some types, say θ_i and θ_i', then the set of allocations dominated by the social choice function is determined locally (around $f(\theta)$) by the preferences of both types. In this case, the hypothesis of Maskin monotonicity may be satisfied and hence become a constraint. The second situation is represented in Figure 4.

Insert Figure 4: Failure of Strict Ex Post Incentive Compatibility

A recent contribution by Arya, Glover & Rajan (2000) explores in the private value environment the relationship between Maskin and Bayesian monotonicity and the single crossing condition. Interestingly, their sufficiency result regarding Bayesian monotonicity, also requires that the incentive compatibility conditions in the direct mechanism are satisfied as strict inequalities. They also present an example which shows that weak incentive compatibility and the single crossing condition alone do not guarantee Bayesian monotonicity. Incidentally, and for the same reason as in Arya, Glover & Rajan (2000) for the Bayesian incentive constraints, the strict ex post incentive compatibility condition actually allows a slightly stronger statement than actually stated in Proposition 4. The single crossing and strict ex post incentive constraints also imply a strict ex post monotonicity condition, where all the weak inequalities are replaced by strict inequalities.

The role of the interior assumption for the monotonicity condition has already been emphasized in work by Hurwicz, Maskin & Postlewaite (1995). They presented an example of a Walrasian social choice correspondence where the Walrasian allocation for a given preference profile is on the boundary of the feasibility set. Naturally then, the indifference curves generated by a second and distinct set of preferences, intersect at the boundary. The crucial implication of the intersection at the boundary is that the set of allocations for which the Maskin monotonicity hypothesis fails is outside the feasible set, and hence Maskin monotonicity fails to hold.

6 Direct Mechanisms

In general, the ex post monotonicity conditions guarantee the existence of an incentive compatible reward. With the preference reversal environment and the strict ex post incentive constraints, we could establish that local changes in the report of the *types* are sufficient to establish the ex post monotonicity conditions. This in fact suggests that preference reversal and strictness of the ex post incentive constraints jointly guarantee implementation in the direct mechanism. We pursue this insight and show that in many economically important circumstances ex post implementation can be achieved in the direct mechanism. In consequence, the implementation does not have to rely on the augmentation on which much of the positive implementation results in the literature rest.

By definition, a direct mechanism cannot coordinate the selection of a particular social choice function f from a set F. Hence, we restrict our attention in this section to the implementation of a given social function f. The basic problem of implementation in the direct mechanism is that an agent must be able to claim the reward y by a report, possibly a misreport, of his type. A sufficient condition for direct implementation is therefore that for every allocation in the reward set $Y_i(\theta_{-i})$ of agent i, another allocation y' which is weakly preferred to y by agent i can be obtained by an appropriate report of agent i's type. In particular, if agent i can induce the choice of every y by the social choice function f through an appropriate reward, then the direct mechanism offers all the feasible rewards.

Definition 16 (Full Range)
Social choice function f satisfies full range if for all i, all $\theta_{-i} \in \Theta_{-i}$ and $y \in Y$, there exists $\hat{\theta}_i$ such that $f\left(\hat{\theta}_i, \theta_{-i}\right) = y$.

Obviously, the full range condition is a very strong condition, but we shall now show that much weaker conditions will suffice in many environments. A common feature of many mechanism design models is that the allocation problem has two dimensions, the first is the assignment of the object and the second the monetary or quasi-monetary transfer. Within this two dimensional framework we can obtain positive results for ex post implementation in the direct mechanism. We thus suppose for the moment that the utility function of each agent permits the following representation:

$$u_i(y, \theta) = \widehat{u}_i(y_0, y_i, \theta), \tag{11}$$

where $\widehat{u}_i$ is strictly increasing in y_i, generalizing the monetary aspect in the quasilinear model. In the remainder of this section we hence investigate the implementation of a given social function f with $f = (f_0, f_1, f_2, ...f_I)$, where the pair (f_0, f_i) represents the relevant two dimensions of the allocation problem for agent i with $Y_i \subset \mathbb{R}$ for every i. We can now restrict the full range condition to the single dimension of every agent i.

Definition 17 (One Dimensional Full Range)
Social choice function f satisfies one dimensional full range if for all i, $y_0 \in Y_0$, $\theta_{-i} \in \Theta_{-i}$, there exists $\hat{\theta}_i$ such that $f_0\left(\hat{\theta}_i, \theta_{-i}\right) = y_0$.

With the monotone utility $\hat{u}_i$ in y_i and the one dimensional full range condition we can indeed guarantee direct implementation.

Proposition 1 (Direct Implementation with One Dimensional Full Range)
If f satisfies $(EPIC)$, (EM) and one dimensional full range, then direct ex post implementation is possible.

Proof. We defined the reward set as:

$$Y_i(\theta_{-i}) = \left\{ y : u_i\left(f\left((\theta_i', \theta_{-i})\right), (\theta_i', \theta_{-i})\right) \geq u_i\left(y, (\theta_i', \theta_{-i})\right) \text{ for all } \theta_i' \right\},$$

and we define the set of allocations attainable for i in the direct mechanism by:

$$Y_i^*(\theta_{-i}) = \left\{ y : y = f\left(\theta_i', \theta_{-i}\right) \text{ for some } \theta_i' \right\}.$$

We now want to show that $y \in Y_i(\theta_{-i}) \Rightarrow \exists y' \in Y_i^*(\theta_{-i})$ such that $u_i(y', \theta) \geq u_i(y, \theta)$ for all θ. To do this, fix any $y \in Y_i(\theta_{-i})$. By one dimensional full range, there exists θ_i' such that $f_0\left(\theta_i', \theta_{-i}\right) = y_0$. If $f_i\left(\theta_i', \theta_{-i}\right) < y_i$, then $u_i(y, (\theta_i, \theta_{-i})) > u_i\left(f\left(\theta_i', \theta_{-i}\right), (\theta_i', \theta_{-i})\right)$, contradicting $y \in Y_i(\theta_{-i})$. So $f_i\left(\theta_i', \theta_{-i}\right) \geq y_i$. So $u_i\left(f\left(\theta_i', \theta_{-i}\right), \hat{\theta}\right) \geq u_i\left(y, \hat{\theta}\right)$ for all $\hat{\theta}$. ■

The full range condition together with the monotonicity in the utility essentially guarantees that the agent can make a sufficiently large misreport to find an appropriate reward. The public good example on Section 3 satisfies the monotonicity condition and the one dimensional full range condition if $\Theta_i = \mathbb{R}_+$. If we replace the monotonicity condition by the single crossing condition, then a local change in the report is sufficient to guarantee the reward to the whistle-blower. The type space Θ_i for every agent i now has to be an open set so that a local change in the report is always feasible.

Proposition 2 (Direct Implementation with Single Crossing)
In a single crossing environment, if Θ_i is an open set for every i, and f is interior, continuous in θ, and satisfies strict EPIC, then f can be ex post implemented in the direct mechanism.

Proof. We first observe that with continuity, strict ex post incentive compatibility, and monotonicity in the second argument, it must be the case that for all θ_i, θ_i' and $\theta_i \neq \theta_i'$ $f_0(\theta_i, \theta_{-i}) \neq f_0\left(\theta_i', \theta_{-i}\right)$. Suppose not, then by strict ex post incentive compatibility the allocations have at least to differ in the second dimension, or, $f_i(\theta_i, \theta_{-i}) \neq f_i\left(\theta_i', \theta_{-i}\right)$. But since $u_i(\cdot)$ is strictly increasing in y_i, it follows that this would violate the ex post incentive constraint for either θ_i or θ_i'. It now follows from continuity that $f_0(\theta_i, \theta_{-i})$ must be monotone in θ_i for every θ_{-i}.

We next show that the inequalities of ex post monotonicity can be satisfied for all θ, θ' and that a reward y can always be obtained by means of an allocation which is generated by the social choice function for some report $\widetilde{\theta}_i$ of agent i. Thus consider the indifference curve for θ and $\widehat{\theta}$ at $f(\theta)$. Since the preferences are single crossing, there is a sequence of allocations $\{y_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} y_n = f(\theta)$ such that for all y_n along the sequence

$$u_i(f(\theta), \theta) > u_i(y_n, \theta),$$

and

$$u_i\left(f(\theta), \widehat{\theta}\right) < u_i\left(y_n, \widehat{\theta}\right). \tag{12}$$

As in the earlier argument, we now argue that there exists an N such that for all $n \geq N$ and all y_n, we have

$$u_i\left(f\left(\theta_i', \theta_{-i}\right), \left(\theta_i', \theta_{-i}\right)\right) \geq u_i\left(y_n, \left(\theta_i', \theta_{-i}\right)\right). \tag{13}$$

The proof is by contrapositive. Suppose now that (13) were not to hold, and that there exists θ_i' such that for all N, we can find $n \geq N$ and y_n such that:

$$u_i\left(f\left(\theta_i', \theta_{-i}\right), \left(\theta_i', \theta_{-i}\right)\right) < u_i\left(y_n, \left(\theta_i', \theta_{-i}\right)\right),$$

then it would follow from continuity of the utility function that:

$$u_i\left(f\left(\theta_i', \theta_{-i}\right), \left(\theta_i', \theta_{-i}\right)\right) \leq u_i\left(f\left(\theta_i, \theta_{-i}\right), \left(\theta_i', \theta_{-i}\right)\right).$$

But this violates the hypothesis of strict ex post incentive compatibility. We have thus shown that for an appropriately chosen subsequence of $\{y_n\}_{n=1}^{\infty}$, converging to $y = f(\theta)$, all the elements satisfy (12) and (13).

By continuity and monotonicity of $f(\cdot)$, it follows that we can find a type $\hat{\theta}_i$ of agent i an element $\hat{y} \triangleq y_n$, such that $f_0\left(\hat{\theta}_i, \theta_{-i}\right) = \hat{y}_0$. By monotonicity of $u_i(\cdot)$ in y_i and the validity of (13), it follows that the corresponding component satisfies $f_i\left(\hat{\theta}_i, \theta_{-i}\right) \geq \hat{y}_i$. Again by monotonicity of $u_i(\cdot)$ in y_i, it now follows that $f\left(\hat{\theta}_i, \theta_{-i}\right)$ is at least as desirable for agent i as $\hat{y}$. Thus agent i weakly prefers to claim $f\left(\hat{\theta}_i, \theta_{-i}\right)$ to $\hat{y}$. But now it follows that agent i can claim the reward simply by reporting $\hat{\theta}_i$ in the direct mechanism without appealing to the augmented mechanism where he would claim $\hat{y}$. ■

The public good example on Section 3 satisfies the condition of Proposition 2 provided the type space Θ_i is open for every i.

The openness condition on the type space Θ_i simply guarantees that agent i can always "downward" and "upward" misreport and obtain a suitable reward y. It follows that we can easily relax the openness condition and obtain a quasi direct implementation by means of the following construction. For any given type space Θ_i, if we can find an open set $\overline{\Theta}_i$ such that $\Theta_i \subseteq \overline{\Theta}_i$ and the single crossing conditions extend to $\overline{\Theta}_i$, then we can directly apply the argument of Proposition 2 to the extended space $\overline{\Theta} = \times_{i=1}^{I} \overline{\Theta}_i$. Moreover, in equilibrium the agents will report only types $\theta \in \Theta$ belonging to the original type space.

Finally, we can directly capture the idea that a small change in the report is sufficient to claim a successful reward. For every agent i, let the type space be a subset of the real line, or $\Theta_i \subset \mathbb{R}$.

We say that an agent i has reactive preferences if all agents weakly overstate their type, and at least one agent strictly overstates his type, then we can find an agent who would like to report his true type rather than overstate his type.

Definition 18 (Reactive Preferences)
The agents have reactive preferences if for all $\theta \leq \theta'$, there exists i such that

$$u_i\left(f\left(\theta'_i, \theta'_{-i}\right), \theta\right) < u_i\left(f\left(\theta_i, \theta'_{-i}\right), \theta\right); \tag{14}$$

and symmetrically if for all $\theta \geq \theta'$, there exists i such that

$$u_i\left(f\left(\theta'_i, \theta'_{-i}\right), \theta\right) < u_i\left(f\left(\theta_i, \theta'_{-i}\right), \theta\right).$$

The notion of reactive preferences might be weakened to simply require that for all $\theta \leq \theta'$, there exists i and θ''_i with $\theta''_i < \theta'_i$ such that the inequality,

$$u_i\left(f\left(\theta'_i, \theta'_{-i}\right), \theta\right) < u_i\left(f\left(\theta''_i, \theta'_{-i}\right), \theta\right),$$

holds. The notion of reactive preferences is stated for general utility functions, and does not require the earlier monotonicity or separability conditions of (11).

Proposition 3 (Direct Implementation with Reactive Preferences)
If the agents have reactive preferences and the type space is compact, then every strict ex post incentive compatible f can be ex post implemented in the direct mechanism.

Proof. The argument is by contradiction. Wlog we suppose that the type space of each agent i, $\Theta_i = \left[\underline{\theta}_i, \overline{\theta}_i\right]$, is given by a compact interval. Suppose in the direct mechanism there is an ex post equilibrium $s^* = (s^*_1, ..., s^*_I)$ such that for some agent i and some type θ_i, we have $s^*_i(\theta_i) \neq \theta_i$. Without loss of generality, we assume that $s_i(\theta_i) > \theta_i$. Consider then the type profile:

$$\left(\theta_i, \underline{\theta}_{-i}\right) = \left(\underline{\theta}_1, ...\underline{\theta}_{i-1}, \theta_i, \underline{\theta}_{i+1}, ..., \underline{\theta}_I\right),$$

at which all agents, but not necessarily i, have the lowest possible type. By strict ex post incentive compatibility, it follows that at $\underline{\theta}^i$, there must at least exist one more agent, say j, who misreports his type. But at $\theta_j = \underline{\theta}_j$, the misreport will have to be upwards, or $s_j\left(\underline{\theta}_j\right) > \underline{\theta}_j$. But with reactive preferences (see inequality (14)), this reporting profile cannot form an ex post equilibrium. ■

It can again be verified that the public good example of Section 3 satisfies the condition of Proposition 3 provided the type space Θ_i is compact for every i.

