GOLD, FIAT AND CREDIT

An Elementary Discussion of Commodity Money, Fiat Money and Credit

PART II

By

Thomas Quint and Martin Shubik

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COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

YALE UNIVERSITY

Box 208281

New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

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Thomas Quint
Department of Mathematics, University of Nevada at Reno
and
Martin Shubik
Cowles Foundation, Yale University

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Abstract

In this paper we present a series of models, all within the context of a simple two-good economy, which bring out the distinctions between the different types of money and financial institutions. The models emphasize the physical properties of the economic goods, moneys, and trading systems. In Part 1 of the paper, we covered models in which the money is a consumable storable; here in Part 2 we consider economies using durable money, fiat money, or credit. Under this framework we are able to successfully contrast the role of private money lenders, banks, bilateral credit systems, and credit clearinghouses. We are also able to model the importance of the bankruptcy or default penalty in supporting the use of fiat.

Keywords: Barley, Gold, Fiat and credit: Evolution of Money

JEL Classification: C28, C72,C91, D52, D84, E41, E43, E51, E58, G21, K12, L12, N20, P10

Contents

1 From a Consumable Storable to Gold

In Part 1 our basic concern was with the use of a storable consumable as a means of payment. In this part we repeat our analysis in the case where the money is a durable asset which provides a stream of services. The properties of the durable together with consideration of the timing of markets provide a means to illustrate the relationship among consumable storable, durable and fiat monies.[1]

2 Models of Markets with Gold

Gold, silver and other metals have all been utilized as a durable money. They all have the property (which is reflected in the mathematical formulation) that one can use the durable as a money in transactions or can use its services for consumption or production and at the end of the period still have the asset.

Table 1

	Cash only	money mkt	Central Bank
Trader 1	$(a, 0, [m, \dot{m}])$	$(a, 0, [m, \dot{m}])$	$(a, 0, [m, \dot{m}])$
Trader 2	$(0, a, [m, \dot{m}])$	$(0, a, [m, \dot{m}])$	$(0, a, [m, \dot{m}])$
Money-lender	no	$(0, 0, [(M–2m), (\dot{M}–2\dot{m})])$	no
Central bank	no	no	$(0, 0, [(M–2m), (\dot{M}–2\dot{m})])$

Table 1 shows the different treatments for trade with a durable money. Once again the columns represent the models we cover, while the rows represent types of agents. A cell with a "no" means that the corresponding type of agent is not present in the corresponding economy. Otherwise, the triples show the initial endowments of the two types of good and the durable money.

2.1 Two Trader Types trading with Gold

We consider two types of traders, trading in two goods, in a one-period game. As in Part 1, we assume that a) the traders of Type 1 are endowed only with good 1;

[1] Work on a topic as broad, basic and diffuse as the theory of money contains large elements of synthesis as well as analysis. Thus it rarely if ever can be attributed to a single individual as in a theorem in pure mathematics. We cannot do justice to the work of the many authors whose work has influenced our thoughts. However the authors have been influenced in particular by Allais [1], Baumol [3], Bewley [4], Friedman [15], Grandmont [16], Grandmont and Younes [17], Gurley and Shaw [18], Hahn [19], Lucas [24], Magill and Quinzi [25], Samuelson [28], Tobin [42], and many others, in particular many collaborations with Dubey, Geanakoplos and Shapley.

A recent study of referencing in economics papers indicates that almost all of the references are to works of the last two to three years. Yet economic theory calls for a process of accretion. We cannot spend too much of our time constantly referencing the distant past but it behooves us to remember the basic contributions of many great minds. In monetary theory, as approached here our intellectual debt to the following historical figures is considerable. They are: Cournot, Edgeworth, Jevons, Ricardo, Wicksell, Böhm-Bawerk, Walras, Fisher, Hicks and Keynes.

b) the traders of Type 2 only with good 2; and c) there is a continuum of each trader type. A strategy for the Type 1 traders is denoted by (q, b) where $q =$ the amount of good 1 offered for sale and $b =$ the amount of gold bid for good 2. The notation for the Type 2 traders is $(\bar{q}, \bar{b})$. We remind the reader that these strategies represent aggregations over the continuum of individual traders; for details see Section 4.2.1 of Part 1.

The traders of Type 1 have total endowment $(a, 0, [m, \dot{m}])$, while those of Type 2 have $(0, a, [m, \dot{m}])$. The reader will notice that these endowments are the same as in Part 1, except that here we distinguish between the asset gold and the stream of services it yields. The asset is measured as a "stock variable," measured in some physical units. The services, by contrast, are a "flow variable," and so would be measured in units per unit of time. To distinguish the two types of variables, we use the "dot" notation, so m denotes the initial endowment of the asset gold, while $\dot{m}$ denotes the initial endowment of the gold's services. The dot notation is also used for other variables; hence, for example $\dot{b}$ represents the amount of gold services the Type 1 traders bid in an effort to obtain good 2.[2]

We note that the asset and the service are bundled together, and unless we introduce a rental market they are sold together. In this paper we do not have such a rental market; hence in what follows, necessarily always $b = \dot{b}$, $\bar{b} = \dot{\bar{b}}$, $q = \dot{q}$, $\bar{q} = \dot{\bar{q}}$, and $m = \dot{m}$.[3]

Finally, we assume that both trader types have the same utility function $\varphi(x, y) + \dot{z}$, where φ is an increasing concave function, x and y represent the amounts of good 1 and good 2 consumed, and $\dot{z}$ stands for the consumption services of (z units of) the asset. There is also the possibility that the traders will assign a salvage value to any leftover asset at the end of the game; we model this as well below.

2.1.1 The Treatment of the Service Value of Gold

In this model there are three aspects of gold to which value is attached. They are its value as an ornament, its value in transactions and its residual value as an asset with a possible salvage value at the end of the game [35]. Hence, in order to fully define the optimization over the period we must indicate who gets to use the services of the asset, when transactions are made, and how the asset is evaluated at the end.

Traders of Type 1 attempt to maximize:

$$\max_{b,q} \varphi\left(a - q, \frac{b}{\bar{p}}\right) + k_2\dot{m} + (k_3 - k_2)(\dot{m} - \dot{b})$$

$$+ (1 - k_3)(\dot{m} - \dot{b} + p\dot{q}) + \Pi(m + pq - b)$$

$$\text{s.t. } m - b \geq 0 \qquad (\lambda) \qquad (1)$$

$$m + pq - b \geq 0 \qquad (\mu) \qquad (2)$$

[2]It should be noted that there are strategic operational differences among buying an asset and utilizing its services then selling it in a buy-back or in the market and renting the asset to use its services.

[3]More precisely, we can select units such that one unit of the asset gold yields one unit of gold services in one unit of time, i.e., $m = \int_{t=0}^{1} \dot{m} dt$.

$$b \geq 0,\ 0 \leq q \leq a$$

Here k_2 is the proportion of the time period that passes before the goods market opens; $k_3 - k_2$ is the proportion of the period taken up by the market, when the consumption use of the gold used in transactions is lost; and $1 - k_3$ is the proportion of the period after the market, when the consumption use of gold is enjoyed by the final owner of the asset (k_1 will be defined later). The parameter Π is the salvage value for the gold at the end of the game.

We note that in expression (1) that the two arguments for φ are quantities of point consumption perishables; hence there is no need to distinguish between their stock and flow. However, the distinction must be made for the services of the gold. The numbers k_t have the dimension of time, thus $k_2\dot{m}$ (more precisely $\int_{t=0}^{k_2} \dot{m}dt$) stands for the consumption services of gold rendered over the interval $[0, k_2]$. The amount pq is the amount of gold earned as income from the sale of good 1, credited at time k_3 . The amount $p\dot{q}$ indicates the service stream available from this gold.

In fact the transactions costs involved in going from gold coin to jewelry and vice-versa are high and take time. Somewhat fancifully we could imagine wearing the coins as jewelry attached to a bracelet and taking them off to make payments. Probably more realistically, $k_2 = 0$ and $k_3 = 1$ covers the situation where, if gold is used for money it has only transactions use in that period.

The values $k_2 = 0$ and $k_3 \approx 0$ model the situation where the market meets for some brief time Δ at the very start of the period, all deliveries are made on the spot and all final recipients obtain the use of all of the goods including the gold for essentially the whole of the period.

Figure 1 shows the general case and the two special cases just noted. These two extreme parameter settings enable us to start to bridge the gulf between a storable consumption good money and a fiat money; see Sections 2.1.2 and 2.1.3 below.

We will also consider the case $k_2 = k_3 = 1$.[4]

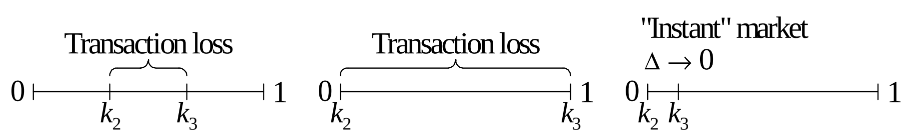

Figure 1

Finally, we note that in the optimization above, the last constraint (2) is irrelevant because it is satisfied whenever (1) is. Thus one may ignore it when doing the mathematical analysis. However, we leave it in the model simply because it is the

[4]It must be noted that the division of the day into the pre-market, market and post-market segments with the consumption and transactions uses of gold differentiated is almost a parody of physical reality in the sense that the considerable costs and time delays in minting and the manufacture of jewelry are left out. However the explicit introduction of market mechanisms is critical to understanding the relationship among consumption value, transaction value and asset disposal value. For this reason we concentrate on providing an explicit well-defined mechanism to produce a playable game and get the logic straight. Generalization to large classes of mechanism must come later.

"budget constraint," and later variants of the game will have budget constraints which are not redundant

The traders of Type 2 face a similar optimization program, namely

$$\max_{\bar{b},\bar{q}} \varphi\left(\frac{\bar{b}}{p}, a-\bar{q}\right) + k_2 \dot{m} + (k_3 - k_2)(\bar{m} - \dot{\bar{b}})$$

$$+(1-k_3)(\dot{m} - \bar{b} + \bar{p}\,\bar{q}) + \Pi(m + \bar{p}\,\bar{q} - \bar{b}) \tag{3}$$

$$\text{s.t. } m - \bar{b} \geq (0) \qquad (\bar{\lambda}) \tag{4}$$

$$m + p\bar{q} - \bar{b} \geq 0 \qquad (\bar{\mu})$$

$$0 \leq \bar{q} \leq a \text{ and } \bar{b} \geq 0 \tag{5}$$

Finally, as in Part 1, price is formed by:

$$p = \frac{\bar{b}}{p} \text{ and } \bar{p} = \frac{b}{\bar{q}}. \tag{6}$$

Again, we remind the reader that the quantities $b, q, \bar{b}$, and $\bar{q}$ (as well as the variables d and $\bar{d}$ defined later on) are all definite integrals taken over the continuum of traders of Type 1 or 2; the quantities p and $\bar{p}$ are just quotients of these values.

We solve the optimization for the case with $\Pi = 0$, $\varphi(x, y) = 2\sqrt{xy}$ and general values for k_2 and k_3. The full solution is given in Appendix A. A sensitivity analysis based on this solution is given below in Table 2.

The condition of "enough money" means that m is high enough so that the cash flow constraints (λ) and $(\bar{\lambda})$ are nonbinding. Economically, it means that the volume of the money supply at equilibrium is sufficient to cover the volume of competitive trade. In this instance, the condition turns out to be

$$m \geq \frac{a\sqrt{1-k_3}}{(2-k_2-k_3)\sqrt{1-k_2}}. \tag{7}$$

In the case where (7) holds, price is given by $p = \bar{p} = \sqrt{\frac{1}{(1-k_2)(1-k_3)}}$ and the physical amount of trade by $q = \bar{q} = \frac{(1-k_3)a}{2-k_2-k_3}$. The bids are $b = \bar{b} = pq = \frac{a\sqrt{1-k_3}}{(2-k_2-k_3)\sqrt{1-k_2}}$. Comparing these results with those from the storable consumable money case (Part 1), we find that the prices of the goods have gone up to reflect the lessening of the consumption utility of gold. Somewhat surprisingly, the meeting time of the market makes a difference, even if there is no transactions loss in the market.

When $k_2 = k_3 = 0$, then $p = 1$ and the consumption services of a unit of gold bid are completely lost for the lender if it is used in transactions. The amount of money needed to finance trade is $a/2$.

When $k_2 = k_3 \approx 1$ price becomes infinite and an infinite amount of gold is called for because as the lender gives up almost no service value to the borrower who intends to use it for transactions, it will have little value in consumption to the final recipient. At the end of the game as there is no salvage value for the asset, the more its service value to the final recipient approaches zero, the more it starts to look like fiat.

When $k_2 \neq k_3$ there is a loss of utility in trade. Hence the act of using a durable is an implicit tax or rate of interest drain on trade; thus efficient exchange is not achieved even though there is enough money to avoid a trading constraint.

When there is "not enough money" (i.e., (7) doesn't hold), we find that prices, bids, and trade all decrease, just as in the models of Part 1.

A sensitivity analysis

We consider $k_2 = 1/2$, $k_3 = 3/4$. These values give service losses for 1/4 of the period when the market meets at mid-day. Just as in Part 1, we examine the model as m ranges from ∞ down to 0:

Table 2
Trading with a transactions loss of services

m	p	q	b	λ
$m > \frac{4a}{3\sqrt{2}}$	$2\sqrt{2}$	$a/3$	$\frac{4a}{3\sqrt{2}}$	0
$m = \frac{4a}{3\sqrt{2}}$	$2\sqrt{2}$	$a/3$	$\frac{4a}{3\sqrt{2}}$	0
$m = \frac{2a}{5}$	2	$\frac{a}{5}$	$\frac{2a}{5}$	$\frac{1}{2}$
$m \to 0$	$\to 0$	$\to 0$	$\to 0$	$\to \infty$

Case 1: $m > 4a/3\sqrt{2}$. When there is more than enough money, the shadow price of the transactions value of money remains at $\lambda = 0$. However, efficient trade of the perishables is not achieved (i.e., each type of trader consumes $2a/3$ of her good and $a/3$ of the other). This is due to the loss of 1/4 of a period of the services of gold used for transactions. Note that the price level is considerably higher than in the storable consumable money case. Surplus gold (above transactions needs) is kept for the full consumption of its services.

