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By

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May 2003

COWLES FOUNDATION DISCUSSION PAPER NO. 1423

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

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Multidimensional Private Value Auctions*

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First Draft: May 2002
This Draft: May 2003

Abstract

We consider parametric examples of two-bidder private value auctions in which each bidder observes her own private valuation as well as noisy signals about her opponent's private valuation. In such multidimensional private value auction environments, we show that the revenue equivalence between the first and second price auctions breaks down and there is no definite revenue ranking; while the second price auction is always efficient allocatively, the first price auction may be inefficient and the inefficiency may increase as the signal becomes more informative; equilibria may fail to exist for the first price auction. We also show that auction mechanisms provide different incentives for bidders to acquire costly information about opponents' valuation.

Keywords: Multidimensional Auctions, Revenue Equivalence, Allocative Efficiency, Information Acquisition.

JEL Classification Numbers: C70, D44, D82.

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*We are grateful to David J. Cooper, Sergio Parreiras and Andrew Postlewaite for helpful conversations. All errors are our own.

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1 Introduction

The paradigm of symmetric independent private value (IPV) auctions assumes that each bidder's valuation of an object is independently drawn from an identical distribution (Myerson 1981, Riley and Samuelson 1981). Each bidder observes her own valuation, and has no information about her opponent's valuation except for the distribution from which it is drawn. An important implication of this assumption is that each bidder's belief about her opponent's valuation is independent of her own valuation and thus it is common knowledge.

In actual auctions, it is probable that bidders may have, or may have incentives to acquire, information about their opponents' valuations. Moreover, such information may often be noisy and privately observed. This point can be made for both private and common value auction environments. For example, timber firms in the U.S. Forest Service timber auctions can cruise the tract being auctioned and form estimates of its characteristics (Baldwin, Marshall and Richard 1997). It is possible that a timber firm can obtain some noisy information about its opponents' estimates via insider rumors and industrial espionage. In highway construction procurement auctions, the capacity constraints of the bidders are an important determinant of their costs (Jofre-Bonet and Pesendorfer 2002). Even when the actual cost of a firm, say firm A, is its private information, other firms may still obtain signals of A's cost based on their noisy observations of how much firm A's capacity is stretched. Importantly, each bidder is aware that her opponents may have some signals about her valuation or cost, but does not actually know the signals observed by her opponents.

In this paper, we assume that bidders have noisy information about opponents' valuations and explore its consequences under the first-price (FPA) and second-price (SPA) auction mechanisms. Specifically, we consider parametric examples of two-bidder private value auctions in which each bidder's private valuation of the object is independently drawn from an identical distribution, and each bidder observes a noisy signal about her opponent's valuation. Thus, each bidder has a two-dimensional type that include her own valuation (the *valuation* type) and the signal about her opponent's valuation (the *information* type). A bidder's information about her opponents' valuation is not known by her opponent. In such a multidimensional auction environment, we show the following results. First, revenue equivalence of standard one-dimensional symmetric IPV auctions breaks down. However, our examples demonstrate that there is no general revenue ranking between the FPA and the SPA. Second, the equilibrium allocation of the object could be inefficient in the FPA but is always efficient in the SPA. Moreover, the revenue and allocative efficiency may not coincide: on the one hand, an inefficient FPA may generate a higher expected revenue for the seller; on the other hand, the seller's expected revenue could be higher in the SPA even when the object is efficiently allocated in both auctions. The inefficiency in the FPA will typically be non-monotonic in the accuracy of information, since with either complete information or no information, efficiency is obtained in the first price auctions. Third, while the SPA always admits equilibrium in weakly dominant strategies, the FPA may not have any equilibrium. Finally, we illustrate that different auction mechanisms provide different incentives for bidders to acquire costly information about opponents' private valuations. We illustrate all these results in simple examples that we can solve explicitly. However, we also argue in each case why the key features of the examples should be expected to occur more generally.