We also observe that Proposition 3 does not require the single crossing environment nor the interiority condition. In fact, the proof relies on the strictness of the ex post incentive constraints and the fact that with a compact type space there is a smallest and largest type for every agent. The notion of reactive preferences is of special interest as it includes the supermodular environment with quasilinear utility. An environment is supermodular if for all i, j:

$$\frac{\partial u_i}{\partial y_0 \partial \theta_j} \geq 0.$$

The notion of supermodularity adopted here is slightly stronger than typically used in the mechanism design literature, where the cross derivative is supposed to hold for agent i only with respect to his own signal, but not with respect to the signal of the other agents. With the supermodular environment, every ex post incentive compatible allocation x has to be (weakly) increasing in θ_i. The extension to all agent-signal pairs (i, j) is important in our context to guarantee the reactiveness of the preferences. By supermodularity of utility i and signal θ_i, and strict ex post incentive compatibility we have:

$$\frac{\partial u_i \left(y_0 \left(\theta_i', \theta_{-i}'\right), \left(\theta_i, \theta_{-i}'\right)\right)}{\partial y_0} \frac{\partial y_0 \left(\theta_i', \theta_{-i}'\right)}{\partial \theta_i} \leq 0,$$

for all $\theta_i < \theta_i'$. By supermodularity we have:

$$\frac{\partial u_i \left(y_0 \left(\theta_i', \theta_{-i}'\right), \left(\theta_i, \theta_{-i}\right)\right)}{\partial y_0} < \frac{\partial u_i \left(y_0 \left(\theta_i', \theta_{-i}'\right), \left(\theta_i, \theta_{-i}'\right)\right)}{\partial y_0},$$

from which the inequality (14) follows directly.

7 Single Unit Auction

We consider the efficient social choice rule in the single unit auction with interdependent values as in Dasgupta & Maskin (2000). The auction model presents an interesting environment as it fails both the strict ex post incentive compatibility condition as well as the interiority condition. The assignment of the object among the agents changes only at pivotal types. As it stays constant for many reported types, it satisfies weak but not strict ex post incentive compatibility. The efficient assignment problem is also a canonical example of an exterior social choice function as for almost all preferences profiles, one agent receives the object with probability one and all other agents receive the object with probability zero. Despite the failure of the model to satisfy the conditions of Theorem 4, we will show that the local violations of strict ex post incentive compatibility and interior allocation can be overcome to establish implementation in ex post equilibrium, in particular by the direct mechanism. Incidentally, the local failure of these conditions leads to a failure of Maskin monotonicity when ex post monotonicity is still guaranteed.

7.1 Model

The utility function of agent i in the assignment problem is given by:

$$u_i(x_i, t_i, \theta) = x_i v_i(\theta) + t_i,$$

where x_i is the probability that agent i receives the object, t_i is his monetary payment and $v_i(\theta)$ is his interdependent valuation. We assume that $v_i(\theta)$ is continuously differentiable in θ_i and that:

$$\frac{\partial v_i(\theta)}{\partial \theta_i} > 0 \text{ and } \frac{\partial v_i(\theta)}{\partial \theta_j} \neq 0. \tag{15}$$

The first condition simply says that a higher signal by i leads to a higher value of i and the second condition guarantees that we are in an interdependent rather than private value environment. The single crossing condition is, as in Dasgupta & Maskin (2000), that for all i, j and θ, if

$$v_i(\theta) = v_j(\theta) = \max_k \{v_k(\theta)\},$$

then

$$\frac{\partial v_i(\theta)}{\partial \theta_i} > \frac{\partial v_j(\theta)}{\partial \theta_i}. \tag{16}$$

In other words, if i and j tie for the largest value, then the marginal value of a higher type θ_i is higher for agent i than for agent j. In addition, we shall restrict our attention to a symmetric environment and a compact type space $\Theta_i = [0, 1]$. In other words, for all i, j and all θ, we assume

$$v_i(\theta_i, \theta_j, \theta_{-ij}) = v_j(\theta_j, \theta_i, \theta_{-ij}),$$

We consider the efficient allocation rule and in case of a tie at the top, we assign the object with equal probability among the agents with the highest valuation. The ex post incentive compatible transfer rule is of the form:

$$t_i(\theta) = -v_i(\underline{\theta}_i, \theta_{-i}), \tag{17}$$

where $\underline{\theta}_i$ is determined by:

$$\underline{\theta}_i = \min\left\{\theta_i' \in \Theta_i \left| v_i\left(\theta_i', \theta_{-i}\right) \geq v_j\left(\theta_i', \theta_{-i}\right), j \neq i\right.\right\}.$$

We refer to the efficient direct mechanism satisfying (17) as the generalized Vickrey Clark Groves (VCG) mechanism.

7.2 Monotonicity and the VCG Mechanism

We present three results in this section. We first show that the generalized VCG mechanism satisfies the ex post monotonicity condition. In fact the positive result is strengthened to obtain ex post implementation in the direct mechanism. We then show that even though ex post monotonicity is satisfied, Maskin monotonicity fails.

Proposition 4 (Ex Post Monotonicity)

1. *For $I \geq 3$, the generalized VCG mechanism satisfies ex post monotonicity.*

2. *For $I \geq 3$, the direct generalized VCG mechanism has a unique pure ex post equilibrium.*

Proof. By Theorem 1, ex post monotonicity is a necessary condition for ex post implementation. It is therefore sufficient to show that the generalized VCG mechanism can be ex post implemented. We show that the unique equilibrium in the direct mechanism is the truthtelling equilibrium. Suppose not and there exists another ex post equilibrium. It follows that for some agent i and some type profile θ_i, we have that $s_i(\theta_i) \neq \theta_i$. We define the highest possible type across all agents for which we observe a report different from truthtelling:

$$\overline{\theta} = \max_{i \in \mathcal{I}} \sup \{\theta_i \in \Theta_i \,|s_i(\theta_i) \neq \theta_i\}. \tag{18}$$

We suppose initially that

$$\sup \{\theta_i \in \Theta_i \,|s_i(\theta_i) \neq \theta_i\} = \max \{\theta_i \in \Theta_i \,|s_i(\theta_i) \neq \theta_i\},$$

for all $i \in \mathcal{I}$, which we shall later relax.

Consider first $\overline{\theta} < 1$ and $s_i(\overline{\theta}) < \overline{\theta}$. We take agent i with $\theta_i = \overline{\theta}$ and $s_i(\overline{\theta})$ and consider for all other agents $j \neq i$, a type profile $\theta_j = \overline{\theta} + \varepsilon$ for some arbitrarily small $\varepsilon > 0$ so that $s_i(\overline{\theta}) < \overline{\theta} < \theta_j$. At θ_j, we know from (18), that $s_j(\theta_j) = \theta_j$. It follows that at the type profile (θ_i, θ_{-i}) and associated reports, all agents $j \neq i$ receive the object with the same probability but due to the misreport of agent i at a transfer $t_j(\cdot)$ (per unit of the object)

$$t_j\left(s_i(\overline{\theta}), \theta_j = \overline{\theta} + \varepsilon\right) > -v_j\left(\theta_i = \overline{\theta}, \theta_j = \overline{\theta} + \varepsilon\right), \tag{19}$$

which is strictly below the value of the object for agent j. It follows that every agent $j \neq i$ has a unilateral profitable deviation by reporting a higher type $\theta_j' > \theta_j$. With the higher report, he will still pay the same transfer (per unit) by the VCG mechanism, but agent j will then receive the object with probability one. By (19), the net utility of the transaction is strictly positive and hence agent j strictly increases his payoff with the deviation.

Consider next $\overline{\theta} < 1$ and $s_i\left(\overline{\theta}\right) > \overline{\theta}$. We now take agent i at $\theta_i = \overline{\theta}$ with $s_i\left(\overline{\theta}\right)$ and consider for all other agents $j \neq i$, $\theta_j = \overline{\theta} + \varepsilon$ for some arbitrarily small $\varepsilon > 0$. At this profile $\theta_i = \overline{\theta}$ and $\theta_j = \overline{\theta} + \varepsilon$, agent i receives the object under the deception but at a transfer which is larger than the value of the object to him:

$$t_i\left(s_i\left(\overline{\theta}\right), \theta_j = \overline{\theta} + \varepsilon\right) < -v_i\left(\theta_i = \overline{\theta}, \theta_j = \overline{\theta} + \varepsilon\right). \tag{20}$$

It follows that by reporting a sufficiently low type profile so that he will not receive the object and receive a zero transfer, he can guarantee himself a zero net utility which is a strict improvement about his candidate negative net utility as displayed in (20).

Consider next $\overline{\theta} = 1$, then $s_i\left(\overline{\theta}\right) < \overline{\theta}$ has to hold. We observe first that at most one agent i can offer a downward biased report in equilibrium. If more than one agent would downward report at $\theta_j = 1$, then each one of the downward reporting agents would have a strict incentive to report the maximal type $\theta_j = 1$, as the monetary transfer (per unit) would be strictly less than the value of the object. It thus follows that all agents $j \neq i$ report truthfully at $\theta_j = 1$, for otherwise they would downward report and then agent i would always have a strict incentive to recover his bid and report truthfully.

By the same argument, it also follows that no agent other than i can ever downward report at any type profile $\theta_j \in \Theta_j$. Suppose to the contrary and that agent j downwards report at some θ_j' with $s_j\left(\theta_j'\right) < \theta_j'$. Consider then the true profile $\theta = (\theta_j, 1, 1, ..., 1)$. With the downward report $s_j\left(\theta_j'\right) < \theta_j$, agent i would again have a strict incentive to report truthfully as the downward report by agent j leads to a transfer payment strictly less than the true value of the object. It hence follows that for all agents $j \neq i$ we only have to consider truthtelling or upward deceptions.

With agent i downward reporting at $\theta_i = 1$, we now argue that at least one agent $j \neq i$ has to consistently upward report for all $\theta_j \in (s_i(1), 1)$. For suppose not, then we can find some type profile θ_{-i} with $\theta_j \in (s_i(1), 1)$ for all $j \neq i$, such that all agents j report truthfully. But at such a type profile agent i would loose with his report of $s_i(1)$, when he values the object higher than everybody else and would have to pay a transfer strictly less than the value of the object to him. As this cannot be an equilibrium strategy profile, it follows that at least one agent most consistently misreport upwardly.

We finally argue that this cannot be an equilibrium strategy profile for agent j either. To see this consider the true profile $\theta_i = 0$ for agent i. At $\theta_i = 0$, agent i cannot downward report, at least he has to report truthfully, and by the earlier argument all the other agents also report at least truthfully. But now consider agent j with an upward report at some θ_j' and $s_j\left(\theta_j'\right) > \theta_j'$. Consider now the type profile $\theta = \left(\theta_i = 0, \theta_j = \theta_j', s_j\left(\theta_j'\right), ..., s_j\left(\theta_j'\right)\right)$. In other words, at type profile θ all agents but i and j have a true type exactly equal to the reported profile of agent j. It follows that either agent j receives the object with positive probability at θ or that he looses out as some other agent, say k, different from i and j also misreports upwardly. But in either case, agent j or agent k will have to pay more than the object is worth to them. It follows that either the candidate strategy of agent j or k offers a profitable deviation by sufficiently lowering the report so that either j or k fails to get the object, receives a zero transfer and guarantees himself a zero net utility. We thus have obtained a contradiction to a candidate equilibrium involving a downward report at $\overline{\theta} = 1$.

It remains to consider the situation where

$$\overline{\theta} = \max_{i \in \mathcal{I}} \sup \{\theta_i \in \Theta_i \,| s_i(\theta_i) \neq \theta_i\} \neq \max_{i \in \mathcal{I}} \max \{\theta_i \in \Theta_i \,| s_i(\theta_i) \neq \theta_i\},$$

for some i. By definition of $\overline{\theta}$, for every $\varepsilon > 0$, we can then find an agent i and a type θ_i, such that $\overline{\theta} - \varepsilon < \theta_i < \overline{\theta}$, and $s_i(\theta_i) \neq \theta_i$. Now we can repeat the above arguments for $\overline{\theta} < 1$ with $s_i\left(\overline{\theta}\right) < \overline{\theta}$ or $s_i\left(\overline{\theta}\right) > \overline{\theta}$ at the type profile $\theta = \left(\theta_i, \overline{\theta}, \overline{\theta}, ..., \overline{\theta}\right)$.

