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Absenteeism, Substitutes, and Complements in Simple Games*

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Abstract

A voting with absenteeism game is defined as a pair $(G;r)$ where G is an n-player (monotonic) simple game and r is an n-vector for which r_i is the probability that player i attends a vote. We define a power index for such games, called the absentee index. We axiomatize the absentee index and provide a multilinear extension formula for it. Using this analysis we re-derive Myerson's (1977, 1980) "balanced contributions" property for the Shapley-Shubik power index. In fact, we derive a formula which quantitatively gives the amount of the "balanced contributions" in terms of the coefficients of the multilinear extension of the game.

Finally, we define the notion of substitutes and complements in simple games. We compare these concepts with the familiar concepts of dummy player, veto player, and master player.

Keywords: simple game, Shapley-Shubik power index, absenteeism, multilinear extension, balanced contributions, substitute, complement

JEL Classification: C7, C71, D72
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Introduction

Perhaps there is no more appealing use of cooperative game theory than the use of simple games to model decision-making bodies. A subset of players is a "winning coalition" if it can enforce its will on the rest of the voters, no matter what those other voters do; otherwise it is a "losing coalition." In terms of the characteristic function V of a transferable utility game, a winning coalition S has $V(S) = 1$, while $V(T) = 0$ if T is losing.

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The beauty of this simple model is that one may apply the mathematical theory of games to derive interesting political insights. In 1954, Shapley and Shubik applied the Shapley value (Shapley, 1953) to simple games, obtaining the Shapley–Shubik power index (SSPI). The SSPI is now widely regarded as a measure of players' power in a voting system, based solely on the structure of the set of winning coalitions.

In this paper we add absenteeism to the model. It is well-known that often absenteeism is high in such non-compulsory votes as presidential midterm elections, corporate shareholder elections or departmental faculty votes. There is clearly a (generally small) cost to voting which many individuals do not wish to incur. Even if costless, voters often feel as if their vote won't make a difference. Finally, an individual may be too ill to go to the polls, or be away on vacation and have neglected to obtain an absentee ballot.

At this point we emphasize the difference between absenteeism and abstention. Abstention is the willful act of attending a vote and then not voting "yea" or "nay." There is a small literature in which abstention is modelled. The main idea is that there are more than two choices when voting on an issue — with "yea" being the highest level of support and "nay" being the highest level of opposition. Abstention corresponds to one of the "middle" choices (see, e.g., Felsenthal–Machover (1997, 1998) or Freixas–Zwicker (2002)).

On the other hand, we view absenteeism as the failure to attend a vote. It is not strategic in any way. However, it *can* make a difference in two ways. First, if enough players are absent there will fail to be a quorum present. In the familiar terminology of simple games, we mean that there is no minimal winning coalition of players present.[1] Second, certain players being absent can affect the abilities of those remaining to form winning coalitions, hence affecting their power. In our analysis below we attempt to capture both of these effects.

We begin by assuming that each player i ($i = 1, ..., n$) has an independent probability r_i of being present (= not absent) for a vote. A voting-with-absenteeism game, then, is defined by the vector r together with an underlying simple game G.

We then define a power index for such games. This index, which we call the absentee index, is essentially a weighted average of the SSPI's of all 2^n possible subgames. The weights are the probabilities that each such subgame will form, given the vector of absentee probabilities r.

It then turns out that there is a natural axiomatization for our index, much in the spirit of Dubey's (1975) axiomatization of the SSPI. In addition, we prove a multilinear extension formula for our index, which generalizes Owen's (1972) formula for the Shapley value in the special case of simple games.

Finally, we use these results to re-prove an interesting property of the SSPI, first found by Myerson (1977, 1980). This is the "balanced contributions" property. Take any simple game G, and consider any two players i and j. Then, *the change in i's SSPI caused by absenting player j from G is exactly the same as the change in j's SSPI caused by absenting player i from G.* Myerson proved this result by using a

[1] Of course we freely admit that a truly realistic model of real world quorums would be much more complex than this. But we feel this is a start!

symmetry argrument, never giving a formula for the magnitude of the changes in i and j's SSPI. Our proof does give such a formula, in terms of the coefficients of the multilinear extension of the game.

Put another way, balanced contributions means that i's valuation of j's presence in the game is exactly j's valuation of i's presence. If the two players' valuation of each other's presence is positive, we call them **complements**; if negative they are **substitutes**. We close the paper by proving some theorems linking the concepts of substitutes/complements with the more familiar concepts of master player, veto player, and dummy player from the theory of simple games.

1 Games, Simple Games, and Multilinear Extensions

A **transferable utility (TU) game** is a pair $G = (N, V)$ in which $N = \{1, ..., n\}$ is the **player set** and $V : 2^N \to \mathbb{R}$ is the **characteristic function**. Here 2^N denotes the set of **coalitions**, i.e., the set of subsets of N. $V(S)$ represents the "worth" of coalition S, i.e., the maximum total surplus the members of S could generate for themselves no matter the players of S^c do. [We will use the notation "S^c" to mean the complement of coalition S. The notation S/i means the set of players in S except for i.]

A **simple game** is a TU game for which
1) The range of V is contained in the two element set $\{0, 1\}$,
2) $V(S) \leq V(T)$ whenever $S \subseteq T$ (monotonicity), and
3) $V(\emptyset) = 0$.

Note that we do *not* require that $V(N) = 1$. Hence it is possible for $V(S) = 0$ for all $S \subseteq N$ — we call such a game the **zero game** and denote it by Z^n. Let $\mathcal{G}^n$ be the set of all simple games with n players. Also, let $\tilde{\mathcal{G}}^n = \{G \in \mathcal{G}^n : G \neq Z^n\}$.

Suppose $G = (N, V) \in \mathcal{G}^n$. A coalition S is **winning** if $V(S) = 1$; otherwise (if $V(S) = 0$) it is **losing**. Let W be the set of winning coalitions. A member of W is a **minimal winning coalition** if each of its proper subsets is losing. Let MWC be the set of minimal winning coalitions. The reader will note that any element of $\mathcal{G}^n$ can be defined by either W or MWC in lieu of its characteristic function V. Hence we use the notation "$G = (N, W)$" or "$G = (N, MWC)$" without ambiguity.