Our examples add to the list of departures from the standard symmetric IPV auction environments in which the revenue equivalence between the first and second-price auctions fails. Maskin and Riley (2000a) consider private value auctions in which bidders are *ex ante* asymmetric in the sense that different bidders' valuations are drawn from different distributions. They show that the revenue ranking between the first and the second-price auctions is ambiguous even though the SPA is at least as efficient as the FPA (see also Arozamena and Cantillon 2002 and Cantillon 2003). In their model, bidders' types are one dimensional and the asymmetry among bidders is common knowledge. Holt (1980) and Matthews (1987) show that, when bidders are symmetrically risk averse, the seller's expected revenue in the FPA is higher than that in the SPA. Che and Gale (1998) compare the standard auctions with financially constrained bidders, and show that the seller's expected revenue in the FPA is higher than that in the SPA. The bidders in their paper are privately informed of both their valuation of the object and their financial capacity, and thus have multidimensional types. However, both the bidders' valuation and financial capacity are assumed to be independently drawn from identical distributions, hence the common knowledge assumption is maintained. This paper is related to the affiliated value framework of Wilson (1977) and Milgrom and Weber (1985) in which they showed that when bidders' valuations are affiliated, the seller's expected revenue is higher in the SPA than in the FPA (more generally, the so-called "linkage principle"). The main difference between the auction environment studied in this paper and the affiliated private value (APV) model is as follows: in the APV model, bidder i's belief about j's valuation depends on i's own private valuation, but does not depend on j's actual valuation; while in our model, bidder j's private valuation influences bidder i's belief about j's valuation through a noisy signal. In other words, in the APV model bidders' types are one-dimensional (i.e., a bidder's valuation type and her information type coincide), and as a result the standard auctions are all efficient in the APV model despite their revenue ranking. A closely related paper is Kim and Che (2002) which considers private value auction environments in which subgroups of bidders may perfectly observe the valuations of others within the group but have no information about bidders outside of the subgroup. They show that the seller's expected revenue is higher in the SPA than that in the FPA and the FPA is allocatively inefficient with positive probability. While their model is more general, our paper is different from theirs in assuming that bidders' signals about opponents' valuations are noisy and private. Che and Gale (2002) propose a general methodology for comparing the seller's expected revenues from different auction mechanisms in environments where bidders have multidimensional types. But they assume that bidders' types are drawn from independent distributions.

We restrict the seller to two possible mechanisms for allocating the object: first and second price auctions. With more general mechanisms, in our setting, sellers could fully extract the surplus, exploiting the correlation between bidders' multidimensional types, using the type of argument employed in Cremer and McLean (1985).[1] Such mechanisms rely on very strong common knowledge assumptions among the seller and the bidders and would not work on more realistic type spaces (see Neeman 2001 and Bergemann and Morris 2003). For this reason, we restrict attention to simple mechanisms. Our work is an attempt to make a first step at relaxing the standard (but unfortunate) assumption in auction theory of identifying players' beliefs with their payoff types.[2] An alternative way of allowing richer beliefs into standard independent private value auctions is to introduce strategic uncertainty by relaxing the solution concept from equilibrium to rationalizability. This avenue has been pursued by Battigalli and Siniscalchi (2000) and Dekel and Wolinsky (2003), for first price auctions, while maintaining the assumption of no private information about others' values.

The remainder of the paper is structured as follows. Section 2 presents the parametric auction environment we examine; Section 3 shows the revenue non-equivalence between the first and second-price auctions in our auction environment; Section 4 shows the possible inefficiency of the FPA; Section 5 shows that there may exist no equilibrium in the FPA; Section 6 provides examples that reverse the revenue ranking between the FPA and the SPA and illustrate the incentives of information acquisition under different auction mechanisms; and Section 7 concludes.

[1]The literature on general mechanisms with multidimensional types focusses on efficiency questions. Jehiel and Moldovanu (2001) show that, generically, there are no efficient auction mechanisms when bidders have independent multidimensional signals and interdependent valuations. McLean and Postlewaite (2003) study situations in which bidders' valuations consist of both common and idiosyncratic components. Bidders privately observe their idiosyncratic component of the valuation, and some signal regarding the common component. They show that a modification of the Vickrey auction is efficient under quite general conditions in their settings.

[2]See Feinberg and Skrypacz (2002) pursue the same relaxation in the context of models of bargaining under incomplete information.

2 The Model

Two bidders, $i = 1, 2$, compete for an object. Bidders' valuations of the object are private and independently drawn from identical distributions. We assume that bidders' valuation of the object takes on three possible values $\{V_l, V_m, V_h\}$ where $V_l < V_m < V_h$.[3] The *ex ante* probability of bidder i's valuation v_i taking on value V_k is denoted by $p_k \in [0, 1)$ where $k \in \{l, m, h\}$. Of course $\sum_{k \in \{l,m,h\}} p_k = 1$. To ease exposition, we will refer to bidder 1 as "she" and bidder 2 as "he", and refer to a generic bidder as "she" when no confusion shall arise.