Case 2: $m = 4a/3\sqrt{2}$. When there is precisely enough money for transactions, and it is distributed in a manner that every individual does not encounter her cash constraints, the shadow price of the transactions value of money is $\lambda = 0$.

Case 3: $0 < m < 4a/3\sqrt{2}$, for example $m = 2a/5$. When there is "not enough money," the shadow price of the transactions value of money is positive as is reflected by $\lambda = 1/2$. Efficient trade of perishables is not achieved for two reasons, the transactions loss of gold and the constraint on trade. The price level is decreasing monotonically as m decreases from $4a/3\sqrt{2}$ to 0.

Case 4: $m \to 0$. When there is little or no money in the economy there is next to no trade. In reality, if this happened, other means of payment would be used and other avenues of trade would spring up. The societal "trust substitutes" would be substituted by local trust substitutes if they became too expensive.

2.1.2 The Link from Gold to a Storable Consumable

We observe that with the special case of the values $k_2 = 0$ and $k_3 = 0$, we obtain the same results as in the Part 1 for the storable consumable. Here we have the implausible proposition that the time taken to operate a market is so small that as a reasonable approximation it can be ignored. In this instance the strict consumption yield of a can of beans or unit of gold are the same. With a "timeless" market at the start of the period, both the consumption of the beans and the full services of the gold belong to the ultimate owner. (With fiat there is *only* transactions value.)

In this one period game with $k_2 = 0$ and $k_3 = 0$, the markets are active regardless of the valuation attached to the salvage value of gold or beans carried over voluntarily or involuntarily to the next period.[5]

2.1.3 The Link from Gold to Fiat

When we consider $k_2 = 0$ and $k_3 \approx 1$ we obtain an interesting but not full transition zone towards fiat.[6] There will be a specific amount of gold which will be enough to satisfy optimal trade and have none utilized as jewelry. Once more we assume that the salvage price for the asset is zero Although the consumption services of the gold used in transactions are lost, there is still utilitarian value to be obtained from the services of gold not used in transactions.[7] This supports the value of the gold as a source of consumption services, but setting $k_3 = 1$ when $\Pi = 0$ destroys the use of gold in both the goods market and the money market. The Hahn paradox (see Section 4.2.1) holds. No lender or seller is motivated to supply any item to the markets as she will be paid in worthless gold after it has exhausted its stream of services, leaving only a valueless asset. The only difference between the gold and the fiat is that all owners of gold can derive (completely non-strategically) the consumption value of the gold; but, in general, there is no consumption value attached to fiat.[8] Both for gold with the markets at the end of period and for fiat their value as a money requires as a necessary condition that a salvage utilitarian value of some sort be attached to the surviving asset. We return to a discussion of fiat in Section 4.

The important case distinction between $k_2 = 0$ and $k_3 = 0$, vs. $k_2 = 0$ and $k_3 = 1$ is seen in the incentives to sell. When $k_3 = 0$, the income from goods sold is paid in gold at the start of the period and will yield almost full service for the period. When $k_3 = 1$, the income from goods sold for gold yields nothing.

[5] Note that there is nevertheless an important strategic difference between the storable consumable and the durable. The choice to not carry the asset over to the next period is available for the first but not the second.

[6] To formally make fiat money a specific case of the model, we would first need to define a parameter α as the per-unit-time service utility of the monetary asset. We then would modify the objective functions for the traders by multiplying the second, third, and fourth terms by α. The case of the consumable storable money would then be where $k_2 = 0$, $k_3 = 0$, and $\alpha = 1$(normalized). The fiat case would be where $k_2 = 0$, $k_3 = 1$, and $\alpha = 0$.

[7] In this instance this is merely a constant of no strategic significance. It is completely autarchic. In the one period game its yield is only to the owner as the loss of services in the trading period equal its service value.

[8] Although a scriptophile might debate this conventional view.

2.2 Trade in Gold with a Money Market

Suppose the types have nonsymmetric endowments of $(a, 0, m_1)$ and $(0, a, m_2)$ where $m_1 > m_2$. As in Part 1, we introduce a money market.

A strategy for the traders of Type 1 is denoted by (g, q, b), where $g =$ the amount of money offered in the money market, $q =$ the amount of good 1 offered for sale and $b =$ the amount of money bid for good 2. The notation for the Type 2 traders is $(\bar{d}, \bar{q}, \bar{b})$ where $\bar{d}$ is the amount of personal IOU notes bid for the gold offered. From considerations of a dimensional analysis it must be considered as a new separate financial instrument, the personal IOU note which is monetized by exchanging it for money. This is consistent with the observation in banking practice that a loan to an individual can be interpreted as "monetizing her personal credit".

2.2.1 The Interday Loan Market

Before we are in a position to write down the payoff function we must decide upon whether the functioning of the financial market for loans takes up any time and/or other resources. We assume (somewhat counterfactually and against "bankers hours") that the financing takes place in a very fast and time efficient market at the start of the day. Thus we break the single period up into four parts with two markets, a financial market and a goods market. Let the length of time for the financial market be $k_1 = \Delta$, where Δ is small as is shown in Figure 2.

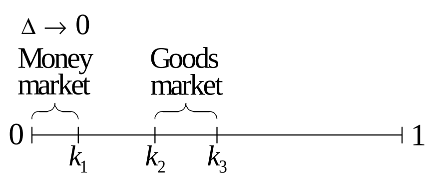

Figure 2. A money market and a goods market

The maximization for the lenders is

$$\max_{q,b,g} \varphi\left(a - q, \frac{b}{p}\right) + k_1(\dot{m}_1 - \dot{g}) \tag{8}$$
$$+ (k_2 - k_1)(\dot{m}_1 - \dot{g}) + (k_3 - k_2)(\dot{m}_1 - \dot{g} - \dot{b})$$
$$+ (1 - k_3)(\dot{m}_1 - \dot{g} - \dot{b} + p\dot{q})$$
$$\text{s.t. } m_1 - g - b \geq 0, \qquad (\lambda)$$
$$\Pi(m_1 + \rho g - b + pq) \geq 0 \qquad (\mu) \tag{9}$$
$$0 \leq q \leq a, \text{and } b, g \geq 0 \tag{10}$$

and for the borrowers it is

$$\max_{\bar{b},\bar{q},\bar{d}} \varphi\left(\frac{\bar{b}}{p}, a - \bar{q}\right) + k_1\dot{m}_2$$
$$+ (k_2 - k_1)\left(\dot{m}_2 + \frac{\dot{\bar{d}}}{1 + \rho}\right) \tag{11}$$

$$+ (k_3 - k_2)\left(\dot{m}_2 + \frac{\bar{\dot{d}}}{1+\rho} - \bar{\dot{b}}\right)$$

$$+ (1 - k_3)\left(\dot{m}_2 + \frac{\bar{\dot{d}}}{1+\rho} + \bar{p}\bar{\dot{q}} - \bar{\dot{b}}\right)$$

$$\text{s.t. } m_2 - \bar{b} + \frac{\bar{d}}{1+\rho} \geq 0, \qquad (\bar{\lambda}) \tag{12}$$

$$\Pi(m_1 - \bar{d} - \bar{b} + \bar{p}\,\bar{q}) \geq 0 \qquad (\bar{\mu}) \tag{13}$$

$$0 \leq \bar{q} \leq a, \text{and } \bar{b}, \bar{d} \geq 0. \tag{14}$$

Price formation is as before

$$p = \frac{\bar{b}}{q} \text{ and } \bar{p} = \frac{b}{\bar{q}}. \tag{15}$$

The endogenous rate of interest is formed as follows:

$$1 + \rho = \frac{\int_\alpha d^\alpha}{\int_\alpha g^\alpha} = \frac{d}{g}. \tag{16}$$

Remember we are considering all individuals as being part of a continuum; thus a more proper notation for the formation of the interest rate is as indicated.[9] The lack of a "dot" on top of the variables in the interest equations is to remind us that the interest rate is formed by an asset (gold) and by an IOU note denominated in this asset, and similarly the price is formed by assets.[10] Also implicit is the presence of a bankruptcy penalty (denominated in "disutility" or punishment per unit of gold defaulted) that is sufficiently steep that strategic bankruptcy is not worthwhile.

Viewing the last term of the objective function for the lenders a modeling problem must be faced. Do the borrowers pay the lenders back immediately at time k_3 or is their borrowing an interday loan? If it is an interday loan it will be paid back "tomorrow morning"; but this apparently minor distinction means that the lender obtains no services from the gold she lent this period. Thus the last term in the objective function does not contain the term $(1+\rho)\dot{g}$, and this implies that there is never an incentive for the lender to lend. Hence $g = 0$ and the money market is inactive.

2.2.2 The Intraday Loan Market

If the loan considered above were an intraday loan which had to be paid back immediately after the goods market terminated at time $k_3 < 1$, then the lenders' problem

[9] In the valuation of the services of gold the use of the time divisions k_1 and k_2 appear to be a distinction without an operational difference for the lenders; but not for the borrowers. This is the zone for lawyers, brokers and accountants. The lender is giving up the use of the gold, but the borrower has not yet obtained its use.

[10] Thus the rate of interest is a dimensionless number.

would have the form:

$$\max_{q,b,g} \varphi\left(a-q,\frac{b}{\bar{p}}\right)+k_1(\dot{m}_1-\dot{g}) \tag{17}$$

$$+(k_2-k_1)(\dot{m}_1-\dot{g})+(k_3-k_2)(\dot{m}_1-\dot{g}-\dot{b}) \tag{18}$$

$$+(1-k_3)(\dot{m}_1+\rho\dot{g}-\dot{b}+p\dot{q}) \tag{19}$$

$$\text{s.t. } m_1-g-b\geq 0, \qquad (\lambda)$$

$$\Pi(m_1+\rho g-b+pq)\geq 0 \qquad (\mu) \tag{20}$$

$$0\leq q\leq a \text{ and } b,g\geq 0. \tag{21}$$

Now it is possible that the intraday money market would go active, as some residual services may remain in this period after time k_3.

2.2.3 Enough Money

As our purpose is to illustrate the basic qualitative nuances of these many structures, we are able to observe immediately (without calculation) the level of "enough money" in the money market.

In part 1, we found that when there is enough of a storable consumable money one can lend it at an endogenous interest rate of $\rho=0$ in the morning, have it used in the market at noon, be repaid in the afternoon and consume one's stock in the evening. We assumed the money entered the utility function as a linearly separable term. It was easy to see that "enough" is given by an amount which precisely equals the value of the volume of trade. In particular, we considered an example with utility functions $u=2\sqrt{xy}+z$, and "enough money" turned out to be the condition $m_1+m_2\geq a$.

When we switch to a durable money, the analysis changes. While a unit of storable consumable money will yield both transaction value and full consumption value when it is consumed, a unit of durable money used for transactions might lose some fraction of its consumption value. On the other hand, at the end of the period the durable money will remain an asset, which may have some salvage value.

Again consider an example with $u=2\sqrt{xy}+z$, but this time with a durable money having a salvage value of $\Pi=0$. Then, instead of a being enough money, the agents will require $m_1+m_2\geq\frac{a\sqrt{1-k_3}}{(2-k_2-k_3)\sqrt{1-k_2}}$. If $k_1=0$, $k_2=0$ and $k_3=1$, the markets close. If $k_1=k_2=1$, there will never be enough money as $k_3\to 1$. At $k_3=1$ the market closes down as there is no incentive to sell consumables for a worthless asset whose services have already been rendered. The consumption marginal utility of the services of gold is one, but the value of gold at the end is 0.

2.2.4 Not Enough Money

If there is not enough money in the system as a whole, trade will be diminished and the price of the monetary commodity goes above its marginal value as a commodity to reflect the shadow price of the cash flow constraint. We omit this case calculation as it is qualitatively similar to the money market with a storable consumable.

2.2.5 Expectations or Can Gold Have a Salvage Value in This Game?

We observe that as long as the value of the services of gold to a recipient is bounded away from zero and some after-trade value of the services remains, no backward induction killing trade will occur. If we regard the one period game as being a simple metaphor for an ongoing process, it is reasonable to regard durable assets as carrying over to the next stage of the game. Following dynamic programming practices, we may wish to attach a salvage value to the left over durables. If we are concerned with being able to enlarge our model structure to encompass a multistage economy, then attaching such a salvage value may be regarded as reflecting the expectations for the value of the assets in the markets of the future.[11]

The attachment of a salvage value to left over gold is necessary and sufficient to give a somewhat different solution to the game posed in Section 2.2.1 above. Instead of the optimization illustrated in expression (8) we must consider, for the lenders

$$\begin{aligned}
&\max_{q,b,g} \varphi\left(a-q, \frac{b}{\bar{p}}\right) + k_1(\dot{m}-\dot{g}) \\
&+ (k_2-k_1)(\dot{m}-\dot{g}) + (k_3-k_2)(\dot{m}-\dot{g}-\dot{b}) && (22)\\
&+ (1-k_3)(\dot{m}-\dot{g}-\dot{b}+p\dot{q}) + \Pi(m+\rho g+pq-b) \\
&\text{s.t. } m-g-b \geq 0, \qquad (\lambda) && (23)\\
&\qquad m+\rho g+pq-b \geq 0 \qquad (\mu) && (24)\\
&\qquad 0 \leq q \leq a, \text{ and } b,g \geq 0 && (25)
\end{aligned}$$

and similarly for the borrowers. With this change the interday loan market will go active for a large enough Π.

>From the point of view of experimental gaming it is easy and straightforward to reward the players in proportion to their "scores" including the salvage value over which each has some strategic control in the selection of g and $\bar{d}$. An interpretation to assigning a salvage value is that the asset will have value in a future game. The existence of a positive salvage value to the remaining assets is necessary to support value, but there are many different ways in which this can be done.

2.2.6 Outside Money with No Outside Agent?

When we consider a money market as contrasted with a large private bank or with a central bank, three basic distinctions must be made. They are:

(1) The number and nature of the financial instruments,

(2) The monopolistic power of the agents, and

(3) The goals and motivation of the agents.