Fix $G \in \mathcal{G}^n$. A player i is a **dummy player** (in G) if $V(S \cup \{i\}) = V(S)$ for all $S \in 2^N$; equivalently, i is a dummy player if he is a member of no minimal winning coalition. If $V(\{i\}) = 1$ we call i a **master player**, while if $i \notin S \Rightarrow V(S) = 0$ we call i a **veto player**. Finally, a **dictator** is any player who is simultaneously a master player and a veto player.

Suppose G_1 and G_2 are elements of $\mathcal{G}^n$, with characteristic functions V_1 and V_2 respectively. Then the **join** of G_1 and G_2, denoted $G_1 \vee G_2$, is the element of $\mathcal{G}^n$ with characteristic function $\max(V_1, V_2)$. Dually, the **meet** of G_1 and G_2, denoted $G_1 \wedge G_2$, is the element of $\mathcal{G}^n$ with characteristic function $\min(V_1, V_2)$. Finally, if $G \in \mathcal{G}^n$ and π is a permutation of N, we define $\pi G \in \mathcal{G}^n$ as $(N, \pi V)$, where $\pi V(S) = V(\pi^{-1}(S))\ \forall S \in 2^N$.

Suppose $G = (N, V)$ is a TU game. An **ordering** is any permutation of the players in N. Given an ordering R and a player i, the **predecessors of** i, denoted $P_R(i)$, is defined as the set of players who precede i in R. The **Shapley value** $\psi(G)$ (Shapley, 1953) is the n-vector defined by

$$\psi_i(G) = \frac{1}{n!} \times \sum_{\text{orderings } R} [V(P_R(i) \cup \{i\}) - V(P_R(i))].$$

Hence the Shapley value measures the average marginal worth of the players over all possible orderings of the players.

Again suppose $G = (N, V)$ is a TU game. The **multilinear extension** of G (Owen 1972) is the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x_1, ..., x_n) = \sum_{S \in 2^N} \prod_{i \in S} x_i \prod_{i \in S^c} (1 - x_i)\ V(S)$. Owen proved the following formula, which often simplifies the calculation of the Shapley value:

$$\psi_i(G) = \int_0^1 \frac{\partial f}{\partial x_i}(t, t, ..., t) dt. \tag{1.1}$$

In words: The Shapley value is the integral of the partial derivative of the multilinear extension, taken along the line from the origin to $(1, ..., 1)$.

In the special case of simple games, the Shapley value is called the **Shapley–Shubik power index** (SSPI). In this setting, $\psi_i(G)$ measures the power of player i within the voting system where the set of coalitions able to enforce their will is precisely W. To get a formula for $\psi(G)$, we consider two cases. First, if G is the zero-game, then the quantity $V(P_R(i) \cup \{i\}) - V(P_R(i))$ will be zero for all R and i, and so $\psi(G) = (0, ..., 0)$. This is an important case in what follows, because it says that if no winning coalitions can form then there is no payoff for anyone in terms of power.

Second, if G is not the zero-game, consider any ordering R. Then there will be precisely one player, called the **swing player** of R, for whom $V(P_R(i) \cup \{i\}) - V(P_R(i)) = 1$; for all other players this quantity is 0. The SSPI for player i ($i = 1, ..., n$) is simply the probability that player i is a swing player, assuming that all orderings are equiprobable. Mathematically, we have

$$\psi_i(G) = \frac{1}{n!} \sum_{S \in W : S/i \notin W} (|S| - 1)!(n - |S|)!.$$

Dubey (1975) axiomatized the SSPI, via the theorem below:

Theorem 1.1: *The function $h : \tilde{\mathcal{G}}^n \to \mathbb{R}^n$ satisfies the following axioms:*

1) *If i is a dummy player in G, then $h_i(G) = 0$.* [Dummy]
2) $\sum_{i=1}^n h_i(G) = 1$ *for any $G \in \tilde{\mathcal{G}}^n$.* [Efficiency]
3) *If π is a permutation of N and $i \in N$, then $h_{\pi(i)}(\pi G) = h_i(G)$.* [Symmetry]
4) *For any $G_1, G_2 \in \tilde{\mathcal{G}}^n$,*

$$h(G_1) + h(G_2) = h(G_1 \vee G_2) + h(G_1 \wedge G_2). \quad \text{[Additivity]} \tag{1.2}$$

if and only if $h(G) = \psi(G)\ \forall G \in \tilde{\mathcal{G}}^n$.

Remark 1.2: One may easily extend Dubey's result to the case where the domain of h is $\mathcal{G}^n$, not $\tilde{\mathcal{G}}^n$. Besides changing all of the "$\tilde{\mathcal{G}}$"s to "$\mathcal{G}$"s, the only change in the statement of the theorem is to replace the "1" with "$V(N)$" in the Efficiency Axiom.

For the multilinear extension in the case of simple games, we have

$$f(x_1, ..., x_n) = \sum_{S \in W} \prod_{j \in S} x_j \prod_{j \in S^c} (1 - x_j),$$

so

$$\begin{aligned}
\frac{\partial f}{\partial x_i} &= \sum_{S \in W : S \ni i} \prod_{j \in S : j \neq i} x_j \prod_{j \in S^c} (1 - x_j) - \sum_{S \in W : S \not\ni i} \prod_{j \in S} x_j \prod_{j \in S^c : j \neq i} (1 - x_j) \\
&= \sum_{\substack{S \in W : S \ni i \\ \text{and } S/i \notin W}} \prod_{j \in S : j \neq i} x_j \prod_{j \in S^c} (1 - x_j) + \sum_{\substack{S \in W : S \ni i \\ \text{and } S/i \in W}} \prod_{j \in S : j \neq i} x_j \prod_{j \in S^c} (1 - x_j) \\
&\quad - \sum_{S \in W : S \not\ni i} \prod_{j \in S} x_j \prod_{j \in S^c : j \neq i} (1 - x_j).
\end{aligned}$$

The last two terms cancel out here, so we have

$$\frac{\partial f}{\partial x_i} = \sum_{\substack{S \in W : S \ni i \\ \text{and } S/i \notin W}} \prod_{j \in S : j \neq i} x_j \prod_{j \in S^c} (1 - x_j). \tag{1.3}$$

2 The Model of Absenteeism and the Absentee Index

We now assume that each player has an independent random probability of being absent from the game. Formally, we define r_i to be the probability that i is "present" in the game (and so $1 - r_i$ is the probability that he is "absent"). A **voting-with-absenteeism** game is thus defined by the quantities $(G; r)$, where $G \in \mathcal{G}^n$ is called the "underlying simple game" and r is an n-vector of probabilities. The class of all n-player voting-with-absenteeism games is denoted by $\mathcal{A}^n$.