As in standard private value auction models, bidder i observes her private valuation $v_i \in \{V_l, V_m, V_h\}$. The novel feature of this paper is as follows: we assume that each bidder also observes a noisy signal about her opponent's valuation. For tractability, we assume that the noisy signal takes on two possible qualitative categories $\{L, H\}$. Bidder i's signal $s_i \in \{L, H\}$ about j's valuation v_j is generated as follows. For $k \in \{l, m, h\}$, and $i, j \in \{1, 2\}, i \neq j$,

$$\Pr(s_i = L | v_j = V_k) = q_k, \quad \Pr(s_i = H | v_j = V_k) = 1 - q_k, \tag{1}$$

where $q_k \in [0, 1]$. We assume that $q_l \geq q_m \geq q_h$. Note that when $q_l = q_m = q_h$, the signals are completely uninformative about the opponent's valuation.[4] We assume that bidders' signals s_1 and s_2 are independent. To summarize, each bidder has a two-dimensional type $(v_i, s_i) \in \{V_l, V_m, V_h\} \times \{L, H\}$ where v_i is called bidder i's *valuation* type and s_i her *information* type.

To summarize, the primitives of our model are a tuple of nine parameters as follows:

$$\mathcal{E} = \left\{ \langle V_k, p_k, q_k \rangle_{k \in \{l,m,h\}} : V_l < V_m < V_h, p_k \in [0, 1], \sum_k p_k = 1, q_k \in [0, 1] \right\}.$$

Any element $e \in \mathcal{E}$ is called an *auction environment.*

We first compare the seller's expected revenue and the allocative efficiency of the standard auctions. Since we are in a two-person private value environment, Dutch and English auctions are strategically equivalent to the FPA and the SPA, respectively. Thus we will only analyze the FPA and the SPA: Bidders simultaneously submit bids; the high bidder wins the object. In the event of a tie, we assume that the bidder with higher valuation wins the object if the bidders' valuations are different; and the tie-breaking can be arbitrary if the bidders' valuations are the same.[5]

[3] Wang (1991) and Campbell and Levin (2000) studied common value auctions with discrete valuations.

[4] Because completely uninformative signals are the same as no signals at all, this special case corresponds to the standard one-dimensional IPV model.

[5] It is well known now that tie-breaking rules are important in guaranteeing equilibrium existence in first-price auctions. Notice that this tie-breaking rule is endogenous. Kim and Che (2002) and Maskin and Riley (2000b) used a similar assumption.

As usual, we will analyze the auctions as a Bayesian game of incomplete information between two bidders in which the type space for each bidder is $T \equiv \{V_l, V_m, V_h\} \times \{L, H\}$. Bidder i's generic type is $t_i = (v_i, s_i) \in T$. Given her information type s_i, bidder i updates her belief about j's valuation type v_j according to Bayes rule as follows. For $s_i \in \{L, H\}$, and $k \in \{l, m, h\}$,

$$\Pr(v_j = V_k | s_i = L) = \frac{p_k q_k}{\sum_{k' \in \{l,m,h\}} p_{k'} q_{k'}}, \qquad \Pr(v_j = V_k | s_i = H) = \frac{p_k (1 - q_k)}{\sum_{k' \in \{l,m,h\}} p_{k'} (1 - q_{k'})}. \quad (2)$$

Analogously, given her valuation type v_i, bidder i updates her belief about j's information type s_j according to the signal technology specified by (1). For any $(t_1, t_2) = ((v_1, s_1), (v_2, s_2)) \in T^2$, the joint probability mass is

$$\Pr(t_1, t_2) = \Pr(v_1) \Pr(s_1 | v_2) \times \Pr(v_2) \Pr(s_2 | v_1),$$

and the conditional probability is

$$\Pr(t_i | t_j) = \Pr(v_i | s_j) \Pr(s_i | v_j) \text{ where } i \neq j. \quad (3)$$

3 Seller's Expected Revenue

We first show that the celebrated revenue equivalence result for the standard one-dimensional IPV auctions breaks down in our multidimensional setting. To demonstrate this result in the simplest possible fashion, we consider a special case of the above model:

- $p_m = 0$, $p_l \in (0, 1)$, $p_h \in (0, 1)$. That is, the bidders' valuations are only of two possible types, $\{V_l, V_h\}$.
- $q_l = 1 - q_h = q \in [1/2, 1]$. That is, signal L is equally indicative of value V_l as signal H is of value V_h. The parameter q measures the accuracy of the signal: when $q = 1/2$, the signals are completely uninformative; and when $q = 1$, the signals are perfectly informative.