For $\overline{\theta} = 1$, we can still find for every $\varepsilon > 0$, an agent i and a type θ_i, such that $1 - \varepsilon < \theta_i < 1$, and $s_i(\theta_i) \neq \theta_i$. If for all such θ_i, we have $s_i(\theta_i) < \theta_i$, then the above argument goes through without modifications. The remaining possibility is that for all $\varepsilon > 0$, and θ_i satisfying $1 - \varepsilon < \theta_i < 1$, we have $s_i(\theta_i) > \theta_i$. But this cannot be part of an equilibrium either because at the profile $\theta = (\theta_i, 1, 1, ..., 1)$, all agents $j \neq i$ report by assumption truthfully, but with $s_i(\theta_i) > \theta_i$, every agent j would pay a transfer strictly exceeding the value of the object. Thus, we have excluded all other candidate ex post equilibria which involve misreporting for some agents at some type profiles.
■

The idea behind the proof is quite simply and relies essentially on interdependent rather than private valuations. Essentially we used two reward schemes. We either gave a currently winning agent the object with probability one at the current price or we released the current winner, i.e. did not give him the object and gave him a zero transfer. These rewards can be (implicitly) claimed in the direct mechanism. If an agent k finds it profitable to receive the object with probability one, then he can do so in the direct mechanism by slightly increasing his reported type. If an agent k would like not to receive the object, then he could always guarantee this by lowering his announced type profile.

This basic argument highlights the role of pivotal profiles at which an increase or decrease in the reported type leads to a change in the allocation. At type profiles at which the auctioneer is indifferent between assigning the object among two or more agents, the pivotal or competitive profiles, two important things happened. First, the allocation rule is now in the interior as the auctioneer awards the object to competitive bidders with the same probability. Second, even though at the competitive profile, the ex post incentive constraints are only weak inequalities, in any neighborhood of the competitive profile, we can find strict ex post incentive constraints. In light of the earlier results on the single crossing environment, notably Theorem 4, we then find that it is the existence of pivotal profiles which matters for ex post implementation rather than the everywhere strictness of the ex post incentive constraints.

In this respect, we should point out that the role of the symmetric valuations is precisely to facilitate the location of pivotal profiles θ. The proof of Proposition 4 would go through unchanged if we were to abandon symmetry everywhere except at the lowest and highest type profiles, or at $\theta = (0, ..., 0)$ and $\theta = (1, ..., 1)$. In fact, the only property of the symmetry at the bottom and the top we really need, is that for every type profile θ_i of agent i, there exists another agent j and type profile θ_{-i} such that j is competitive with respect to i at $\theta = (\theta_i, \theta_{-i})$. Hence a different sufficient condition for ex post implementation in the single unit auction model would be the full range condition introduced in Section 5.

Despite the positive ex post implementation results, the generalized VCG mechanism fails Maskin monotonicity. Indeed, the generalized VCG mechanism is an important example where the two monotonicity notions fail to coincide.

Proposition 5 (Maskin Monotonicity)
The generalized VCG mechanism fails Maskin monotonicity.

Proof. Consider a profile θ such that

$$v_i(\theta) > v_j(\theta), \ \forall j \neq i.$$

Then by the single crossing conditions (15) and (16) we can find θ' with

$$\theta'_i > \theta_i, \quad \theta'_j < \theta_j$$

such that

$$v_i(\theta') > v_i(\theta), \quad v_j(\theta') < v_j(\theta).$$

It then follows that

$$t_i(\theta) > t_i(\theta').$$

We now recall that a social choice function f is Maskin monotone, if for all $\theta, \theta' \in \Theta$:

$$u_i(f(\theta), \theta) \geq u_i(y, \theta) \Rightarrow u_i(f(\theta), \theta') \geq u_i(y, \theta') \tag{21}$$

for all i and y, then

$$f(\theta) = f(\theta'). \tag{22}$$

We can verify that the implication (21) holds for all i and y, but the conclusion to be drawn, $f(\theta) = f(\theta')$, obviously fails as the transfers offered to agent i have to be different to guarantee incentive compatibility. ■

We should add that there are transfer rules which satisfy Maskin monotonicity, but necessarily fail ex post incentive compatibility. For example, a constant transfer rule satisfies Maskin monotonicity, but obviously is not ex post incentive compatible. Yet as Maskin monotonicity is concerned with complete information environments, this may be less of a concern for the notion of Maskin monotonicity.

The existence rather than ubiquitousness of the pivotal profiles also underlies the difference between ex post and Maskin monotonicity. With ex post monotonicity we can search for a competitive profile among all types of agent i given the type profile of the other agents, whereas with Maskin monotonicity and the inherent synchronicity of complete information, we cannot do that. In particular, the fact that for most type profiles we are at the exterior of the allocation space is a problem for Maskin, but not necessarily for ex post monotonicity. Provided that the social choice function is at least sometimes in the interior we can use the single crossing condition at the interior profiles.

This basic intuition as to why ex post monotonicity holds, but Maskin monotonicity fails are next presented graphically. For simplicity, the illustrations are based on the following linear model:

$$v_i(\theta) = \theta_i + \gamma \sum_{j \neq i} \theta_j, \quad 0 < \gamma < 1, \tag{23}$$

and $I = 3$.

We begin with Maskin monotonicity and consider the profile $\theta = (1, 0, 0)$ and associated deception $\alpha(\theta) = \theta' = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. In this deception, agent 1 understates his type and agent 2 and 3 overstate their type, yet even under the deception, agent 1 receives the object and pays $t_1(\theta') = -\left(\frac{1}{4} + \gamma\left(\frac{1}{4} + \frac{1}{4}\right)\right)$. The reward sets for agent 1 and agent 2 and 3 are depicted below in Figure 5A and 5B respectively. We then ask whether at the true type profile $\theta = (1, 0, 0)$ we can offer a successful reward when constrained to the reward sets $Y_i(\theta')$. But at true profile $\theta = (1, 0, 0)$, agent 1 is even more keen on winning, and agents 2 and 3 are even less keen on winning, so that there is no allocation in the reward set which could be used as a successful reward. Pictorially, the indifference curve through the allocation $f(\theta')$ for the type profile θ is strictly below the indifference curve for the type profile θ' for all three agents. In consequence Maskin monotonicity fails for the generalized VCG mechanism.

Insert Figure 5: Maskin Monotonicity In Single Unit Auction

We contrast this negative result for Maskin monotonicity with a positive result for ex post monotonicity. As we start with the earlier observation that the reward set $Y_i\left(\theta'_{-i}\right)$ is smaller with ex post monotonicity than with Maskin monotonicity, we do not seem to be on a promising track. Indeed if we consider the true type $\theta = (1,0,0)$ and the deception $\alpha(\theta) = \theta' = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$, then we find that the reward sets for all agents are now strictly smaller. The reward sets $Y_i\left(\theta'_{-i}\right)$ are identified below as the sets which are delineated by the 45° line and thus strictly smaller than before.

Insert Figure 6: Ex Post Monotonicity In Single Unit Auction (I)

The difference between Maskin and ex post monotonicity emerges as we observe that the deceptive strategy $\alpha_i : \Theta_i \to \Theta_i$ is determined by agent i independently of the other agents. In particular, his deception $\alpha_i(1) = \frac{1}{2}$ is constant with respect to the type profile and the reports of the other agents. It is this independence which we can then exploit. For example, we may consider the true type profile $\widehat{\theta} = (1,1,1)$. At this profile agent 1 still reports $\alpha_i(1) = \frac{1}{2}$. We have not yet said what the deception of agents 2 and 3 at this profile might be. But for any report of agents 2 and 3, the monetary transfer of the winning agent will be strictly less than his value for the object. Conversely, all loosing agents, would be willing to pay strictly more than the winning transfer conditional on the reported types. But this means that we can assign a currently loosing agent the object at a transfer which is higher than the reported but lower than his valuation of the object. For simplicity, we depict below the case when agents 2 and 3 report their true value. In this case, all three agents can be considered losers as neither of them is receiving the object with probability one. It follows that we could offer every single one of the three agents a successful reward.

Insert Figure 7: Ex Post Monotonicity In Single Unit Auction (II)

7.3 Private versus Interdependent Values

We are now in the position to make an interesting observation regarding the implementation of the single unit auction. Suppose we consider the Vickrey auction (for the private value model) and the generalized Vickrey-Clark-Groves mechanism (for the interdependent value model). With private values, we have a failure of Maskin and ex post monotonicity. However with interdependent values, this coincidence ceases to exist and while Maskin monotonicity continues to fail, ex post monotonicity can be reestablished. While it is well-known that the second price auction cannot be Nash implemented (see Saijo, Sjostrom & Yamato (2004)), the positive results here regarding interdependent valuations are new in the literature.

The intuition for the divergence between private and interdependent values goes as follows. In a private value model, if one agent submits the highest possible report and all other agents submit the lowest possible report, then the former receives the object at the lowest possible price. Any attempt to reward a whistle-blower will then inevitably violate the ex post incentive compatibility constraint. In contrast, consider the exact same reporting strategy with interdependent valuations. Now we can reward a loosing agent in all those instances where the loosing agents all report the lowest value but in fact all have a higher valuation. We can reward every loosing agent by giving him the object and asking him to pay only as much as the reported value would suggest. This satisfies ex post incentive compatibility, but at the same time provides the reward to break the undesired equilibrium. Yet, it is clear that the argument relies essentially on interdependent rather than private values. The distinction in the ex post implementation result between private and interdependent values continues to exist for any arbitrary small amount of interdependence.

The distinction between private and interdependent values also becomes apparent in the role of the three or more agent condition in Proposition 4. In a recent paper, Birulin (2003) shows that with two agents there is a continuum of inefficient undominated ex post equilibria in the single unit auction model with the generalized VCG mechanism. The basic insight in Birulin (2003) is easiest explained within the earlier linear set-up (see (23)). In this environment, the truthtelling strategy certainly forms an efficient ex post equilibrium. But Birulin (2003) shows there are other equilibria which involve monotone, but discontinuous, reporting strategies, which lead to inefficient equilibria. The idea behind the construction is that over an arbitrary interval of profiles, agent i overstates and agent j understates his type. The following pair of deceptions illustrate how an inefficient equilibrium may arise. Suppose agent i and j misreport over the same interval:

$$\left(\underline{\theta}_i, \overline{\theta}_i\right) = \left(\underline{\theta}_j, \overline{\theta}_j\right) \subset [0,1],$$

and otherwise report truthfully. On the interval $\left(\underline{\theta}_i, \overline{\theta}_i\right)$ agent i upward misreports:

$$s_i(\theta_i) = \begin{cases} \theta_i, & \text{if} \quad \theta_i \in [0, \underline{\theta}_i] \\ \frac{\overline{\theta}_i + \gamma\theta_i}{1+\gamma}, & \text{if} \quad \theta_i \in (\underline{\theta}_i, \overline{\theta}_i) \\ \theta_i, & \text{if} \quad \theta_i \in [\overline{\theta}_i, 1] \end{cases} ; \tag{24}$$

whereas agent j downward misreports over the interval $\left(\underline{\theta}_j, \overline{\theta}_j\right)$:

$$s_j(\theta_j) = \begin{cases} \theta_j, & \text{if} \quad \theta_j \in \left[0, \underline{\theta}_j\right] \\ \frac{\underline{\theta}_j + \gamma\theta_j,}{1+\gamma}, & \text{if} \quad \theta_j \in \left(\underline{\theta}_j, \overline{\theta}_j\right) \\ \theta_j, & \text{if} \quad \theta_j \in \left[\overline{\theta}_j, 1\right] \end{cases} . \tag{25}$$

The reporting strategy of each agent is monotone, but discontinuous at $\underline{\theta}_i$ and $\overline{\theta}_j$, respectively. With the above deception strategy, agent i will win the auction whenever $\theta = (\theta_i, \theta_j) \in \left(\underline{\theta}_i, \overline{\theta}_i\right) \times \left(\underline{\theta}_j, \overline{\theta}_j\right)$, irrespective of whether it is efficient to allocate the object to i or not. In contrast for any $\theta \notin \left(\underline{\theta}_i, \overline{\theta}_i\right) \times \left(\underline{\theta}_j, \overline{\theta}_j\right)$, the equilibrium with the deception strategies (α_i, α_j) will lead to the efficient assignment of the object. The reporting strategies inside the open interval are chosen such that they do not prevent truthtelling outside of the interval. In fact, the reporting strategies in (24) and (25) are constructed precisely on the basis of the indifference conditions of i and j at the critical profiles, $\theta_i = \underline{\theta}_i$ and $\theta_j = \overline{\theta}_j$. The misreport of agent j is chosen such that the resulting transfer payment of the generalized VCG mechanism makes agent i at $\theta_i = \underline{\theta}_i$ just indifferent between receiving the object and not receiving it:

$$\underline{\theta}_i + \gamma\theta_j = s_j\left(\theta_j\right) + \gamma s_j\left(\theta_j\right). \tag{26}$$

The rhs of the equality is the true value of the object to i and the lhs is the transfer agent i would have to pay if, at the reported type $s_j\left(\theta_j\right)$, agent i were to receive the object. The above equality determines the deception strategy of agent j which makes truthtelling for agent i ex post incentive compatible even under the misreport of agent j and leads precisely to the deception strategy displayed in (25). The related indifference condition for agent j is given by

$$\overline{\theta}_j + \gamma\theta_i = s_i\left(\theta_i\right) + \gamma s_i\left(\theta_i\right),$$

and its solution leads again to the deception strategy displayed in (24).

We notice that the deception strategy of agent i is basically determined by agent j's true valuation and vice versa. This indicates that the same construction is not feasible anymore when there are more than two agents. With more than two agents, the valuation of agent j will depend on the type of agent i, j and k. In consequence the derivation of the deception strategy of agent i will not only depend on agent $j's$ critical type, but the *exact* type of agent k. But of course the strategy of agent i cannot depend simultaneously on the type of agent j and k. In fact, Proposition 4 showed that there will be no construction of a deception strategy which will lead to an equilibrium different from the truthtelling equilibrium.