Our goal is to define a power index for these games. To this end, suppose we are given a voting-with-absenteeism game $(G; r)$, in which $G = (N, MWC)$. For $S \in 2^N$, let $p_S = \prod_{i \in S} r_i \prod_{i \in S^c} (1 - r_i)$. Thus p_S is the probability that the set of voters present is precisely S. Next, define $G^S = (N, MWC^S)$, where $MWC^S = \{T \in MWC : T \subseteq S\}$. G^S is the game that occurs if the set of voters present is S; the only minimal winning coalitions are those from the underlying game which are completely composed of members of S. Note that if S contains no minimal winning coalitions, then G^S is the zero-game.

Proposition 2.1: Suppose $G_1 = (N, MWC_1)$, $G_2 = (N, MWC_2)$, and $S \in 2^N$. Then a) $(G_1 \vee G_2)^S = G_1^S \vee G_2^S$ and b) $(G_1 \wedge G_2)^S = G_1^S \wedge G_2^S$.

Proof: Before we start, let us define the notation $W(G)$ to mean the set of winning coalitions of G, and 2^S to be the set of subsets of S.

To prove a), we aim to show that $W((G_1 \vee G_2)^S) = W(G_1^S \vee G_2^S)$. First suppose $T \in W(G_1^S \vee G_2^S)$. Without loss of generality $T \in W(G_1^S)$. This means that $\exists X \subseteq T : X \in MWC_1 \cap 2^S$. But $X \in MWC_1$ implies X is winning in $(G_1 \vee G_2)$, and $X \in 2^S$ further implies that X is winning in $(G_1 \vee G_2)^S$. Hence $T \in W((G_1 \vee G_2)^S)$. Now suppose $T \in W(G_1 \vee G_2)^S$. This implies $\exists X \subseteq T : X \in W(G_1 \vee G_2) \cap 2^S$. But this implies $X \in W(G_1) \cap 2^S$ or $X \in W(G_2) \cap 2^S$, which implies $X \in W(G_1^S)$ or $X \in W(G_2^S)$. Thus $X \in W(G_1^S \vee G_2^S)$, which gives $T \in W(G_1^S \vee G_2^S)$.

We prove b) similarly. First suppose $T \in W(G_1^S \wedge G_2^S)$. This means $\exists X, Y \subseteq T : X \in W(G_1) \cap 2^S$ and $Y \in W(G_2) \cap 2^S$. But then $X \cup Y$ is

1) winning in G_1 and in G_2, hence winning in $G_1 \wedge G_2$;

2) an element of 2^S; and

3) a subset of T.

Hence $T \in W((G_1 \wedge G_2)^S)$.

Finally suppose $T \in W((G_1 \wedge G_2)^S)$. This means that $\exists X \subseteq T : X \in 2^S$ and is a minimal winning coalition of $G_1 \wedge G_2$. But because $X \in W(G_1) \cap 2^S$ we have $T \in W(G_1^S)$. And, since $X \in W(G_2) \cap 2^S$ we have $T \in W(G_2^S)$. Hence $T \in W(G_1^S \wedge G_2^S)$, proving b). ■

Finally, suppose ψ represents the SSPI operator. We define the **absentee index** ϕ by

$$\phi(G; r) = \sum_{S \in 2^N} p_S \psi(G^S).$$

Hence the absentee index is simply the weighted average of SSPI's over all possible G^S's, where the weights are the probabilities that the set of players present is S.[2]

Example 2.2: Suppose $N = \{1, 2, 3\}$, $MWC = \{\{1\}, \{23\}\}$, and $r = (\frac{1}{2}, 1, \frac{1}{4})$. Then

$$\begin{aligned}\phi(G; r) &= p_\emptyset \psi(G^\emptyset) + p_{\{1\}} \psi(G^{\{1\}}) + p_{\{2\}} \psi(G^{\{2\}}) + p_{\{3\}} \psi(G^{\{3\}}) \\ &\quad + p_{\{12\}} \psi(G^{\{12\}}) + p_{\{13\}} \psi(G^{\{13\}}) + p_{\{23\}} \psi(G^{\{23\}}) + p_N \psi(G) \\ &= 0 * (0,0,0) + 0 * (1,0,0) + \frac{3}{8} * (0,0,0) + 0 * (0,0,0) \\ &\quad + \frac{3}{8} * (1,0,0) + 0 * (1,0,0) + \frac{1}{8} * \left(0, \frac{1}{2}, \frac{1}{2}\right) + \frac{1}{8} * \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right) \\ &= \left(\frac{11}{24}, \frac{1}{12}, \frac{1}{12}\right).\end{aligned}$$

[2] Straffin (1988) provides an interesting "probability model" interpretation of the SSPI. Every player is assumed to favor a bill with probability p and to be opposed with probability $1 - p$, where the value of p is drawn from the uniform distribution on $[0, 1]$. Then ψ_i is exactly the probability that i's vote (either for or against) will be pivotal. Under this framework one may see that our absentee index measures the exact same probability, with the proviso that absent players are assumed to be "opposed" with Probability 1.

Remark 2.3: Note that the sum of the payoffs from ϕ is *not* 1. Instead it is 5/8, which happens to be the probability that either the minimal winning coalition $\{1\}$ or the minimal winning coalition $\{23\}$ is present.

Remark 2.4: The reader might wonder why ϕ assigns equal amounts of power to players 2 and 3, because $r_2 > r_3$. However, upon further thought we realize that Player 2 is a dummy unless Player 3 is present, and Player 3 is a dummy unless Player 2 is present. Hence both players' ability to exert positive power is dependent on *both* being present, so their power is the same.