3.1 Second-Price Auction

In the SPA, it is routine to show that the unique equilibrium in weakly dominant strategies in this multidimensional setting is for a bidder of type (v_i, s_i) to bid her private value v_i regardless of her information type. That is, the equilibrium bidding strategy of bidder i in the SPA, denoted by B_i^{SPA}, is:

$$B_i^{\text{SPA}}(v_i, s_i) = v_i \text{ for } i = 1, 2, \text{ and } (v_i, s_i) \in \{V_l, V_h\} \times \{L, H\}. \quad (4)$$

In fact, this equilibrium characterization for the SPA is completely general to any private value auction environment and does not depend on the number of bidders, discrete valuation and signal types. We thus conclude that the multidimensional SPA is efficient; and the seller's expected revenue is independent of accuracy of the signals, hence equal to that in the standard environment where bidders only observe their own valuations.

3.2 First-Price Auction

Now we analyze the equilibrium of the first-price auction. We will proceed via some intermediate lemmas and summarize the characterization of the equilibrium of the FPA in Proposition 1.

Lemma 1 *In any equilibrium of the FPA, type-*(V_l, s) *bidders bid* V_l *in pure strategies for* $s \in \{L, H\}$. *That is, for* $i = 1, 2$,

$$B_i^{\text{FPA}}(V_l, s) = V_l \text{ for } s \in \{L, H\}.$$

Proof. We first argue that bidders with valuation V_l must bid in pure strategies in equilibrium. Suppose that type-(V_l, H) bidders plays a mixed strategy equilibrium on support $\left[\underline{b}, \bar{b}\right]$ with $\underline{b} < \bar{b}$. (The lower limit of the interval may be open, but this is not important for the argument.) Clearly $\bar{b} \leq V_l$. Since the bid $\left(\underline{b} + \bar{b}\right)/2$ wins positive probability, it yields a positive surplus for type-(V_l, H) bidder. However, bids close to $\underline{b}$ will win with probability almost zero, hence the expected surplus will approach zero. A contradiction for the indifference condition required for the mixed strategy. Hence type-(V_l, H) bidders must bid in pure strategies. Identical arguments show that type-(V_l, L) bidders must also bid in pure strategy. Now we argue that, if type-(V_l, L) and (V_l, H) bidders must bid their valuation V_l in pure strategy. To see this, suppose that type-(V_l, L) and (V_l, H) bidder 2 bids less than V_l. Then bidder 1 of these types can deviate by bidding ε more than bidder 2, which will be a profitable deviation if ε is made arbitrarily close to zero. A contradiction. ■

Next we show that the following mixed strategies for type-(V_h, L) and (V_h, H) bidders constitute a symmetric equilibrium. We will later argue that this is the unique equilibrium of the FPA.

Lemma 2 *Together with the strategies specified in Lemma 1 for bidders with valuation* V_l, *the following constitute a symmetric equilibrium:*

1. *Type-*(V_h, L) *bidders play a mixed strategy on* $\left[V_l, \bar{b}_{(V_h,L)}\right]$ *according to CDF* $G_{(V_h,L)}(\cdot)$ *given by*

$$G_{(V_h,L)}(b) = \frac{p_l q (b - V_l)}{p_h (1-q)^2 (V_h - b)}, \tag{5}$$

where

$$\bar{b}_{(V_h,L)} = \frac{p_h (1-q)^2 V_h + p_l q V_l}{p_h (1-q)^2 + p_l q}; \tag{6}$$

2. *Type-*(V_h, H) *bidders play a mixed strategy on* $\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ *according to CDF* $G_{(V_h,H)}(\cdot)$ *given by*

$$G_{(V_h,H)}(b) = \frac{(p_l + p_h q)(1-q)\left(b - \bar{b}_{(V_h,L)}\right)}{p_h q^2 (V_h - b)}, \tag{7}$$

where

$$\bar{b}_{(V_h,H)} = \frac{p_h q^2 V_h + (p_l + p_h q)(1-q)\bar{b}_{(V_h,L)}}{p_h q^2 + (p_l + p_h q)(1-q)}. \tag{8}$$

Proof. Suppose that bidder 2 bids according to the postulated strategies.