8 Social Choice Sets

In the initial discussion of ex post and Maskin monotonicity we argued that the notions diverge in two aspects: (i) the set of profiles at which rewards could be offered and (ii) the size of the reward set. The single unit auction demonstrated the relevance of the first aspect. The set of profiles at which the designer could offer a reward to the whistle-blower was larger with ex post monotonicity. In consequence, we could satisfy ex post monotonicity yet fail Maskin monotonicity. This section considers the reverse case, in which Maskin monotonicity is satisfied but ex post monotonicity fails.

At the end of this section, we shall also discuss the relationship between functions, correspondences and sets in the context of ex post implementation.

8.1 Pareto Correspondence

Maskin (1999) observes that many prominent social choice correspondences, among them the Pareto, the Condorcet and the Walrasian correspondences, satisfy the complete information monotonicity notion. We now show with a specific Pareto correspondence that ex post monotonicity fails to share this property. Similar examples can be constructed for the Condorcet and the Walrasian correspondence. As we would expect, the divergence between the two notions arises from the difference in the respective reward sets.

The Pareto correspondence is generally defined by:

$$PO(\theta) = \{y \in Y \mid \forall z \in Y, \exists i \text{ s.t. } u_i(y, \theta) \geq u_i(z, \theta)\}.$$

We consider an example with three agents, $i = 1, 2, 3$ and each agent has two possible types: $\theta_i \in \Theta_i = \{0, 1\}$. A type profile is then given by $\theta = (\theta_1, \theta_2, \theta_3) \in \Theta = \times_{i=1}^{3} \Theta_i$. The set of allocations, Y, has the same cardinality as the type space, Θ. For simplicity, allocations and type profiles carry the same labels, but allocations are described as strings rather than vectors:

$$Y = \{000, 001, 010, 011, 100, 101, 110, 111\}.$$

The payoffs of the agents are described for every true type profile $\theta = (\theta_1, \theta_2, \theta_3)$ below. In each matrix, each cell identifies the utility from a specific allocation. As the type set coincides with the allocation set, the described utilities also represent the payoffs arising in the direct mechanism for any reported type profile by the agents:

$\theta = (0, 0, 0)$:

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$3, 0, 0$	$0, 0, 0$
$\theta_1 = 1$	$1, 1, 1$	$0, 0, 0$

$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$0, 0, 0$	$0, \varepsilon, 0$
$\theta_1 = 1$	$0, 0, 0$	$\varepsilon, \varepsilon, \varepsilon$

$\theta = (0, 0, 1)$:

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$0, 0, 0$	$0, 0, 0$
$\theta_1 = 1$	$0, 0, 0$	$\varepsilon, \varepsilon, \varepsilon$

$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$0, 3, 0$	$1, 1, 1$
$\theta_1 = 1$	$0, 0, 0$	$\varepsilon, 0, 0$

$\theta = (0, 1, 0)$:

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$1, 1, 1$	$0, 3, 0$
$\theta_1 = 1$	$0, 0, \varepsilon$	$0, 0, \varepsilon$

$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$0, 0, 0$	$0, 0, 0$
$\theta_1 = 1$	$\varepsilon, \varepsilon, \varepsilon$	$0, 0, 0$

$\theta = (0,1,1)$:

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$\varepsilon, 0, 0$	$1, 1, 1$	$\theta_1 = 0$	$0, 0, 0$	$0, 0, 3$
$\theta_1 = 1$	$\varepsilon, \varepsilon, \varepsilon$	$\varepsilon, \varepsilon, 0$	$\theta_1 = 1$	$0, 0, 0$	$0, \varepsilon, 0$

$\theta = (1,0,0)$:

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$0, 0, \varepsilon$	$\varepsilon, 0, \varepsilon$	$\theta_1 = 0$	$0, 0, 0$	$\varepsilon, \varepsilon, \varepsilon$
$\theta_1 = 1$	$0, 3, 0$	$1, 1, 1$	$\theta_1 = 1$	$0, 0, 0$	$\varepsilon, 0, 0$

$\theta = (1,0,1)$:

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$\varepsilon, 0, 0$	$\varepsilon, \varepsilon, \varepsilon$	$\theta_1 = 0$	$0, 0, 0$	$0, 0, 0$
$\theta_1 = 1$	$1, 1, 1$	$0, 0, 0$	$\theta_1 = 1$	$0, 0, 3$	$0, 0, 0$

$\theta = (1,1,0)$:

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$0, 0, 0$	$0, \varepsilon, 0$	$\theta_1 = 0$	$\varepsilon, \varepsilon, \varepsilon$	$\varepsilon, \varepsilon, 0$
$\theta_1 = 1$	$\varepsilon, 0, 0$	$0, 0, 1$	$\theta_1 = 1$	$0, 0, 0$	$1, 1, 1$

$\theta = (1,1,1)$:

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$\varepsilon, \varepsilon, \varepsilon$	$0, \varepsilon, 0$	$\theta_1 = 0$	$0, 0, 0$	$1, 1, 1$
$\theta_1 = 1$	$0, 0, \varepsilon$	$0, 0, 0$	$\theta_1 = 1$	$0, 0, 0$	$3, 0, 0$

The above example has the property that in every state θ, there exist exactly two Pareto efficient allocations. The first Pareto allocation corresponds to the true state: $y = \theta$ and it favors one agent with payoff 3 and leaves the remaining two agents with payoff 0. The identity of the favored agent is determined by $1 + \sum_i \theta_i \pmod 3$. The second Pareto allocation generates a uniform payoff of 1 across agents. The remaining allocations are all Pareto inferior. Besides the Pareto allocations, there is one more important allocation in this example, given by $y = \theta'$ with $\theta_i \neq \theta'_i$ for all i. It generates a payoff of ε for all agents in all states and is obviously Pareto dominated. Yet, we will show that it can be obtained as an ex post equilibrium under the deception $\alpha(\theta) = \theta'$. The remaining payoff vectors are combination of 0 and ε entries. In each one of the vectors, the 0 entries serve to support (always) truthtelling and (always) misreporting as ex post equilibria in the direct mechanism, whereas the ε entries serve to shrink the ex post reward set.

In this example, the Pareto correspondence is described by $PO : \Theta \to Y$:

$\theta_3 = 0$	$\theta_2 = 0$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$\{000, 100\}$	$\{010, 000\}$	$\theta_1 = 0$	$\{001, 011\}$	$\{011, 010\}$
$\theta_1 = 1$	$\{100, 110\}$	$\{110, 111\}$	$\theta_1 = 1$	$\{101, 010\}$	$\{111, 011\}$

(27)

The matrices describe the set of Pareto efficient allocations as a function of the true type profiles of the agents. The corresponding social choice set F is the set of all functions which satisfy $f(\theta) \in PO(\theta)$ for all $\theta \in \Theta$.

Maskin Monotonicity Maskin (1999) showed that the Pareto correspondence satisfies complete information monotonicity. For a better grasp of the difference between Maskin and ex post monotonicity, it may be instructive to see how these differences play out in this example. We therefore verify first that the Pareto selection $f(\theta) = \theta$ for all $\theta \in \Theta$, which assigns asymmetric utilities, is Maskin monotone. The only relevant deception is the complete deception with:

$$\alpha_i(\theta_i) \neq \theta_i, \forall i, \forall \theta_i.$$

Without loss of generality we may consider $\theta = (0,0,0)$ and $\alpha(0,0,0) = (1,1,1)$. By setting $y = 000$ and $i = 1$, we clearly satisfy Maskin monotonicity as:

$$3 = u_1(y,(0,0,0)) > u_1(f(\alpha(0,0,0)),(0,0,0)) = \varepsilon,$$

and

$$3 = u_1(f(\alpha(0,0,0)),\alpha(0,0,0)) \geq u_1(y,\alpha(0,0,0)) = \varepsilon.$$

Ex Post Monotonicity We continue with ex post monotonicity and again consider $\theta = (0,0,0)$ and $\alpha(0,0,0) = (1,1,1)$. We first observe that the candidate allocation $y = 000$, which we used in the complete information setting is not in the ex post reward set $Y_1(\theta_{-1})$ anymore. More precisely, if the true type of agent 1 is $\theta_1 = 0$, then agent 1 has an incentive to claim the reward $y = 000$ given the true type profile of the remaining two agents is $\theta_{-1} = (1,1)$, or:

$$0 = u_1(f(0,\alpha_{-1}(0,0)),(0,\alpha_{-1}(0,0))) < u_1(000,(0,\alpha_{-1}(0,0))) = \varepsilon.$$

At the true profile $\theta = (0,0,0)$ and deception $\alpha(0,0,0) = (1,1,1)$, the designer could alternatively offer the reward $y = 100$, which is the second Pareto allocation. For every agent i, this allocation satisfies the reward equality:

$$1 = u_i(100,(0,0,0)) > u_i(f(\alpha(0,0,0)),(0,0,0)) = \varepsilon.$$

It also satisfies the ex post incentive compatibility at the true type profile $\alpha(0,0,0) = (1,1,1)$ for agent 1 and 2, but fails for agent 3, as:

$$0 = u_3(f(\alpha(0,0,0)),\alpha(0,0,0)) < u_3(100,\alpha(0,0,0)) = \varepsilon.$$

Moreover, for agent 1 and 2, it fails to be satisfied at $\theta_i' \neq \theta_i$ with:

$$0 = u_1\left(f\left(0, \alpha_{-1}\left(0,0\right)\right), \left(0, \alpha_{-1}\left(0,0\right)\right)\right) < u_1\left(100, \left(0, \alpha_{-1}\left(0,0\right)\right)\right) = \varepsilon,$$

and

$$0 = u_2\left(f\left(0, \alpha_{-2}\left(0,0\right)\right), \left(0, \alpha_{-2}\left(0,0\right)\right)\right) < u_2\left(100, \left(0, \alpha_{-2}\left(0,0\right)\right)\right) = 1.$$

By construction, the same argument goes through at every type profile and in consequence, ex post monotonicity fails in this example. The failure of ex post monotonicity comes as the reward set $Y_i\left(\theta_{-i}\right)$ is strictly smaller than $Y_i\left(\theta\right)$ and in particular, eliminates all rewards which could satisfy the reward inequality.[12]

8.2 Functions, Sets and Correspondences

The Pareto set is an example of a social choice set rather than a social choice function. This naturally leads us to inquire the relationship between functions and sets in the context of ex post implementation. In particular, we can ask whether implementation of a social choice set F is equivalent to the implementation of every social choice function $f \in F$ separately. The obvious direction is that if every $f \in F$ can be (ex post) implemented, then the social choice set F can be implemented. The more difficult direction is easy to understand with the following example. Let the social choice set F be the set of *all* ex post incentive compatible *plans*, i.e. mappings from profiles to allocations. By construction, it follows that F is ex post monotone and can be ex post implemented. But of course a single element $f \in F$ may not be ex post implementable by itself as there might be multiple equilibria under the social choice function f which do not correspond to f under some profiles. By the revelation principle, any such distinct equilibrium will also be an equilibrium under the corresponding direct mechanism. Hence a deception α which forms an equilibrium in f is an element in F and by contrast would not harm the implementability of F. It follows more generally that the implementation of a social choice set F does not imply that every social choice function $f \in F$ can be implemented separately.

The Pareto set was defined as a correspondence from the set of profiles to the set of allocations. On the basis of the Pareto correspondence, we naturally defined an associated social choice set.

[12] The example is complicated as the Pareto correspondence is a set rather than a point everywhere. Yet, to display a difference between ex post and Maskin monotonicity, this appears to be necessary. First, observe, that if the payoffs were symmetric, then (generically) a Pareto efficient allocation would also constitute a strictly Pareto dominant allocation. The strictly dominant allocation clearly constitutes an ex post equilibrium and every deception α could be fended off by simply reestablishing the social choice allocation. The same argument continues to go through without symmetry if there is a unique Pareto allocation in every state. It follows that for a (generic) discrepancy between ex post and Maskin monotonicity, we need multiple Pareto efficient allocations, which (generically) have to display some asymmetries in the way they affect the utilities of the agents.

More generally, given a social choice correspondence $\phi : \Theta \to Y$, we can define an associated social choice set $F = \{f \,|f : \Theta \to Y\}$ by including all social choice functions f which select at all profiles allocations in the image of the correspondence:

$$F = \{f \,|f(\theta) \in \phi(\theta), \ \forall \theta \in \Theta\}.$$

Similarly, we can start with a social choice set F and define an associated social choice correspondence by including all allocations y at a profile θ which can be obtained by some selection f at θ from the social choice set F :

$$\phi(\theta) = \{y \,|y = f(\theta), \ f \in F\}.$$

With the above associations, we can then relate ex post implementation of sets and correspondences. For the purpose of this discussion, it might be useful to keep in mind the class of social choice problems in which a designer faces agents with quasilinear utility and wishes to implement the social efficient allocation without any balanced budget considerations. The social choice set of efficient and ex post incentive compatible allocations is then very large as the transfers to the agents are essentially only determined up to a constant. However if we consider the associated social choice correspondence, then we will typically loose the ex post incentive compatibility as arbitrary combinations of transfers across profiles will not satisfy ex post incentive compatibility. For this reason, incomplete information implementation typically considers sets rather than correspondences and even though we analyze ex post rather than Bayesian equilibrium, social choice correspondences typically still lack ex post incentive compatibility.