Remark 2.5: The reader will note that voting-with-absenteeism game $(G; (1, ..., 1))$ is equivalent to the original game G. We have $\phi(G; (1, ..., 1)) = \psi(G)$. For the rest of the paper, we abbreviate the game $(G; (1, ..., 1))$ by $(G; 1)$. In addition the notation $(G; 1_i)$ $(i = 1, ..., n)$ will refer to the game in which $r_i = 0$ but $r_j = 1$ for all $j \neq i$.

3 An Axiomitization

Now consider the following axioms, which we feel are natural for any solution concept $h : \mathcal{A}^n \to \Re^n$.

1) NONNEGATIVITY: For any $(G; r) \in \mathcal{A}^n$, $h(G; r) \geq 0$.
2) DUMMY: If i is a dummy player in G, then $h_i(G; r) = 0 \ \forall r$.
3) EFFICIENCY: $\sum_i^n h_i(G; r) = \text{Prob}_r$(at least one minimal winning coalition has all members present).
4) SYMMETRY: If π is a permutation of N and $i \in N$, then $h_{\pi(i)}(\pi G; \pi r) = h_i(G; r)$. [$\pi r \in \Re^n$ is defined by $(\pi r)_i = r_{\pi^{-1}(i)}$.]
5) ADDITIVITY: For any r, $G_1 \in \mathcal{G}^n$, $G_2 \in \mathcal{G}^n$, $h(G_1; r) + h(G_2; r) = h(G_1 \vee G_2; r) + h(G_1 \wedge G_2; r)$.
6) LINEARITY: For any $i, j \in N$, $h_i(G; r_1, ..., r_n)$ is a linear (affine) function of r_j.

The nonnegativity axiom is simply a very weak form of individual rationality, while the dummy axiom says that players who don't add worth to any coalition should get payoff zero. The symmetry axiom says that a player's payoff does not depend on his identity, only upon his role in the game. In particular, if players i and j are "role-substitutes" (i.e., $r_i = r_j$ and $V(S \cup \{i\}) = V(S \cup \{j\}) \ \forall S \in N/i, j$), then i and j should receive the same payoff.

The efficiency axiom should be contrasted with that for the SSPI in simple games without absenteeism. In those games the sum of players' payoffs is to be 1. But here it is also possible for absenteeism to cause there to be no minimal winning coalitions of players present. If this happens, everybody loses, because in the resulting zero-game everyone receives zero. Hence the sum of payoffs for the players should be equal to the probability that at least one minimal winning coalition of players is present. In a sense we are modelling the possibility that absenteeism can result in a quorum not being present, with deleterious effects on everyone in terms of power.[3,4]

Finally, the additivity axiom is the natural extension of the Dubey additivity axiom (1.2), while the linearity axiom requires the simplest functional form for our index.

Theorem 3.1: *The unique function h which satisfies axioms* 1)–6) *is* ϕ.

Proof: First we show that ϕ does indeed satisfy 1)–6). Nonnegativity clearly follows from the monotonicity of V. Next, if i is a dummy player in G, then i is a dummy player in G^S for all S. Hence $\psi_i(G^S) = 0$ for all S, which gives $\phi_i(G;r) = 0$. For Axiom 3), let us define $M_1 = \{S \in 2^N : \exists T \in MWC \text{ with } T \subseteq S\}$ and $M_0 = \{S \in 2^N : \nexists T \in MWC \text{ with } T \subseteq S\}$. Then

$$\phi(G;r) = \sum_{S \in M_0} p_S \psi(G^S) + \sum_{S \in M_1} p_S \psi(G^S).$$

Now note that $\psi(G^S) = (0, ..., 0)$ for all $S \in M_0$, and that $\sum_{i=1}^n \psi_i(G^S) = 1$ for all $S \in M_1$. Thus $\sum_{i=1}^n \phi(G;r) = \sum_{i=1}^n \sum_{S \in M_1} p_S \psi_i(G^S) = \sum_{S \in M_1} p_S$, which is precisely the probability that a minimal winning coalition forms.

The symmetry axiom follows simply because the process of forming $\phi(G;r)$ (see Section 2) is "anonymous," i.e., depends only upon the players' roles in the game and not upon their identities.

For the additivity axiom, suppose $G_1, G_2 \in \mathcal{G}^n$. We have

$$\begin{aligned}
\phi(G_1;r) + \phi(G_2;r) &= \sum_{S \in 2^N} p_S \psi(G_1^S) + \sum_{S \in 2^N} p_S \psi(G_2^S) = \sum_{S \in 2^N} p_S (\psi(G_1^S) + \psi(G_2^S)) \\
&= \sum_{S \in 2^N} p_S (\psi(G_1^S \vee G_2^S) + \psi(G_1^S \wedge G_2^S)) \qquad \text{(by (1.2))} \\
&= \sum_{S \in 2^N} p_S (\psi((G_1 \vee G_2)^S) + \psi((G_1 \wedge G_2)^S)) \qquad \text{(Prop 2.1)} \\
&= \sum_{S \in 2^N} p_S \psi((G_1 \vee G_2)^S) + \sum_{S \in 2^N} p_S \psi((G_1 \wedge G_2)^S) \\
&= \phi(G_1 \vee G_2; r) + \phi(G_1 \wedge G_2; r).
\end{aligned}$$

[3] Hence we are assuming that if an issue fails because a quorum of players is not present, it is not possible to immediately go and recruit the players necessary to make a quorum.

[4] Another way to look at the efficiency axiom here is if we view the legislature as meeting repeatedly. In this case $h_i(G)$ represents i's "average payoff" from a meeting. Then the axiom says that the sum of the players' average payoffs should be equal to the proportion of the time that a quorum is present.

Finally, for linearity, we note that for any i, we have

$$\phi_i(G;r) = \sum_{S\in 2^N} p_S \psi_i(G^S) = \sum_{S\in 2^N} \prod_{k\in S} r_k \prod_{k\in S^c} (1-r_k)\ \psi_i(G^S).$$

The terms $\psi_i(G^S)$ don't have any r_j's in them — so the expression for $\phi_i(G;r)$ is linear in any one particular r_j.

We have finally showed that the function ϕ satisfies axioms 1)–6). So, to prove the theorem all we need to do is show that any h satisfying axioms 1) to 6) is unique.