The instruments In either the money market or the economy with a central bank, there is an initial supply of commodity money or an injection of fiat at the start of

[11] If we are concerned with a stationary economy in terms of real resources we might wish to invoke the "rational expectations" values associated with an infinite horizon stationary state.

the game. If we wish to let individuals obtain more money from the government or from other individuals, there are two types of "inside money" which must be considered. In institutional fact, in the United States further fiat money may enter or leave the economy in exchange for government bonds, or taxes and subsidies, but not for individual IOU notes as modeled here. In general, individual IOU notes or even bank drafts or clearinghouse credits are "credit" and may be regarded as money substitutes that need considerable credit evaluation.

The monopolistic power of the agents In our analysis we have purposely contrasted a monopolistic money lender with a central bank. Both are monopolists, but with different motivations and sources of power. The sources of power appear to involve a mixture of law, network and reputation considerations (see Shubik [36]). In the one period games considered here, there is no need to vary the money supply from period to period. When three or more periods are considered the banking system may be called on to both shrink and expand the money supply and a further distinction between the roles of the inside or private and the outside or central bank must be made.

The goals and motivation of the agents The institution of a central bank is around 300 years old. It is, in essence, a public good and a creature of the state. Money lenders have been in existence for several thousand years. Some may have had a more or less monopoly position in a limited domain. But they had goals of profit maximization and consumption, and in general had less power than the whole state. One could argue that families such as the Medici came close to being money lenders, rulers and a central bank. But the central bank control problem is somewhat different from the strategies and intent of the money lender. The money lender wants to make sure that there is never enough money, even if he has to keep a sufficient supply in his vault, but the central bank wants to see that there is an adequate supply.

3 The Central Bank Control Problem

We turn to the central bank control problem and to the precise formulation of what might be meant by an altruistic or public servant central bank.

3.1 Trade with Gold: Two Trader Types with an Altruistic Central Bank

In Part 1 we considered both the roles of a profit maximizing monopolistic money lender and an altruistic central bank. As the qualitative distinction between the two has already been discussed in the context of a storable consumable money and the contrast here appears to be substantially the same, we confine our remarks to the central bank confronting two symmetric agent types.

We replace the monopolistic money-lender with an altruistic agency who is only interested in breaking even, thus it is happy to lend its resources to facilitate trade.

We consider the symmetric situation where $m_1 = m_2 = m$ and the central bank has $B = M - 2m$ units of gold.

Suppose that the bank offers an amount g for loan to the traders. The optimization for the traders of Type 1 is then

$$\begin{aligned}
&\max_{q,b,d} \varphi\left(a-q, \frac{b}{\bar{p}}\right) + k_1\dot{m} + (k_2-k_1)\left(\dot{m} + \frac{\dot{d}}{1+\rho}\right) \\
&+(k_3-k_2)\left(\dot{m} + \frac{\dot{d}}{1+\rho} - \dot{b}\right) && (26)\\
&+(1-k_3)\left(\dot{m} + \frac{\dot{d}}{1+\rho} + p\dot{q} - \dot{b}\right) \\
&\text{s.t. } m - b + \frac{d}{1+\rho} \geq 0, \qquad (\lambda) && (27)\\
&\quad \Pi(m-d-b+pq) \geq 0 \qquad (\mu) && (28)\\
&\quad 0 \leq q \leq a \text{ and } b, d \geq 0 && (29)
\end{aligned}$$

Let us denote this problem by Γ_g. There is a similar optimization for traders of Type 2, which we call $\overline{\Gamma}_g$. We remark that as in the models of Part 1, the traders consider ρ to be an exogenously given constant.

Although it appears that the variable g plays no role in the traders optimizations, we remark that the interest rate ρ is formed via

$$1+\rho = \frac{\int_\alpha d^\alpha \mu(\alpha) + \int_\gamma \bar{d}^\gamma \mu(\gamma)}{g} = \frac{d+\bar{d}}{g}. \tag{30}$$

In addition, prices are formed as before:

$$p = \frac{\bar{b}}{q} \text{and } \bar{p} = \frac{b}{\bar{q}}. \tag{31}$$

Now consider the bank's problem. The initial endowment of the bank is $(0, 0, M - 2m)$. The strategy of the bank is to select a number g, where $0 \leq g \leq M - 2m$, in such a manner that it optimizes some welfare function of the trading economy. Formally, the bank solves

$$\max_{0\leq g\leq M-2m} \Psi(\arg\max_{d,b,q\geq 0} \Gamma_g, \arg\max_{\bar{d},\bar{b},\bar{q}\geq 0} \overline{\Gamma}_g) \tag{32}$$

where Ψ indicates that the central bank requires a criterion for the valuation of the outcome of the game, which, in general is vector valued, with at least a component for each type.

3.1.1 Controlling the Money Supply

In this purposely selected simple example with symmetric types, suppose that the one dimensional goal Ψ for the central bank is to minimize the level of the endogenously formed interest rate, i.e., to minimize the cost to the agents of financing trade. A simple example illustrates the conditions for the central bank's actions and resource needs.

We consider four initial conditions, for the instance where $k_1 = k_2 = 0$, $k_3 = 1$. This is the case where the consumption value of the services of gold is completely lost when used for a transaction. The conditions are displayed in Table 3. We note in each instance the central bank's resources and the initial resources of the trader types. The bank always has $(0, 0, 2[a, \dot{a}])$; it has a considerable supply of gold and nothing else.

Table 3

Bank	Trader 1	Trader 2	Init. Val.	CE Val.	MU trade
$(0,0,2[a,\dot{a}])$	$(a,0,[0,0])$	$(0,a,[0,0])$	0	a	∞
$(0,0,2[a,\dot{a}])$	$(a,0,[\frac{a}{2},\frac{\dot{a}}{2}])$	$(0,a,[\frac{a}{2},\frac{\dot{a}}{2}])$	$\frac{a}{2}$	$\frac{3a}{2}$	∞
$(0,0,2[a,\dot{a}])$	$(\frac{3a}{4},\frac{a}{4},[\frac{a}{2},\frac{\dot{a}}{2}])$	$(\frac{a}{4},\frac{3a}{4},[\frac{a}{2},\frac{\dot{a}}{2}])$	$\frac{(\sqrt{3}+1)a}{2}$	$\frac{3a}{2}$	4
$(0,0,2[a,\dot{a}])$	$(\frac{a}{2},\frac{a}{2},[\frac{a}{2},\frac{\dot{a}}{2}])$	$(\frac{a}{2},\frac{a}{2},[\frac{a}{2},\frac{\dot{a}}{2}])$	$\frac{3a}{2}$	$\frac{3a}{2}$	1

The values and their trade implications are shown in Figure 3.

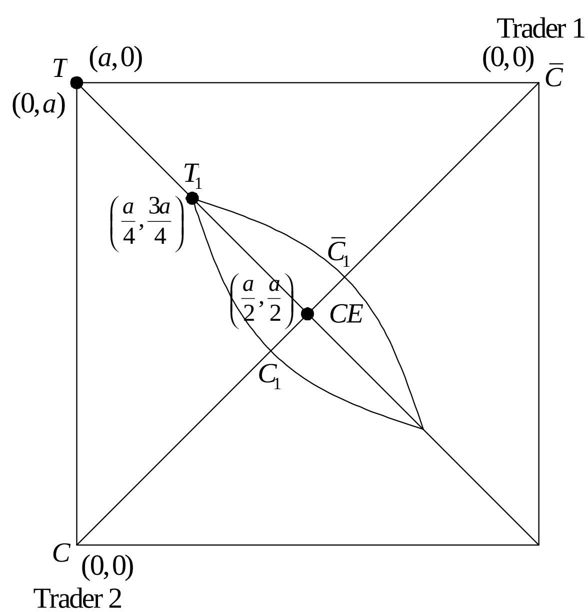

Figure 3

When the initial conditions are $(a, 0, 0)$, $(0, a, 0)$ the contract curve is $C\bar{C}$ and the marginal value of trade is unbounded. A monopolistic banker could remove all resources. The altruistic banker can put up $g = a$ units of its gold for loan There is a noncooperative equilibrium where the agents offer to pay back precisely $d + \bar{d} = g$, but dynamically a coordination problem is encountered. By providing the a the government enables the individuals to obtain the CE outcome by using its gold for trade. They return the asset which unless it has a salvage value is valueless. The government has wasted the service value of its gold in reserve and has given away the service value of its gold in circulation. The traders did not use the service value, but instead used its transactions value.

If initial conditions are $(a, 0, [\frac{a}{2}, \frac{\dot{a}}{2}])$, $(0, a, [\frac{a}{2}, \frac{\dot{a}}{2}])$ the traders have enough money to finance efficient trade, but they lose all of the service value of the gold in doing so. Instead they are able to obtain a gift of an interest free loan $g = a$ from the bank which covers their transactions needs. The system has still wasted the consumption value of $2a$ units of gold.

If initial conditions are $(\frac{3a}{4}, \frac{a}{4}, [\frac{a}{2}, \frac{\dot{a}}{2}])$, $(\frac{a}{4}, \frac{3a}{4}, [\frac{a}{2}, \frac{\dot{a}}{2}])$ the traders have more than enough money to finance efficient trade. Their initial supplies and valuations of the two trading commodities are shown in Figure 3 at the point T_1. The new contract curve is given by $C_1\bar{C}_1$. The government bank need only offer $g = a/2$ to enable the traders to achieve the full value of $3a/2$. The system has still wasted the consumption value of $2a$ units of gold.

If initial conditions are $(\frac{a}{2}, \frac{a}{2}, [\frac{a}{2}, \frac{\dot{a}}{2}]), (\frac{a}{2}, \frac{a}{2}, [\frac{a}{2}, \frac{\dot{a}}{2}])$ the traders are at a point which requires no trade. Their initial supplies and valuations of the two trading commodities are shown in Figure 3 at the point CE. The new contract curve is just the point CE. The government bank makes no loans. The traders obtain the full value of $3a/2$. The system has still wasted the consumption value of $2a$ units of gold.

As the means of exchange is gold the bank, by storing its reserves uselessly imposes a deadweight loss on the society.

3.1.2 Controlling the Rate of Interest

Instead of utilizing the quantity of money as its control variable, the bank could use the rate of interest. The traders' optimization problems are then denoted by Γ_ρ and $\bar{\Gamma}_\rho$, defined in the obvious way. The bank then solves

$$\max_{\rho \geq 0} \ \Psi(\arg\max_{d,b,q \geq 0} \Gamma_\rho, \arg\max_{\bar{d},\bar{b},\bar{q} \geq 0} \bar{\Gamma}_\rho). \tag{33}$$

The nonnegativity of ρ guarantees that the bank will not operate at a loss.

But if it were to announce $\rho = 0$ the demand by the traders would be unbounded because the services of gold are being given away.

3.1.3 The 100% Backed Credit Solution: Fort Knox or the Bank of Amsterdam?

When there is a great deal of gold in the system, trade between the consumables can be maximized. But there is a social accounting problem. Suppose the gold has services that enters the utility function in such a way that its marginal utility is always bounded away from zero.[12] Then, if the bank has more than enough gold, we need to specify what are the goals of the central bank with respect to its assets which might

[12] The selection of a model with a gold entering the utility function as a linearly separable term is made for several reasons:

have value elsewhere. This problem shows up in the constraints of the optimization problem for the individuals (who have a value for more services of the gold) and in the motivation of the bank who has more gold than is needed for transactions.

Historically the Bank of Amsterdam wrote paper against the gold it held. Physically the paper was easier to transport and to protect against theft In the context of a world with exogenous uncertainty (which we do not deal with here) reserves are required to protect against fluctuations. But here they are not needed. Hence if transactions costs were low enough, from the view point of an altruistic central bank (in control of its treasury), rather than keep its gold in a Fort Knox it could lend gold for consumption, thereby raising consumption levels. In reality the ever pervasive cost of process makes this implausible.

4 Models of Markets with Fiat, Clearing and Credit

Retailer to Wholesaler: "People ate your sardines and were poisoned."
Wholesaler to Retailer: "You fool those were trading sardines not eating sardines."
Old New York Story (abbreviated)

4.1 Two Trader Types trading with Fiat

Table 4
Two treatments for trade with fiat money and credit

	Cash only	Altruistic central bank
Trader 1	$(a, 0, m)$	$(a, 0, 0)$
Trader 2	$(0, a, m)$	$(0, a, 0)$
Central bank	no	$(0, 0, B)$

We now move to models with fiat money. When we consider a fiat money, there is no immediate consumption service value to the recipient. Hence, if no salvage value is attached to the surviving assets the Hahn paradox applies. No one offers goods for sale because they will end up with worthless paper. Many writers have pointed out that in order for the money to support trade there must be an expectation that a terminal supply of the asset has value. Hence, for a simple one period game where payment in cash is a required rule, the existence of a positive salvage value for fiat is a necessary condition for trade. We discuss this more fully below, in Section 4.2.1.

1. It is easy to illustrate and compute.
2. It provides a simple welfare measure.
3. It helps to provide the contrast between the use of gold and fiat.
4. It looks like the terminal wealth constraint in a competitive exchange model.
5. It provides a link (not utilized here) between strategic market games and market games studied in cooperative game theory.

4.1.1 The Treatment of the Service Value of Fiat

The transactions aspects of the services of the fiat are different in many ways from, often superior to, and sometimes inferior to those of gold.[13] In Section 2 we noted three aspects of gold to which value is attached. They are its value as an ornament, its value in transactions, and its residual value as an asset with a possible salvage value at the end of the game. Fiat has only two values, its value in transactions and its residual value as an asset with a possible salvage value at the end of the game.

Let us recall the way the service value of gold money was modelled in Section 2.1. Type 1 traders attempt to solve:

$$\begin{aligned} &\max_{q,b} \varphi\left(a-q, \frac{b}{\bar{p}}\right) + k_2\dot{m} + (k_3-k_2)(\dot{m}-\dot{b}) \\ &+ (1-k_3)(\dot{m}-\dot{b}+\dot{p}q) + \Pi(m+pq-b) && (34) \\ &\text{s.t. } m-b \geq 0 \qquad\qquad (\lambda) && (35) \\ &\qquad m+pq-b \geq 0 \qquad (\mu) && (36) \\ &\qquad b \geq 0,\ 0 \leq q \leq a && (37) \end{aligned}$$

Here k_2 is the proportion of the period before the goods market opens, $k_3 - k_2$ is the proportion of time spent during the market where the consumption use of the gold used as money is lost, and $1-k_3$ is the proportion of time after the market where the consumption use of gold held by the final owner of the asset is realized. But when the money is fiat, consumption value is irrelevant.