First, consider type-(V_h, L) bidder 1. Her expected payoff from submitting a bid $b \in \left[V_l, \bar{b}_{(V_h,L)}\right]$ is

$$(V_h - b)\left\{\frac{p_l q}{p_l q + p_h(1-q)} + \frac{p_h(1-q)^2}{p_l q + p_h(1-q)} G_{(V_h,L)}(b)\right\}, \tag{9}$$

where:

- the term $p_l q / \left[p_l q + p_h (1-q)\right]$ is the probability that bidder 2 has a valuation type V_l conditional on bidder 1's own information type L [recall formula (2)]. By Lemma 1, bidder 2 with valuation type V_l bids V_l with probability one. Thus bidder 1 wins with probability 1 against such an opponent with any bid in the interval $\left[V_l, \bar{b}_{(V_h,L)}\right]$ (note that the tie-breaking rule is applied at the bid V_l);

- the term $p_h (1-q)^2 / \left[p_l q + p_h (1-q)\right]$ is the probability that bidder 2 is of type (V_h, L) conditional on bidder 1's own type (V_h, L) [recall formula (3)]. Since type-(V_h, L) bidder 2 is postulated to bid in mixed strategies according to $G_{(V_h,L)}(\cdot)$, bidder 1's bid of b wins against such an opponent with probability $G_{(V_h,L)}(b)$.

Plugging $G_{(V_h,L)}(\cdot)$ as described by (5) into (9) yields a positive constant, denoted by $K_{(V_h,L)}$, given by

$$K_{(V_h,L)} = \frac{p_l q}{p_h (1-q) + p_l q}(V_h - V_l), \tag{10}$$

which is type-(V_h, L) bidder's expected surplus. Therefore type-(V_h, L) bidder 1 indeed is indifferent between any bids in the interval $\left[V_l, \bar{b}_{(V_h,L)}\right]$ provided that bidder 2 follows the postulated strategy.

Now we check that type-(V_h, L) bidder 1 does not have incentive to deviate to other bids. First, she clearly does not have incentive to deviate to bids lower than or equal to V_l, since it would have yielded her a zero surplus instead of a positive $K_{(V_h,L)}$. Now suppose that she deviates to $\bar{b}_{(V_h,L)} < b \leq \bar{b}_{(V_h,H)}$, her expected payoff would be

$$(V_h - b)\left\{\frac{p_l q}{p_l q + p_h(1-q)} + \frac{p_h(1-q)^2}{p_l q + p_h(1-q)} + \frac{p_h(1-q)q}{p_l q + p_h(1-q)} G_{(V_h,H)}(b)\right\} \tag{11}$$

where the term $p_h(1-q)q/[p_h(1-q)+p_l q]$ the probability that bidder 2 is of type (V_h, H) conditional on bidder 1's own type (V_h, L); and $G_{(V_h,H)}(b)$ is the probability that a bid $b \in \left(\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ wins against such an opponent. Plugging $G_{(V_h,H)}(\cdot)$ as described by (7) into (11), we obtain

$$\begin{aligned} & \frac{p_l q + p_h(1-q)^2}{p_l q + p_h(1-q)}(V_h - b) + \frac{p_h(1-q)q}{p_l q + p_h(1-q)} \frac{(p_l + p_h q)(1-q)\left(b - \bar{b}_{(V_h,L)}\right)}{p_h q^2} \\ = & \frac{\left\{\left[p_l q + p_h(1-q)^2\right] V_h - (p_l + p_h q)(1-q)^2 \bar{b}_{(V_h,L)}\right\} + p_l\left[(1-q)^2 - q^2\right] b/q}{p_l q + p_h(1-q)} \end{aligned}$$

which is non-increasing in b since $q \geq 1/2$. Hence type-(V_h, L) bidder 1 does not have incentive to deviate to bids in the interval $\left(\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$; which also implies that her expected payoff would be even smaller if she bids more than $\bar{b}_{(V_h,H)}$.