First, we note that in the zero-game all players are dummies. Hence, the assumption that h satisfies the dummy axiom implies $h(Z^n;r) = (0,...,0)$ for any r. Thus $h(G;r)$ is uniquely determined in the case where $G = Z^n$.

So now our task is to show $h(G;r)$ is uniquely determined for any $G \in \tilde{\mathcal{G}}^n$ and any r. To do this, define the game $G_T \in \tilde{\mathcal{G}}^n$ for $T \subseteq N$ as the game in which the only minimal winning coalition is T.

Lemma 3.2: If h satisfies Axioms 1)–6), $h(G_T;r)$ is uniquely defined for any T and r.

Proof: Without loss of generality, suppose $T = \{1,...,t\}$. We start with a simple case.

Proposition 3.3: If $r_i = 0$ for at least one $i \in T$, then $h(G_T;r) = (0,...,0)$.

Proof: If $r_i = 0$ for some $i \in T$, then the probability that a minimal winning coalition will form is zero. Hence the Efficiency Axiom gives $\sum_{i=1}^n h_i(G_T;r) = 0$, and then applying Nonnegativity gives $h(G_T;r) = (0,...,0)$. ■

Now consider $h(G_T;(1,...,1,r_{t+1},...,r_n))$. We know

a) $h_i(G_T;(1,...,1,r_{t+1},...,r_n)) = 0$ if $i > t$, because of the dummy axiom;

b) $\sum_{i=1}^n h_i(G_T;(1,...,1,r_{t+1},...,r_n)) = 1$ because of the efficiency axiom; and

c) $h_i(G_T;(1,...,1,r_{t+1},...,r_n)) = h_j(G_T;(1,...,1,r_{t+1},...,r_n))$ for $i,j \le t$ because i and j are "role substitutes" (symmetry axiom).

Together, a), b), and c) imply that

$$h_i(G_T;(1,...,1,r_{t+1},...,r_n)) = \begin{cases} \frac{1}{t} & \text{if } i \in T \\ 0 & \text{if } i \notin T \end{cases}.$$

We also know from Proposition 3.3 that

$$h(G_T;(0,1,...,1,r_{t+1},...,r_n)) = (0,...,0);$$

hence for any r_1 the linearity axiom (applied separately for each $i \in T$) gives us

$$h_i(G_T;(r_1,1,...,1,r_{t+1},...,r_n)) = \begin{cases} \frac{1}{t}r_1 & \text{if } i \in T \\ 0 & \text{if } i \notin T \end{cases}.$$

Now again by Proposition 3.3, we have

$$h(G_T;(r_1,0,1,...,1,r_{t+1},...,r_n)) = (0,...,0).$$

So we may again apply linearity to get

$$h_i(G_T;(r_1,r_2,1,...,1,r_{t+1},...,r_n)) = \begin{cases} \frac{1}{t}r_1 r_2 & \text{if } i \in T \\ 0 & \text{if } i \notin T \end{cases}.$$

Continuing in this way, we finally arrive at, for any r,

$$h_i(G_T;(r_1,...,r_t,r_{t+1},...,r_n)) = \begin{cases} \frac{1}{t}r_1 r_2,...,r_n & \text{if } i \in T \\ 0 & \text{if } i \notin T \end{cases}.$$

This proves Lemma 3.2. ■

Finally, using this last equation and the Additivity axiom repeatedly, we may determine $h(G;r)$ for *any* $G \in \tilde{\mathcal{G}}^n$, much in the manner of Dubey (1975) or Dubey-Shapley (1979). ■ ■

4 A Formula for ϕ

It is clear from our description that $\phi_i(G;r)$ is just the probability that i is a swing player, given that a) a random order of the players is chosen and, independently, b) the predecessors of i are those players before i in the ordering who are present (i.e., not "absent"). Hence we have the following:

$$\begin{aligned}
\phi_i(G;r) &= \sum_{\substack{S\subseteq W:S\ni i \\ \text{and } S/i\notin W}} P\left(\begin{array}{c} i \text{ is present, and in a random ordering} \\ \text{the set of present players who precede } i \text{ is } S/i \end{array}\right) \\
&= \sum_{\substack{S\subseteq W:S\ni i \\ \text{and } S/i\notin W}} \sum_{T\subseteq S^c} P\left(\begin{array}{c} \text{The only players present are those in } S\cup T\text{, and, in} \\ \text{a random ordering of those players, those in } S/i \\ \text{come first, then } i\text{, and then the players in } T \end{array}\right) \\
&= \sum_{\substack{S\subseteq W:S\ni i \\ \text{and } S/i\notin W}} \sum_{T\subseteq S^c} \prod_{j\in S\cup T} r_j \left(\prod_{j\in(S\cup T)^c}(1-r_j)\right)\frac{(|S|-1)!|T|!}{(|S|+|T|)!} \\
&= \sum_{\substack{S\subseteq W:S\ni i \\ \text{and } S/i\notin W}} \sum_{T\subseteq S^c} \prod_{j\in S\cup T} r_j \left(\sum_{Q\subseteq(S\cup T)^c}(-1)^{|Q|}\prod_{j\in Q} r_j\right)\frac{(|S|-1)!|T|!}{(|S|+|T|)!}.
\end{aligned}$$

We wish to write the above sum in the form $\sum_S \sum_{X\subseteq S^c} a_X \prod_{j\in S\cup X} r_j$, for some coefficients $\{a_X\}$. To do this, for each X we need to collect all of the "$\prod_{j\in S\cup X} r_j$-terms". There will be $2^{|X|}$ such terms, one for each $T \subseteq X$. For each such T, Q will be equal to X/T, and so a_X will be equal to

$$\sum_{T\subseteq X} (-1)^{|X/T|} \frac{(|S|-1)!|T|!}{(|S|+|T|)!}.$$

Hence,

$$\phi_i(G;r) = \sum_{\substack{S\subseteq W: S\ni i \\ \text{and } S/i\notin W}} \sum_{X\subseteq S^c} \sum_{T\subseteq X} (-1)^{|X/T|} \frac{(|S|-1)!|T|!}{(|S|+|T|)!} \prod_{j\in S\cup X} r_j.$$