A bridging case between gold and fiat is obtained by setting $k_2 = 0$ and $k_3 = 1$ (see Footnote 6). No consumption value is left for gold if it is used for transactions. The distinction between fiat and gold here is that whereas fiat never has consumption worth, gold has worth either in consumption or transactions but not both.

Note that in the gold money model above, if we set the salvage value parameter $\Pi = 0$ the full force of the Hahn paradox is encountered. In a one period economy gold, like fiat, is worthless if left over unless someone pays something of worth for it.

4.1.2 The Treatment of the Salvage Value of Fiat

Because fiat is an ersatz real durable, it has different physical properties concerning its utilization, creation and destruction when compared with other physical assets. In particular, a finite economy can be considered as "cash consuming" with an outside government agency initially financing the float through a free issue of outside money, then via a nonzero rate of interest the central bank extracts all of the outside money from the system. In essence the rate of interest and borrowing serve as a backward operator through time so that at the start of the final settlement day $T+1$ (after a

[13] Transportability is easier and much faster; durability is extremely different; cognizability raises different questions and divisibility is easier. In a decaying or war-ridden society, gold may be more trustworthy than the political legal and societal guarantees of its paper. (See Jevons [21] and Shubik [34].)

T period economy has closed down), the fiat in the system is just adequate to pay the debt owed to the central bank (see Shubik [33]).[14]

4.2 The Hahn Paradox Revisited: Many Feasible Fixes

Before we can fully define a viable economic model with fiat we need to discuss the treatment of the terminal boundary conditions. This represents the value attached to the durable assets, left over after the game.

4.2.1 The paradox and a necessary condition for the fix

The Hahn paradox deals with why there are problems with introducing fiat money[15] (with no salvage value) into the general equilibrium system. Hahn [19] observed that if such money were introduced into a general equilibrium model with a duration of a finite time T, then one could work a backward induction as follows. First, at time $T+1$ any of the money left over is worthless; hence there is no point in selling any item at time T if one is going to be paid in a durable with no future worth and no current consumption service value. Hence no individual small agent will offer any item of value for sale in period T. If this happens the markets are inactive at time T and we may repeat the argument for time $T-1$; eventually we conclude that the markets never go active.

The mathematics and logic point out the problem. But the solution lies in formulating the appropriate models of economic activity. As a good approximation, we live embedded in time. There has been a yesterday and with a high probability there will be a tomorrow. The finite horizon economic model is embedded in society and time. In order to make the model more plausible we must approximate the conditions posed by the existence of history and the future. It is necessary to be specific about how initial or "pre game" conditions are specified and to provide explicit instructions for the terminal conditions of the game.

The mathematical treatments linking infinite horizon and finite horizon dynamic programming appear to be appropriate. In more or less standard operations research the initial conditions specify the ownership claims, including items such as goods-in-process which represent the eventual output into the process of items whose control variables have been selected prior to the start of the process, thus they are parameters of a new game rather than control variables.

At the end of time what are the conditions which appropriately reflect the possibility that the actual process continues but is being approximated by a finite horizon? A reasonably sophisticated way to handle this problem is to introduce and justify a salvage value.

[14] This construction has recently been used by Dubey and Geanakoplos [9] in introducing a strategic dummy player central bank.

[15] We say "fiat" here, but in the original work of Debreu [7] even commodity money cannot be made to fit without explicitly introducing period by period cash flow conditions. The model was the right abstraction to get away from transactions problems and the host of boundary condition and inequalities their control system structure generates. A model that is excellent for one purpose may blind one from the facts needed for other purposes.

4.2.2 The Many Sufficient Conditions for a Fix

It has been suggested elsewhere [34] that there are many plausible ways to achieve a fix. We must pay attention to not only the meaning of the terminal conditions, but how they depend on the initial conditions.

Initial conditions: Debt and ownership >From the viewpoint of the economist the initial conditions specify property rights and ownership claims. These conditions may also include debts owed to agents outside of the game, or who pre-existed but still exist in the game. Examples are the government and the first generation in an overlapping generations model.

Terminal conditions: Salvage values and expectations

> There are six and thirty ways
> of constructing Tribal lays
> and every single one of them is right!
> R. Kipling

Aesthetics and simplicity call for us to seek the basic "cleanest" condition. The whole philosophy of the approach here has been that of "minimal institutions." Yet in a complex world sometimes more than one set of conditions are both plausible and sufficient. Several different ways for taking care of terminal conditions are noted: The detailed discussions of most of them have been given elsewhere by many scholars of monetary theory. As these specifics are not germane to the main thrust of these papers they are not discussed in detail here and no attempt is made to provide an exhaustive treatment. The list includes:

1. The existence of a randomly distributed initial amount of outside money which must, by the rules of the game be returned at the end (passed to the next generation, or back to society in some other way) [38]

2. The existence of a private bank of such reputation that only the paper it issues is treated as a money. It is trusted to the extent that it may print its money in the sense that it can pay for goods and services or monetize the credit of others with its notes.[16]

3. There may be oligopolistic private banks who control much of the supply of the outside money or gold (see Shubik [32], [30])[17].

[16] If a private merchant bank's paper is univesally accepted, some of it's previously issued paper, such as "gift certificates" could circulate as the equivalent of government outside money where, in general, they are never called. If they are called they are redeemable in "goods at the store." In such an economy real assets back the "outside money" the interest rate can be positive representing the merchant banker's revenues which may be consumed thereby balancing all books internally.

[17] A completely different way of modeling the trust problem is via cooperative game theory, this was attempted[30], but abandoned for the more institutional strategic form approach. The utilization of the core concept to a monetary economy to illustrate the properties of trust appears to be promising and is treated elsewhere.

4. There may be an outside central bank which, by charging a positive rate of interest in what would physically be a stationary state will be able to "consume all of the original free issue of fiat by the end of any finite horizon [33].

5. There is an outside bank which offers loans and imposes default conditions on those who fail to cover their debts at the end of the game [38].

6. Any of the above can be combined with salvage values attached to any left over assets attributed to any type of expectations proposed.

7. The valuation could be based on some form of rational expectations.[18]

8. One can assume that as the processes are intrinsically dynamic, the correct fix is to deal directly with the infinite horizon, thus a backward induction is not relevant [24], [31], [22].

9. Overlapping generations may be postulated as a better model than either the finite termination or the dynasty economies [2].

10. One can introduce taxes, subsidies and public goods in many different ways providing a variety of realistic "fixes" for the terminal conditions.

11. All agents issue their own credit notes, there is a bankruptcy law specified and a credit agency evaluates each agent's credit so that she conforms with her budget constraints (see Sorin [41]). This imaginative but somewhat counterfactual solution stresses the ever-growing importance of credit evaluation in a mass anonymous market economy. The evaluation poses problems in coordination (for a connection with related problems in physics, see Smith and Shubik [40], [37]).

12. A wedge between buying and selling costs intermediated by a money can be introduced in several ways [20], [14], [23].

13. Credit considerations.

14. Legal considerations [5].

15. More or less any combination of the above list can be considered. Any of them can be used to set up a playable experimental game.

All are sufficient but none appear to be necessary, even the government bank. The government bank can use a rate of interest to remove the outside fiat money which is the basis for an essentially costless financing of the transactions float [33]. Even with an outside bank, in a finite horizon game it is unreasonable to attach zero worth to all other durables such as land and other highly durable resources which will physically be there at time $T+1$.

We consider the central bank, money market and salvage value for other assets here. In Part 3 private banks are considered.

[18] An excellent discussion of expectations is given by Grandmont [16]. For a treatment of rational expectations with non-symmetric information see Dubey, Geanakoplos and Shubik [10].

4.3 Two Trader Types trading with Fiat: The Model

Now we modify the models of Section 2 for trade in fiat.

We again consider two types of traders trading in two goods, with initial endowments of $(a, 0, [m, \dot{m}])$ and $(0, a, [m, \dot{m}])$. However, the common utility function for the types is now $\varphi(x, y) + \Pi\hat{w}$. The $\hat{w}$ stands for the expected terminal amount of fiat held by the traders of either type, while the parameter Π is the end-of-game salvage value for the fiat.

As before, there is a continuum of each trader type. A strategy of the Type 1 traders is denoted by (q, b), where q is the amount of good 1 offered for sale and b is the amount of fiat bid for good 2. The notation for traders of Type 2 is $(\bar{q}, \bar{b})$.

The traders of Type 1 attempt to solve:

$$\max_{q,b} \varphi\left(a - q, \frac{b}{\bar{p}}\right) + \Pi(m + pq - b) \tag{38}$$

$$\text{s.t. } m - b \geq 0 \qquad (\lambda) \tag{39}$$

$$b \geq 0,\ 0 \leq q \leq a. \tag{40}$$

The Type 2 traders face a similar maximization problem. Price is formed by:

$$p = \frac{\bar{b}}{q} \text{and } \bar{p} = \frac{b}{\bar{q}}. \tag{41}$$

If our stress is on stationarity we can impose an extra "rational expectations" condition and impose extra constraints on expectations. Thus here if we feel that predictions of future prices are absolutely precise we require that $\Pi = 1/\hat{p} = 1/\bar{\hat{p}}$, i.e., that money is expected to preserves its purchasing power. $\hat{p} = p$ implies that the predicted price will be as the actual price. When there is more than one durable the rational expectations condition requires values for each durable and if we desire to reflect different levels of uncertainty this can be done by placing constraints such as Π_j, the expected price of the jth durable lies within a range $p_j - \Delta \leq \Pi_j \leq p_j + \Delta$ and a subjective probability distribution over the range is supplied as common knowledge.[19]

We note that the combination of the rational expectations together with the implicit "no bankruptcy law" in the constraint reduces this problem to being identical with the model for gold with $k_1 = 0$, $k_2 = 0$, $k_3 = 1$ and $\Pi_i > 0$.

The concept of enough fiat money cannot be defined without specifying the bankruptcy rules and the salvage conditions. But once both bankruptcy and salvage conditions are given then fiat becomes "as good as gold."

The solution to the above model, in the case where $\varphi(x, y) = 2\sqrt{xy}$, is given in Appendix B.

[19] Refinements of the information set of the subjective probability distribution may be regarded as reflecting expertise. This way of reflecting expertise is suggested in Dubey, Geanakoplos and Shubik [10].

4.4 Two Trader Types trading with Fiat: The Model with a Central Bank

4.4.1 A Strategic Dummy Central Bank

We first consider altruistic agency or strategic dummy bank which has a fixed strategy; hence we do not even need to ascribe a payoff function for it, much less attribute profit maximization.[20]

A strategy for the Type 1 traders is as before, and is denoted by (d, q, b) where $d =$ the amount of money offered in the money market, $q =$ the amount of good 1 offered for sale and $b =$ the amount of money bid for good 2. The notation for traders of Type 2 is $(\bar{d}, \bar{q}, \bar{b})$ where $\bar{d}$ is the amount of personal IOU notes bid for the fiat money offered by the central bank.[21]

Thus Trader Type 1's optimization problem is

$$\max_{d,q,b} \varphi\left(a - q, \frac{b}{\bar{p}}\right) + \Pi\left(m + \frac{d}{1+\rho} - b + p\bar{q} - d\right)$$

$$\text{s.t. } m + \frac{\bar{d}}{1+\rho} - b \geq 0 \qquad (\lambda) \qquad (42)$$

$$m + \frac{d}{1+\rho} - b + p\bar{q} - d \geq 0 \qquad (\mu) \qquad (43)$$

$$d, b \geq 0, \ 0 \leq q \leq a \qquad (44)$$

with a similar optimization for the Type 2 traders. The endogenous rate of interest is formed as before:

$$1 + \rho = \frac{d + \bar{d}}{g}$$

where $0 \leq g \leq M - 2m$ is the amount of money offered for loan by the central bank who could face one of two optimization problems

Controlling the money supply The initial endowment of the bank is $(0, 0, M - 2m)$. If the bank were not a dummy the strategy of the bank would be to select a number g, where $0 \leq g \leq M - 2m = B$, in such a manner that it optimizes some welfare function of the trading economy. Here we assume it has a fixed strategy which is to simply lend out B no matter what the other players do.

We solve this model completely in Appendix B.

The distinction between the bank control problem with gold and with fiat is that there is no foregone loss of the consumption services component of the fiat money as there is no services component to lose. For the parameterization of $k_1 = 0$, $k_2 = 0$, $k_3 = 1$, the only difference between the two models is the lack of consumption services for fiat. If the gold is used for transactions it fully loses its consumption value, but maintains it if only used for consumption. In both instances some institutional fix

[20] In institutional fact central banks have often made a profit.

[21] In an economy involving several periods we would wish to consider the central bank a both a lender and borrower thus the strategy sets would involve g and d on one side and a matching d and g on the other. We do not deal with this here.

of the Hahn paradox must be provided. In a one person game, a salvage value will do and is realistic when the embedding in time is considered. The extraction of the outside money suggested in a discussion of financing the float [33] and considered in the light of gains to trade [9] provides a natural way to introduce a central bank to perform this function. The introduction of oligopolistic or monopolistic bankers will also serve. If the central bank is introduced, if it is strategically granted the right of issue there is no clear physical meaning to the upper bound B other than a legally or societally imposed limit. Although if $2m$ of fiat is held by the public we can observe that the upper bound on the debt issue of the bank for any ρ must be $g = 2m/\rho$. The desired borrowing of the traders will however be determined by their transactions needs and thus the equating of the money supply and demand may be interior or lie on a boundary.

If the bank is a strategic dummy we may assume that in the instance of utilizing the money supply as its strategic variable it is constrained to offer B units of money to the loan market.

We are now in a position to consider the situation where both there is a strategic dummy central bank and expectations concerning the future value of money exist. Suppose $\Pi = 1$ and $B + 2m = a$ We obtain efficiency, $\rho = 0$, $q = \bar{q} = a/2$, $p = \bar{p} = 1$ and each trader ends up with the same amount m of money she started with. The presence of the expectations of the future have weakened the power of the central bank. The Appendix for the more general calculations is in Part 3.