Now consider type-(V_h, H) bidder 1. Given that bidder 2 plays according to the postulated strategies, her expected payoff from bidding $b \in \left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ is given by

$$(V_H - b)\left\{\frac{p_l(1-q)}{p_h q + p_l(1-q)} + \frac{p_h q(1-q)}{p_h q + p_l(1-q)} + \frac{p_h q^2}{p_h q + p_l(1-q)} G_{(V_h,H)}(b)\right\}, \tag{12}$$

where:

- the term $p_l(1-q)/[p_h q + p_l(1-q)]$ is the probability that bidder 2 has valuation V_l conditional on bidder 1's signal H; and the term $[p_h q(1-q)]/[p_h q + p_l(1-q)]$ is the probability that bidder 2 is of type (V_h, L). In both events, a bid $b \in \left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ wins against such opponents with probability one under the postulated strategies by bidder 2;
- the term $p_h q^2/[p_h q + p_l(1-q)]$ is probability that bidder 2 is of type (V_h, H) conditional on bidder 1's own type (V_h, H). In this case, a bid $b \in \left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ wins with probability $G_{(V_h,H)}(b)$.

Plugging $G_{(V_h,H)}(\cdot)$ as described by (7) into (12), we obtain a positive constant, denoted by $K_{(V_h,H)}$, given by

$$K_{(V_h,H)} = \frac{(p_l + p_h q)(1-q)}{p_h q + p_l(1-q)}\left(V_h - \bar{b}_{(V_h,L)}\right). \tag{13}$$

Hence type-(V_h, H) bidder 1 is indeed indifferent between any bids in the interval $\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$.

Now we check that type-(V_h, H) bidder 1 does not have incentive to deviate to other bids. First, she does not have incentive to bid more than $\bar{b}_{(V_h,H)}$, since bidding $\bar{b}_{(V_h,H)}$ strictly dominates any higher bid given bidder $2'$s strategies; second, she does not have incentive to bid less than or equal to V_l since such bids will yield a zero surplus. Now we show that she does not have incentive to bid in the interval $\left(V_l, \bar{b}_{(V_h,L)}\right)$. Her expected payoff from a bid $b \in \left(V_l, \bar{b}_{(V_h,L)}\right)$ is given by

$$(V_h - b)\left\{\frac{p_l (1-q)}{p_h q + p_l (1-q)} + \frac{p_h q (1-q)}{p_h q + p_l (1-q)} G_{(V_h,L)}(b)\right\} \tag{14}$$

since such a bid loses to type-(V_h, H) opponent with probability one and win against a type-(V_h, L) opponent with probability $G_{(V_h,L)}(b)$. Plugging $G\left(_{V_h,L}\right)(\cdot)$ as described by (5) into (14), we get

$$\begin{aligned} & \frac{p_l (1-q)}{p_h q + p_l (1-q)}(V_h - b) + \frac{p_h q (1-q)}{p_h q + p_l (1-q)} \frac{p_l q (b - V_l)}{p_h (1-q)^2} \\ = & \frac{\left[p_l (1-q)^2 V_h - p_l q^2 V_l\right] + p_l \left[q^2 - (1-q)^2\right] b}{\left[p_h q + p_l (1-q)\right](1-q)}, \end{aligned}$$

which is non-decreasing in b since $q \geq 1/2$. Hence type-(V_h, H) bidder 1 does not have incentive to deviate to bids in the interval $\left(V_l, \bar{b}_{(V_h,L)}\right)$.

Finally, note that the expressions for $\bar{b}_{(V_h,L)}$ and $\bar{b}_{(V_h,H)}$ respectively satisfy $G_{(V_h,L)}\left(\bar{b}_{(V_h,L)}\right) = 1$ and $G_{(V_h,H)}\left(\bar{b}_{(V_h,H)}\right) = 1$. This concludes the proof that the postulated bidding strategies constitute a symmetric equilibrium. ■

Now we show that the FPA admits no other symmetric equilibrium.

Lemma 3 *The symmetric equilibrium described in Lemma 2 is the unique symmetric equilibrium of the FPA.*

Proof. The argument proceeds in three steps.

Step 1. We show that in any symmetric equilibrium type-(V_h, L) and type-(V_h, H) bidders must bid in mixed strategies. For example, suppose to the contrary that, say, a type-(V_h, L) bidder 2 bids in pure strategy an amount $\tilde{b} < V_h$, then type-(V_h, L) bidder 1 can profitably deviate by bidding $\tilde{b} + \varepsilon$ where $\varepsilon > 0$ is arbitrarily small. Such a deviation will provide a discrete positive jump in type-(V_h, L) bidder 1's probability of winning, hence it is profitable. The argument for type-(V_h, H) bidders is analogous.