Next, for each $T \subseteq X$ with $|T| = t$ ($t = 0, ..., |X|$), the number of such T's is $\binom{|X|}{t}$; hence we can rewrite the sum as

$$\begin{aligned}
\phi_i(G;r) &= \sum_{\substack{S\subseteq W: S\ni i \\ \text{and } S/i\notin W}} \sum_{X\subseteq S^c} \sum_{t=0}^{|X|} \frac{|X|!}{t!(|X|-t)!} (-1)^{|X|-t} \frac{(|S|-1)!t!}{(|S|+t)!} \prod_{j\in S\cup X} r_j \\
&= \sum_{\substack{S\subseteq W: S\ni i \\ \text{and } S/i\notin W}} \sum_{X\subseteq S^c} \sum_{m=0}^{|X|} (-1)^m \frac{|X|!(|S|-1)!}{m!(|S|+|X|-m)!} \prod_{j\in S\cup X} r_j \\
&= \sum_{\substack{S\subseteq W: S\ni i \\ \text{and } S/i\notin W}} \sum_{X\subseteq S^c} \frac{|X|!(|S|-1)!}{(|S|+|X|)!} \sum_{m=0}^{|X|} (-1)^m \binom{|S|+|X|}{m} \prod_{j\in S\cup X} r_j.
\end{aligned}$$

We can now use a well-known combinatorics identity (see e.g. Lovasz (1993), problem 42h) to evaluate $\sum_{m=0}^{|X|} (-1)^m \binom{|S|+|X|}{m}$; the result is

$$\begin{aligned}
\phi_i(G;r) &= \sum_{\substack{S\subseteq W: S\ni i \\ \text{and } S/i\notin W}} \sum_{X\subseteq S^c} \frac{|X|!(|S|-1)!}{(|S|+|X|)!} (-1)^{|X|} \frac{(|S|+|X|-1)!}{|X|!(|S|-1)!} \prod_{j\in S\cup X} r_j \\
&= \sum_{\substack{S\subseteq W: S\ni i \\ \text{and } S/i\notin W}} \sum_{X\subseteq S^c} (-1)^{|X|} \frac{1}{|S|+|X|} \prod_{j\in S\cup X} r_j. \qquad (4.1)
\end{aligned}$$

Example 4.1: Let us calculate $\phi_1(G;r)$ using formula (4.1), in the case where G is the game from Example 2.1 ($N = \{1, 2, 3\}$ and $MWC = \{\{1\}, \{23\}\}$).

Note that in this case $\{S \subseteq W : S \ni 1 \text{ and } S/1 \not\in W\} = \{\{1\}, \{12\}, \{13\}\}$, and so

$$\begin{aligned}
\phi_1(G;r) &= (-1)^0 \frac{1}{1+0} r_1 + (-1)^1 \frac{1}{1+1} r_1 r_2 + (-1)^1 \frac{1}{1+1} r_1 r_3 + (-1)^2 \frac{1}{1+2} r_1 r_2 r_3 \\
&\quad + (-1)^0 \frac{1}{2+0} r_1 r_2 + (-1)^1 \frac{1}{2+1} r_1 r_2 r_3
\end{aligned}$$

$$
\begin{aligned}
& + (-1)^0 \frac{1}{2+0} r_1 r_3 + (-1)^1 \frac{1}{2+1} r_1 r_2 r_3 \\
= \quad & r_1 - \frac{1}{3} r_1 r_2 r_3.
\end{aligned}
$$

If $r = (1/2, 1, 1/4)$ this is equal to $11/24$, which agrees with our answer from Example 2.1.

5 A Multilinear Extension Formula for ϕ

The formula (4.1) is really not meant to be a "stand-alone" result[5] — instead its purpose is to be used to prove a generalization of Owen's formula (1.1).

Theorem 5.1: *Let $(G; r)$ be a voting-with-absenteeism game, and let f be the multilinear extension of the simple game G. Then, for any i, we have*

$$\phi_i(G; r) = \int_0^1 \frac{\partial f}{\partial x_i}(r_1 t, ..., r_n t) r_i dt$$

In words: The absentee index (with parameter r) is the integral of the partial derivative of the multilinear extension, taken along the line from the origin to r.

Remark 5.2: It is clear that in the case where $r = (1, ..., 1)$ we get Owen's result (1.1).

Remark 5.3: Again let us revisit Example 2.1/4.1 (where $N = \{1,2,3\}$ and $MWC = \{\{1\},\{23\}\}$). The multilinear extension is $f(x_1, x_2, x_3) = x_1(1-x_2)(1-x_3) + x_1 x_2 (1-x_3) + x_1 x_3 (1-x_2) + x_2 x_3 (1-x_1) + x_1 x_2 x_3 = x_1 + x_2 x_3 - x_1 x_2 x_3$. So $\frac{\partial f}{\partial x_1} = 1 - x_2 x_3$, and $\frac{\partial f}{\partial x_1}(r_1 t, r_2 t, r_3 t) = 1 - r_2 r_3 t^2$. This in turn gives $\int_0^1 \frac{\partial f}{\partial x_1}(r_1 t, r_2 t, r_3 t) r_1 dt = \int_0^1 r_1 (1 - r_2 r_3 t^2) dt = r_1 - \frac{1}{3} r_1 r_2 r_3$. This agrees with the formula for $\phi_1(G; r)$ that we found in Example 4.1.

Proof of Theorem 5.1: We start by recalling formula (1.3), namely

$$\frac{\partial f}{\partial x_i} = \sum_{\substack{S \in W : S \ni i \\ \text{and } S/i \notin W}} \prod_{j \in S : j \neq i} x_j \prod_{j \in S^c} (1 - x_j).$$

At $x_i = r_i t$ $(i = 1, ..., n)$, this is

$$\frac{\partial f}{\partial x_i}(r_1 t, ..., r_n t) \quad = \quad \sum_{\substack{S \in W : S \ni i \\ \text{and } S/i \notin W}} \left(\prod_{j \in S/i} r_j \right) t^{|S|-1} \prod_{j \in S^c} (1 - r_j t)$$

[5] We are not aware of any discovery of formula (4.1) in the literature, even for the case where $r = (1, ..., 1)$.