Controlling the rate of interest Instead of utilizing the quantity of money as its control variable, the bank could use the rate of interest. The constraint on $\rho \geq 0$ guarantees that the bank will not operate at a loss. But the use of the rate of interest without the specification of a bound on bank issue produces an endogenous demand for bank funds that could be unbounded.

An aside on clearing houses, credit evaluation and individual credit In all of the games described so far, we expected the players to stick to the rules of monetary usage. In fact, for the most part anyone can trade with anyone in more or less any way that they want. Thus the ability of any rule making is limited by custom and individual acceptance. Some years ago, one of us coined the phrase "mathematical institutional economics" [29] to reflect the importance of institutional detail in economic dynamics. There is and should be a vast literature on intra-day credit arrangements which we do not cover here. This is a zone where current the micro-micro-detail financial practises intersect with economic theory.

4.4.2 Expectations or the Outside Passive Bank?

We are now in a position to comment on two completely different "fixes" for the Hahn paradox and to compare them. With the central bank alone we require that $B \to \infty$ for an approach to efficiency. This is not so with expectations. If expectations are stationary then in a stationary state the bank will be able to offer non-inflationary financing at whatever the natural discount rate in a (nonstochastic) economy happens to be; the injection of outside money finances the float and the expectations induce the carry forward of the fixed amount of outside money.

4.4.3 A Profit Maximizing Monopolistic Bank?

Suppose we give most of the means of payment to a private individual. We assume endowments of the two perishables as before, however a third type of agent, the money lender has been introduced. The money lender is a profit maximizer who will consume real resources and thus provide a measure of the worth of transactions in terms of resources extracted.

As before, the two types of traders have symmetric endowments of $(a, 0, m)$ and $(0, a, m)$. We introduce a money lender with resources $(0, 0, B)$, where the total amount of "money" in the system is $M = B + 2m$.

A digression on currency, reputation and credit Before analyzing this model we note that the initial conditions could have a mix of currencies. One currency could be the free issue of government fiat in the hands of the public and another, the outstanding paper from the merchant bankers store redeemable at his store in some form of "value received" such as coffee, beans or fiat. The outside currency does not need an outside bank. It needs to be honored by all in acceptance, recognition, trust and enforcement and it needs a monopolist in control to guarantee that there is a shortage.

In a game where there is no free ride for the banker, the actions of the banker must be constrained so that he cannot spend more than the revenues he nets from lending money.

With this condition if he controls much of the money supply he can extract real resources from the economy. Conservation is preserved by the fact that the inside banker consumes the resources and the imaginary assets, the fiat money and the circulating merchant bank debt held by the public, being used as a means of payment held disappears at the end of the period into the bank. The value of the transactions technology is given by the worth of resources that the monopolist banker can extract. This is, in turn reflected in the trust and custom levels of the society, where individuals would abandon the currency if they felt that they were being unfairly taxed.

The formal model resumed We formulate the simplest model where there is just the outside fiat in circulation.

A strategy for the Type 1 traders is denoted by (d, q, b) where $d =$ the amount of money offered in the money market $q =$ the amount of good 1 offered for sale and $b =$ the amount of money bid for good 2. The notation for the Type 2 traders is $(\bar{d}, \bar{q}, \bar{b})$ where $\bar{d}$ is the amount of personal IOU notes bid for the personal money offered by the money lender.

Thus for Type 1 we consider

$$\max_{d,b,q} \varphi\left(a - q, \frac{b}{\bar{p}}\right) + \Pi\left(\frac{d}{1+\rho} - b + p\bar{q} - d\right)$$

$$\text{s.t. } m + \frac{d}{1+\rho} - b \geq 0 \tag{45}$$

$$m + \frac{d}{1+\rho} - b + p\bar{q} - d \geq 0.$$

$$d, b \geq 0,\ 0 \leq q \leq a \tag{46}$$

The endogenous rate of interest is formed as before:

$$1 + \rho = \frac{d + \overline{d}}{g}.$$

where $0 \leq g \leq 2m$ is the amount of money offered for loan by the monopolistic money lender. The lender's problem is

$$\max_{b_1^*, b_2^*, g} \varphi\left(\frac{b_1^*}{p}, \frac{b_2^*}{\overline{p}}\right)$$

$$\text{s.t. } \rho g - b_1^* - b_2^* \geq 0 \qquad (\mu^*) \tag{47}$$

$$\text{where } 0 \leq g \leq B,\ b_1^*, b_2^* \geq 0. \tag{48}$$

This constraint requires that the monopolistic banker cannot spend more than his profits.

The bankruptcy law can be interpreted as implicitly included in the non-negativity constraint on the final budgets of the traders. We do not need a rational expectations condition which would have us set the appropriate price for a left over durable gold because there is no real durable in this model. The full amount of money in this system is a fiction as the bank can write its own paper, possibly up to some bound B imposed by law or custom or intrinsic in the dynamics but the combination of the no bankruptcy constraint and not being permitted to spend more than its booked profits balances the books.

Qualitatively the details of a solution to this model are along the lines of the model of the monopolistic bank presented in Part 1. The analysis is considered in Part 3.

4.4.4 A Digression on Bankruptcy

In institutional fact the bankruptcy laws are not changed daily by an all-seeing government who knows how to solve a mammoth general stochastic strategic market game. The law is hopefully "good enough" and, in a world with uncertainty should accommodate to a positive level of bankruptcy (see Dubey, Geanakoplos and Shubik [11], Shubik [34], Chs. 11, 12). An adequate discussion of the structure of bankruptcy concerns its treatment as a public good and its relationship to society's risk propensity for innovation. This is not dealt with here.

There are two important distinctions in the treatment of bankruptcy which illustrate a difference both in economic analysis and in administrative and informational reality. If we consider a budget constraint such as:

$$\frac{d}{1+\rho} - b + \bar{p}q - d \geq 0 \qquad (\mu) \tag{49}$$

where we treat μ as a Lagrangian variable, in essence we are offering an immediate adjustment of the bankruptcy penalty to the actions in the economy. By doing so we avoid the host of analytical, computational, informational and administrative difficulties. An alternative way to introduce a bankruptcy penalty is to consider that μ is a societally legally fixed parameter, say μ^*, which is adjusted every few decades by socio-legal processes. In such an instance we replace constraint (μ) by subtracting from the traders' objectives a penalty function of the form:[22]

$$\mu^* \min\left(\frac{d}{1+\rho} - b + \bar{p}q - d,\ 0\right). \tag{50}$$

An economy is embedded in its polity and society, thus in spite of the aesthetic appeal of a model of exchange such as the general equilibrium formulation, it is fruitful to seek formulations at a high level of abstraction, where outside parameters connecting naturally to the polity and society fit, without attempting to find a premature endogenization of these parameters. This is especially true when we observe that the dynamics of the society, polity and economy tend to be on considerable different time scales.

5 Trade with Fiat and Demonetized Gold, the Role of Expectations

The models with gold and two consumables have only one real durable and that durable was the money The models with fiat and two consumables have only one ersatz or "constructive" durable and that is the fiat money which we may destroy by the backward operation of debt. The gold however is not destroyed but if it has a monetary salvage value of say, Π then the surviving asset can also be utilized to extinguish debt.

Suppose that our simple economy was using gold as its money and now switches from gold to fiat. This economy is now is trading in four commodities; two perishable consumables, one durable and one constructive durable. We set up the market trade model without financial institutions as a demonstration of the nature of the change in the demonetization. We consider two types of traders trading in three goods, each with the same utility function $\varphi(x, y) + \dot{z}$, where $\dot{z}$ stands for the consumption services of the asset. The trader types are distinguished by their endowments of $(a, 0, [c, \dot{c}], [m, \dot{m}])$ and $(0, a, [c, \dot{c}], [m, \dot{m}])$.

As before there is a continuum of each trader type. A general strategy of the Type 1 traders is denoted by (q_j, b_j) where $q_j =$ the amount of good j $(j = 1, ..., 3)$ offered for sale and $b_j =$ the amount of fiat bid for good j. The notation for a typical trader of Type 2 is $(\bar{q}_j, \bar{b}_j)$. We introduce the distinction between the commodities gold and fiat and the stream of services they yield. Given the existence of c units of

[22] Although the parameter μ^* may seem somewhat abrupt and not as nice as a smoothly adjusting Lagrangian, in insitutional fact there are only a few aggregate parameters for bankruptcy; but the more or less "smoothness" be it efficient or inefficient is taken care of by the lawyers, accountants and courts who personalize the highly aggregated parameters.

the asset and the flow rate of $\dot{c}$ units of its services, we note this ownership by $[c, \dot{c}]$. We have changed notation slightly, denoting the amount of gold by c rather than by m indicating that it is no longer the money. We denote the amount of fiat by m. As our purpose is to qualitatively indicate the new phenomenon introduced by the demonetization we purposely stay with a highly symmetric example.

As noted previously we must indicate who gets to use the services of the asset and how the asset is evaluated at the end. However we now must be careful concerning services. Previously monetary gold had three values: (1) its consumption use as, jewelry; (2) its transactions use as a money; and (3) the salvage value of the asset. Now it only has two values as it has been demonetized. The fiat has two values: (1) its transactions use as a money and (2) the salvage value of the asset.

The Type 1 traders attempt to maximize:[23]

$$\max_{q_j, b_j} \varphi\left(a - q_1, \frac{b_2}{p_2}\right) + \dot{c}k_2 + (k_3 - k_2)(\dot{c} - \dot{q}_3) + (1 - k_3)\left(\dot{c} + \frac{\dot{b}_3}{p_3} - \dot{q}_3\right)$$
$$+ \Pi\left(m + \sum_{j=1}^{3} p_j q_j - \sum_{j=1}^{3} b_j\right) + \Pi_3\left(c + \frac{b_3}{p_3} - q_3\right) \tag{51}$$

$$\text{s.t. } m - \sum_{j=1}^{3} b_j \geq 0 \tag{52}$$

$$\Pi\left(m + \sum_{j=1}^{3} p_j q_j - \sum_{j=1}^{3} b_j\right) \geq 0 \text{ and } \Pi_3\left(c + \frac{b_3}{p_3} - q_3\right) \geq 0 \tag{53}$$

$$0 \leq q_1 \leq a,\ 0 \leq q_2 \leq a,\ 0 \leq q_3 \leq c,\ b_1, b_2, b_3 \geq 0 \tag{54}$$

where k_2 is the percentage of the period before the goods market opens, $k_3 - k_2$ is the percentage time spent during the market where the consumption services use of the gold is lost, and $1 - k_3$ is the percentage of time after the market where the consumption services use of gold held by the final owner of the asset is realized. No services loss of the consumables is encountered as they can be consumed at a point anytime during the period (and are not lost unless the trading period is of length $[0, 1]$). A similar optimization problem holds for the Type 2 traders.

[23] In the previous examples our reasons for selecting gold as appearing in the utility function as a linear separable term were two fold. First and foremost it was to make the comparison between gold and fiat as monies; second it makes the calculations far easier and more transparent. For the simple point about expectations and salvage values with no extra difficulty we could consider a utility function of the form

$$\varphi\left((a - q_1), \frac{b_2}{p_2}, \left(\dot{c}k_2 + \left(c + \frac{\dot{b}_3}{p_3} - q_3\right)(1 - k_3)\right)\right)$$
$$+\Pi\left(m + \sum_{j=1}^{3} p_j q_j - \sum_{j=1}^{3} b_j\right)$$

requiring only that the marginal utility of the services of gold be positive in the whole domain.

Prices are formed as before, but now there are three prices:

$$p_j = \frac{b_j + \bar{b}_j}{q_j + \bar{q}_j}, \ j = 1, 2, 3 \tag{55}$$

where, in each instance $b_j q_j = \bar{b}_j \bar{q}_j = 0$ This is a no "wash sales" condition.[24]

Without calculating we may illustrate the fundamental role that expectations must play. When the expected values of the two durables (gold and fiat) are zero, there is no trade. When the expected salvage value prices are set at $\Pi = 1$ and $\Pi_3 = \hat{p}_3$ and it turns out that the actual price p_3 is equal to $\hat{p}_3$, i.e., the expected and formed prices match. We would have a noninflationary solution in which if $m = a$ there is enough money, and if $c \geq a$, there would be enough (if it were held appropriately) for a 100% gold backed currency where all of the gold is nevertheless used for consumption.

6 Clearinghouses, Credit and Control

This case contrasts directly with the model in Section 2.1 and stresses the importance of the expectations valuations.

6.1 Some Comments on the Trading Structure

In all of the models presented here and in Part 1, transactions take time and may use resources. All individuals are locked in to using the pieces of paper or ciphers which are defined as means of payment in the game. But although the use of a specified means of payment in a classroom game is given by the referee enforcing the rules as in a Monopoly game, in an actual economy when the price of the government enforced credit arrangements become too high, alternative means are sought. In particular netting arrangements between well known traders can easily replace cash. Furthermore new markets spring up. New instruments are invented. Legal, social and technological change influence the costs of running a trading technology. Market meeting frequencies and hours of trade are all in flux.

Even in a payments models as simple as the ones considered here, assuming a fixed trading technology individual informal and formal exchange relations can circumvent the formal trading relations and change both the velocity and volume of transactions. At best a government's control over prices, the velocity and the volume of trade is limited.

In the remaining models we are concerned only with economies using a fiat or symbolic money.

[24] At this level of modeling the phenomenon of wash sales where one both simultaneously sells and buys back an item is ruled out if there is a continuum of agents. A more complex model is required with nonsymmetric information to catch the actual use of wash sales.

6.2 Variations on the Fisher Equation: $PQ = MV$

In the classical Cambridge-Cambridge controversy (see for example, Modigliani [26]). The pristine simple "physics-like" Fisher equation has been considered in many ways. Utilizing the structure of strategic market games we are now in a position to revisit this equation in a somewhat more precise micro-economic modeling form provided by the models constructed above. A reasonably complete treatment of this topic requires models with several alternate markets within the same period, in order to reflect velocity as a strategic choice.[25] Our concern here is restricted to a one period, one goods market structure.

