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Limit Theory for Moderate Deviations from a Unit Root[1]

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Abstract

An asymptotic theory is given for autoregressive time series with a root of the form $\rho_n = 1 + c/n^\alpha$, which represents moderate deviations from unity when $\alpha \in (0,1)$. The limit theory is obtained using a combination of a functional law to a diffusion on $D[0,\infty)$ and a central limit law to a scalar normal variate. For $c < 0$, the results provide a $n^{(1+\alpha)/2}$ rate of convergence and asymptotic normality for the first order serial correlation, partially bridging the $\sqrt{n}$ and n convergence rates for the stationary $(\alpha = 0)$ and conventional $(\alpha = 1)$ local to unity cases. For $c > 0$, the serial correlation coefficient is shown to have a $n^\alpha \rho_n^n$ convergence rate and a Cauchy limit distribution without assuming Gaussian errors, so an invariance principle applies when $\rho_n > 1$. This result links moderate deviation asymptotics to earlier results on the explosive autoregression proved under Gaussian errors for $\alpha = 0$, where the convergence rate of the serial correlation coefficient is $(1+c)^n$ and no invariance principle applies.

Keywords: Central limit theory; Diffusion; Explosive autoregression, Local to unity; Moderate deviations, Unit root distribution.

AMS 1991 subject classification: 62M10; JEL classification: C22

1. Introduction

Regression asymptotics with roots at or near unity have played an important role in time series econometrics over the last two decades. The limit theory makes extensive use of functional laws of partial sums to Brownian motion, functional laws of weighted partial sums to linear diffusions and weak convergence of discrete martingales to stochastic integrals. Almost all this theory involves time series with autoregressive roots that are at unity (or on the unit circle) or roots that are local to unity in the sense that they have the form $\rho = 1 + c/n$, where n is the sample size. In the latter case, the situation of primary importance occurs when $c < 0$, so that $\rho < 1$ and the local asymptotics therefore seek to characterize alternatives to a unit root that lie in the stationary region. The asymptotic theory turns out to be similar whether $c = 0$ or $c < 0$, and the same rate of convergence in terms of the sample size n applies in both cases. These results have been useful in power evaluations and in confidence interval construction.

To characterize greater deviations from unity we can allow the parameter c to be large and negative or even consider limits as $c \to -\infty$ (Phillips, 1987; Chan and Wei, 1988). While such analysis has proved insightful, it does not resolve all difficulties of the discontinuities of unit root asymptotics. In particular, it does not effectively bridge the very different convergence rates of the stationary and unit root cases.

The present paper takes another approach and provides an asymptotic theory for time series with an autoregressive root of the form $\rho_n = 1 + c/n^\alpha$, where the exponent α lies on $(0, 1)$. Such roots represent moderate deviations from unity in the sense that they belong to larger neighborhoods of one than conventional local to unity roots. The boundary value as $\alpha \to 1$ includes the conventional local to unity case, whereas the boundary value as $\alpha \to 0$ includes the stationary or explosive AR(1) process, depending on the sign of c. The limit theory for such time series is developed here using a combination of a functional law to a diffusion and a central limit law.

The paper provides limit results for a standardized version of such time series, for various sample moments in both the near-stationary ($c < 0$) and the near-explosive ($c > 0$) cases, and for the serial correlation coefficient. When there are near-stationary moderate deviations from unity, the centred first order serial correlation coefficient $\widehat{\rho}_n - \rho_n$ is shown to have a $n^{(1+\alpha)/2}$ rate of convergence and a limit normal distribution, bridging the $\sqrt{n}$ and n asymptotics of the stationary ($\alpha = 0$) and conventional local to unity ($\alpha = 1$) cases. For near-explosive moderate deviations from unity, the rate of convergence of $\widehat{\rho}_n - \rho_n$ is $n^\alpha \rho_n^n$, which increases with α from $O(n)$ when $\alpha \to 1$ to $O((1+c)^n)$ when $\alpha \to 0$, thereby bridging the asymptotics of local to unity and explosive autoregressions. An interesting feature of the moderate deviation explosive case ($c > 0$) is that the limit distribution theory is Cauchy even for non-Gaussian errors. This result differs from conventional theory for the explosive case where the limit distribution is dependent on the distribution of the errors and no invariance principle applies (Anderson, 1959).

After these results were obtained, we learnt of some independent, related work by Park (2003) on weak unit root asymptotics. Park considers autoregressive processes with a root that can be written in the form $\rho = 1 - m/n$ where $m,\ n \to \infty$. This (weak unit root) setup is analogous to our formulation (see (1) below) of moderate deviations from unity of the form $\rho_n = 1 + \frac{c}{n^\alpha}$ for $\alpha \in (0,1)$. However, the weak unit root specification considers only the stationary side of unity. Using different methods and among some other results, Park shows a rate of convergence of $n/\sqrt{m}$ and asymptotic normality for the serial correlation coefficient in autoregressions with independent identically distributed errors when $\frac{1}{m} + \frac{m}{n} \to 0$. Theorem 3.1(d) of the present paper also establishes asymptotic normality of the serial correlation coefficient with a rate of convergence $n^{\frac{1}{2}+\frac{\alpha}{2}}$, which corresponds to $n/\sqrt{m}$, on the stationary side of unity ($c < 0$). As discussed above, this paper also provides a limit theory for the explosive side of unity ($c > 0$).

2. The moderate deviations from unity model

Consider the time series

$$y_t = \rho_n y_{t-1} + u_t, \quad t = 1, ..., n; \quad \rho_n = 1 + \frac{c}{n^\alpha},\ \alpha \in (0,1) \tag{1}$$

initialized at some $y_0 = o_p\left(n^{\alpha/2}\right)$ and where u_t is a sequence of independent and identically distributed $(0, \sigma^2)$ random variables with finite ν'th absolute moment

$$E\left|u_t\right|^\nu < \infty \text{ for some } \nu \geq \frac{2}{\alpha}. \tag{2}$$

These conditions suffice (cf. Phillips and Solo, 1992) to ensure that partial sums $S_t = \sum_{i=1}^t u_i$ of u_t satisfy the functional law

$$B_{k_n}(\cdot) := \frac{S_{\lfloor k_n \cdot \rfloor}}{\sqrt{k_n}} = \frac{\sum_{i=1}^{\lfloor k_n \cdot \rfloor} u_i}{\sqrt{k_n}} \Longrightarrow B(\cdot) \tag{3}$$

for any sequence $(k_n)_{n\in\mathbb{N}}$ increasing to infinity, where $\lfloor \cdot \rfloor$ signifies integer part and $B(\cdot)$ is Brownian motion with variance σ^2.

A strong approximation (e.g. Csörgõ and Horváth, 1993) to S_t is also possible, according to which we can construct an expanded probability space with a Brownian motion $B(\cdot)$ for which

$$\sup_{0\leq i\leq n} \left|S_i - B(i)\right| = o_{a.s.}(n^{\frac{1}{\nu}}) \quad \text{as } n \to \infty. \tag{4}$$

A straightforward calculation (given in the Appendix) then shows that for each $\alpha \in (0,1)$

$$\sup_{t\in[0,n^{1-\alpha}]} |B_{n^\alpha}(t) - B(t)| = o_{a.s.}\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \quad \text{as } n\to\infty. \tag{5}$$

In what follows, we will assume that the probability space has been expanded as necessary in order for (5) to apply. Note that the moment condition $\nu \geq \frac{2}{\alpha}$ in (2) ensures that $o_{a.s.}\left(1/n^{\frac{\alpha}{2}-\frac{1}{\nu}}\right) = o_{a.s.}(1)$ in (5).

Our approach to developing a limit theory for statistics arising from model (1) is to segment the series into blocks. Specifically, we write the chronological sequence $\{t = 1, ..., n\}$ in blocks of size $\lfloor n^\alpha \rfloor$ as follows. Set $t = \lfloor n^\alpha j \rfloor + k$ for $k = 1, ..., \lfloor n^\alpha \rfloor$ and $j = 0, ..., \lfloor n^{1-\alpha} \rfloor - 1$, so that

$$y_{\lfloor n^\alpha j\rfloor + k} = \sum_{i=1}^{\lfloor n^\alpha j\rfloor + k} \left(1+\frac{c}{n^\alpha}\right)^{\lfloor n^\alpha j\rfloor + k - i} u_i + \left(1+\frac{c}{n^\alpha}\right)^{\lfloor n^\alpha j\rfloor + k} y_0.$$

This arrangement effectively partitions the sample size into $\lfloor n^{1-\alpha} \rfloor$ blocks each containing $\lfloor n^\alpha \rfloor$ sample points. Since the last element of each block is asymptotically equivalent to the first element of the next block, it is possible to study the asymptotic behavior of the time series $\{y_t : t = 1, ..., n\}$ via the asymptotic properties of the time series $\{y_{\lfloor n^\alpha j\rfloor + k} : j = 0, ..., \lfloor n^{1-\alpha} \rfloor - 1,\ k = 1, ..., \lfloor n^\alpha \rfloor\}$.

Letting $k = \lfloor n^\alpha p \rfloor$, for some $p \in [0, 1]$, we obtain

$$\begin{aligned}\frac{1}{n^{\alpha/2}} y_{\lfloor n^\alpha j\rfloor + \lfloor n^\alpha p\rfloor} &= \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor n^\alpha j\rfloor + \lfloor n^\alpha p\rfloor} \left(1+\frac{c}{n^\alpha}\right)^{\lfloor n^\alpha j\rfloor + \lfloor n^\alpha p\rfloor - i} u_i \\ &\quad + \left(1+\frac{c}{n^\alpha}\right)^{\lfloor n^\alpha j\rfloor + \lfloor n^\alpha p\rfloor} \frac{y_0}{n^{\alpha/2}}.\end{aligned}$$

The random element $y_{\lfloor n^\alpha j\rfloor + \lfloor n^\alpha p\rfloor}$ is central in the blocking method adopted in this paper. Most statistics of interest such as the sample variance and the sample covariance can be expressed as functionals of $y_{\lfloor n^\alpha j\rfloor + \lfloor n^\alpha p\rfloor}$, and it will be convenient to characterize its asymptotic behavior.