Step 2. We show that in any symmetric mixed strategy equilibrium, the supports of $G_{(V_h,L)}(\cdot)$ and $G_{(V_h,H)}(\cdot)$ are contiguous and non-overlapping. That the supports should be contiguous follows from the same ε-deviation argument as the one to rule out pure strategies. Now suppose that the supports of $G_{(V_h,L)}(\cdot)$ and $G_{(V_h,H)}(\cdot)$ overlap in an interval $[b_1, b_2]$ with $b_2 > b_1$. To be consistent with mixed strategies, it must be the case that, the expected surplus for both types from any bid $b \in [b_1, b_2]$ is constant. That is, for some constants $\tilde{K}_{(V_h,L)}$ and $\tilde{K}_{(V_h,H)}$,

$$(V_h - b)\left\{\frac{p_l(1-q)}{p_h q + p_l(1-q)} + \frac{p_h q^2 G_{(V_h,H)}(b)}{p_h q + p_l(1-q)} + \frac{p_h q(1-q) G_{(V_h,L)}(b)}{p_h q + p_l(1-q)}\right\} = \tilde{K}_{(V_h,H)}, \quad (15)$$

$$(V_h - b)\left\{\frac{p_l q}{p_h(1-q) + p_l q} + \frac{p_h(1-q) q G_{(V_h,H)}(b)}{p_h(1-q) + p_l q} + \frac{p_h(1-q)^2 G_{(V_h,L)}(b)}{p_h(1-q) + p_l q}\right\} = \tilde{K}_{(V_h,L)}. \quad (16)$$

Multiplying Eq. (15) by $(1-q)[p_h q + p_l(1-q)]$, and Eq. (16) by $q[p_h(1-q) + p_l q]$, and summing up, we obtain:

$$\tilde{K}_{(V_h,H)}(1-q)[p_h q + p_l(1-q)] - \tilde{K}_{(V_h,L)} q [p_h(1-q) + p_l q] = (V_h - b) p_l \left[(1-q)^2 - q^2\right]. \quad (17)$$

Because the left hand side of Eq. (17) is a constant, this equation holds only for a single value of b unless $q = 1/2$. Therefore , the supports of the symmetric equilibrium mixed strategies of type-(V_h, L) and (V_h, H) bidders must be non-overlapping. The same argument also shows that the supports of the symmetric equilibrium mixed strategies of type-(V_h, L) and type-(V_h, H) bidders can not overlap at more than one point.

Step 3. We show that the support of type-(V_h, L) bidders' mixed strategy must be lower than that of type-(V_h, H) bidders. Suppose to the contrary. Let $\left[V_l, \tilde{b}\right]$ be the support of type-(V_h, H) bidder and $\left[\tilde{b}, \hat{b}\right]$ be the support of type-(V_h, L) bidder, for some $\hat{b} > \tilde{b}$. For type-(V_h, L) bidders to randomize on $\left[\tilde{b}, \hat{b}\right]$, it must be the case that

$$\begin{aligned} &(V_h - b)\left\{\frac{p_l q}{p_h(1-q) + p_l q} + \frac{p_h(1-q) q}{p_h(1-q) + p_l q} + \frac{p_h(1-q)^2}{p_h(1-q) + p_l q}\tilde{G}_{(V_h,L)}(b)\right\} \\ = &\left(V_h - \tilde{b}\right)\frac{p_l q + p_h q(1-q)}{p_h(1-q) + p_l q}, \end{aligned}$$

from which, after solving for $\tilde{G}_{(V_h,L)}(b)$, we obtain

$$\tilde{G}_{(V_h,L)}(b) = \frac{q(1-p_h q)\left(b - \tilde{b}\right)}{p_h(1-q)^2(V_h - b)}. \quad (18)$$