Even without going into mathematical detail it is useful to utilize a shorthand or symbolic description of a strategic market game with a given transactions and financial structure. Thus we use $\Gamma(\varphi, a, m|\Theta_j)$ to stand for the game with N types of agents with utility functions, φ and endowments of goods and outside money (a, m) embedded in a transactions and financial structure Θ_j.

Let Θ_1 stand for a storable consumable cash buy-sell economy where the market meets once. In such an economy $\Gamma(\varphi, a, m|\Theta_1)$

$$PQ = MV \text{ is modified to } PQ \leq M \tag{56}$$

as in a one market period world without any form of banking or money market the velocity is at most one, but there could be a hoarding or storage of the barley, beans or salt, in which instance the velocity would be less than one.

Suppose Θ_2 stands for the sell-all structure but is otherwise the same. In $\Gamma(\varphi, a, m|\Theta_2)$ the Q has been considerably enlarged to include all assets except the money.

Let Θ_3 stand for a fiat cash buy-sell economy where the goods market meets once but there is an outside bank which arranges for loans and which meets once and precedes the goods market . In such an economy $\Gamma(\varphi, a, m|\Theta_3)$ In such an economy

$$PQ = MV \text{ is modified to } PQ = M + g. \tag{57}$$

Let Θ_4 stand for a gold cash buy-sell economy with salvage value $\Pi = 0$ where the goods market meets once and $(k_1 = k_2 = k_3 = 0)$ This means that the money market meets at the start of the period and the goods market meets immediately after and all settlement takes place immediately after that meeting. Furthermore there is precisely enough money in the economy for efficient trade but it is badly distributed. In such an economy $\Gamma(\varphi, a, m|\Theta_4)$ the amount of money required for there to be enough is as though the gold were a storable consumable. The endogenous rate of interest will be $\rho = 0$

$$PQ = MV \text{ is modified to } PQ = M. \tag{58}$$

This describes the velocity of money in the goods market. But there will also be a velocity in the money market given by $(m_1 - m_2)/2M$ where $m_1 > m_2$ are the initial holdings of the two types of traders and $m_1 + m_2 = M$ (see Part 1, Section 4.3.1).

These few examples are given merely in order to indicate that it is necessary and feasible to develope a scheme to classify and specify market structures to reflect their

[25] This was done in part by Dubey, Sahi and Shubik [12].

need for money or trust. The concept of "enough money" can be made meaningful and can be calculated but with considerable specification.

6.3 Clearinghouses, Fiat, Credit and Credit Evaluation

> In Utopia all greed for money was entirely removed with the use of money. What a mass of troubles were then cut away!... Who does not know that fear, anxiety, worries, toils and sleepless nights will also perish at the same time as money? What is more, poverty, which alone money seemed to make poor, forthwith would itself dwindle and disappear if money were entirely done away with everywhere.
>
> Thomas Moore, *Utopia*, Book II, p. 149

We conclude our sequence of models with a final model. We have progressed from a storable consumable money (such as the barley of Mesopotamia or cans of beans) to a consumer durable money (such as gold) and thence to a paper money backed in many different ways (gold in Fort Knox, convention, law, expectations, debt to be repaid and bankruptcy penalties). The last stage is required to take us back to the sublime equilibrium with no worries, no time, no constraints on the system and no outside money whatsoever.

We consider an economy with two types of traders, a profit maximizing monopolist banker,[26] and an initial supply of outside fiat money.

As in Section 2.2.1 we consider a situation where the financial market meets to establish borrowing and lending for the period, after which the trade markets meet and clear before or after the end of the period. In order to be completely specific we resort to the proposition that the financial markets meet at the start of the period and engage in establishing a single interday loan which is paid back at the start of next period. The trade markets open for some hours in the middle of the period and all settlement takes place immediately after they close.

We add a clearinghouses a credit evaluation agency and the courts to complete the market structure as is shown in Figure 4.

[26] There are at least eight reasonable models which can be mathematically specified, involving a private or a public bank.

Four cases involve the choice of (1) a public bank, (2) a profit maximizing monopolist bank individually held (3) a profit maximizing monopolist bank owned by stockholders, (3) an overall utility maximizing monopolist bank.

These four choices are multiplied by two. The strategic alternatives are choice of the rate of interest or quantity of money as the strategic variable.

We limit our analysis to one of the models as the essential demonstration of the disappearance of outside money can be demonstrated with any of the models.

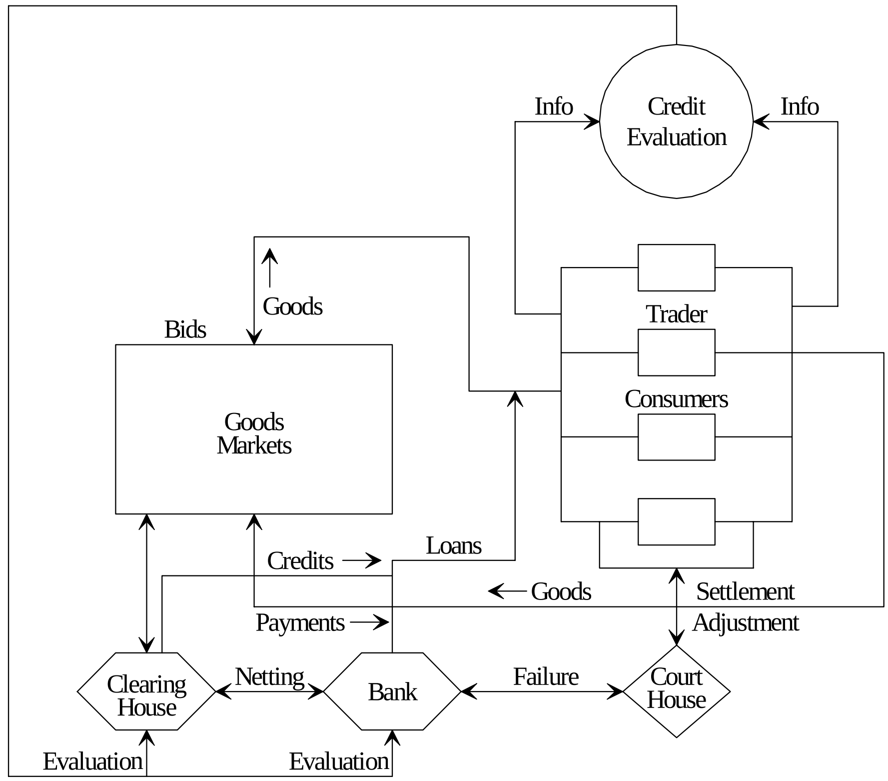

Figure 4

Before we present the formal symbolic model an interpretation of Figure 4 is given. The traders have goods and a certain amount of cash or fiat money. We look upon them as a composite of two individuals, consumers and traders. They inform the credit evaluation agency of their total holdings. The credit agency informs the banks about personal cash positions and the clearinghouse about goods for sale and its estimate of what they are worth. The banks issue one period loans and the clearinghouse extends clearinghouse credits. The traders may go to the bank.[27] At the bank they can exchange their IOU notes for the bank's money. They are required to redeem their IOU notes at the start of the next period with whatever interest payments promised.

The clearinghouse gives each agent a credit line based on three parameters. They are the assessment of the valuation of prices supplied by the credit agency, which we call $\hat{p}$ and $\overline{\hat{p}}$ and the valuation of the individuals assets which will act as security[28] which we call $\hat{a}$ and $\overline{\hat{a}}$ The clearing house extends a credit to the individual based on a "haircut" of its credit information. We call the haircut or credit limit θ where

[27] In this simple one period model this may be the government bank or a private monopolist bank with the same powers of issue as the central bank.

[28] Even here another nasty institutional and technological fact enters into our modeling. If the goods for sale are an ounce of caviar which can be eaten before the bailiffs close in the "secured loan aspects of the caviar" are somewhat less than an immoveable durable such as an acre of prime land.

the credit line extended by the clearinghouse to a trader of Type 1 is $\theta\hat{p}q$.[29] If the clearinghouse managers believe the security assessment is completely accurate and face no expenses or uncertainties the θ could be close to 1.

The traders of Type 1, with initial assets of $(a, 0, m)$, may regard their purchasing power as being $(a, 0, m + \theta\hat{p}q)$. Assuming the "usual" utility function of $\varphi(x, y) = 2\sqrt{xy}$, their optimization problem becomes

$$2\sqrt{(a-q)\frac{b}{\overline{p}}} + \Pi\left(m - b + pq - \frac{\rho d}{1+\rho}\right) \tag{59}$$

$$\text{s.t. } m - b + \theta\hat{p}q + \frac{d}{1+\rho} \geq 0 \qquad (\lambda) \tag{60}$$

$$m - b + \theta\hat{p}q - \frac{\rho d}{1+\rho} \geq 0 \qquad (\mu) \tag{61}$$

$$0 \leq q \leq a, \text{ and } q, b \geq 0.$$

Let λ and μ be the Lagrangian multipliers for constraints (λ) and (μ) respectively.

The first order conditions wrt b, q, d, λ and μ (respectively) yield

$$\frac{1}{\sqrt{\overline{p}}}\sqrt{\frac{(a-q)}{b}} - \Pi - \lambda - \mu = 0 \tag{62}$$

$$\frac{1}{\sqrt{\overline{p}}}\sqrt{\frac{b}{(a-q)}} + \Pi p + \lambda\theta\hat{p} + \mu p = 0 \tag{63}$$

$$-\Pi\frac{\rho}{1+\rho} + \lambda\frac{1}{1+\rho} + \frac{\rho}{1+\rho}\mu = 0 \tag{64}$$

$$m - b + \theta\hat{p}q + \frac{d}{1+\rho} = 0 \text{ or } \lambda = 0 \tag{65}$$

$$m - b + \theta\hat{p}q - \frac{\rho d}{1+\rho} = 0 \text{ or } \mu = 0. \tag{66}$$

Similarly, Type 2 traders face an optimization described by:

$$2\sqrt{(a-\bar{q})\frac{\bar{b}}{p}} + \Pi\left(m - \bar{b} + \overline{p}\bar{q} - \frac{\rho\bar{d}}{1+\rho}\right) \tag{67}$$

$$\text{s.t. } m - \bar{b} + \theta\hat{p}\bar{q} + \frac{\bar{d}}{1+\rho} \geq 0 \qquad (\bar{\lambda}) \tag{68}$$

$$m - \bar{b} + \theta\hat{p}\bar{q} - \frac{\rho\bar{d}}{1+\rho} \geq 0 \qquad (\bar{\mu}) \tag{69}$$

$$0 \leq \bar{q} \leq a, \text{ and } \bar{d}, \bar{b} \geq 0.$$

[29] The bound on the credit line is set by the information of the amount a (if there is truthful disclosure $\hat{a} = a$) available for sale, but the actual credit line realized would be based on realized sales thus will be $\theta\hat{p}q$.

The first order conditions wrt $\bar{b}$, $\bar{q}$, $\bar{d}$, $\bar{\lambda}$ and $\bar{\mu}$ yield

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{(a-\bar{q})}{\bar{b}}} - \Pi - \bar{\lambda} - \bar{\mu} = 0 \tag{70}$$

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{\bar{b}}{(a-\bar{q})}} + \Pi\bar{p} + \lambda\theta\hat{p} + \bar{\mu}\bar{p} = 0 \tag{71}$$

$$-\Pi\frac{\rho}{1+\rho} + \bar{\lambda}\frac{1}{1+\rho} + \frac{\rho}{1+\rho}\bar{\mu} = 0 \tag{72}$$

$$m - \bar{b} + \theta\hat{p}\bar{q} + \frac{\bar{d}}{1+\rho} = 0 \text{ or } \bar{\lambda} = 0 \tag{73}$$

$$m - \bar{b} + \theta\hat{p}\bar{q} - \frac{\rho\bar{d}}{1+\rho} = 0 \text{ or } \bar{\mu} = 0. \tag{74}$$

The banker's optimization is given by:

$$\max_{\rho} \rho g(\rho) \tag{75}$$

$$\text{s.t. } M - 2m - g \geq 0 \tag{76}$$

$$g \geq 0 \text{ or } \rho = 0 \tag{77}$$

and balance conditions are $p = \bar{b}/q$ and $\bar{p} = b/\bar{q}$ and $1 + \rho = (d + \bar{d})/g$.

A simple case Suppose that $\hat{p} = p$ and $\theta = 1$. [Π can be any positive value.]

It is clear that since $\hat{p} = p$, we have $\hat{p}q = pq = \bar{b} = b$ (from symmetry). Hence the conditions (λ) and ($\bar{\lambda}$) must hold loosely, i.e., $\lambda = \bar{\lambda} = 0$.

Since $b = \hat{p}q$ the constraint (μ) is $m - \rho d/(1+\rho) \geq 0$. But then it is clear that there is no need for borrowing, i.e., $d = 0 = \bar{d}$ (any solution with d or $\bar{d} > 0$ can be improved upon by lowering d or $\bar{d}$ to zero). In words, the amount of outside money called for is zero. But this then implies that (μ) (and ($\bar{\mu}$)) holds loosely. Hence $\mu = 0 = \bar{\mu}$.

Next, since $\lambda = \bar{\lambda} = \mu = \bar{\mu} = 0$, equation 64 gives $-\Pi\rho/(1+\rho) = 0 \Rightarrow \rho = 0$. Hence the bank cannot make a profit, no matter what it does.

The final values for this case are: $b = \bar{b} = a/2\Pi$, $p = \bar{p} = 1/\Pi$, $q = \bar{q} = a/2$, $\lambda = \mu = \rho = 0$, $d = \bar{d} = 0$.

The Utopian and general equilibrium world does not need outside money as it has total trust and accurate forecasting thus it runs on perfect individual credit.[30]

7 Concluding Remarks

In God we trust!
All others pay cash.
Old American folk saying

[30] In a world without outside money there are problems in fixing the price level which are discussed elsewhere (see [41], [40], [37]).

Outside money is "free trust" and is a (context dependent) credit information evaluation substitute.