9 Mixed Strategy Implementation

Finally we extend the ex post implementation results to cover pure as well as mixed strategy equilibria. In the process we shall also propose significantly weaker sufficient conditions for ex post implementation. The proof strategy follows the argument for complete information implementation with mixed strategies presented in Maskin (1999) and refined in Maskin & Sjostrom (2004). The idea of their proof is to enlarge the strategy space of each agent by allowing him to make a contingent rather than deterministic reward proposal. In addition, they allow the agent to quote integers in the augmented mechanism to prevent the possibility of further equilibria arising in the augmented mechanism. We shall use the same basic idea but in addition use the integers to create lotteries in the augmented mechanism. The introduction of lotteries is natural in an environment which allows for mixed strategies. We then show that the use of lotteries has the additional advantage that we can dispense with the ex post monotonicity no veto hypothesis as a sufficient condition and replace it by a much weaker condition, called value distinction.[13]

The idea of using lotteries to weaken the sufficient condition also appears in a recent contribution by Benoit & Ok (2004). In the complete information environment, they show that by using simple lotteries the no veto condition can be replaced by a much weaker top-coincidence notion. As they focus on pure strategy implementation, their augmented mechanism makes use only of modulo rather than integer games. In sum, the use of random mechanism allows us to extend the earlier implementation result from pure to mixed strategy implementation and to substantially weaken the sufficiency conditions.

A *mixed* strategy for agent i is $\sigma_i : \Theta_i \to \Delta(M_i)$ and we denote the probability that type θ_i sends message m_i under strategy σ_i by $\sigma_i(m_i|\theta_i)$. The set of feasible allocations $\mathcal{Y}$ is now understood to be the set of all lotteries over a set of finite deterministic outcomes Y, or $\mathcal{Y} = \Delta(Y)$.

Definition 19 (Ex Post Equilibrium in Mixed Strategies)
A mixed strategy profile $\sigma^* = (\sigma_1^*, ..., \sigma_I^*)$ is an ex post equilibrium if

$$\sum_{m \in M} u_i(g(m), \theta)\,\sigma^*(m|\theta) \geq \sum_{m_{-i} \in M_{-i}} u_i\left(g\left(m_i', m_{-i}\right), \theta\right)\sigma_{-i}^*(m_{-i}|\theta_{-i}),$$

for all i, θ and $m_i' \in M_i$.

The mixed strategy ex post equilibrium maintains all the features of the pure strategy ex post equilibrium. In particular, we observe that the no regret property is maintained conditional on the true type profile (but not conditional on every possible realization of messages). The notions of ex post monotonicity and ex post implementation remain unchanged with the understanding that the allocation y is possibly a lottery.

The necessary conditions for ex post implementation clearly remain necessary with the extension to mixed strategy equilibria and stochastic mechanisms. The focus of the remainder of this section is therefore on the sufficiency conditions. The no veto condition on the social choice set is replaced by a very weak condition on the preferences of the agents, referred to as value distinction.[14]

Definition 20 (Value Distinction)
The environment satisfies value distinction if for all $\theta \in \Theta$ and all $y, y' \in Y$, there exists i such that $u_i(y, \theta) \neq u_i(y', \theta)$.

[13] We would like to thank Andy Postlewaite and Phil Reny for comments during a Cowles Foundation Conference on "Robust Mechanism Design" which prompted us to pursue this argument.

[14] The notion of value distinction is different from value distinguished types as defined by Palfrey & Srivastava (1989*b*). Their notion requires that for every pair, θ_i and θ_i', by agent i, there exists an allocation y which is valued differently by the two types, θ_i and θ_i'.

The basic novelty is the introduction of a contingent reward in the augmented mechanism. Each agent sends a message of the form $m_i = (\theta_i, f_i, z_i, y_i)$, where $\theta_i \in \Theta_i$, $f_i : \Theta \to \mathcal{Y}$ is the social choice function suggested by i, $z_i \in \mathbb{N} = \{1, 2, ...\}$, and $y_i \in \mathcal{Y}$. The set of feasible messages for agent i is given by

$$M_i = (\Theta_i \times \mathcal{F} \times \mathbb{N} \times \mathcal{Y}) . \tag{28}$$

A strategy profile in this game is a collection $\sigma = (\sigma_1, ..., \sigma_I)$, with $\sigma_i : \Theta_i \to \Delta(M_i)$.

The proposal is required to satisfy:

$$f_i(\theta) \in \left\{ y' \in \mathcal{Y} \left| u_i \left(f\left(\theta_i', \theta_{-i}\right), \left(\theta_i', \theta_{-i}\right)\right) \geq u_i\left(y', \left(\theta_i', \theta_{-i}\right)\right), \ \forall \theta_i' \in \Theta_i \right. \right\} . \tag{29}$$

The outcome function $g : M \to \mathcal{Y}$ is defined by three rules:

Rule 1. If at m (and reported type profile θ), we have for all i, $f_i = f$ for some $f \in F$, then

$$g(m) = f(\theta) . \tag{30}$$

Rule 2. If at m (and reported type profile θ), there exists $j \in N$ such that $f_i(\theta) = f_k(\theta)$, $\forall i, k \neq j$ and $f_i(\theta) \neq f_j(\theta)$, then

$$g(y|m) = \begin{cases} 1 - \frac{1}{z_j+1} & \text{if} \quad y = f_j(\theta), \\ \frac{1}{z_j+1} & \text{if} \quad y = f_i(\theta), \\ 0 & \text{if} \quad \text{otherwise.} \end{cases} \tag{31}$$

Rule 3. In all other cases, the agent j with the highest integer z_j is the winner (and, in the event of a tie, the agent with the highest label), and with probability $\left(1 - \frac{1}{z_j+1}\right)$ pick y_j, and with probability $\frac{1}{\#Y}\frac{1}{z_j+1}$ pick $y \in Y$, or

$$g(y|m) = \begin{cases} 1 - \frac{\#Y-1}{\#Y}\frac{1}{z_j+1} & \text{if} \quad y = y_j \text{ and } z_j > z_i, \\ 1 - \frac{\#Y-1}{\#Y}\frac{1}{z_j+1} & \text{if} \quad y = y_j \text{ and } z_j = \max_{i \neq j} z_i \text{ and } \forall i \text{ s.th. } z_i = z_j, j > i, \\ \frac{1}{z_j+1}\frac{1}{\#Y} & \text{otherwise.} \end{cases} \tag{32}$$

The randomization in Rule 3 is simply a uniform randomization over the set of deterministic outcomes, and $\#Y$ is the cardinality of the set of deterministic outcomes.

We refer to the mechanism described by the message space $M = \times_{i=1}^{I} M_i$, described by (28), and the outcome function $g : M \to \mathcal{Y}$, described by (30)-(32), as the augmented mechanism. In contrast to the augmented mechanism presented in Section 4, the integer game is now defined on the natural numbers rather than a finite set of numbers.

Theorem 5 (Mixed Strategy Implementation)
For $I \geq 3$, if the environment satisfies value-distinction and F satisfies $(EPIC)$ and (EM), then F is ex post implementable.

Proof. We use the augmented mechanism described by (28)-(32).

Claim 1. Every $f \in F$ can be realized as an ex post equilibrium with $\sigma_i(m_i|\theta_i) > 0 \Rightarrow m_i = (\theta_i, f, \cdot, \cdot)$ for all $\theta_i \in \Theta_i$ and all i. Thus suppose that all agents but j pursue the above "truthtelling" strategy: $\sigma_i(m_i|\theta_i) > 0 \Rightarrow m_i = (\theta_i, f, \cdot, \cdot)$, for all $i \neq j$. By $(EPIC)$, it follows that given $f_j = f$, truthtelling for agent j, is a best response. It remains to argue that it remains a best response if the restriction of $f_j = f$ is removed. But by definition of $f_j(\theta)$ and Rule 2, a deviation to $f_j(\theta) \neq f(\theta)$ does not increase the utility of agent j, and may even decrease the utility of agent j, and hence it follows that every $f \in F$ can be realized as an ex post equilibrium.

Claim 2. In any ex post equilibrium with $f_i = f, \forall i, \forall \theta$, $g(m) = f(\theta)$ for some $f \in F$. Suppose not, then by ex post monotonicity, there exists an agent i, a type profile θ, and an allocation y which strictly improves the utility of agent i, and under Rule 2, he can obtain this improvement with arbitrarily large probability.

Claim 3. In any ex post equilibrium, $\sigma_i(m_i|\theta_i) > 0 \Rightarrow m_i = (\cdot, f, \cdot, \cdot)$. Suppose not and hence there exists an ex post equilibrium with $\sigma_j(\cdot, f, \cdot, \cdot|\cdot) < 1$ for some j at some $\theta \in \Theta$ and $\sigma_i(\cdot, f, \cdot, \cdot|\cdot) = 1$ for all other i. Then with positive probability the allocation will be either $f_i(\theta)$ or $f_j(\theta)$. By value distinction, there exists an agent k who assigns different utilities to these two different allocations. If k happens to be j, then by proposing a sufficiently large integer z_j, he guarantees himself a higher payoff. If $k \neq j$, then k can make a proposal $f_k(\theta) \neq f(\theta)$ such that Rule 3 will be applied in which k can guarantee himself to be the winner with arbitrarily high probability, and thus lower the probability of receiving the low utility arbitrarily close to zero.

Consider finally an ex post equilibrium with i, j such that for $f_i, f_j \neq f$:

$$\sigma_i(\cdot, f_i, \cdot, \cdot|\cdot)\, \sigma_j(\cdot, f_j, \cdot, \cdot|\cdot) > 0.$$

Now the above argument applies again and it follows that there cannot exist an ex post equilibrium where Rule 3 is applied with positive probability. ■

Benoit & Ok (2004) show that by using "simple" stochastic mechanisms, the sufficient conditions for Nash implementation in pure strategies can be substantially weakened. In particular, the no veto condition of the social choice set can be replaced by much weaker conditions on (i) the social choice function, namely weak unanimity and (ii) the preferences, namely a top coincidence condition. Our Theorem 5 does not require weak unanimity and the value distinction condition is strictly weaker than their top coincidence condition. Their top coincidence condition says that if for every profile θ and every i there exists at most a single allocation z such that:

$$u_j(z, \theta) \geq u_j(y, \theta), \; \forall j \neq i, \; \forall y \in Y. \tag{33}$$

To see why value distinction is (strictly) weaker, fix any arbitrary θ and i, and suppose that there does not exist an allocation z which is the preferred allocation for all agents $j \neq i$. Then there exist at least two agents, j and k, which differ in their most preferred allocation, call them y_j, y_k, respectively. In consequence, for at least one of the agents, say j, we have $u_j(y_j, \theta) \neq u_j(y_k, \theta)$. The same argument goes through if there does exist a single allocation z satisfying (33). By implication, there must exist another allocation y which is not the most preferred allocation for all $j \neq i$. For the allocation y and a particular agent j, we then have value distinction again: $u_j(z, \theta) \neq u_j(y, \theta)$. It is further immediate that the two conditions, top coincidence and value distinction, do not coincide, which establishes the strict implication.

We can replace top coincidence by the weaker value distinction condition because we allow for more than "simple" stochastic allocations. In the augmented mechanism of Benoit & Ok (2004), the whistle-blower i claims the reward by choosing a lottery which selects the reward y and the social choice $f(\theta)$ with equal probability, the "simple" stochastic allocation. As $y \neq f(\theta)$, the top coincidence condition then implies that there will be another agent j who has a strict preference between y and $f(\theta)$, and can impose his choice by appealing to Rule 3. In our augmented mechanism, the whistle-blower i can always increase the probability of receiving the reward by increasing the integer z_i, and hence it suffices that for the pair of allocations, y and $f(\theta)$, there exists an agent who values the two allocations differently. The second condition in Benoit & Ok (2004), weak unanimity, is not necessary in our augmented mechanism either. The use of random allocations in Rule 3 excludes the possibility of equilibria with a strategy profile in which Rule 3 applies. Consequently, we can use the condition of value distinction as in Theorem 5 to obtain an even more permissive result for Nash implementation with mixed strategies, which we simply report here.

Corollary 2 (Nash Implementation with Mixed Strategies)
For $I \geq 3$, if the environment satisfies value distinction and if F satisfies Maskin monotonicity, then F is Nash implementable in mixed strategies.

10 Conclusion

In this paper we reported a comprehensive set of results on the possibility of ex post implementation. The general necessary and sufficient conditions for ex post implementation have a similar structure as the well-known conditions for Nash and Bayesian Nash implementation. Ex post equilibrium requires that every strategy profile remains an equilibrium choice even if a given agent would know the true type profile of all the remaining agents. The informational assumptions underlying the notion of an ex post equilibrium are hence closer to complete information, i.e. Nash implementation. In consequence, we pursued a close comparison of the conditions for Nash and ex post implementation. We showed that the respective necessary and sufficient conditions are not nested, and that neither Nash nor ex post notions imply the other.

However, in the important class of single crossing environments, we showed that ex post monotonicity is given virtually for free as it is known to be true for Maskin monotonicity. Yet in the single crossing environment, for Maskin monotonicity to hold , the social choice function must be everywhere in the interior of the set of feasible alternatives. Ex post monotonicity however can already be guaranteed even if the social choice function is rarely in the interior of the feasible set. As an important example we showed that the single unit auction with interdependent values can be ex post, but not Nash implemented. Interestingly, the positive implementation results relied on interdependent values and do not hold for private values. Additionally, we showed that in the single crossing environment ex post implementation is possible in the direct mechanism and does not have to rely on the augmented mechanisms which have been frequently criticized for various unrealistic features, such as modulo or integer games.