$$= \sum_{\substack{S\in W: S\ni i \\ \text{and } S/i\notin W}} \left(\prod_{j\in S/i} r_j\right) t^{|S|-1} \left(\sum_{X\subseteq S^c} (-1)^{|X|} \left(\prod_{j\in X} r_j\right) t^{|X|}\right)$$

$$= \sum_{\substack{S\in W: S\ni i \\ \text{and } S/i\notin W}} \sum_{X\subseteq S^c} (-1)^{|X|} \left(\prod_{j\in S\cup X/i} r_j\right) t^{|S|+|X|-1}.$$

Hence $\frac{\partial f}{\partial x_i}(r_1 t, ..., r_n t) r_i$ is equal to

$$\sum_{\substack{S\in W: S\ni i \\ \text{and } S/i\notin W}} \sum_{X\subseteq S^c} (-1)^{|X|} \left(\prod_{j\in S\cup X} r_j\right) t^{|S|+|X|-1}, \text{ and}$$

$$\begin{aligned} \int_0^1 \frac{\partial f}{\partial x_i}(r_1 t, ..., r_n t) r_i dt &= \Big|_0^1 \sum_{\substack{S\in W: S\ni i \\ \text{and } S/i\notin W}} \sum_{X\subseteq S^c} (-1)^{|X|} \left(\prod_{j\in S\cup X} r_j\right) \frac{t^{|S|+|X|}}{|S|+|X|} \\ &= \sum_{\substack{S\in W: S\ni i \\ \text{and } S/i\notin W}} \sum_{X\subseteq S^c} (-1)^{|X|} \left(\prod_{j\in S\cup X} r_j\right) \frac{1}{|S|+|X|} \\ &= \phi(G; r) \text{ from formula (4.1)} \end{aligned}$$

■

6 Substitutes and Complements in Simple Games

Consider the following simple example.[6] Let G be the three-player simple game in which the set of minimal winning coalitions is $\{\{12\}, \{13\}\}$. The SSPI for this game is $\psi(G) = \phi(G; 1) = (2/3, 1/6, 1/6)$.

Now lets consider the question of whether or not Player 1 views Player 2's presence in the game as a good thing. To do this, we note that if Player 2 were to be absent, the game would reduce to a two-player game (just Players 1 and 3) in which the sole MWC is $\{13\}$. Hence ϕ_1 would decrease, from 2/3 to 1/2. We say Player 2 **is complementary for** Player 1, because Player 2's absence is bad for Player 1 (and hence his *presence* is *good* for Player 1).

Next, lets examine if Player 1 is complementary for Player 2. If Player 1 is absent, then the game reduces to a two player game (Players 2 and 3) which is a zero-game. Hence Player 2's SSPI[7] would also decrease, from 1/6 to 0. So Player 1 is also

[6] The following discussion parallels that in Myerson (1991, Section 9.5).

[7] When we use the term "SSPI" here, we mean the SSPI in the original 3-person-game and in the resultant two-person subgame of the original game. We alternatively could have used the term "absentee index" here, meaning the absentee index of the original 3-person-game with $r = (1, 1, 1)$ and with $r = (0, 1, 1)$.

complementary for Player 2. Since 1 is complementary for 2 and 2 is complementary for 1, we say simply that Players 1 and 2 are **complements**.

Next, we can perform the same analysis for Players 2 and 3. If Player 3 were absent, Player 2's SSPI would *increase*, from 1/6 to 1/2. Likewise, if Player 2 were absent, ϕ_3 would increase from $\frac{1}{6}$ to $\frac{1}{2}$. Hence Players 2 and 3 are **substitutes**, because the presence of one of these players is bad for the other.[8]

It turns out that if one player is complementary (resp. substitutionary) for another, then the second player is also complementary (resp. substitutionary) for the first player. Hence every pair of players are either a) substitutes; b) complements; or c) neutrals. [Two players are **neutrals** if the presence or absence of one of them does not affect the SSPI for the other.]

In fact, one may show more than this. In the above example, we see that Players 1 and 2's SSPI both decrease by 1/6 if the other doesn't show up, while Players 2 and 3's SSPI both increase by 1/3. Indeed, the "balanced contributions" property of Myerson (1977, 1980) says that *for any pair of players i and j, the change in ψ_i caused by j's absence is exactly the same as the change in ψ_j caused by i's absence.* Hence we can order the pairwise relationships in the game, from "strongest complements" to "strongest substitutes." This suggests a natural heirarchy of partnership and opposition within a legislature, based solely on the voting structure of the game.[9]

We can state and prove Myerson's property by using the absenteeism terminology developed earlier:

Theorem 6.1: *Let G be a simple game, and let i and j be any two players in the game. Suppose further that the multilinear extension of G is given by $f(x_1, ..., x_n) = \sum_{S \in 2^N} a_S \prod_{k \in S} x_k$, for some constants $\{a_S\}$. Then*

$$\phi_i(G;1) - \phi_i(G;1_j) = \phi_j(G;1) - \phi_j(G;1_i) = - \sum_{S \in 2^N : S \ni i,j} a_S \frac{1}{|S|}. \qquad (6.1)$$

[Note: for the notation $(G;1)$ and $(G;1_k)$, see Remark 2.5.]

Remark 6.2: The first equality in Theorem 6.1 is just Myerson's property. The second equality, which gives the "amount" of the balanced contributions, is new. We comment here that a similar formula, quantifying the balanced contributions for the Banzhaf index, is proved in Quint (2003).

Proof: Since ϕ_i is a linear function of r_j, the left hand side of (6.1) is equal to $(0-1) * \frac{\partial \phi_i}{\partial r_j}(1, ..., 1)$. Similarly, since ϕ_j is a linear function of r_i, the center of (6.1)

[8] This meaning of "substitutes" is taken from the theory of supply and demand, where two goods are called substitutes if the presence of one lowers the demand for the other. Also, in the theory of two-sided matching, Shapley (1962) called two players "substitutes" if the addition of one player caused a lowering of the core payoffs of the other.