We start with the near stationary case $c < 0$. Noting that $j + p \in [0, \lfloor n^{1-\alpha} \rfloor]$, $\frac{1}{n^{\alpha/2}} y_{\lfloor n^\alpha j\rfloor + \lfloor n^\alpha p\rfloor}$ can be written in terms of the Stieltjes integral

$$V_{n^\alpha}(t) := \int_0^t e^{c(t-r)} dB_{n^\alpha}(r) = \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor n^\alpha t\rfloor} e^{\frac{c}{n^\alpha}(n^\alpha t - i)} u_i$$

in the following way: for each $\alpha \in (0, 1)$ and $c < 0$,

$$\sup_{t\in[0,n^{1-\alpha}]} \left| \frac{1}{n^{\alpha/2}} y_{\lfloor n^\alpha t\rfloor} - V_{n^\alpha}(t) \right| = o_p(1), \tag{6}$$

as shown in the Appendix. (5) and (6) together imply that $V_{n^\alpha}(t)$ converges weakly to the linear diffusion

$$J_c(t) := \int_0^t e^{c(t-s)} dB(s)$$

on the Skorohod space $D[0,M]$ for every $M > 0$ and hence (e.g. Pollard, 1984, Theorem V.23) on $D[0,\infty)$. In particular, we have the following strong approximation of $V_{n^\alpha}(t)$ in terms of $J_c(t)$.

2.1 Lemma. *For each $\alpha \in (0,1)$ and $c < 0$*

$$\sup_{t\in[0,n^{1-\alpha}]} |V_{n^\alpha}(t) - J_c(t)| = o_{a.s.}\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \quad \text{as } n \to \infty \tag{7}$$

on the same probability space that (5) holds.

An immediate consequence of Lemma 2.1 and (6) is that

$$\sup_{t\in[0,n^{1-\alpha}]} \left| \frac{1}{n^{\alpha/2}} y_{\lfloor n^\alpha t \rfloor} - J_c(t) \right| = o_p(1). \tag{8}$$

Therefore, for the original random variables $y_{\lfloor n^\alpha \cdot \rfloor}$ (rather than their distributionally equivalent copies for which (5) and (8) hold) we obtain

$$\frac{1}{n^{\alpha/2}} y_{\lfloor n^\alpha j \rfloor + \lfloor n^\alpha p \rfloor} \Longrightarrow \int_0^{j+p} e^{c(j+p-r)} dB(r) \quad \text{as } n \to \infty \tag{9}$$

for all $j = 0, ..., \lfloor n^{1-\alpha} \rfloor - 1$ and $p \in [0,1]$. Result (8) enables us to proceed with a limit theory for the near stationary case where $c < 0$.

3. Limit theory for the near stationary case

This section develops the asymptotic properties of the serial correlation coefficient

$$\hat{\rho}_n - \rho_n = \frac{\sum_{t=1}^n y_{t-1} u_t}{\sum_{t=1}^n y_{t-1}^2} \tag{10}$$

when $\rho_n = 1 + \frac{c}{n^\alpha}$ and $c < 0$. Our approach is to use a segmentation of the series into blocks in which we may utilize the embedding (8) and apply law of large numbers and central limit arguments to the denominator and numerator of (10).

We start by considering the sample variance $\sum_{t=1}^n y_t^2$. Using Proposition A3 and the identity

$$\int_0^1 y_{\lfloor n^\alpha j \rfloor + \lfloor n^\alpha p \rfloor}^2 dp = \frac{1}{n^\alpha} \sum_{k=1}^{\lfloor n^\alpha \rfloor} y_{\lfloor n^\alpha j \rfloor + k}^2,$$

the sample variance can be written as

$$\begin{aligned}\frac{1}{n^{1+\alpha}}\sum_{t=1}^{n}y_t^2 &= \frac{1}{n^{1+\alpha}}\sum_{j=0}^{\lfloor n^{1-\alpha}\rfloor-1}\sum_{k=1}^{\lfloor n^{\alpha}\rfloor}y_{\lfloor n^{\alpha}j\rfloor+k}^2+O_p\left(\frac{1}{n^{1-\alpha}}\right)\\ &= \frac{1}{n^{1-\alpha}}\sum_{j=0}^{\lfloor n^{1-\alpha}\rfloor-1}\frac{1}{n^{2\alpha}}\sum_{k=1}^{\lfloor n^{\alpha}\rfloor}y_{\lfloor n^{\alpha}j\rfloor+k}^2\\ &= \frac{1}{n^{1-\alpha}}\sum_{j=0}^{\lfloor n^{1-\alpha}\rfloor-1}\int_0^1\left(\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}j\rfloor+\lfloor n^{\alpha}p\rfloor}\right)^2dp\\ &= \frac{1}{n^{1-\alpha}}\int_0^{\lfloor n^{1-\alpha}\rfloor}\left(\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}r\rfloor}\right)^2dr+o_p(1).\end{aligned}$$

By (8) and Proposition A2 we obtain

$$\begin{aligned}&\frac{1}{n^{1-\alpha}}\left|\int_0^{\lfloor n^{1-\alpha}\rfloor}\left(\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}r\rfloor}\right)^2dr-\int_0^{\lfloor n^{1-\alpha}\rfloor}J_c(r)^2\,dr\right|\\ \leq\ &\frac{1}{n^{1-\alpha}}\int_0^{\lfloor n^{1-\alpha}\rfloor}\left|\left(\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}r\rfloor}\right)^2-J_c(r)^2\right|dr\\ =\ &\frac{1}{n^{1-\alpha}}\int_0^{\lfloor n^{1-\alpha}\rfloor}\left|\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}r\rfloor}-J_c(r)\right|\left|\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}r\rfloor}+J_c(r)\right|dr\\ \leq\ &\frac{\lfloor n^{1-\alpha}\rfloor}{n^{1-\alpha}}\sup_{r\in[0,\lfloor n^{1-\alpha}\rfloor]}\left|\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}r\rfloor}-J_c(r)\right|\left(\sup_{r\in[0,\lfloor n^{1-\alpha}\rfloor]}\left|\frac{1}{n^{\alpha/2}}y_{\lfloor n^{\alpha}r\rfloor}\right|+\sup_{r\geq 0}|J_c(r)|\right)\\ =\ &2\sup_{r\geq 0}|J_c(r)|\,o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right)+o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right)=o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right).\end{aligned}$$

Hence, the sample variance becomes

$$\frac{1}{n^{1+\alpha}}\sum_{t=1}^{n}y_t^2=\frac{1}{n^{1-\alpha}}\int_0^{\lfloor n^{1-\alpha}\rfloor}J_c(r)^2\,dr+o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right). \tag{11}$$

At this point, it is convenient to approximate the Ornstein-Uhlenbeck process $J_c(t)$ by its stationary version

$$J_c^*(t):=e^{ct}J_c^*(0)+\int_0^t e^{c(t-s)}dB(s)=e^{ct}J_c^*(0)+J_c(t),$$

where $J_c^*(0)$ is a random variable independent of $B(.)$ that follows a $N\left(0,\frac{\sigma^2}{-2c}\right)$ distribution. It is well known that $J_c^*(t)$ is a strictly stationary process with autocovariance function given by

$$\gamma_{J_c^*}(h)=\frac{\sigma^2}{-2c}e^{c|h|}\quad h\in\mathbb{Z}.$$

Moreover, the following approximation of J_c by J_c^* is established in the Appendix

$$\frac{1}{n^{1-\alpha}}\int_0^{\lfloor n^{1-\alpha}\rfloor} J_c(r)^2\,dr = \frac{1}{n^{1-\alpha}}\int_0^{\lfloor n^{1-\alpha}\rfloor} J_c^*(r)^2\,dr + O_p\left(n^{-(1-\alpha)}\right). \tag{12}$$

Combining (11) and (12) the sample variance in the near-stationary case becomes

$$\begin{aligned}\frac{1}{n^{1+\alpha}}\sum_{t=1}^{n} y_t^2 &= \frac{1}{n^{1-\alpha}}\int_0^{\lfloor n^{1-\alpha}\rfloor} J_c^*(r)^2\,dr + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right)\\ &= \frac{1}{n^{1-\alpha}}\sum_{j=0}^{\lfloor n^{1-\alpha}\rfloor-1}\int_j^{j+1} J_c^*(r)^2\,dr + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right)\\ &= \frac{\sigma^2}{-2c} + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \end{aligned} \tag{13}$$

by the weak law of large numbers for stationary processes, since $\gamma_{J_c^*}(0) = \sigma^2/-2c$.

The limiting distribution of the sample covariance can be obtained by using the fact that, as in the case of stationary asymptotics, the standardized sample variance has a constant (non random) probability limit. Defining $\xi_t = n^{-\frac{1+\alpha}{2}} y_{t-1}u_t$, $(\xi_t)_{t\in\mathbb{N}}$ is a martingale difference sequence with respect to the filtration $\mathcal{F}_t = \sigma(y_0, u_1, ..., u_t)$. The conditional variance of the martingale $\sum_{t=1}^{n}\xi_t$ is given by

$$\begin{aligned}\sum_{t=1}^{n} E_{\mathcal{F}_{t-1}}\left(\xi_t^2\right) &= \frac{1}{n^{1+\alpha}}\sum_{t=1}^{n} E_{\mathcal{F}_{t-1}}\left(y_{t-1}^2 u_t^2\right) = \frac{1}{n^{1+\alpha}}\sum_{t=1}^{n} y_{t-1}^2 E_{\mathcal{F}_{t-1}}\left(u_t^2\right)\\ &= \sigma^2\frac{1}{n^{1+\alpha}}\sum_{t=1}^{n} y_{t-1}^2 = \frac{\sigma^4}{-2c} + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right)\end{aligned}$$

by (13), since y_{t-1} is $\mathcal{F}_{t-1}$ measurable. By virtue of the Lindeberg condition

$$\sum_{t=1}^{n} E_{\mathcal{F}_{t-1}}\left(\xi_t^2\mathbf{1}\{|\xi_t| > \eta\}\right) = o_p(1), \quad \eta > 0 \tag{14}$$

established in the Appendix, the martingale central limit theorem (e.g. Pollard (1984), Theorem VIII.1) yields

$$\frac{1}{n^{\frac{1+\alpha}{2}}}\sum_{t=1}^{n} y_{t-1}u_t \Longrightarrow N\left(0, \frac{\sigma^4}{-2c}\right). \tag{15}$$

Finally, the asymptotic distribution of the centred least squares estimator $\widehat{\rho}_n - \rho_n = \sum_{t=1}^{n} y_{t-1}u_t / \sum_{t=1}^{n} y_{t-1}^2$ can be derived by combining (13) and (15):

$$n^{\frac{1}{2}+\frac{\alpha}{2}}\left(\widehat{\rho}_n - \rho_n\right) \Longrightarrow N(0, -2c) \quad \text{as } n \to \infty.$$

We collect these results together as follows.