Given the initial ownership claims, the sell-all model of exchange where all individuals are required to send all assets to market, maintaining only their ownership claims, and cash places a natural upper bound on the need for trust and evaluation. The buy-sell model [13] reflects the acceptance of immediate individual control over the established individual property rights hence is not as strategically limiting as the sell-all model and more closely provides the context for the study of an individual ownership competitive price economy.

Perfect trust can be achieved in a Utopian world with completely accurate price and trade volume assessments leading to the extension of perfect clearinghouse credits to all agents.

The minimal level of trust is where, in a sell-all economy there is no credit whatsoever and the transformation of ownership bundles is achieved via a minimal financial structure, the single simultaneous move price-formation market. All trades require cash. This amounts to the complete monetization of all wealth which equals the volume of transactions in this extreme instance.

The buy-sell model of Dubey and Shubik [13] is, in essence, not totally institution-free but requires a single mechanism, a minimal market structure, to map single simultaneous move strategies into trades. This credit free operation requires the implicit financing of the difference between the initial resource distribution and the final.

Depending on the initial conditions the required amount of trust varies from zero to the value of all assets. An imputation of wealth on the Pareto optimal surface is a point at which all individual gains have been achieved. No trade is required and the need for trust is zero. If the initial point is not on the Pareto surface trade may be desired and the need for trust will be a function of the potential gains from trade and the nature of the trading process, This calls for an appropriate measurement of the potential gains from trade and a specific description of the trading process

There have been three suggested measures of the gains from trade [6], [9], [39],[31] and the two simplest fully defined trading mechanisms to which they can be applied appear to be the sell-all or buy-sell strategic market games.

The one period interest rate ρ is nothing more than a tax imposable due to lack of trust. As the parameter $\theta \rightarrow 1$,the tax goes to zero and the power of ρ as a control variable utilized by a large monied agent is lost.

Trading economies, to avoid the need for individual trust, can use either a storable consumable good or a durable as a means of exchange. If the commodity were to enter the individual utility functions as a linearly separable term it would immediately supply an easy measure for the value of trade. Unfortunately this does not appear to be the case. Even if it were, there are still many physical drawbacks to utilizing either storable consumable or durables as a money. The storable consumable has the advantage that it always avoids the Hahn paradox, but this is not necessarily true for a durable, even in a one period economy.

[31] They all raise problems concerning the relationship between local and global measures.

As has been argued above, if transactions take time a societal resource loss can be incurred by utilizing a commodity money. A way to avoid this loss is to replace a real commodity with a societally constructed symbolic commodity. But doing this, this requires as a necessary condition that it be acceptable by a buyer in exchange. As is noted in Section 4.2 there are many different institutionally meaningful minimal ways to provide a sufficient way out of the Hahn paradox.

One of the natural ways out of the Hahn paradox is the positive valuation given by expectations concerning the future value of durable capital assets in the economy.

The store of value function of a money within a single period can be obtained via a monopolistic banker who purposely restricts the amount of money he issues to make sure that there is not sufficient outside money to avert all cash flow constraints. Custom, law and modern society conspire to limit the power of the individual merchant-banker-king, thus, although feasible, it is usually not individual monopolistic power that provides the support of the value of the currency. The goal of the merchant banker is to extract resources for herself and this provides a measure of the worth of trust, translated mathematically into the relevance of the tightness of the cash flow constraints given the transactions structure of the economy.

An alternative to the monopolistic banker is to introduce a central bank whose role is to promote efficient trade. The central bank can avoid the waste of natural resources by choosing a symbolic rather than a commodity money. But the choice of the symbolic money as opposed to, say gold, poses the question of trust in the central government and in the expectations of its performance.

One of the few reasoned political economy defenses of the use of gold was provided by Ricardo ([27], p. 24) "Experience ... shows that neither a state nor a bank ever have had the unrestricted power of issuing paper without abusing that power; in all states therefore, the issue of paper money ought to be under some check and control; and none seems so proper for that purpose as that of subjecting the issuers of paper money to the obligation of paying their notes either in gold coin or bullion."

Time, population, society and technology have changed since Ricardo, but the essence of the monetary problem remains. The need for individual trust or for substitutes for trust is central. The trust and the substitutes for trust are generated in the context of the polity and society as a whole, not just the economy. The measurement of the economic value of trust in exchange requires a mathematical institutional economics which reflects the institutional richness of the society and its transactions structure, yet is amenable to a formal quantitative analysis as is reflected in the structure of the constraints on the economic optimization.[32]

[32] The mathematics of the general equilibrium system may be looked at as that of optimization subject to equalities. Monetary control and finance are more congenially considered in terms of optimization subject to several constraints where the pressures on the constraints reflect the potential value of trust and the opportunity for control. Further investigation of trust calls for network analysis (see Shubik, 2003 [36]).

8 Appendix A: Trade with Gold

8.1 Two Trader Types: No Financial Markets

We present the calculations for the Example 2.1, where the traders are symmetrically endowed.[33]

The Type 1 traders face an optimization described by:

$$\max_{b,q} 2\sqrt{(a-q)\frac{b}{\bar{p}}} + k_2\dot{m} + (k_3-k_2)(\dot{m}-\dot{b})$$
$$+(1-k_3)(\dot{m}-\dot{b}+p\dot{q}) \tag{78}$$
$$\text{s.t. } m-b \geq 0 \ (\lambda) \tag{79}$$
$$m+pq-b \geq 0 \qquad (\mu)$$
$$0 \leq q \leq a \text{ and } b \geq 0 \qquad (\lambda) \tag{80}$$

We introduce the Lagrangian multipliers λ and μ, for the constraints (λ) and (μ) respectively.

The first order conditions wrt b, q, λ and μ yield

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{a-q}{b}} = 1-k_2+\lambda+\mu \tag{81}$$
$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{b}{a-q}} = p(1-k_3+\mu) \tag{82}$$

$$m-b = 0 \text{ or } \lambda = 0. \tag{83}$$
$$m+pq-b = 0 \text{ or } \mu = 0. \tag{84}$$

Similarly the Type 2 traders face

$$\max_{\bar{b},\bar{q}} 2\sqrt{(a-\bar{q})\frac{\bar{b}}{p}} + k_2\dot{m} + (k_3-k_2)(\bar{m}-\dot{\bar{b}}) + (1-k_3)(\dot{m}-\bar{b}+\bar{p}\,\bar{q}) \tag{85}$$
$$\text{s.t. } m-\bar{b} \geq (0) \qquad (\bar{\lambda}) \tag{86}$$
$$m+p\bar{q}-\bar{b} \geq 0 \qquad (\bar{\mu}) \tag{87}$$
$$0 \leq \bar{q} \leq a \text{ and } \bar{b} \geq 0. \tag{88}$$

We introduce the Lagrangian multipliers $\bar{\lambda}$ and $\bar{\mu}$.

The first order conditions wrt $\bar{b}$, $\bar{q}$, $\bar{\lambda}$ and $\bar{\mu}$ yield

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-\bar{q}}{\bar{b}}} = 1-k_2+\bar{\lambda}+\bar{\mu} \tag{89}$$
$$\frac{1}{\sqrt{p}}\sqrt{\frac{\bar{b}}{a-\bar{q}}} = \bar{p}(1-k_3+\bar{\mu}) \tag{90}$$

[33] For the computations we (somewhat sloppily) do not call explicit attention to the important distinction between the asset and its service.

$$m - \bar{b} \quad = \quad 0 \text{ or } \bar{\lambda} = 0 \tag{91}$$

$$m + p\bar{q} - \bar{b} \quad = \quad 0 \text{ or } \bar{\mu} = 0. \tag{92}$$

The balance conditions are $p = \bar{b}/q$ and $\overline{p} = b/\bar{q}$

Before continuing we note that the problems for Type 1 and Type 2 players are isomorphic; hence we may assume that there is a symmetric solution; i.e., a solution in which $p = \bar{p}$, $q = \bar{q}$, and $b = \bar{b}$

Case 1: m is high.

If m is large we may assume that $\lambda = 0$ and $\bar{\lambda} = 0$ Then necessarily (μ) also holds loosely, hence μ (and $\bar{\mu}$) $= 0$. Condition (82) becomes

$$\frac{1}{\sqrt{\overline{p}}}\sqrt{\frac{b}{a-q}} = (1-k_3)p \Rightarrow \sqrt{\frac{b}{a-q}} = (1-k_3)p\sqrt{\overline{p}} \tag{93}$$

and so

$$\sqrt{\frac{a-q}{b}} = \frac{1}{p\sqrt{\overline{p}}(1-k_3)}. \tag{94}$$

Substituting this into (81) gives

$$\frac{1}{\sqrt{\overline{p}}}\frac{1}{(1-k_3)p\sqrt{\overline{p}}} = 1-k_2 \Rightarrow p\overline{p} = \frac{1}{(1-k_2)(1-k_3)} \tag{95}$$

hence from symmetry

$$p = \overline{p} = \sqrt{\frac{1}{(1-k_2)(1-k_3)}}. \tag{96}$$

Next, from (81) we obtain

$$\frac{1}{\overline{p}}\frac{(a-q)}{b} = (1-k_2)^2 \Rightarrow \frac{a-q}{b} = (1-k_2)^{\frac{3}{2}}(1-k_3)^{-1/2}. \tag{97}$$

But $b = pq$; hence we may substitute for b then p and solve for q

$$\frac{a-q}{q} = \frac{a}{q} - 1 = (1-k_2)(1-k_3)^{-1} \tag{98}$$

which simplifies to

$$q = \frac{a(1-k_3)}{2-k_2-k_3} \tag{99}$$

and

$$b = pq = \frac{a}{2-k_2-k_3}\sqrt{\frac{1-k_3}{1-k_2}}. \tag{100}$$

These results hold as long as $m - b \geq 0$, i.e.,

$$m \geq \frac{a}{2-k_2-k_3}\sqrt{\frac{1-k_3}{1-k_2}}, \tag{101}$$

so this is what we mean by "m large". The economic meaning of "m large" is that there is enough money to cover efficient trade [34].

We note that if $k_2 = k_3 = 0$ we obtain the same results as with a consumable storable. In particular, the symmetric solution is $p = \bar{p} = 1$, $b = \bar{b} = q = \bar{q} = a/2$ (see Section 4.2 of Part 1). In this instance since $b = a/2$ the cash flow constraint requires that $m \geq a/2$.

Case 2: m is small.

If m is small then $\lambda, \bar{\lambda} > 0$ and $b = m$, $\bar{b} = m$ We still have $\mu = \bar{\mu} = 0$, because constraints (μ) and ($\bar{\mu}$) are redundant. Equations (81) and (82) imply

$$\frac{1}{\sqrt{\bar{p}}} \frac{1}{(p\sqrt{\bar{p}})(1-k_3)} = 1 - k_2 + \lambda \tag{102}$$

which gives

$$p = \bar{p} = \frac{1}{\sqrt{(1+k_2+\lambda)(1-k_3)}}.$$

Hence $q = \bar{b}/p = m\sqrt{(1-k_2+\lambda)(1-k_3)} = \bar{q}$.

Next, condition (81) gives $\frac{1}{p}\frac{a-q}{b} = (1-k_2+\lambda)^2$. Substituting in our expressions for $\bar{p}$, q, and b gives

$$\sqrt{(1-k_2+\lambda)(1-k_3)} * \frac{a - m\sqrt{(1-k_2+\lambda)(1-k_3)}}{m} = (1-k_2+\lambda)^2, \tag{103}$$

from which we may obtain

$$\frac{(1-k_2+\lambda)^{3/2}}{(1-k_3)^{1/2}} + (1-k_2+\lambda)^{1/2}(1-k_3)^{1/2} = \frac{a}{m}. \tag{104}$$

It is not hard to see that we may rewrite (104) as

$$\frac{1}{p^3(1-k_3)^2} + \frac{1}{p} = \frac{a}{m}. \tag{105}$$

This shows how price varies with the money supply. We remark that equations (104) and (105) reduce to equations (26) and (27) from Part 1, in the case where $k_2 = k_3 = 0$, i.e., the case where the gold is essentially a storable consumable.

Finally, if $k_3 = 1$ the Hahn paradox will hold.

9 Appendix B: Trade with Fiat

9.1 Trade with No Bank but with a Salvage Value

Our first model has trade where left over fiat has a salvage value.

Traders of Type 1 attempt to solve:

$$\max_{q,b} \varphi\left((a-q), \frac{b}{\bar{p}}\right) + \Pi(m + pq - b) \tag{106}$$

$$\text{s.t. } m - b \geq 0 \qquad\qquad (\lambda) \tag{107}$$

$$b \geq 0,\ 0 \leq q \leq a. \tag{108}$$

First order conditions wrt b, q, and λ give

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{(a-q)}{b}} = \Pi + \lambda \tag{109}$$

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{b}{(a-q)}} = \Pi p \tag{110}$$

$$m - b = 0 \text{ or } \lambda = 0. \tag{111}$$

Traders of Type 2 attempt to solve:

$$\max_{\bar{q},\bar{b}} \varphi\left((a-\bar{q}), \frac{\bar{b}}{p}\right) + \Pi(m + \bar{p}\bar{q} - \bar{b}) \tag{112}$$

$$\text{s.t. } m - \bar{b} \geq 0 \qquad (\bar{\lambda}) \tag{113}$$

$$\bar{b} \geq 0, 0 \leq \bar{q} \leq a. \tag{114}$$

First order conditions give

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-\bar{q}}{\bar{b}}} = \Pi + \lambda \tag{115}$$

$$\frac{1}{\sqrt{p}}\sqrt{\frac{\bar{b}}{a-\bar{q}}} = \Pi \bar{p} \tag{116}$$

$$m - \bar{b} = 0 \text{ or } \lambda = 0. \tag{117}$$

Price is formed by:

$$p = \frac{\bar{b}}{q} \text{ and } \bar{p} = \frac{b}{\bar{q}}. \tag{118}$$

Case 1: m is large.

In this case $\lambda = \bar{\lambda} = 0$. Then equation (109) gives $\Pi = \frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{a-q}{b}}$, from which $\sqrt{\frac{b}{a-q}} = \frac{1}{\Pi\sqrt{\bar{p}}}$. Substituting into (110) gives $\Pi p = \frac{1}{\sqrt{\bar{p}}}\frac{1}{\Pi\sqrt{\bar{p}}}$, which is $\Pi^2 p\bar{p} = 1$. From symmetry, $p = \bar{p} = \frac{1}{\Pi}$.