[9] In Quint (2003), one of us presents a five-player example in which the ten player-pairs in the game are quantitatively ranked from "strongest complements" to "strongest substitutes."

is $-\frac{\partial \phi_j}{\partial r_1}(1,...,1)$. Hence the theorem will be proved if we can show

$$\frac{\partial \phi_i}{\partial r_j}(1,...,1) = \frac{\partial \phi_j}{\partial r_i}(1,...,1) = \sum_{S \in 2^N : S \ni i,j} a_S \frac{1}{|S|}. \tag{6.2}$$

To show (6.2), we recall from Theorem 5.1 that $\phi_i = \int_0^1 \frac{\partial f}{\partial x_i}(r_1 t,...,r_n t) r_i dt$. Here $\frac{\partial f}{\partial x_i} = \sum_{S \in 2^N : S \ni i} a_S \prod_{k \in S/i} x_k$, so $\frac{\partial f}{\partial x_i}(r_1 t,...,r_n t) = \sum_{S \in 2^N : S \ni i} a_S (\prod_{k \in S/i} r_k) t^{|S|-1}$. This gives

$$\phi_i = \int_0^1 \sum_{S \in 2^N : S \ni i} a_S \left(\prod_{k \in S} r_k \right) t^{|S|-1} dt = \sum_{S \in 2^N : S \ni i} a_S \left(\prod_{k \in S} r_k \right) \frac{1}{|S|}.$$

>From here we obtain

$$\frac{\partial \phi_i}{\partial r_j} = \sum_{S \in 2^N : S \ni i,j} a_S \left(\prod_{k \in S/j} r_k \right) \frac{1}{|S|},$$

from which finally

$$\frac{\partial \phi_i}{\partial r_j}(1,...,1) = \sum_{S \in 2^N : S \ni i,j} a_S \frac{1}{|S|}. \tag{6.3}$$

>From the symmetric nature of the right hand side of (6.3) with regard to the variables i and j, we see that we must also have

$$\frac{\partial \phi_j}{\partial r_i}(1,...,1) = \sum_{S \in 2^N : S \ni i,j} a_S \frac{1}{|S|}. \tag{6.4}$$

But together (6.3) and (6.4) imply (6.2), and so the theorem is proven. ■

Definition 6.3: Let G be a simple game, and let i and j be any two players in the game. Then i and j are

a) **substitutes** if $\phi_i(G;1) - \phi_i(G;1_j) = \phi_j(G;1) - \phi_j(G;1_i) < 0$;
b) **complements** if $\phi_i(G;1) - \phi_i(G;1_j) = \phi_j(G;1) - \phi_j(G;1_i) > 0$; and
c) **neutrals** if $\phi_i(G;1) - \phi_i(G;1_j) = \phi_j(G;1) - \phi_j(G;1_i) = 0$.

We close this section by stating some simple results concerning substitutes and complements.

Proposition 6.4: Suppose i is a dummy player or a dictator in G, and let j be any other player. Then i and j are neutrals.

Proof: If i is a dummy player in G, his SSPI is zero, and this doesn't change if any other player becomes absent. Similarly, if i is a dictator, his SSPI is one and this doesn't change if any other player becomes absent. ■

Remark 6.5: It is *not* true that i, j neutral $\Rightarrow$ either i or j is a dummy. For, consider the game $G = (N, MWC)$ in which $N = \{1, 2, 3, 4\}$ and $MWC = \{\{12\}, \{13\}, \{24\}\}$. Then $\psi(G) = (1/3, 1/3, 1/6, 1/6)$. If Player 4 is absent we see that we have a three player game in which the minimal winning coalitions are $\{12\}$ and $\{13\}$, for which Player 3's SSPI is again 1/6. Hence Players 3 and 4 are neutrals in G, even though neither is a dummy player.

$>$From the above analysis, we see that in any game with dummy players, each dummy player is neutral with all other players. In addition, none of the complement/substitute/ neutral relationships between pairs of non-dummy players would be altered if we simply remove all dummy players from a game. Hence in what follows we consider games in which all dummy players have been removed.

Proposition 6.6: Suppose G has no dummy players. Then i is a veto player if and only if i is a complement to every other player.

Proof: First, suppose i is a veto player, and let j be any other player. Since j is not a dummy, $\phi_j(G; 1) > 0$. But if i becomes absent, we are left with a zero-game, so $\phi_j(G; 1_i) = 0$. Hence i and j are complements.

For the converse, consider any non-veto player i. Since i is not a dummy, we have $\sum_{j \neq i} \phi_j(G; 1) < 1$. Now suppose i becomes absent. The fact that i is not a veto player means the resulting game is not a zero-game; hence $\sum_{j \neq i} \phi_j(G; 1_i) = 1$. Hence there must be a player j^* for whom $\phi_{j^*}(G; 1) < \phi_{j^*}(G; 1_i)$, i.e., j^* and i are substitutes. ■

Proposition 6.7: Suppose G has no dummy players. If i is a master player in G, then i is a substitute with every other player.

Proof: Let j be any other player in the game. Consider the games $(G; 1)$ and $(G; 1_j)$, and take any ordering R. Suppose i is the swing player of R in $(G; 1)$. Since i is a master player, this is equivalent to saying that $V(P_R(i)) = 0$. But absenting player j will not change this; hence in game $(G; 1_j)$ we still have $V(P_R(i)) = 0$, and so i is again the swing player for R. This in turn implies

$$\phi_i(G, 1) \leq \phi_i(G; 1_j). \tag{6.5}$$

To get strict inequality in (6.5), we need to find at least one ordering for which i is *not* the swing player in $(G; 1)$ but *is* the swing player in $(G; 1_j)$. To do this, note that since j is not a dummy player, there must be an ordering R for which j is the swing player in G. It must be that i comes *after* j in R; otherwise master player i would be the swing player. But j's swinging doesn't not depend on the the order of players after j in R; hence there is an ordering R^i in which j swings and i comes immediately after j. This then is the ordering for which i is not the swing player in $(G; 1)$ but is the swing player in $(G; 1_j)$. Thus (6.5) holds with strict inequality, and so i and j are substitutes. ■

Remark 6.8: The converse of Proposition 6.7 is *not* necessarily true. For instance, consider the n-player simple game in which the winning coalitions are those containing m or more players $(1 < m < n)$. Then each player is a substitute with all others, yet no player is a master player.

7 An Absentee Banzhaf Index

We remark here that a similar analysis for an "absentee Banzhaf index" has been carried out. These results will be presented in another paper (Quint, 2003).

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