3.1 Theorem. *For model (1) with $\rho_n = 1+c/n^a$, $c<0$ and $\alpha \in (0,1)$, the following limits apply as $n \to \infty$:*

(a) $n^{-\alpha/2} y_{\lfloor n^\alpha j \rfloor + \lfloor n^\alpha p \rfloor} \Longrightarrow \int_0^{j+p} e^{c(j+p-r)} dB(r)$,

(b) $n^{-1-\alpha} \sum_{t=1}^n y_t^2 \longrightarrow_p \frac{\sigma^2}{-2c}$,

(c) $n^{-\frac{1}{2}-\frac{\alpha}{2}} \sum_{t=1}^n y_{t-1} u_t \Longrightarrow N\left(0, \frac{\sigma^4}{-2c}\right)$,

(d) $n^{\frac{1}{2}+\frac{\alpha}{2}} \left(\widehat{\rho}_n - \rho_n\right) \Longrightarrow N(0, -2c)$,

where B is Brownian motion with variance σ^2.

3.2 Remarks

(i) When there are moderate deviations from unity, the proofs above reveal that both a functional law to a diffusion (part (a)) and central limit theory (parts (b), (c) and (d)) play a role in the derivation of the results. The functional law provides in each case a limiting subsidiary process whose elements form the components that upon further summation satisfy a law of large numbers and a central limit law. While there is only one limiting process involved as $n \to \infty$, it is convenient to think of the functional law operating within blocks of length $\lfloor n^\alpha \rfloor$ and the law of large numbers and central limit laws operating across the $\lfloor n^{1-\alpha} \rfloor$ blocks. The moment condition in (2) ensures the validity of the embedding argument that makes this segmentation rigorous as $n \to \infty$.

(ii) Results (b), (c) and (d) match the standard stationary limit theory for fixed $|\rho| < 1$. In particular,

$$n^{-1} \sum_{t=1}^n y_t^2 \longrightarrow_p \frac{\sigma^2}{1-\rho^2},$$
$$n^{-\frac{1}{2}} \sum_{t=1}^n y_{t-1} u_t \Longrightarrow N\left(0, \frac{\sigma^4}{1-\rho^2}\right),$$
$$n^{\frac{1}{2}} \left(\hat{\rho}_n - \rho\right) \Longrightarrow N\left(0, 1-\rho^2\right).$$

A heuristic argument for the correspondence is that upon replacing ρ by $1+c/n^a$ in each of the above results, a simple rescaling of the first order approximation delivers (b)-(d) of Theorem 3.1. Thus, for the serial correlation coefficient $\hat{\rho}_n$, substituting $1-\rho^2 = -\frac{2c}{n^\alpha}[1+o(1)]$ into the limit distribution of $n^{\frac{1}{2}}(\hat{\rho}_n - \rho)$ gives the asymptotic approximation

$$n^{\frac{1}{2}} \left(\hat{\rho}_n - \rho\right) \sim_d N\left(0, -\frac{2c}{n^\alpha}\right) \text{ or } n^{\frac{1}{2}+\frac{\alpha}{2}} \left(\hat{\rho}_n - \rho\right) \sim_d N(0, -2c),$$

just as in part (d) of the theorem.

4. Limit theory for the near explosive case

This section considers the limit behavior of the serial correlation coefficient $\hat{\rho}_n - \rho_n$ when $\rho_n = 1 + c/n^a$ and $c > 0$. In this case the weak convergence of $V_{n^\alpha}(t)$ to $J_c(t)$ still holds on $D[0,\infty)$. However, the random element $J_c(t) \equiv N\left(0, \frac{\sigma^2}{2c}\left(e^{2ct}-1\right)\right)$ is no longer bounded in probability as $t \to \infty$. For $t \in [0, n^{1-\alpha}]$, a further normalization of $O\left(\exp\left\{-cn^{1-\alpha}\right\}\right)$ is needed as $n \to \infty$ to achieve a weak limit for $V_{n^\alpha}(t)$. It turns out that a similar normalization is needed for $n^{-\alpha/2} y_{\lfloor n^\alpha t \rfloor}$, namely ρ_n^{-n}.

For notational convenience in what follows we define $\kappa_n = n^\alpha \lfloor n^{1-\alpha} \rfloor$ and $q = n^{1-\alpha} - \lfloor n^{1-\alpha} \rfloor \in [0,1)$. Two useful approximation results for the near explosive case follow.

4.1 Lemma. *For each $\alpha \in (0,1)$ and $c > 0$*

$$\sup_{t \in [0, n^{1-\alpha}]} \left| \int_0^t \rho_n^{-n^\alpha s} dB_{n^\alpha}(s) - \int_0^t e^{-cs} dB(s) \right| = o_p\left(\frac{1}{n^{\frac{\alpha}{2} - \frac{1}{\nu}}}\right) \quad \textit{as } n \to \infty$$

on the same probability space that (5) holds.

4.2 Lemma. *For each $\alpha \in (0,1)$ and $c > 0$*

$$\sup_{t \in [0, n^{1-\alpha}]} \left| \int_0^t \rho_n^{-(\lfloor n^\alpha t \rfloor - \lfloor n^\alpha s \rfloor)} dB_{n^\alpha}(s) - J_{-c}(t) \right| = o_p\left(\frac{1}{n^{\frac{\alpha}{2} - \frac{1}{\nu}}}\right) \quad \textit{as } n \to \infty$$

on the same probability space that (5) holds.

For the sample variance, note first that, unlike the near-stationary case, the limit theory is not determined exclusively from the blocks $\{y^2_{\lfloor n^\alpha j \rfloor + k} : j = 0, ..., \lfloor n^{1-\alpha} \rfloor - 1$, $k = 1, ..., \lfloor n^\alpha \rfloor\}$. Using (30) in the Appendix, we can write the sample variance as

$$\frac{\rho_n^{-2\kappa_n}}{n^{2\alpha}} \sum_{t=1}^n y_t^2 = \frac{\rho_n^{-2\kappa_n}}{n^{2\alpha}} \sum_{j=0}^{\lfloor n^{1-\alpha} \rfloor - 1} \sum_{k=1}^{\lfloor n^\alpha \rfloor} y^2_{\lfloor n^\alpha j \rfloor + k} + \frac{\rho_n^{-2\kappa_n}}{n^{2\alpha}} \sum_{t=\lfloor \kappa_n \rfloor}^{n} y_t^2 + O_p\left(\frac{1}{n^\alpha}\right). \tag{16}$$

We denote by U_{1n} and U_{2n} the first and second term on the right side of (16) respectively. Since U_{2n} is almost surely positive with limiting expectation $\frac{\sigma^2}{4c^2}\left(e^{2cq}-1\right) > 0$ when $q > 0$, we conclude that it contributes to the limit theory whenever $n^{1-\alpha}$ is not an integer.

We will analyze each of the two terms on the right of (16) separately. The term containing the block components can be written as

$$\begin{aligned} U_{1n} &= \rho_n^{-2\kappa_n} \sum_{j=0}^{\lfloor n^{1-\alpha} \rfloor -1} \frac{1}{n^{2\alpha}} \sum_{k=1}^{\lfloor n^{\alpha} \rfloor} y^2_{\lfloor n^{\alpha} j \rfloor + k} + o_p(1) \\ &= \rho_n^{-2\kappa_n} \sum_{j=0}^{\lfloor n^{1-\alpha} \rfloor -1} \int_0^1 \left(\frac{1}{n^{\alpha/2}} y_{\lfloor n^{\alpha} j \rfloor + \lfloor n^{\alpha} p \rfloor} \right)^2 dp \\ &= \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha} \rfloor} \left(\frac{1}{n^{\alpha/2}} y_{\lfloor n^{\alpha} r \rfloor} \right)^2 dr + o_p(1) \\ &= \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha} \rfloor} \left(\int_0^r \rho_n^{\lfloor n^{\alpha} r \rfloor - n^{\alpha} s} dB_{n^{\alpha}}(s) \right)^2 dr + o_p(1). \end{aligned}$$

Taking the inner integral along $[0, r] = [0, \lfloor n^{1-\alpha} \rfloor] \setminus [r, \lfloor n^{1-\alpha} \rfloor]$ we have, up to $o_p(1)$,

$$U_{1n} = \left(\int_0^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^{\alpha} s} dB_{n^{\alpha}}(s) \right)^2 \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{2\lfloor n^{\alpha} r \rfloor} dr + R_n, \tag{17}$$

where the remainder term R_n is shown in the Appendix to be $o_p(1)$. The second integral on the right side of (17) can be evaluated directly to obtain

$$\int_0^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{2\lfloor n^{\alpha} r \rfloor} dr = \frac{\rho_n^{2\kappa_n}}{2c} (1 + o(1)) \quad as\ n \to \infty, \tag{18}$$

as shown in the Appendix. Using Lemma 4.1, (17) becomes

$$\begin{aligned} U_{1n} &= \frac{1}{2c} \left(\int_0^{\lfloor n^{1-\alpha} \rfloor} e^{-cs} dB(s) \right)^2 + o_p\left(\frac{1}{n^{\frac{\alpha}{2} - \frac{1}{\nu}}} \right) \\ &= \frac{1}{2c} \left(\int_0^{\infty} e^{-cs} dB(s) \right)^2 + o_p\left(\frac{1}{n^{\frac{\alpha}{2} - \frac{1}{\nu}}} \right) \end{aligned} \tag{19}$$

on the same probability space that (5) holds.