We have $\frac{1}{\Pi} = p = \frac{b}{q}$ and also $\frac{b}{a-q} = (\frac{1}{\Pi\sqrt{\bar{p}}})^2 = \frac{1}{\Pi}$. Hence $\frac{b}{q} = \frac{b}{a-q}$, which implies $q = \frac{a}{2}$. Also, $b = \bar{b} = pq = \frac{a}{2\Pi}$.

The implication is that fiat is hoarded and carried forward and the allocations from trade are efficient.

Finally, we remark that the above analysis is valid so long as the cash flow constraint $m - b \geq 0$ holds, i.e. $m \geq \frac{a}{2\Pi}$. So this is formally what we mean by "m is large."

Case 2: m is small.

Here the cash flow constraints are tight, i.e.. $\lambda > 0$ and $\bar{\lambda} > 0$. This further implies $b = m = \bar{b}$. Then (109) implies $\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{a-q}{b}} = \Pi + \lambda$, which is $\sqrt{\frac{b}{a-q}} = \frac{1}{\sqrt{\bar{p}}(\Pi+\lambda)}$.

Substituting into (110) gives $\Pi p = \frac{1}{\sqrt{\bar{p}}}\frac{1}{\sqrt{\bar{p}(\Pi+\lambda)}}$, which is $p\bar{p} = \frac{1}{\Pi(\Pi+\lambda)}$ Hence by symmetry we have

$$p = \bar{p} = \frac{1}{\sqrt{\Pi(\Pi+\lambda)}}.$$

Next, equation (109) gives $\frac{1}{\bar{p}}\frac{a-q}{b} = (\Pi+\lambda)^2$. Substituting in our values for $\bar{p}$ and b, we get $q = a - \bar{p}b(\Pi+\lambda)^2 = a - \frac{1}{\sqrt{\Pi(\Pi+\lambda)}}m(\Pi+\lambda)^2 = a - m\frac{(\Pi+\lambda)^{\frac{3}{2}}}{\sqrt{\Pi}}$. But also $b = \bar{b} = pq$, hence $m = \frac{1}{\sqrt{\Pi(\Pi+\lambda)}}\left(a - m\frac{(\Pi+\lambda)^{\frac{3}{2}}}{\sqrt{\Pi}}\right)$. We can solve this computationally to find λ, and use the value of λ to find p and q.

9.2 Trade with a Strategic Dummy Bank

We consider the market with a bank who is constrained to offer B units of fiat for loan to the market. Fiat is assumed to have a salvage value Π. The traders of Type 1 face

$$\begin{aligned}
&\max_{d,q,b}\ 2\sqrt{(a-q)\frac{b}{\bar{p}}} + \Pi\left(m + \frac{d}{1+\rho} - b + pq - d\right) \\
&\text{s.t. } m + \frac{d}{1+\rho} - b \geq 0 && (\lambda) \\
&\qquad m + \frac{d}{1+\rho} - b + pq - d \geq 0 && (\mu) \qquad (119) \\
&\qquad b, d \geq 0,\ 0 \leq q \leq a. && (120)
\end{aligned}$$

Similarly, for the Type 2 traders we have

$$\begin{aligned}
&\max_{\bar{d},\bar{q},\bar{b}}\ 2\sqrt{(a-\bar{q})\frac{\bar{b}}{p}} + \Pi\left(m + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\bar{q} - \bar{d}\right) \\
&\text{s.t. } m + \frac{\bar{d}}{1+\rho} - \bar{b} \geq 0 && (\lambda) \\
&\qquad m + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\bar{q} - \bar{d} \geq 0 && (\mu) \qquad (121) \\
&\qquad \bar{b}, \bar{d} \geq 0,\ 0 \leq \bar{q} \leq a. && (122)
\end{aligned}$$

The first order conditions for the Type 1 players are:

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{a-q}{b}} = \Pi + \lambda + \mu \qquad (123)$$

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{b}{a-q}} = (\mu + \Pi)p \qquad (124)$$

$$\lambda = (\mu + \Pi)\rho \qquad (125)$$

$$m + \frac{d}{1+\rho} - b = 0 \text{ or } \lambda = 0 \qquad (126)$$

$$m - \frac{\rho d}{1+\rho} - b + pq = 0 \text{ or } \mu = 0 \qquad (127)$$

and similarly, for the Type 2 players we have

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{a-\bar{q}}{\bar{b}}} = \Pi + \bar{\lambda} + \bar{\mu} \tag{128}$$

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{\bar{b}}{a-\bar{q}}} = (\bar{\mu} + \Pi)\bar{p} \tag{129}$$

$$\lambda = (\bar{\mu} + \Pi)\rho \tag{130}$$

$$m + \frac{\bar{d}}{1+\rho} - \bar{b} = 0 \text{ or } \bar{\lambda} = 0 \tag{131}$$

$$m - \frac{\rho\bar{d}}{1+\rho} - \bar{b} + \bar{p}\bar{q} = 0 \text{ or } \bar{\mu} = 0. \tag{132}$$

The strategic dummy bank offers B for loan no matter what the other players do.

The Balance conditions are:

$$p = \frac{\bar{b}}{q},\ \bar{p} = \frac{b}{\bar{q}} \text{ and } 1+\rho = \frac{d+\bar{d}}{B}. \tag{133}$$

Before beginning the analysis, we comment that the two trader types' problems are isomorphic; hence we may make the assumption of "symmetry", i.e., that in a solution $p = \bar{p}$, $q = \bar{q}$, $d = \bar{d}$, and $b = \bar{b}$.

Case 1: m is large ("enough money").

If m is large, it is clear that both the cash flow constraints ((λ) and ($\bar{\lambda}$)) and the budget constraints ((μ) and ($\bar{\mu}$)) will be loose, i.e., $\lambda = \mu = \bar{\lambda} = \bar{\mu} = 0$ Thus equation (124) is $\frac{1}{\sqrt{p}}\sqrt{\frac{b}{a-q}} = \Pi p$, which is $\sqrt{\frac{a-q}{b}} = \frac{1}{\Pi p\sqrt{p}}$. Substitution into (123) gives $\frac{1}{\sqrt{\bar{p}}}\frac{1}{\Pi p\sqrt{\bar{p}}} = \Pi$, or $p\bar{p} = \frac{1}{\Pi^2}$. Symmetry then gives $p = \bar{p} = \frac{1}{\Pi}$.

Next, condition (123) gives $\sqrt{\frac{a-q}{b}} = \Pi\sqrt{\bar{p}} = \sqrt{\Pi}$; hence $\frac{a-q}{b} = \Pi$. But $b = \bar{b} = pq = \frac{q}{\Pi}$, so we have $\frac{a-q}{\frac{q}{\Pi}} = \Pi$. This simplifies to $q = \frac{a}{1+\Pi^2} = \bar{q}$. Also, $b = \bar{b} = pq = \frac{a}{\Pi(1+\Pi^2)}$.

Next, we note that since $\lambda = 0$, equation (125) implies $\rho = 0$ Hence the balance condition (133) implies $d = \bar{d} = \frac{B}{2}$.

We remark that the above analysis will be valid only so long as the cash flow constraints (λ) and ($\bar{\lambda}$) remain satisfied. Since $d = \bar{d} = \frac{B}{2}$, this is $m \geq b - \frac{B}{2}$, or $m \geq \frac{a}{\Pi(1+\Pi^2)} - \frac{B}{2}$. So this is what is meant by "enough money." Finally, note that the requirements for "m is large" become less onerous the higher Π is. This makes sense — Π higher means that the fiat is worth more at the end, meaning that there is more incentive for the traders to hoard their money, which in turn means they are less likely to bump up against their cash flow constraints.

Case 2: m is very small ("very little money").

In this case both the cash flow constraints ((λ) and ($\bar{\lambda}$)) and the budget constraints ((μ) and ($\bar{\mu}$)) will be tight, i.e., $\lambda > 0$, $\mu > 0$, $\bar{\lambda} > 0$, and $\bar{\mu} > 0$.

Since (λ) and (μ) both hold tightly, we have $d = pq$. Also from the balance condition $pq = \bar{b} = b$, hence $b = d$. But then (126) is $m - d + \frac{d}{1+\rho} = 0$, which gives $d = \frac{1+\rho}{\rho}m = b = \bar{b} = \bar{d}$. But then the balance condition for the interest rate gives $1 + \rho = \frac{2\frac{1+\rho}{\rho}m}{B}$, which is $\rho = 2m/B$.

Next, (124) implies that $\sqrt{\frac{a-q}{b}} = \frac{1}{(\mu+\Pi)p\sqrt{\bar{p}}}$. Substituting into (123), we have $\frac{1}{\sqrt{\bar{p}}}\frac{1}{(\mu+\Pi)p\sqrt{\bar{p}}} = \lambda+\mu+\Pi$, which is $p\bar{p} = \frac{1}{(\mu+\Pi)(\lambda+\mu+\Pi)}$. But (125) means $\lambda = (\mu+\Pi)\rho$; hence $p\bar{p} = \frac{1}{(\mu+\Pi)((\mu+\Pi)\rho+\mu+\Pi)} = \frac{1}{(1+\rho)(\mu+\Pi)^2}$. So by symmetry, $p = \bar{p} = \frac{1}{(\mu+\Pi)\sqrt{1+\rho}}$.

But also (124) implies $\frac{a-q}{b} = \frac{1}{(\mu+\Pi)^2p^3}$, which is $q = a - \frac{b}{(\mu+\Pi)^2p^3}$. Substituting in our expressions for b and p gives

$$q = a - \frac{\frac{1+\rho}{\rho}m}{(\mu+\Pi)^2\left(\frac{1}{(\mu+\Pi)\sqrt{1+\rho}}\right)^3} = a - \frac{(1+\rho)^{\frac{5}{2}}(\mu+\Pi)m}{\rho} = \bar{q}. \tag{134}$$

But then the balance equation $b = \bar{b} = pq$ is just $\frac{1+\rho}{\rho}m = \frac{1}{(\mu+\Pi)\sqrt{1+\rho}}\left(a - \frac{(1+\rho)^{\frac{5}{2}}(\mu+\Pi)m}{\rho}\right)$, which simplifies to

$$\mu = \frac{\rho a}{m(1+\rho)^{\frac{3}{2}}(2+\rho)} - \Pi. \tag{135}$$

Substituting (135) into (134) gives

$$q = a - \frac{(1+\rho)^{\frac{5}{2}}\left(\frac{\rho a}{m(1+\rho)^{\frac{3}{2}}(2+\rho)}\right)m}{\rho} = \frac{a}{2+\rho} = \bar{q}. \tag{136}$$

Also $p = \bar{p} = \frac{b}{q} = \frac{(1+\rho)(2+\rho)m}{\rho a}$, where $\rho = 2m/B$.

We remark that the results in Case 2 are valid only so long as the expression for the multiplier μ (or $\bar{\mu}$) remains nonnegative >From (135), this condition is $\frac{\rho a}{m(1+\rho)^{\frac{3}{2}}(2+\rho)} \geq \Pi$. Substituting in $\rho = 2m/B$ and rearranging gives

$$\Pi(B+2m)(B+m) \leq B^{\frac{3}{2}}a. \tag{137}$$

Hence (137) gives the precise mathematical definition of "m very small." In particular we note that the condition is always satisfied if $\Pi = 0$. Intuitively this makes sense, because $\Pi = 0$ means the fiat is worthless at the end of the game. So the traders will spend all they have and borrow to such a degree that any future income will just cover their debts. But this is just another way of saying that their cash flow and budget constraints will be tight.

Case 3: m is intermediate-valued ("a moderate amount of money, but still 'not enough"')

This is the case in which neither Case 1 nor Case 2 holds. Hence it must be that precisely one of the constraints (λ) and (μ) holds tightly, while the other is loose. But it is impossible for (λ) to be loose and (μ) tight in an optimum. This is because if (λ)

were loose and (μ) tight the Type 1 traders could raise b and lower d in such a way that (μ) stays tight, (λ) stays satisfied, and the objective function value is improved.

So, in Case 3 (λ) and $(\bar{\lambda})$ are tight (i.e., $\lambda > 0$ and $\bar{\lambda} > 0$) while (μ) and $(\bar{\mu})$ are loose ($\mu = \bar{\mu} = 0$). Then (124) is $\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{b}{a-q}} = \Pi p$, which is $\sqrt{\frac{a-q}{b}} = \frac{1}{\Pi p\sqrt{\bar{p}}}$. Substituting into (123) gives $\frac{1}{\sqrt{\bar{p}}}\frac{1}{\Pi p\sqrt{\bar{p}}} = \Pi + \lambda$, which gives $p\bar{p} = \frac{1}{\Pi(\Pi+\lambda)}$. Now (125) is $\lambda = \Pi\rho$, so we have $p\bar{p} = \frac{1}{\Pi(\Pi+\Pi\rho)} = \frac{1}{\Pi^2(1+\rho)}$. Symmetry then gives $p = \bar{p} = \frac{1}{\Pi\sqrt{1+\rho}}$

Next, (124) implies $\frac{1}{\bar{p}}\frac{b}{a-q} = \Pi^2 p^2$ But $\frac{b}{\bar{p}} = \bar{q} = q$, so we have $\frac{q}{a-q} = \Pi^2 p^2 = \frac{1}{1+\rho}$. Solving for q gives $q = \frac{a}{2+\rho} = \bar{q}$. And then $b = \bar{b} = pq = \frac{a}{\Pi\sqrt{1+\rho}(2+\rho)}$. Also, $d = (1+\rho)(b-m) = (1+\rho)\left(\frac{a}{\Pi\sqrt{1+\rho}(2+\rho)} - m\right) = \bar{d}$.

So now all that remains is to find ρ. But by the balance equation $1+\rho = \frac{d+\bar{d}}{B} = \frac{2}{B}(1+\rho)\left(\frac{a}{\Pi\sqrt{1+\rho}(2+\rho)} - m\right)$. This means the value of ρ will be that which satisfies $\frac{a}{\Pi\sqrt{1+\rho}(2+\rho)} = \frac{B}{2} + m$. This calculation can easily be done using computational methods.

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