In earlier work, Bergemann & Morris (2004), we investigated the role of ex post equilibrium for robust incentive compatibility. In particular, we showed that the notion of ex post equilibrium can be justified as a robust equilibrium concept in the sense that the social objective can be obtained as a Bayesian equilibrium for every possible type space if and only if it can be obtained as an ex post equilibrium. In a companion paper, Bergemann & Morris (2005), we separately investigate the role of the ex post equilibrium as a robust implementation notion.

11 Appendix

11.1 Private Values

The analysis of this paper applies to interdependent value environments. In the special case of private values, ex post incentive compatibility implies dominant strategy incentive compatibility, under which dominant strategy implementation (and thus ex post implementation) is trivially possible in the direct mechanism. In this section, we very briefly state without proof the simple connections between the properties described in this section in the special case of private values.

Definition 21 (Private Values)
There are private values if

$$u_i\left(y,(\theta_i,\theta_{-i})\right) = u_i\left(y,\left(\theta_i,\theta'_{-i}\right)\right)$$

for all i, y, θ_i, θ_{-i} and θ'_{-i}.

Definition 22 (Dominant Strategies Incentive Compatibility)
Social choice set F is dominant strategies incentive compatible if for every $f \in F$:

$$u_i\left(f\left(\theta\right),\theta\right) \geq u_i\left(f\left(\theta'\right),\theta\right),$$

for all i, θ, θ'.

Lemma 2 *Under private values, F is ex post incentive compatible if and only if F is dominant strategies incentive compatible.*

Definition 23 (Strict Dominant Strategies Incentive Compatibility)
Social choice set F is strictly dominant strategies incentive compatible if for every $f \in F$:

$$u_i\left(f\left(\theta\right),\theta\right) > u_i\left(f\left(\theta'\right),\theta\right),$$

for all i, θ and θ' with $\theta'_i \neq \theta_i$.

Lemma 3 *Under private values, F is strictly ex post incentive compatible if and only if F is strictly dominant strategies incentive compatible.*

Lemma 4 *Under private values, if F satisfies strict dominant strategies incentive compatibility, then F satisfies ex post monotonicity.*

11.2 Ex Post Monotonicity No Veto for Sets

We now state the ex post monotonicity no veto condition ($EMNV$) for the case of a social choice set rather than social choice function. Given a social choice set F and a deception α, we define for each $f \in F$ and i a set $\Phi_i^f \subseteq \Theta_i$ and let $\Phi^f = \times_{i=1}^I \Phi_i^f \subseteq \Theta$. The set Φ^f represents the set of profiles at which the agents all agree to implement the selection $f \in F$. The no veto power condition is now imposed on the complement set to the union of the sets Φ^f:

$$\Phi \triangleq \bigcup_{f \in F} \Phi^f.$$

.

Definition 24 (Ex Post Monotonicity No Veto (For Sets))
F satisfies ex post monotonicity no veto if for any deception α and any product set $\Phi \subseteq \Theta$ the following holds. If the environment is non-economic at each $\theta \in \Theta - \Phi$, then

1. *there exists $\hat{f} \in F$ which satisfies no veto power on $\Theta - \Phi$ and*
2. *if $f(\alpha(\theta)) \neq \hat{f}(\theta)$ for some f and some $\theta \in \Phi^f$ then there exists i, $\theta \in \Phi^f$ and y such that*

$$u_i(y, \theta) > u_i(f(\alpha(\theta)), \theta),$$

while

$$u_i\left(f\left(\theta_i', \alpha_{-i}(\theta_{-i})\right), \left(\theta_i', \alpha_{-i}(\theta_{-i})\right)\right) \geq u_i\left(y, \left(\theta_i', \alpha_{-i}(\theta_{-i})\right)\right), \forall \theta_i' \in \Theta_i.$$

The proof of Theorem 3 now goes through simply by extending the argument from the a single set Φ^f to the union of sets $\bigcup_{f \in F} \Phi^f$. We simply record it is a Corollary.

Corollary 3 (Sufficiency for Social Choice Sets)
For $I \geq 3$, if the social choice set F satisfies ($EPIC$) and ($EMNV$), then F is ex post implementable.

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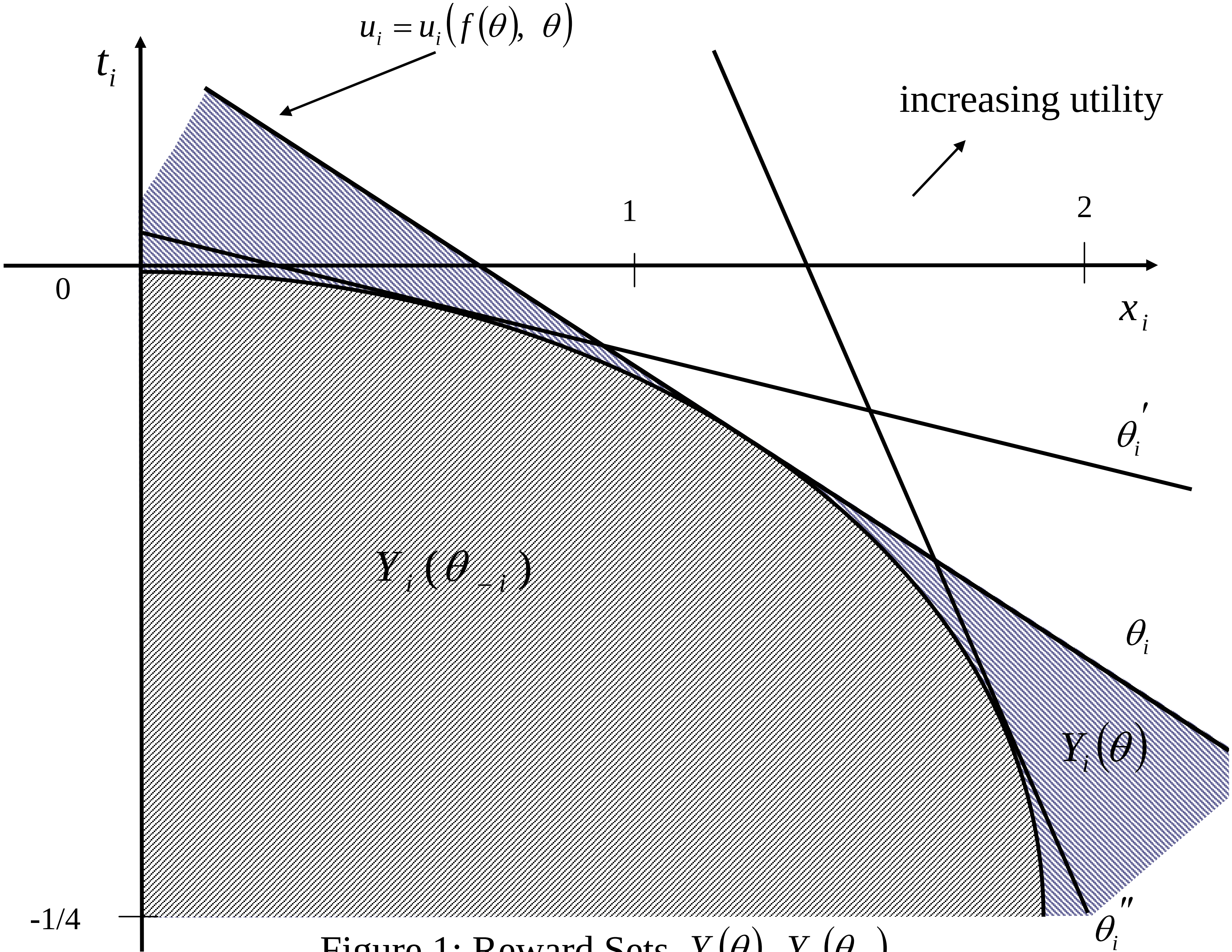