For the second term on the right of (16), noting that $\lfloor n - \kappa_n \rfloor = \lfloor n^{\alpha} q \rfloor$, $q \in [0, 1)$, we obtain

$$\begin{aligned} U_{2n} &= \frac{\rho_n^{-2\kappa_n}}{n^{2\alpha}} \sum_{i=0}^{n - \lfloor \kappa_n \rfloor} y^2_{i + \lfloor \kappa_n \rfloor} \\ &= \frac{\rho_n^{-2\kappa_n}}{n^{2\alpha}} \sum_{i=1}^{\lfloor n^{\alpha} q \rfloor} y^2_{i + \lfloor \kappa_n \rfloor - 1} + O_p\left(\frac{1}{n^{\alpha}} \right) \\ &= \frac{\rho_n^{-2\kappa_n}}{n^{\alpha}} \int_0^q y^2_{\lfloor \kappa_n \rfloor + \lfloor n^{\alpha} p \rfloor} dp - \frac{\rho_n^{-2\kappa_n}}{n^{2\alpha}} \left(q - \frac{\lfloor n^{\alpha} q \rfloor}{n^{\alpha}} \right) y^2_{\lfloor \kappa_n \rfloor + \lfloor n^{\alpha} q \rfloor} \\ &= \int_0^q \left(\frac{\rho_n^{-\kappa_n}}{n^{\alpha/2}} y_{\lfloor \kappa_n \rfloor + \lfloor n^{\alpha} p \rfloor} \right)^2 dp + O_p\left(\frac{1}{n^{2\alpha}} \right). \end{aligned} \tag{20}$$

Now for each $p \in [0,q]$, $q \in [0,1)$, we can show (the details are included in the Appendix) that

$$\frac{\rho_n^{-\kappa_n}}{n^{\alpha/2}} y_{\lfloor \kappa_n \rfloor + \lfloor n^\alpha p \rfloor} = e^{cp} \int_0^\infty e^{-cs} dW(s) + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \tag{21}$$

on the same probability space that (5) holds. Thus, applying the dominated convergence theorem to (20) yields

$$\begin{aligned} U_{2n} &= \left(\int_0^\infty e^{-cs} dW(s)\right)^2 \int_0^q e^{2cp} dp + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \\ &= \frac{1}{2c}\left(\int_0^\infty e^{-cs} dW(s)\right)^2 \left(e^{2cq}-1\right) + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right). \end{aligned} \tag{22}$$

Letting $X := \int_0^\infty e^{-cs} dB(s) \equiv N\left(0, \frac{\sigma^2}{2c}\right)$, we conclude from (16), (19), (22) and the asymptotic equivalence $\rho_n^{-2\kappa_n} e^{-2cq} = \rho_n^{-2n}[1+o(1)]$ that

$$\frac{\rho_n^{-2n}}{n^{2\alpha}} \sum_{t=1}^n y_t^2 = \frac{1}{2c} X^2 + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right),$$

on the same probability space that (5) holds. This implies that the limiting distribution of the sample variance is given by

$$\frac{\rho_n^{-2n}}{n^{2\alpha}} \sum_{t=1}^n y_t^2 \Longrightarrow \frac{1}{2c} X^2 \tag{23}$$

on the original space.

As in the case of the sample variance, the asymptotic behavior of the sample covariance is partly determined by elements of the time series $y_{t-1}u_t$ that do not belong to the block components $\left\{y_{\lfloor n^\alpha j \rfloor + k - 1} u_{\lfloor n^\alpha j \rfloor + k} : j = 0, ..., \lfloor n^{1-\alpha} \rfloor - 1, k = 1, ..., \lfloor n^\alpha \rfloor\right\}$. Obtaining limits for the block components and the remaining time series separately in a method similar to that used for the sample variance will work. It is, however, more efficient to derive the limiting distribution of the sample covariance by using a direct argument on $\frac{\rho_n^{-n}}{n^\alpha} \sum_{t=1}^n y_{t-1} u_t$.

Using the initial condition $y_0 = o_p\left(n^{\alpha/2}\right)$ and (29) in the Appendix we can write

$$\begin{aligned}
\frac{\rho_n^{-n}}{n^\alpha}\sum_{t=1}^{n} y_{t-1}u_t &= \frac{\rho_n^{-n}}{n^\alpha}\sum_{t=1}^{n-1} y_t u_{t+1} + o_p\left(\frac{\rho_n^{-n}}{n^{\alpha/2}}\right) \\
&= \frac{\rho_n^{-n}}{n^\alpha}\sum_{t=1}^{\left\lfloor n^\alpha\left(n^{1-\alpha}-\frac{1}{n^\alpha}\right)\right\rfloor} y_t u_{t+1} \\
&= \rho_n^{-n}\int_0^{n^{1-\alpha}-\frac{1}{n^\alpha}} \frac{1}{n^{\alpha/2}} y_{\lfloor n^\alpha r\rfloor} dB_{n^\alpha}\left(r+\frac{1}{n^\alpha}\right) \\
&= \rho_n^{-n}\int_{\frac{1}{n^\alpha}}^{n^{1-\alpha}} \frac{1}{n^{\alpha/2}} y_{\left\lfloor n^\alpha\left(r-\frac{1}{n^\alpha}\right)\right\rfloor} dB_{n^\alpha}(r) \\
&= \rho_n^{-n}\int_{\frac{1}{n^\alpha}}^{n^{1-\alpha}}\int_0^{r-\frac{1}{n^a}} \rho_n^{\lfloor n^\alpha r\rfloor - n^\alpha s - 1} dB_{n^a}(s)\, dB_{n^\alpha}(r) + o_p(1).
\end{aligned}$$

Taking the inner integral along $\left[0, r-\frac{1}{n^a}\right] = \left[0, n^{1-\alpha}\right] \setminus \left[r-\frac{1}{n^a}, n^{1-\alpha}\right]$ we obtain, up to $o_p(1)$,

$$\frac{\rho_n^{-n}}{n^\alpha}\sum_{t=1}^{n} y_{t-1}u_t = \rho_n^{-1}\int_0^{n^{1-\alpha}} \rho_n^{-n^\alpha s} dB_{n^a}(s)\int_{\frac{1}{n^\alpha}}^{n^{1-\alpha}} \rho_n^{-(n-\lfloor n^\alpha r\rfloor)} dB_{n^a}(r) - I_n, \qquad (24)$$

where the remainder term

$$I_n := \rho_n^{-n-1}\int_{\frac{1}{n^\alpha}}^{n^{1-\alpha}}\int_{r-\frac{1}{n^a}}^{n^{1-\alpha}} \rho_n^{\lfloor n^\alpha r\rfloor - n^\alpha s} dB_{n^a}(s)\, dB_{n^a}(r)$$

is shown in the Appendix to be $o_p(1)$. Now, since $\int_0^{\frac{1}{n^\alpha}} \rho_n^{-(n-\lfloor n^\alpha r\rfloor)} dB_{n^a}(r) = O_p\left(\frac{\rho_n^{-n}}{n^{\alpha/2}}\right)$, Lemma 4.2 implies that

$$\int_{\frac{1}{n^\alpha}}^{n^{1-\alpha}} \rho_n^{-(n-\lfloor n^\alpha r\rfloor)} dB_{n^a}(r) = J_{-c}\left(n^{1-\alpha}\right) + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right).$$

Since $J_{-c}(t)$ is a L^2-bounded martingale on $[0,\infty)$, the martingale convergence theorem ensures the existence of an almost surely finite random variable Y such that

$$J_{-c}\left(n^{1-\alpha}\right) \longrightarrow_{a.s.} Y \quad \textit{as } n \to \infty.$$

Since $J_{-c}\left(n^{1-\alpha}\right) \equiv N\left(0, \frac{\sigma^2}{2c}\left(1-e^{-2cn^{1-\alpha}}\right)\right)$, it is clear that $Y \equiv N\left(0, \frac{\sigma^2}{2c}\right)$. Thus, if $X = \int_0^\infty e^{-cs} dB(s)$ as in (23), (24) yields

$$\frac{\rho_n^{-n}}{n^\alpha}\sum_{t=1}^{n} y_{t-1}u_t = XY + o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \quad \textit{as } n \to \infty$$

on the same probability space that (5) holds. The latter strong approximation implies that the asymptotic distribution of the sample covariance is given in the original space by

$$\frac{\rho_n^{-n}}{n^\alpha}\sum_{t=1}^{n} y_{t-1}u_t \Longrightarrow XY \quad X,Y \equiv N\left(0,\frac{\sigma^2}{2c}\right). \tag{25}$$

The asymptotic behavior of the serial correlation coefficient in the near explosive case is an easy consequence of (23), (25) and the fact that the limiting random variables X and Y are independent.

4.3 Theorem. *For model (1) with $\rho_n = 1 + c/n^a$, $c > 0$ and $\alpha \in (0,1)$*

$$\frac{n^\alpha \rho_n^n}{2c}\left(\hat{\rho}_n - \rho_n\right) \Longrightarrow C \quad \text{as } n \to \infty \tag{26}$$

where C is a standard Cauchy variate.

4.4 Remarks

(i) Theorem 4.3 relates to earlier work (White, 1958; Anderson, 1959; Basawa and Brockwell, 1984) on the explosive Gaussian AR(1) process. For a Gaussian first order autoregressive process with fixed $|\rho| > 1$ and $y_0 = 0$, White showed that

$$\frac{\rho^n}{\rho^2 - 1}\left(\hat{\rho}_n - \rho\right) \Longrightarrow C \quad \text{as } n \to \infty. \tag{27}$$

Replacing ρ by $\rho_n = 1 + c/n^a$, we obtain $\rho^2 - 1 = \frac{2c}{n^\alpha}[1 + o(1)]$. Hence, the normalizations in Theorem 4.3 and (27) are asymptotically equivalent as $n \to \infty$. Anderson (1959) showed that $\frac{\rho^n}{\rho^2-1}\left(\hat{\rho}_n - \rho\right)$ has a limit distribution that depends on the distribution of the errors u_t when $\rho > 1$ and that no central limit theory or invariance principle is applicable.

(ii) The limit theory derived in this section for the moderate deviations case is not restricted to Gaussian processes. In particular, the Cauchy limit result (26) applies for $\rho_n = 1 + c/n^a$ and innovations u_t satisfying (2) with $\alpha > 0$, which includes a much wider class of processes. At the boundary where $\alpha \to 0$, Theorem 4.3 reduces to (27) with $\rho = 1 + c$, and the the errors u_t have infinitely many moments as under Gaussianity.

(iii) The limit theory for near explosive moderate deviations from unity is invariant to the initial condition y_0 being any fixed constant value or random variable of smaller asymptotic order than $n^{\alpha/2}$. This property is not shared by explosive autoregressions where y_0 does influence the limit theory, as shown by Anderson (1959).

5. Discussion

The convergence rates of Theorem 3.1 bridge those for unit root or local to unity processes and those that apply under stationarity. Thus, in part (d) the convergence rate $n^{\frac{1}{2}+\frac{\alpha}{2}}$ ranges over $\left(n^{1/2}, n\right)$ for $\alpha \in (0,1)$. However, the bridging asymptotics are not continuous at the boundaries of α. For example, when $\alpha \to 0$, part (d) becomes $\sqrt{n}\left(\hat{\rho}_n - \rho_n\right) \Longrightarrow N\left(0, -2c\right)$, whereas the correct stationary result when $\rho = 1 + c$ is $\sqrt{n}\left(\widehat{\rho}_n - \rho\right) \Longrightarrow N\left(0, -2c - c^2\right)$. Thus, part (d) as it stands overestimates the variance of $\widehat{\rho}_n$ in the boundary case where $\alpha = 0$. Continuity at this boundary can be achieved (for parts (b)-(d)) through replacement of c by $c + c^2/2n^{\alpha}$, without affecting the asymptotic results for $\alpha > 0$. For the limit as $\alpha \to 1$, we have $n^{1-\alpha} \to 1$, and so $\lfloor n^{1-\alpha} \rfloor = 1$ for $\alpha = 1$, in which case $j = 0$ necessarily and part (a) becomes $n^{-1/2} y_{\lfloor np \rfloor} \Longrightarrow J_c(p)$, the usual local to unity limit result (cf. Phillips, 1987). In that case, part (d) is replaced by the non-normal limit

$$n\left(\widehat{\rho}_n - \rho_n\right) \Longrightarrow \int_0^1 J_c\left(q\right) dB\left(q\right) / \int_0^1 J_c\left(q\right)^2 dq. \tag{28}$$

Similarly, when $c > 0$, the convergence rate of Theorem 4.3 takes values on $\left(n, \left(1+c\right)^n\right)$ as α ranges from 1 to 0. Since $1 + c$ is the autoregressive root of an explosive AR(1) process when $\alpha = 0$, there is a discontinuity due to the discrepancy between $1 - \rho_n^2 = -\frac{2c}{n^{\alpha}} + O\left(n^{-2\alpha}\right)$ when $\alpha \in (0,1)$ and $1 - \rho^2 = 2c + c^2$ when $\alpha = 0$. As in the near stationary case, continuity can be achieved through replacement of c by $c + c^2/2n^{\alpha}$ without affecting Theorem 4.3. However, when $\alpha \to 1$, the blocking scheme is such that $j = 0$ and again the local to unity limit theory (28) applies. Thus, continuity is achieved at the outside boundaries with the stationary and explosive case asymptotics, but not at the inside boundaries with the conventional local to unity asymptotics.

6. Notation

$\lfloor \cdot \rfloor$	integer part of	$\longrightarrow_{a.s.}$	almost sure convergence
$:=$	definitional equality	$\longrightarrow_p$	convergence in probability
$B\left(t\right)$	Brownian motion with variance $\sigma^2 t$	$\longrightarrow_{L^p}$	convergence in L^pnorm
$J_c\left(t\right)$	Ornstein-Uhlenbeck process	$\Longrightarrow$	weak convergence
$\left[X\right]_t$	quadratic variation process of X_t	$\equiv$	distributional equivalence
κ_n	$:=n^{\alpha}\lfloor n^{1-\alpha}\rfloor$	$\sim_d$	asymptotically distributed as
q	$:=n^{1-\alpha} - \lfloor n^{1-\alpha}\rfloor$	$o_p(1)$	tends to zero in probability
$E_{\mathcal{F}}\left(\cdot\right)$	conditional expectation $E\left(\cdot\middle|\mathcal{F}\right)$	$o_{a.s.}(1)$	tends to zero almost surely
$P_{\mathcal{F}}\left(\cdot\right)$	conditional probability $P\left(\cdot\middle|\mathcal{F}\right)$	$\mathbf{1}\left\{\cdot\right\}$	indicator function

7. Technical appendix and proofs

Proposition A1. *For each $x \in [0, M]$, $M > 0$, possibly depending on n, and real valued, measurable function f* on $[0, \infty)$

$$\frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor xn^{\alpha} \rfloor} f\left(\frac{i}{n^{\alpha}}\right) u_i = \int_0^x f(r)\, dB_{n^{\alpha}}(r).$$

Proof. It is convenient to reduce the interval from $[0, M]$ to $[0, 1]$. If $x \in [0, M]$, then $y := \frac{x}{M} \in [0, 1]$ and $m_n := Mn^{\alpha} \to \infty$, so we can write

$$\begin{aligned}
\frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor xn^{\alpha} \rfloor} f\left(\frac{i}{n^{\alpha}}\right) u_i &= \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor yMn^{\alpha} \rfloor} f\left(\frac{i}{n^{\alpha}}\right) u_i \\
&= \sqrt{M} \frac{1}{(Mn^{\alpha})^{1/2}} \sum_{i=1}^{\lfloor yMn^{\alpha} \rfloor} f\left(M \frac{i}{Mn^{\alpha}}\right) u_i \\
&= \sqrt{M} \frac{1}{m_n^{1/2}} \sum_{i=1}^{\lfloor ym_n \rfloor} f\left(M \frac{i}{m_n}\right) u_i \\
&= \sqrt{M} \int_0^y f(Ms)\, dB_{m_n}(s) \\
&= \int_0^x f(r)\, d\left[\sqrt{M} B_{m_n}\left(\frac{r}{M}\right)\right] \\
&= \int_0^x f(r)\, dB_{n^{\alpha}}(r)
\end{aligned}$$

since

$$\begin{aligned}
\sqrt{M} B_{m_n}\left(\frac{r}{M}\right) &= \sqrt{M} \frac{1}{m_n^{1/2}} \sum_{i=1}^{\lfloor m_n \frac{r}{M} \rfloor} u_i = \sqrt{M} \frac{1}{(n^{\alpha}M)^{1/2}} \sum_{i=1}^{\lfloor n^{\alpha} M \frac{r}{M} \rfloor} u_i \\
&= \frac{1}{n^{\frac{\alpha}{2}}} \sum_{i=1}^{\lfloor n^{\alpha} r \rfloor} u_i = B_{n^{\alpha}}(r).
\end{aligned}$$

■

The following integral representation on $[0, M]$ is an immediate corollary of Proposition A1:

$$\frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor xn^{\alpha} \rfloor} f\left(\frac{i}{n^{\alpha}}\right) u_{i+m} = \int_0^x f(r)\, dB_{n^{\alpha}}\left(r + \frac{m}{n^{\alpha}}\right), \quad m \in \mathbb{N}. \tag{29}$$

Proposition A2. *For $c < 0$, $\sup_{t>0} |J_c(t)| < \infty$ a.s.*

Proof. Since $\sup_{t>0} |J_c(t)| > 0$ *a.s.*, it is enough to show that

$$E\left(\sup_{t>0} |J_c(t)|\right) = \int_0^\infty P\left(\sup_{t>0} |J_c(t)| \geq x\right) dx < \infty.$$

Define $\tau = \frac{\sigma^2}{-2c}$. Since $[J_c]_\infty = \tau$ *a.s.*,

$$\begin{aligned} P\left(\sup_{t>0} |J_c(t)| \geq x\right) &= P\left(\sup_{t>0} |J_c(t)| \geq x, [J_c]_\infty = \tau\right) \\ &\leq P\left(\sup_{t>0} |J_c(t)| \geq x, [J_c]_\infty \leq \tau\right) \\ &\leq e^{-\frac{x^2}{2\tau}} \end{aligned}$$

by Bernstein's inequality (cf. Revuz and Yor, 1999, p.153 Exercise 3.16). Thus,

$$E\left(\sup_{t>0} |J_c(t)|\right) \leq \int_0^\infty e^{-\frac{x^2}{2\tau}} dx = \sqrt{\frac{\pi}{2}\tau} < \infty$$

which completes the proof. ■

Proposition A3. *For each $\alpha \in (0,1)$ and $c < 0$*

$$\frac{1}{n^{1+\alpha}} \sum_{t=1}^{n} x_t^2 = \frac{1}{n^{1+\alpha}} \sum_{j=0}^{\lfloor n^{1-\alpha} \rfloor - 1} \sum_{k=1}^{\lfloor n^\alpha \rfloor} x_{\lfloor n^\alpha j \rfloor + k}^2 + O_p\left(\frac{1}{n^{1-\alpha}}\right)$$

as $n \to \infty$, on the same probability space that (5) holds.

Proof. Denoting $\kappa_n = n^\alpha \lfloor n^{1-\alpha} \rfloor$, note that

$$\{y_{\lfloor n^\alpha j \rfloor + k}^2 : j = 0, ..., \lfloor n^{1-\alpha} \rfloor - 1,\ k = 1, ..., \lfloor n^\alpha \rfloor\} \subseteq \{y_t^2 : t = 1, ..., n\},$$

where the maximal subscript $\lfloor (\lfloor n^{1-\alpha} \rfloor - 1) n^\alpha \rfloor + \lfloor n^\alpha \rfloor$ of the block components on the left takes values

$$\lfloor \kappa_n \rfloor - 1 \leq \left\lfloor \left(\lfloor n^{1-\alpha} \rfloor - 1 \right) n^\alpha \right\rfloor + \lfloor n^\alpha \rfloor \leq \lfloor \kappa_n \rfloor. \tag{30}$$

Also, (8) and Proposition A2 give

$$\sup_{0 \leq k \leq n} \frac{|y_k|}{n^{\alpha/2}} = \sup_{r \in [0, n^{1-\alpha}]} \left| \frac{y_{\lfloor n^\alpha r \rfloor}}{n^{\alpha/2}} \right| \leq \sup_{r \in [0, n^{1-\alpha}]} \left| \frac{y_{\lfloor n^\alpha r \rfloor}}{n^{\alpha/2}} - J_c(r) \right| + \sup_{r>0} |J_c(r)| = O_p(1)$$

on the same probability space that (5) holds. Thus, the remainder term E_n of the proposition is given by

$$\begin{aligned} E_n &= \frac{1}{n^{1+\alpha}} \sum_{t=\lfloor (\lfloor n^{1-\alpha} \rfloor - 1) n^\alpha \rfloor + \lfloor n^\alpha \rfloor + 1}^{n} y_t^2 \leq \frac{1}{n^{1+\alpha}} \sum_{t=\lfloor \kappa_n \rfloor}^{n} y_t^2 \\ &\leq \left(\sup_{0 \leq t \leq n} \frac{|y_t|}{n^{\alpha/2}} \right)^2 \frac{n - \lfloor \kappa_n \rfloor}{n} = O_p\left(\frac{1}{n^{1-\alpha}} \right), \end{aligned}$$

which shows the proposition, since $E_n \geq 0$ *a.s.*. ■

Proof of (5). For each $\alpha \in (0,1)$ we have

$$\begin{aligned}
&\sup_{t\in[0,n^{1-\alpha}]} \left|B_{n^\alpha}(t) - B(t)\right| \\
&= n^{-\frac{\alpha}{2}} \sup_{t\in[0,n^{1-\alpha}]} \left|S_{\lfloor n^\alpha t\rfloor} - B(n^\alpha t)\right| \\
&\leq n^{-\frac{\alpha}{2}} \sup_{t\in[0,n^{1-\alpha}]} \left|S_{\lfloor n^\alpha t\rfloor} - B(\lfloor n^\alpha t\rfloor)\right| + n^{-\frac{\alpha}{2}} \sup_{t\in[0,n^{1-\alpha}]} \left|B(n^\alpha t) - B(\lfloor n^\alpha t\rfloor)\right| \\
&\leq n^{-\frac{\alpha}{2}} \sup_{0\leq i\leq n} \left|S_i - B(i)\right| + n^{-\frac{\alpha}{2}} \sup_{t\in[0,n]} \left|B(t) - B(\lfloor t\rfloor)\right|. \qquad (31)
\end{aligned}$$