Figure 1: Reward Sets $Y_i(\theta),\ Y_i(\theta_{-i})$

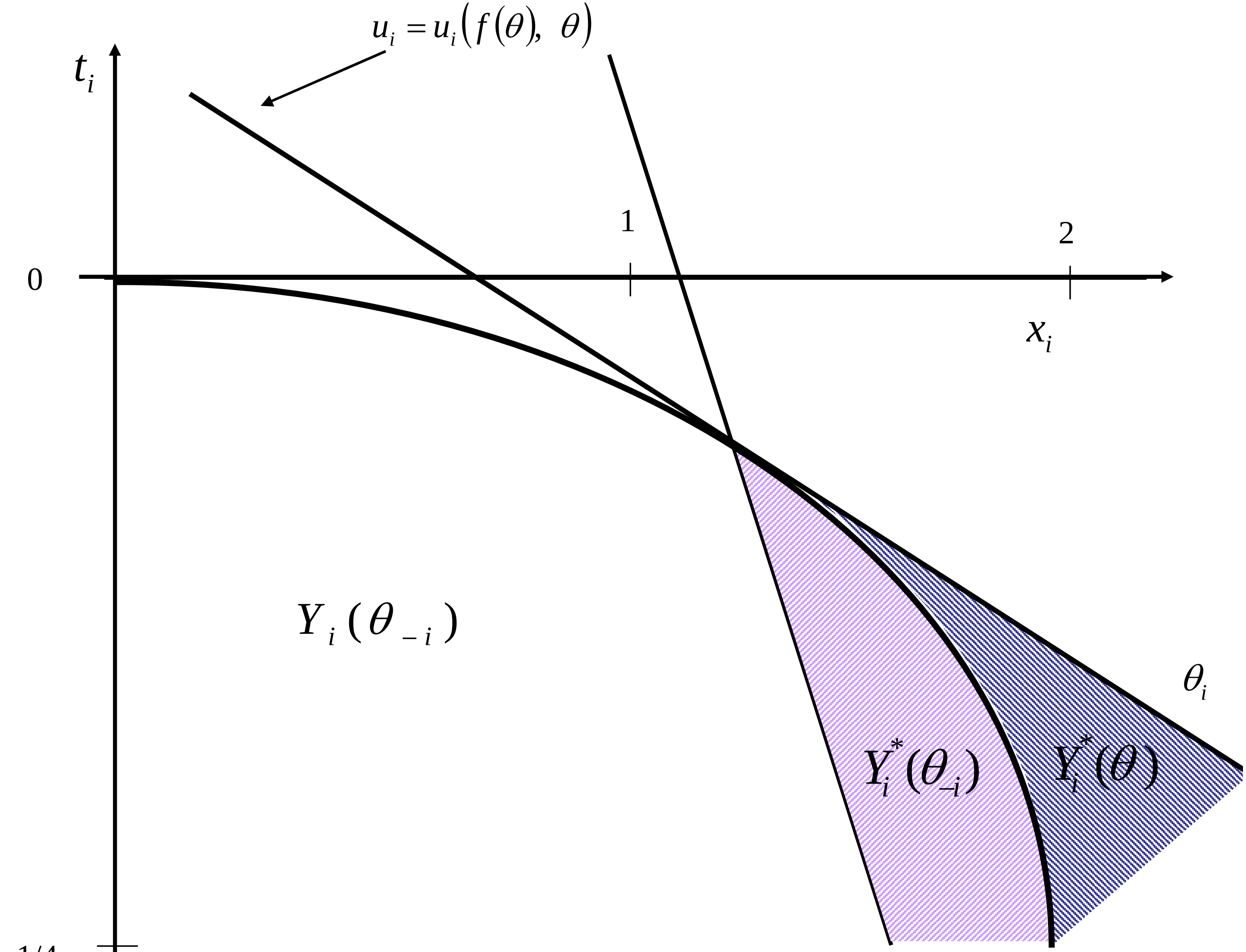

Figure 2: Successful Reward Sets $Y_i^*(\theta), Y_i^*(\theta_{-i})$

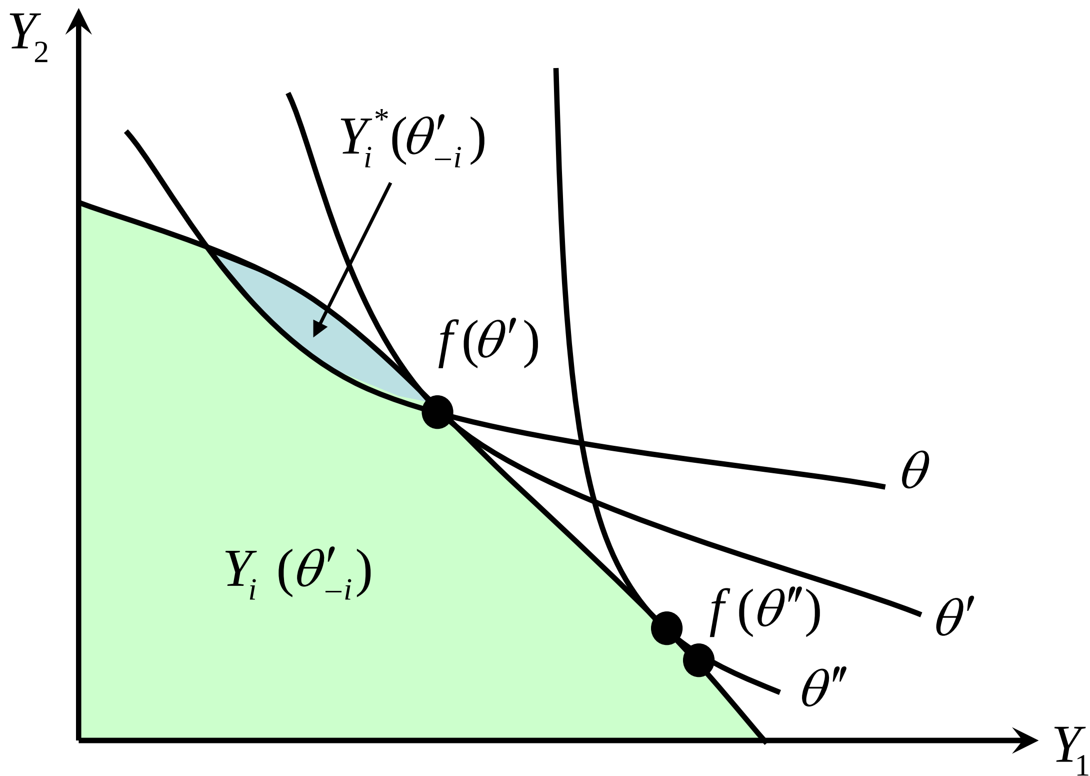

Figure 3: Ex Post Monotonicity with Strictness

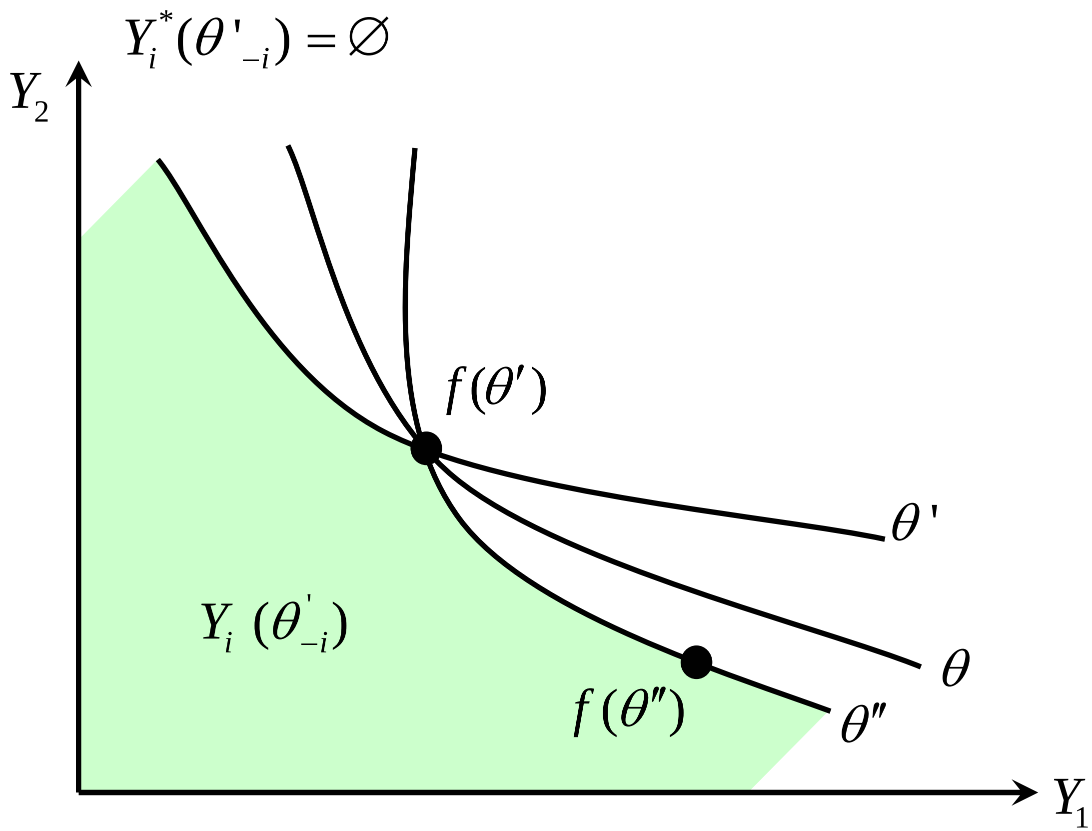

Figure 4: Failure of Strict Ex Post Incentive Compatibility

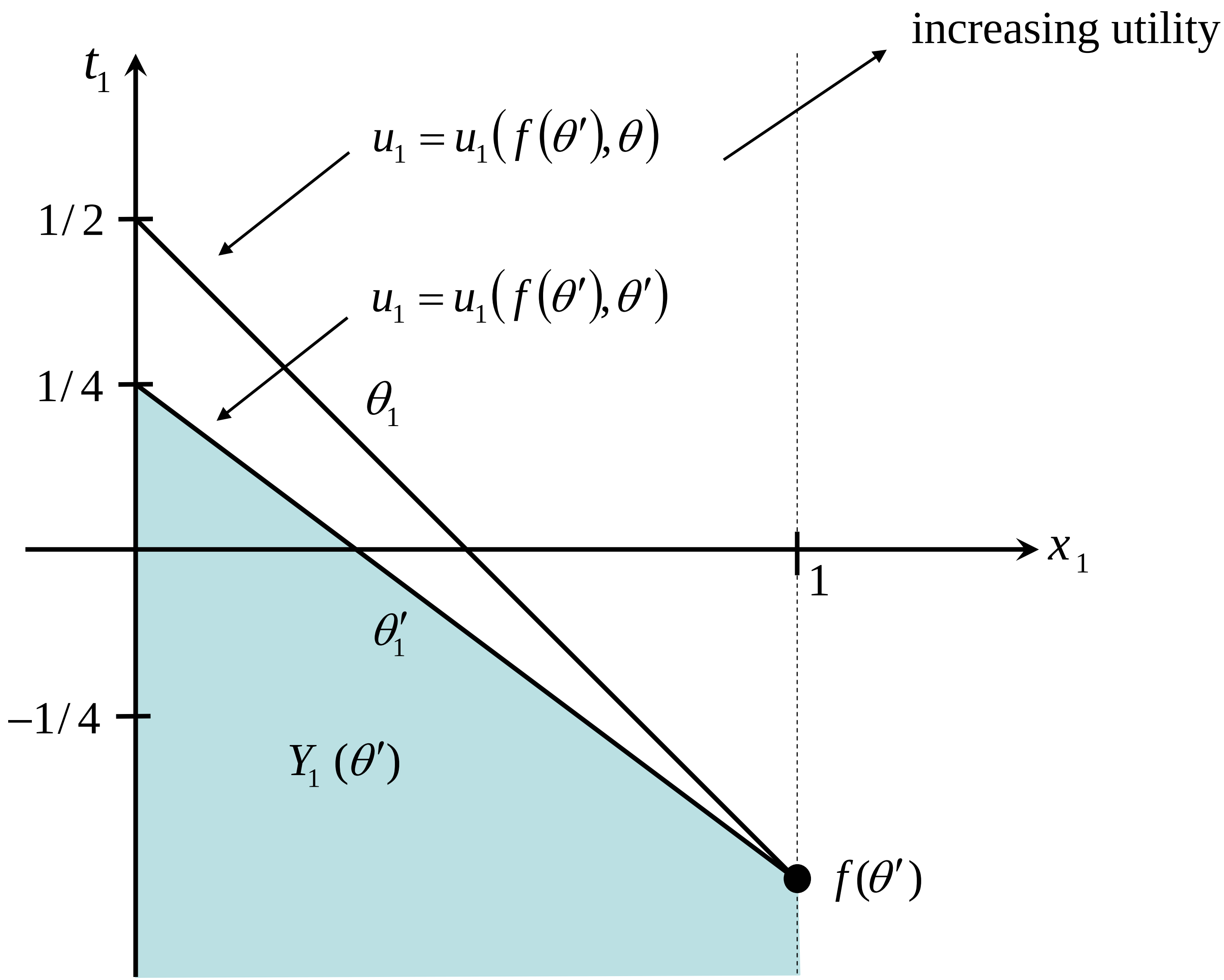

Figure 5a: Maskin Monotonicity in Single Unit Auction: j = 1

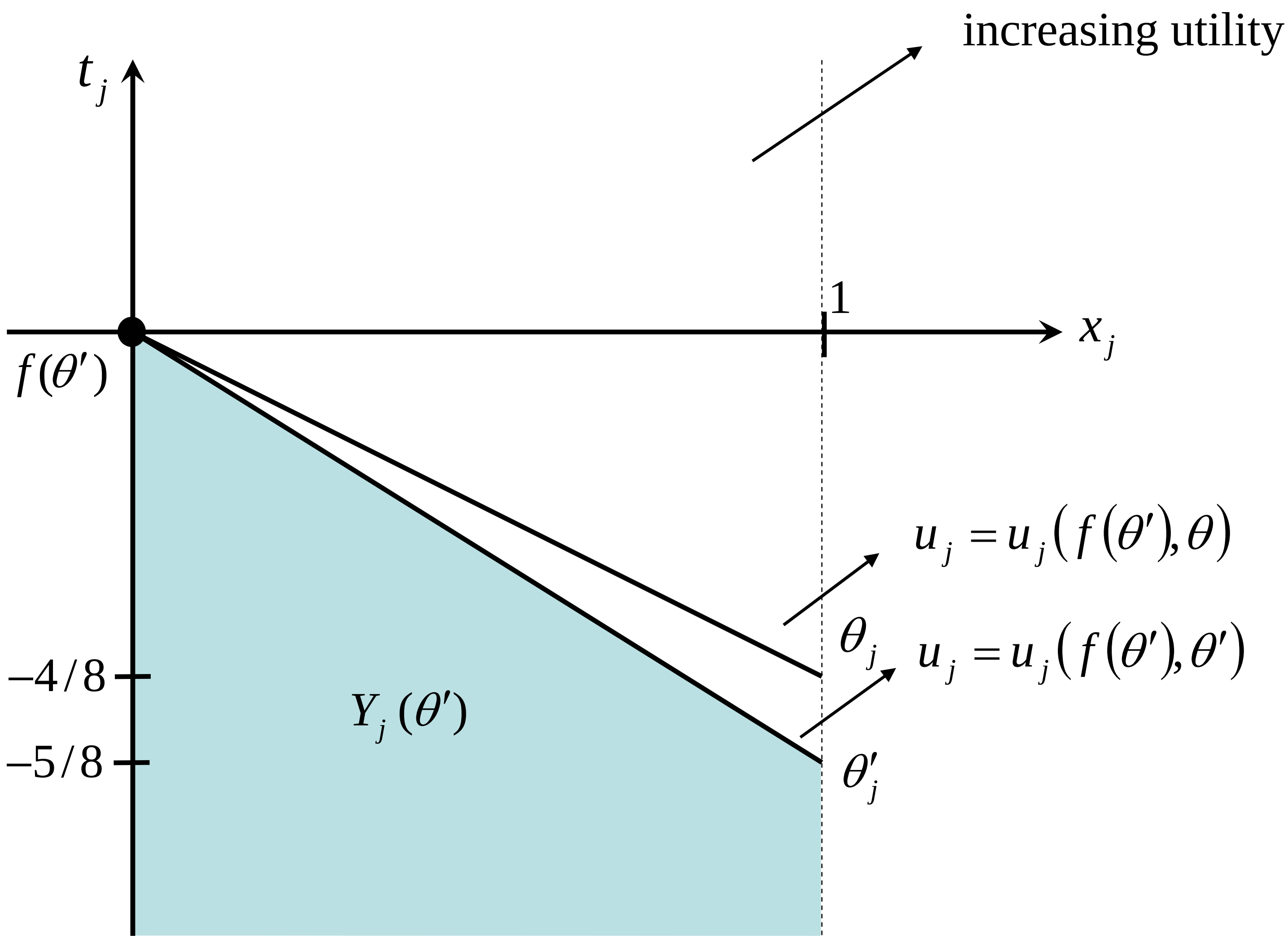