For any $\beta \in \left(0,\frac{1}{2}\right)$ the Hölder continuity property of Brownian motion sample paths (e.g., Revuz and Yor, Theorem 2.2) gives

$$\begin{aligned}
&\sup_{t\in[0,n]} \left|B(t) - B(\lfloor t\rfloor)\right| \\
&= \sup_{\substack{t\in[0,n] \\ t-\lfloor t\rfloor\in(0,1)}} \left|B(t) - B(\lfloor t\rfloor)\right| \\
&\leq \sup_{\substack{t\in[0,n] \\ t-\lfloor t\rfloor\in(0,1)}} \frac{\left|B(t) - B(\lfloor t\rfloor)\right|}{\left|t-\lfloor t\rfloor\right|^\beta} < \infty \quad a.s., \qquad (32)
\end{aligned}$$

so the second term on the right side of (31) is of order $O_{a.s.}\left(n^{-\frac{\alpha}{2}}\right)$. (31), (32) and (4) give the stated result. ■

Proof of (6). Write $\rho_n = 1 + c/n^\alpha$ and note that $\left|\frac{y_0}{n^{\alpha/2}}\right| \sup_{t>0} \left|\rho_n\right|^{\lfloor n^\alpha t\rfloor} = o_p(1)$ since $\sup_{t>0} \left|\rho_n\right|^{\lfloor n^\alpha t\rfloor} < \infty$. Then

$$\begin{aligned}
&\sup_{t\in[0,n^{1-\alpha}]} \left|\frac{1}{n^{\alpha/2}} y_{\lfloor n^\alpha t\rfloor} - V_{n^\alpha}(t)\right| \\
&\leq \sup_{t\in[0,n^{1-\alpha}]} \left|\frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor tn^\alpha\rfloor} \rho_n^{\lfloor tn^\alpha\rfloor - i} u_i - V_{n^\alpha}(t)\right| + o_p(1) \\
&\leq \sup_{t\in[0,n^{1-\alpha}]} \left|\frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor tn^\alpha\rfloor} \left\{\rho_n^{\lfloor tn^\alpha\rfloor - i} - e^{\frac{c}{n^\alpha}(\lfloor tn^\alpha\rfloor - i)}\right\} u_i\right| + o_p(1) \qquad (33)
\end{aligned}$$

Now

$$\begin{aligned}
&\sup_{t\in[0,n^{1-\alpha}]} \left|\frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor tn^\alpha\rfloor} \left\{\rho_n^{\lfloor tn^\alpha\rfloor - i} - e^{\frac{c}{n^\alpha}(\lfloor tn^\alpha\rfloor - i)}\right\} u_i\right| \\
&= \sup_{1\leq m\leq n} \left|\frac{1}{n^{\alpha/2}} \sum_{i=1}^{m} \left\{\rho_n^{m-i} - e^{\frac{c}{n^\alpha}(m-i)}\right\} u_i\right| = o_p(1), \qquad (34)
\end{aligned}$$

since Kolmogorov's inequality shows that for arbitrary $\eta > 0$

$$\begin{aligned} P\left(\sup_{1\le m\le n}\left|\frac{1}{n^{\alpha/2}}\sum_{i=1}^{m}\left\{\rho_n^{m-i}-e^{\frac{c}{n^\alpha}(m-i)}\right\}u_i\right|>\eta\right) &\le \frac{\sigma^2}{\eta^2 n^\alpha}\sum_{i=1}^{n}\left\{\rho_n^{n-i}-e^{\frac{c}{n^\alpha}(n-i)}\right\}^2 \\ &= \frac{\sigma^2}{\eta^2 n^{2\alpha}}\left[\frac{c}{16}+O\left(n^{-\alpha}\right)\right], \end{aligned}$$

as $n \to \infty$, the last line following from direct calculation of the sums. Combining (33) and (34) delivers the required result. ■

Proof of Lemma 2.2. Using the integration by parts formula for Stieltjes integrals we obtain

$$\begin{aligned} & \sup_{t\in[0,n^{1-\alpha}]}\left|V_{n^\alpha}(t)-J_c(t)\right| \\ =\; & \sup_{t\in[0,n^{1-\alpha}]}\left|\int_0^t e^{c(t-r)}dB_{n^\alpha}(s)-\int_0^t e^{c(t-r)}dB(s)\right| \\ =\; & \sup_{t\in[0,n^{1-\alpha}]}\left|B_{n^\alpha}(t)+c\int_0^t e^{c(t-s)}B_{n^\alpha}(s)\,ds-B(t)-c\int_0^t e^{c(t-s)}B(s)\,ds\right| \\ \le\; & \sup_{t\in[0,n^{1-\alpha}]}\left|B_{n^\alpha}(t)-B(t)\right|+|c|\sup_{t\in[0,n^{1-\alpha}]}\int_0^t e^{c(t-s)}\left|B_{n^\alpha}(s)-B(s)\right|ds \\ \le\; & \sup_{t\in[0,n^{1-\alpha}]}\left|B_{n^\alpha}(t)-B(t)\right| \\ & +|c|\left(\sup_{t\in[0,n^{1-\alpha}]}\sup_{s\in[0,t]}\left|B_{n^\alpha}(s)-B(s)\right|\right)\sup_{t\in[0,n^{1-\alpha}]}\int_0^t e^{c(t-s)}ds \\ \le\; & \sup_{t\in[0,n^{1-\alpha}]}\left|B_{n^\alpha}(t)-B(t)\right|+|c|\sup_{t\in[0,n^{1-\alpha}]}\left|B_{n^\alpha}(t)-B(t)\right|\frac{1}{-c}\sup_{t\ge 0}\left(1-e^{ct}\right) \\ =\; & 2\sup_{t\in[0,n^{1-\alpha}]}\left|B_{n^\alpha}(t)-B(t)\right| \\ =\; & o_{a.s.}\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right) \end{aligned}$$

by the strong invariance principle (5). ■

Proof of (12). Squaring $J_c^*(r) = J_c(r) + e^{cr} J_c^*(0)$ we obtain

$$\begin{aligned}
\int_0^{\lfloor n^{1-\alpha} \rfloor} \left| J_c(r)^2 - J_c^*(r)^2 \right| dr &= \int_0^{\lfloor n^{1-\alpha} \rfloor} \left| e^{2cr} J_c^*(0)^2 + 2J_c^*(0) e^{cr} J_c(r) \right| dp \\
&\leq J_c^*(0)^2 \int_0^{\lfloor n^{1-\alpha} \rfloor} e^{2cr} dp + 2 \left| J_c^*(0) \right| \int_0^{\lfloor n^{1-\alpha} \rfloor} e^{cr} \left| J_c(r) \right| dp \\
&\leq \left(J_c^*(0)^2 + 2 \left| J_c^*(0) \right| \sup_{r \geq 0} \left| J_c(r) \right| \right) \int_0^{\lfloor n^{1-\alpha} \rfloor} e^{cr} dp \\
&< \infty \; a.s.
\end{aligned}$$

by Proposition A2. Thus,

$$\begin{aligned}
\frac{1}{n^{1-\alpha}} \left| \int_0^{\lfloor n^{1-\alpha} \rfloor} \left\{ J_c(r)^2 - J_c^*(r)^2 \right\} dr \right| &\leq \frac{1}{n^{1-\alpha}} \int_0^{\lfloor n^{1-\alpha} \rfloor} \left| J_c(r)^2 - J_c^*(r)^2 \right| dr \\
&= O_{a.s.}\left(n^{-(1-\alpha)} \right).
\end{aligned}$$

■

Proof of (14). First, note that since $\frac{1}{n^{1+\alpha}} \sum_{i=1}^n y_{i-1}^2 \longrightarrow_p \sigma^2 / -2c$ and

$$\begin{aligned}
\sum_{t=1}^n E_{\mathcal{F}_{t-1}} \left(\xi_t^2 \mathbf{1} \left\{ |\xi_t| > \eta \right\} \right) &= \frac{1}{n^{1+\alpha}} \sum_{t=1}^n y_{t-1}^2 E_{\mathcal{F}_{t-1}} \left(u_t^2 \mathbf{1} \left\{ |y_{t-1} u_t| > \eta n^{\frac{1+\alpha}{2}} \right\} \right) \\
&\leq \max_{1 \leq t \leq n} E_{\mathcal{F}_{t-1}} \left(u_t^2 \mathbf{1} \left\{ |y_{t-1} u_t| > \eta n^{\frac{1+\alpha}{2}} \right\} \right) \frac{1}{n^{1+\alpha}} \sum_{i=1}^n y_{i-1}^2,
\end{aligned}$$

(14) holds if

$$\max_{1 \leq t \leq n} E_{\mathcal{F}_{t-1}} \left(u_t^2 \mathbf{1} \left\{ |y_{t-1} u_t| > \eta n^{\frac{1+\alpha}{2}} \right\} \right) \longrightarrow_p 0. \tag{35}$$

for each $\eta > 0$. To show (35), recall from the moment condition (2) that the i.i.d. sequence $(u_t)_{t \in \mathbb{N}}$ satisfies $E |u_t|^\nu < \infty$ for some $\nu > \frac{2}{\alpha} > 2$. Using the Chebyshev inequality and the Hölder inequality with $r_1 = \frac{\nu}{2} > 1$, $r_2 = \frac{\nu}{\nu - 2} > 0$, so that $r_1^{-1} + r_2^{-1} = 1$, we obtain

$$\begin{aligned}
E_{\mathcal{F}_{t-1}} \left(u_t^2 \mathbf{1} \left\{ |y_{t-1} u_t| > \eta n^{\frac{1+\alpha}{2}} \right\} \right) &\leq \left(E_{\mathcal{F}_{t-1}} |u_t|^{2r_1} \right)^{\frac{1}{r_1}} \left(E_{\mathcal{F}_{t-1}} \mathbf{1} \left\{ |y_{t-1} u_t| > \eta n^{\frac{1+\alpha}{2}} \right\} \right)^{\frac{1}{r_2}} \\
&= \left(E_{\mathcal{F}_{t-1}} |u_t|^\nu \right)^{\frac{2}{\nu}} \left(P_{\mathcal{F}_{t-1}} \left\{ |y_{t-1} u_t| > \eta n^{\frac{1+\alpha}{2}} \right\} \right)^{\frac{\nu-2}{\nu}} \\
&= \left(E |u_1|^\nu \right)^{\frac{2}{\nu}} \left(P_{\mathcal{F}_{t-1}} \left\{ |y_{t-1} u_t| > \eta n^{\frac{1+\alpha}{2}} \right\} \right)^{\frac{\nu-2}{\nu}} \\
&\leq \left(E |u_1|^\nu \right)^{\frac{2}{\nu}} \left[\frac{E_{\mathcal{F}_{t-1}} \left(y_{t-1}^2 u_t^2 \right)}{\eta^2 n^{1+\alpha}} \right]^{\frac{\nu-2}{\nu}} \\
&= \left(E |u_1|^\nu \right)^{\frac{2}{\nu}} \left(\frac{\sigma^2}{\eta^2} \right)^{\frac{\nu-2}{\nu}} \left(\frac{y_{t-1}^2}{n^{1+\alpha}} \right)^{\frac{\nu-2}{\nu}}.
\end{aligned}$$

Therefore, letting $C := (E|u_1|^\nu)^{\frac{2}{\nu}} \left(\frac{\sigma^2}{\eta^2}\right)^{\frac{\nu-2}{\nu}} < \infty$ we conclude that

$$\begin{aligned}
\max_{1\le t\le n} E_{\mathcal{F}_{t-1}}\left(u_t^2 \mathbf{1}\left\{|y_{t-1}u_t| > \eta n^{\frac{1+\alpha}{2}}\right\}\right) &\le C \max_{1\le t\le n}\left(\frac{y_{t-1}^2}{n^{1+\alpha}}\right)^{\frac{\nu-2}{\nu}} \\
&= C\left[\frac{1}{n}\left(\max_{1\le t\le n}\left|\frac{y_{t-1}}{n^{\alpha/2}}\right|\right)^2\right]^{\frac{\nu-2}{\nu}} = o_p(1),
\end{aligned}$$

since

$$\begin{aligned}
\max_{1\le t\le n}\left|\frac{y_{t-1}}{n^{\alpha/2}}\right| &\le \sup_{s\in[0,n^{1-\alpha}]}\left|\frac{1}{n^{\alpha/2}} y_{\lfloor n^\alpha s\rfloor}\right| \\
&\le \sup_{s\in[0,n^{1-\alpha}]}\left|\frac{1}{n^{\alpha/2}} y_{\lfloor n^\alpha s\rfloor} - J_c(s)\right| + \sup_{s>0}|J_c(s)| = O_p(1)
\end{aligned}$$

by (8) and Proposition A2. ■

Proof of Lemma 4.1.

$$\begin{aligned}
&\sup_{t\in[0,n^{1-\alpha}]}\left|\int_0^t \rho_n^{-n^\alpha s} dB_{n^\alpha}(s) - \int_0^t e^{-cs} dB(s)\right| \\
\le\ &\sup_{t\in[0,n^{1-\alpha}]}\left|\int_0^t \rho_n^{-n^\alpha s} dB_{n^\alpha}(s) - \int_0^t e^{-cs} dB_{n^\alpha}(s)\right| \\
&+ \sup_{t\in[0,n^{1-\alpha}]}\left|\int_0^t e^{-cs} dB_{n^\alpha}(s) - \int_0^t e^{-cs} dB(s)\right| \\
=\ &\sup_{t\in[0,n^{1-\alpha}]}\left|\int_0^t \rho_n^{-n^\alpha s} dB_{n^\alpha}(s) - \int_0^t e^{-cs} dB_{n^\alpha}(s)\right| + o_{a.s.}\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right)
\end{aligned}$$

by an argument identical to that used in the proof of Lemma 2.1. So it is enough to show that

$$\sup_{t\in[0,n^{1-\alpha}]}\left|\int_0^t \rho_n^{-n^\alpha s} dB_{n^\alpha}(s) - \int_0^t e^{-cs} dB_{n^\alpha}(s)\right| = o_p(1). \tag{36}$$

Since

$$\begin{aligned}
\sup_{t\in[0,n^{1-\alpha}]}\left|\int_0^t \rho_n^{-n^\alpha s} dB_{n^\alpha}(s) - \int_0^t e^{-cs} dB_{n^\alpha}(s)\right| &= \sup_{t\in[0,n^{1-\alpha}]}\left|\frac{1}{n^{\frac{\alpha}{2}}}\sum_{i=1}^{\lfloor tn^\alpha\rfloor}\left\{\rho_n^{-i} - e^{-\frac{c}{n^\alpha}i}\right\} u_i\right| \\
&= \sup_{1\le m\le n}\left|\frac{1}{n^{\frac{\alpha}{2}}}\sum_{i=1}^{m}\left\{\rho_n^{-i} - e^{-\frac{c}{n^\alpha}i}\right\} u_i\right|,
\end{aligned}$$

Kolmogorov's inequality gives for any $\varepsilon > 0$

$$\begin{aligned}
P\left(\sup_{1\leq m\leq n}\left|\frac{1}{n^{\frac{\alpha}{2}}}\sum_{i=1}^{m}\left\{\rho_n^{-i}-e^{-\frac{c}{n^{\alpha}}i}\right\}u_i\right|>\varepsilon\right) &\leq \frac{1}{\varepsilon^2 n^{\alpha}}\sum_{i=1}^{n}\left\{\rho_n^{-i}-e^{-\frac{c}{n^{\alpha}}i}\right\}^2 E\left(u_i^2\right) \\
&= \frac{\sigma^2}{\varepsilon^2 n^{\alpha}}\sum_{i=1}^{n}\left\{\rho_n^{-i}-e^{-\frac{c}{n^{\alpha}}i}\right\}^2 \\
&= \frac{\sigma^2}{\varepsilon^2 n^{2\alpha}}\left[\frac{c}{16}+O\left(n^{-\alpha}\right)\right]
\end{aligned}$$

as $n \to \infty$, again by direct calculation of the sums. It follows that

$$\sup_{1\leq m\leq n}\left|\frac{1}{n^{\alpha/2}}\sum_{i=1}^{m}\left\{\rho_n^{-i}-e^{-\frac{c}{n^{\alpha}}i}\right\}u_i\right| = o_p(1), \tag{37}$$

and (36) and the Lemma follow directly. ■

Proof of Lemma 4.2. The argument is similar to that used for Lemma 4.1.

$$\begin{aligned}
&\sup_{t\in[0,n^{1-\alpha}]}\left|\int_0^t \rho_n^{-(\lfloor n^{\alpha}t\rfloor-\lfloor n^{\alpha}s\rfloor)}dB_{n^{\alpha}}(s)-J_{-c}(t)\right| \\
\leq\ &\sup_{t\in[0,n^{1-\alpha}]}\left|\int_0^t \rho_n^{-(\lfloor n^{\alpha}t\rfloor-\lfloor n^{\alpha}s\rfloor)}dB_{n^{\alpha}}(s)-\int_0^t e^{-c(t-s)}dB_{n^{\alpha}}(s)\right| \\
&+\sup_{t\in[0,n^{1-\alpha}]}\left|\int_0^t e^{-c(t-s)}dB_{n^{\alpha}}(s)-J_{-c}(t)\right| \\
=\ &\sup_{t\in[0,n^{1-\alpha}]}\left|\int_0^t \rho_n^{-(\lfloor n^{\alpha}t\rfloor-\lfloor n^{\alpha}s\rfloor)}dB_{n^{\alpha}}(s)-\int_0^t e^{-c(t-s)}dB_{n^{\alpha}}(s)\right|+o_{a.s.}\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right),
\end{aligned}$$

by Lemma 2.1 with $-c < 0$. Since

$$\int_0^t\left\{\rho_n^{-(\lfloor n^{\alpha}t\rfloor-\lfloor n^{\alpha}s\rfloor)}-e^{-c(t-s)}\right\}dB_{n^{\alpha}}(s)=\frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor tn^{\alpha}\rfloor}\left\{\rho_n^{-(\lfloor n^{\alpha}t\rfloor-i)}-e^{-\frac{c}{n^{\alpha}}(n^{\alpha}t-i)}\right\}u_i,$$

it is enough to show that

$$\begin{aligned}
&\sup_{t\in[0,n^{1-\alpha}]}\left|\frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor tn^{\alpha}\rfloor}\left\{\rho_n^{-(\lfloor n^{\alpha}t\rfloor-i)}-e^{-\frac{c}{n^{\alpha}}(n^{\alpha}t-i)}\right\}u_i\right| \\
=\ &\sup_{t\in[0,n^{1-\alpha}]}\left|\frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor tn^{\alpha}\rfloor}\left\{\rho_n^{-(\lfloor n^{\alpha}t\rfloor-i)}-e^{-\frac{c}{n^{\alpha}}(\lfloor n^{\alpha}t\rfloor-i)}\right\}u_i\right|+o_p(1) \\
=\ &\sup_{1\leq m\leq n}\left|\frac{1}{n^{\alpha/2}}\sum_{i=1}^{m}\left\{\rho_n^{-(m-i)}-e^{-\frac{c}{n^{\alpha}}(m-i)}\right\}u_i\right|=o_p(1),
\end{aligned}$$

which is proved in the same way as (37). ■

Proof of (18). Let $s=\frac{r}{\lfloor n^{1-\alpha}\rfloor}$ and since $\rho_n^{2\lfloor\kappa_n\rfloor}=[1+o(1)]\,\rho_n^{2\kappa_n}$ we obtain

$$\begin{aligned}\int_0^{\lfloor n^{1-\alpha}\rfloor}\rho_n^{2\lfloor n^\alpha r\rfloor}dr &= \lfloor n^{1-\alpha}\rfloor\int_0^1\rho_n^{2\lfloor\kappa_n s\rfloor}ds\\ &= \lfloor n^{1-\alpha}\rfloor\left\{\int_0^{\frac{1}{\kappa_n}}+\int_{\frac{1}{\kappa_n}}^{\frac{2}{\kappa_n}}+...+\int_{\frac{\lfloor\kappa_n\rfloor-1}{\kappa_n}}^{\frac{\lfloor\kappa_n\rfloor}{\kappa_n}}+\int_{\frac{\lfloor\kappa_n\rfloor}{\kappa_n}}^{1}\right\}\rho_n^{2\lfloor\kappa_n s\rfloor}ds\\ &= \frac{1}{n^\alpha}\sum_{i=1}^{\lfloor\kappa_n\rfloor}\rho_n^{2(i-1)}+O\left(\frac{\rho_n^{2\lfloor\kappa_n\rfloor}}{n^\alpha}\right)=\frac{\rho_n^{2\kappa_n}}{2c}\left[1+o(1)\right].\end{aligned}$$