Figure 5b: Maskin Monotonicity in Single Unit Auction: j = 2,3

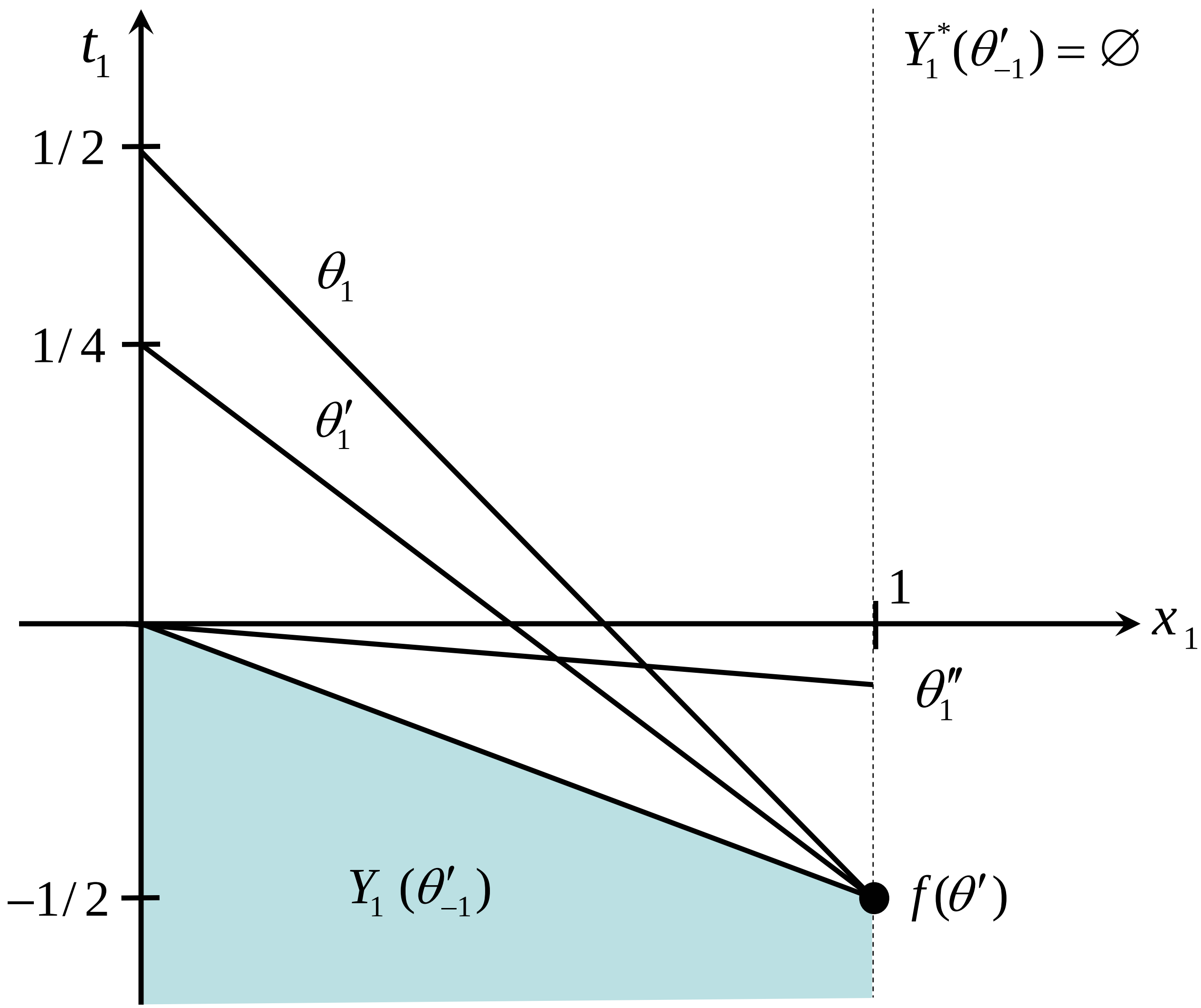

Figure 6a: Ex Post Monotonicity in Single Unit Auction: $j = 1$

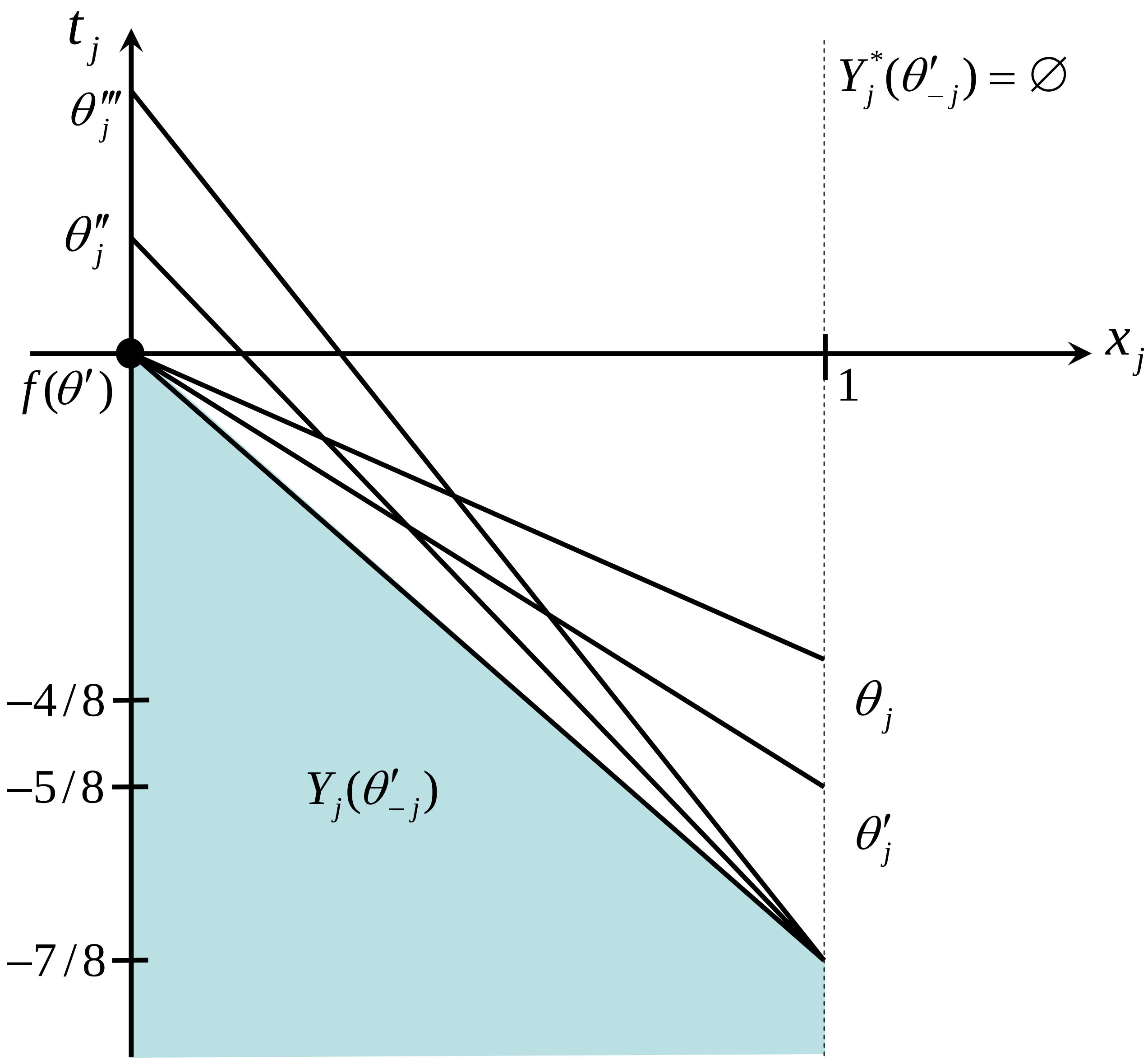

Figure 6b: Ex Post Monotonicity in Single Unit Auction: j = 2,3

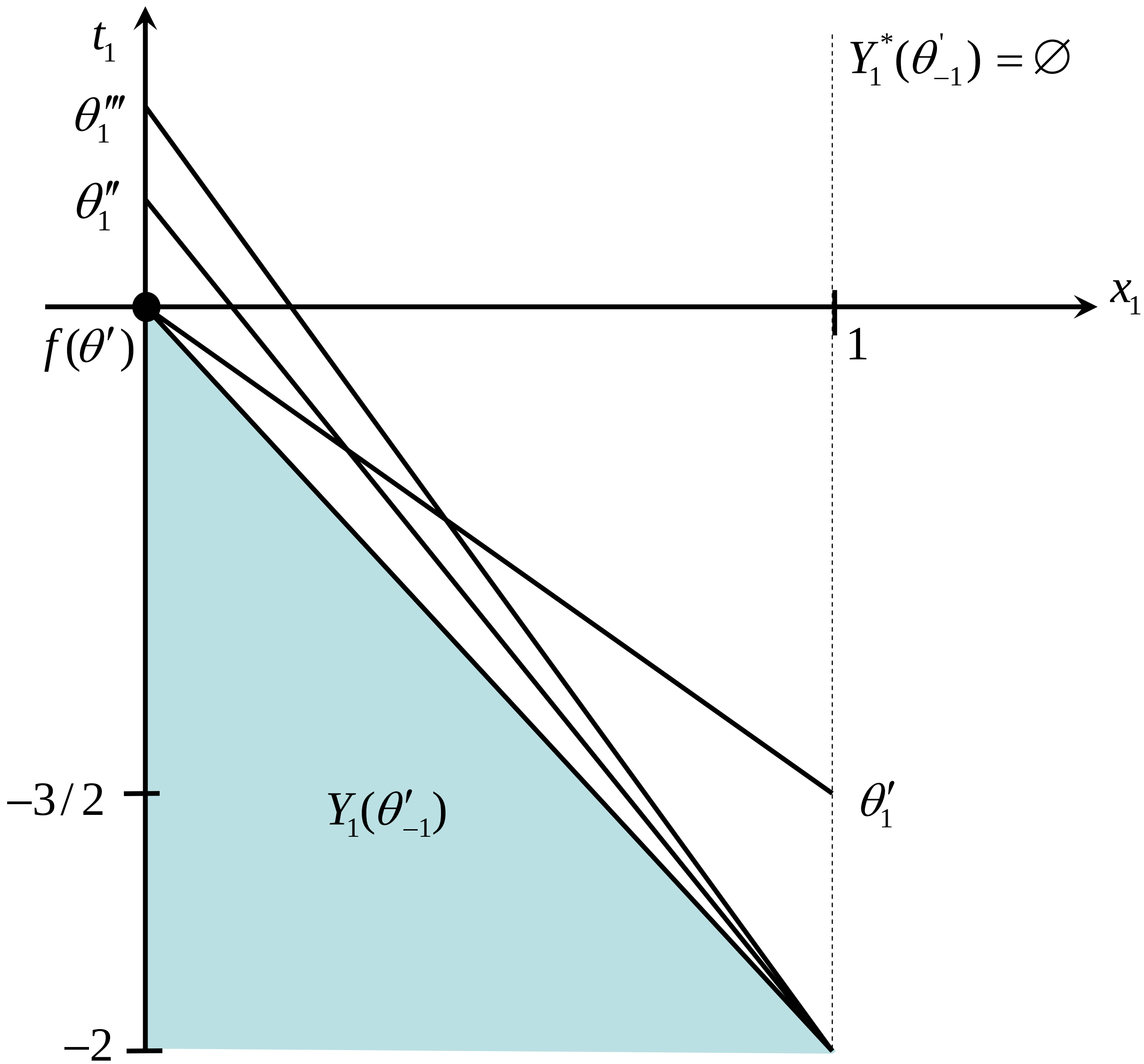

Figure 7a: Ex Post Monotonicity in Single Unit Auction: $j = 1$

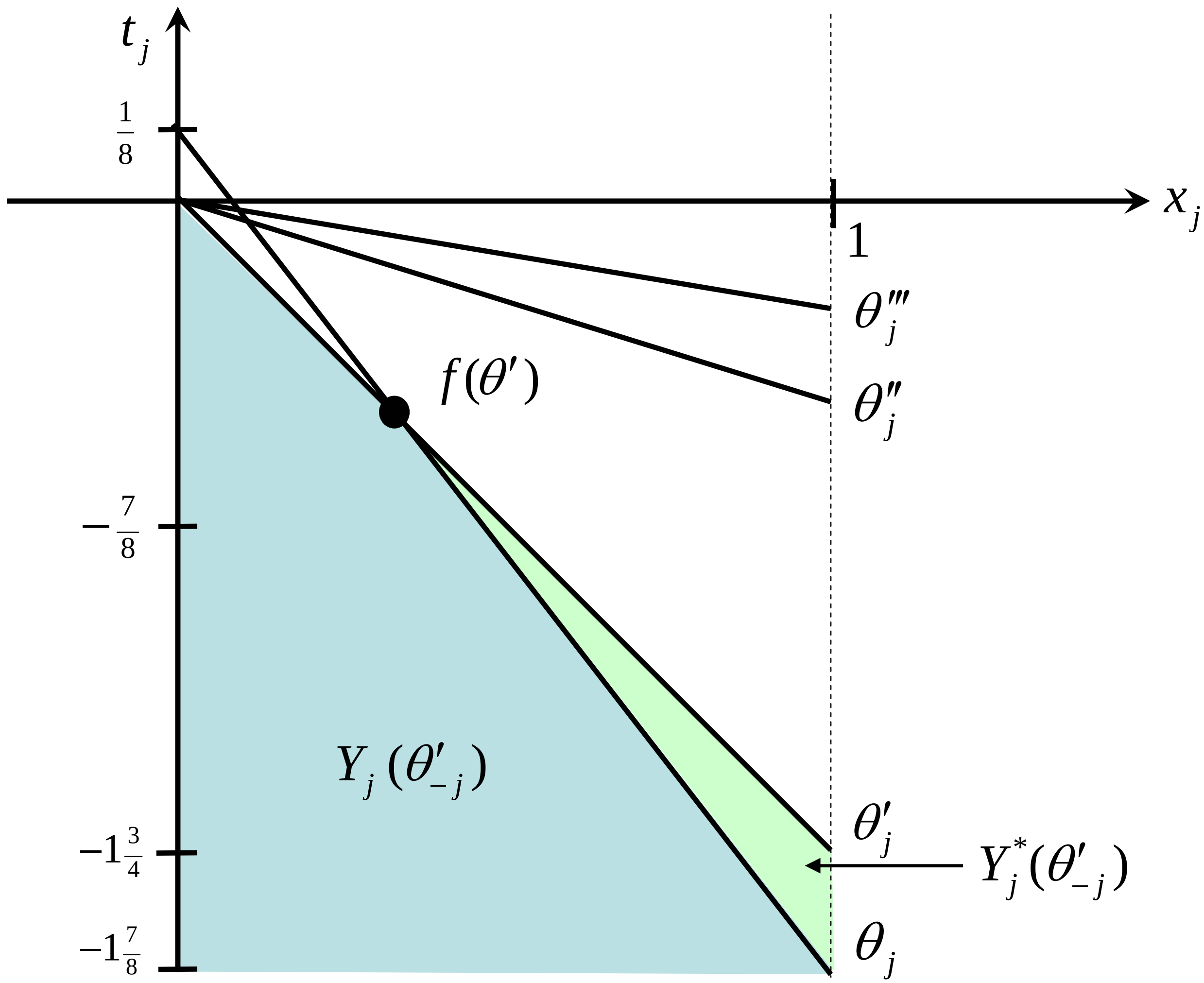

Figure7b: Ex Post Monotonicity in Single Unit Auction: $j = 2,3$