■

Proof of (21). Proposition A1 and Lemma 4.1 give for each $p\in[0,q]$

$$\begin{aligned}\frac{\rho_n^{-\kappa_n}}{n^{\alpha/2}}y_{\lfloor\kappa_n\rfloor+\lfloor n^\alpha p\rfloor} &= \frac{\rho_n^{-\kappa_n}}{n^{\alpha/2}}\sum_{i=1}^{\lfloor\kappa_n\rfloor+\lfloor n^\alpha p\rfloor}\rho_n^{\lfloor\kappa_n\rfloor+\lfloor n^\alpha p\rfloor-i}u_i+\frac{y_0}{n^{\alpha/2}}\rho_n^{\lfloor n^\alpha p\rfloor}\\ &= \frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor\kappa_n+n^\alpha p\rfloor}\rho_n^{\lfloor n^\alpha p\rfloor-i}u_i+o_p(1)\\ &= \frac{1}{n^{\alpha/2}}\sum_{i=1}^{\lfloor n^\alpha\left(\lfloor n^{1-\alpha}\rfloor+p\right)\rfloor}\rho_n^{\lfloor n^\alpha p\rfloor-i}u_i\\ &= \rho_n^{\lfloor n^\alpha p\rfloor}\int_0^{\lfloor n^{1-\alpha}\rfloor+p}\rho_n^{-n^\alpha s}dB_{n^a}(s)\\ &= e^{cp}\int_0^\infty e^{-cs}dB(s)+o_p\left(\frac{1}{n^{\frac{\alpha}{2}-\frac{1}{\nu}}}\right)\end{aligned}$$

on the probability space that (5) holds. ■

Order of $\rho_n^{-\kappa_n}$.

$$\begin{aligned}\log\rho_n^{-\kappa_n}n &= -\kappa_n\log\left(1+\frac{c}{n^\alpha}\right)+\log n\\ &= -\kappa_n\left(\frac{c}{n^\alpha}+O\left(\frac{1}{n^{2\alpha}}\right)\right)+\log n\\ &= -cn^{1-\alpha}(1+o(1))+\log n\\ &= -cn^{1-\alpha}\left[1+o(1)-\frac{1}{c}\frac{\log n}{n^{1-\alpha}}\right]\\ &= -cn^{1-\alpha}(1+o(1)),\end{aligned}$$

since $\log n=o\left(n^\delta\right)$, for all $\delta>0$. Thus, $\rho_n^{-\kappa_n}n=\exp\{-cn^{1-\alpha}(1+o(1))\}=o(1)$ and

$$\rho_n^{-\kappa_n}=o\left(n^{-1}\right)\text{ as }n\to\infty. \tag{38}$$

Proof of asymptotic negligibility of R_n. Write $R_n = R_{1n} - 2R_{2n}$, where

$$\begin{aligned} R_{1n} &= \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha} \rfloor} \left(\int_r^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) \right)^2 dr \\ R_{2n} &= \rho_n^{-2\kappa_n} \left(\int_0^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) \right) \\ &\quad \times \int_0^{\lfloor n^{1-\alpha} \rfloor} \left(\int_r^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) \right) dr \\ &= \left(\int_0^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) \right) \overline{R}_{2n}, \end{aligned}$$

where

$$\overline{R}_{2n} := \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha} \rfloor} \int_r^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s)\, dr.$$

From Proposition A1 $\int_r^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) = \frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor \kappa_n \rfloor - \lfloor n^\alpha r \rfloor} \rho_n^{-i} u_{\lfloor n^\alpha r \rfloor + i}$, and

$$\begin{aligned} E\left(\frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor \kappa_n \rfloor - \lfloor n^\alpha r \rfloor} \rho_n^{-i} u_{\lfloor n^\alpha r \rfloor + i} \right)^2 &= \frac{\sigma^2}{n^\alpha \rho_n} \sum_{i=1}^{\lfloor \kappa_n \rfloor - \lfloor n^\alpha r \rfloor} \rho_n^{-2i} \\ &= \frac{\sigma^2}{1 + \frac{c}{n^\alpha}} \frac{1 - \rho_n^{-2(\lfloor \kappa_n \rfloor - \lfloor n^\alpha r \rfloor)}}{2c + \frac{c^2}{n^\alpha}} \\ &= \frac{\sigma^2}{2c} (1 + o(1))\, \rho_n^{-2(\lfloor \kappa_n \rfloor - \lfloor n^\alpha r \rfloor)} = O(1), \end{aligned}$$

uniformly in $r \in [0, \lfloor n^{1-\alpha} \rfloor]$ because

$$\rho_n^{-2(\lfloor \kappa_n \rfloor - \lfloor n^\alpha r \rfloor)} = \{1 + c/n^\alpha\}^{-2\left(n^\alpha \lfloor n^{1-\alpha} \rfloor - \lfloor n^\alpha r \rfloor\right)} = O\left(e^{-2c\left(\lfloor n^{1-\alpha} \rfloor - r\right)} \right) = O(1).$$

Thus, $\frac{1}{n^{\alpha/2}} \sum_{i=1}^{\lfloor \kappa_n \rfloor - \lfloor n^\alpha r \rfloor} \rho_n^{-i} u_{\lfloor n^\alpha r \rfloor + i} = O_p(1)$ uniformly in $r \leq \lfloor n^{1-\alpha} \rfloor$ and so

$$\int_r^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) = O_p(1), \quad \text{uniformly in } r \leq \lfloor n^{1-\alpha} \rfloor.$$

Then, using (38), we find

$$\begin{aligned} R_{1n} &= \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha} \rfloor} \left(\int_r^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) \right)^2 dr \\ &= O_p(1) \times O\left(\rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha} \rfloor} dr \right) \\ &= O_p\left(\rho_n^{-2\kappa_n} \lfloor n^{1-\alpha} \rfloor \right) = o_p\left(\frac{1}{n^\alpha} \right), \end{aligned}$$

and

$$\overline{R}_{2n} = \rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha} \rfloor} \int_r^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s)\, dr$$

$$= O_p\left(\rho_n^{-2\kappa_n} \int_0^{\lfloor n^{1-\alpha} \rfloor} dr \right) = o_p\left(\frac{1}{n^\alpha} \right),$$

so that

$$R_{2n} = \left(\int_0^{\lfloor n^{1-\alpha} \rfloor} \rho_n^{-n^\alpha (s-r)} dB_{n^\alpha}(s) \right) \overline{R}_{2n} = o_p\left(\frac{1}{n^\alpha} \right),$$

giving the required results. ■

Proof of asymptotic negligibility of I_n. From Proposition A1 we have

$$\int_{r-\frac{1}{n^a}}^{n^{1-\alpha}} \rho_n^{\lfloor n^a r \rfloor - n^a s} dB_{n^a}(s) = \frac{1}{n^{\alpha/2}} \sum_{i=\lfloor n^\alpha r \rfloor}^{n} \rho_n^{\lfloor n^a r \rfloor - i} u_i,$$

and so I_n can be written as

$$I_n = \rho_n^{-n-1} \int_{\frac{1}{n^a}}^{n^{1-\alpha}} \int_{r-\frac{1}{n^a}}^{n^{1-\alpha}} \rho_n^{\lfloor n^a r \rfloor - n^a s} dB_{n^a}(s)\, dB_{n^a}(r)$$

$$= \rho_n^{-n-1} \int_{\frac{1}{n^a}}^{n^{1-\alpha}} \frac{1}{n^{\alpha/2}} \sum_{i=\lfloor n^\alpha r \rfloor}^{n} \rho_n^{\lfloor n^a r \rfloor - i} u_i dB_{n^a}(r)$$

$$= \rho_n^{-n-1} \frac{1}{n^\alpha} \sum_{j=2}^{n} \sum_{i=j}^{n} \rho_n^{j-i} u_i u_j.$$

Using the Cauchy-Schwarz inequality,

$$E|I_n| \leq \rho_n^{-n-1} \frac{1}{n^\alpha} \sum_{j=2}^{n} \sum_{i=j}^{n} \rho_n^{j-i} E|u_i u_j|$$

$$\leq \rho_n^{-n-1} \frac{1}{n^\alpha} \sum_{j=2}^{n} \sum_{i=j}^{n} \rho_n^{j-i} \left(E u_i^2 \right)^{1/2} \left(E u_j^2 \right)^{1/2}$$

$$= \rho_n^{-n-1} \frac{\sigma^2}{n^\alpha} \sum_{j=2}^{n} \sum_{i=j}^{n} \rho_n^{j-i}$$

$$= O\left(\rho_n^{-n} n \right) = o(1),$$

and so $I_n = o_p(1)$ as required. ■

Proof of Theorem 4.3. Having established (23) and (25), the only thing that remains to be proved is that the zero mean Gaussian random variables X and Y are independent, or equivalently, that $E(XY) = 0$. First, note that

$$\begin{aligned} E\left|\int_0^{n^{1-\alpha}} e^{-cs} dB(s)\, J_{-c}\left(n^{1-\alpha}\right)\right| &\leq E^{\frac{1}{2}}\left(\int_0^{n^{1-\alpha}} e^{-cs} dB(s)\right)^2 E^{\frac{1}{2}} J_{-c}\left(n^{1-\alpha}\right)^2 \\ &= \frac{\sigma^2}{2c}\left(1 - e^{-2cn^{1-\alpha}}\right) < \infty. \end{aligned}$$

Since $X = \lim_{n\to\infty} \int_0^{n^{1-\alpha}} e^{-cs} dB(s)$ *a.s.*, $Y = \lim_{n\to\infty} J_{-c}(n^{1-\alpha})$ *a.s.* the dominated convergence theorem yields

$$\begin{aligned} E(XY) &= \lim_{n\to\infty} E\left(\int_0^{n^{1-\alpha}} e^{-cs} dB(s)\, J_{-c}\left(n^{1-\alpha}\right)\right) \\ &= \lim_{n\to\infty} e^{-cn^{1-\alpha}} E\left(\int_0^{n^{1-\alpha}} e^{-cs} dB(s) \int_0^{n^{1-\alpha}} e^{cr} dB(r)\right) \\ &= \sigma^2 \lim_{n\to\infty} e^{-cn^{1-\alpha}} \int_0^{n^{1-\alpha}} dr = \sigma^2 \lim_{n\to\infty} e^{-cn^{1-\alpha}} n^{1-\alpha} = 0. \end{aligned}$$

Hence, X and Y are independent. ■

8. References

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