Suppose that type-(V_h, H) bidder 2 mixes over $\left[V_l, \tilde{b}\right]$, and type-$(V_h, L)$ bidder 2 mixes over $\left[\tilde{b}, \hat{b}\right]$ according to $\tilde{G}_{(V_h,L)}(\cdot)$ as described by (18). Then the expected surplus for type-(V_h, H) bidder 1 from bidding $b \in \left(\tilde{b}, \hat{b}\right)$ is given by

$$\begin{aligned} & \left(V_h - b\right)\left\{\frac{p_l\left(1-q\right)}{p_h q + p_l\left(1-q\right)} + \frac{p_h q^2}{p_h q + p_l\left(1-q\right)} + \frac{p_h q\left(1-q\right)}{p_h q + p_l\left(1-q\right)}\tilde{G}_{\left(V_h, L\right)}\left(b\right)\right\} \\ = & \frac{\left(1-q\right)\left[p_l\left(1-q\right) + p_h q^2\right] V_h - q^2\left(1 - p_h q\right)\tilde{b} + p_l\left[q^2 - \left(1-q\right)^2\right] b}{\left(1-q\right)\left[p_h q + p_l\left(1-q\right)\right]} \end{aligned}$$

which is non-decreasing in b. Therefore, type-(V_h, H) bidder will have an incentive to bid higher than $\tilde{b}$ if her opponent follows the prescribed strategies, a contradiction.

Combining Steps 1-3 and Lemma 1, we know that the equilibrium described in Lemma 2 is the only symmetric equilibrium. ■

Lastly, we show that there is no asymmetric equilibrium.

Lemma 4 *There is no asymmetric equilibrium.*

Proof. First, arguments similar to step 3 in the proof of Lemma 3 can be used to show that in an asymmetric equilibrium, the support of type-(V_h, L) bidders must be lower than that of type-(V_h, H) bidders.

Now suppose that type-(V_h, L) bidder 1 and bidder 2 respectively play a mixed strategy on the support $\left[V_l, \tilde{b}_1\right]$ and $\left[V_l, \tilde{b}_2\right]$, and without loss of generality, suppose that $\tilde{b}_1 > \tilde{b}_2$. Since type-$(V_h, L)$ bidder 1 must be indifferent between any bids in $(V_l, \tilde{b}_2]$, type-(V_h, L) bidder 2's mixed strategy, denoted by $\tilde{G}_{2(V_h,L)}$, must satisfy

$$\left(V_h - b\right)\left\{\frac{p_l q}{p_l q + p_h\left(1-q\right)} + \frac{p_h\left(1-q\right)}{p_l q + p_h\left(1-q\right)}\tilde{G}_{2\left(V_h, L\right)}\left(b\right)\right\} = \left(V_h - V_l\right)\frac{p_l q}{p_l q + p_h\left(1-q\right)}$$

from which we obtain that $\tilde{b}_2 = \overline{b}_{(V_h,L)}$ where $\overline{b}_{(V_h,L)}$ is specified by formula (6). Now since type-(V_h, L) bidder 2 is indifferent between any bids in $\left[V_l, \tilde{b}_2\right]$, type-$(V_h, L)$ bidder 1's mixed strategy CDF, denoted by $\tilde{G}_{1(V_h,L)}$, in the interval $\left[V_l, \tilde{b}_2\right]$ must satisfy

$$\left(V_h - b\right)\left\{\frac{p_l q}{p_l q + p_h\left(1-q\right)} + \frac{p_h\left(1-q\right)}{p_l q + p_h\left(1-q\right)}\tilde{G}_{1\left(V_h, L\right)}\left(b\right)\right\} = \left(V_h - V_l\right)\frac{p_l q}{p_l q + p_h\left(1-q\right)}$$

from which we obtain that

$$\tilde{G}_{1\left(V_h, L\right)}\left(b\right) = \frac{p_l q\left(b - V_l\right)}{p_h\left(1-q\right)^2\left(V_h - b\right)}.$$

But then $\tilde{G}_{1(V_h,L)}\left(\tilde{b}_2\right) = \tilde{G}_{1(V_h,L)}\left(\overline{b}_{(V_h,L)}\right) = 1$. Hence $\tilde{b}_1 = \tilde{b}_2$, a contradiction. ■

The following proposition summarizes Lemma 1-4:

Proposition 1 *If $p_m = 0, q_l = 1 - q_h = q \in [1/2, 1]$, then the unique equilibrium of the first-price auction is symmetric and is described as follows: for $i = 1, 2$,*

1. $B_i^{\text{FPA}}(V_l, s) = V_l$ *for* $s \in \{L, H\}$;
2. *Type-(V_h, L) bidder i mixes over $\left[V_l, \bar{b}_{(V_h,L)}\right]$ according to CDF $G_{(V_h,L)}(\cdot)$ specified by (5) where $\bar{b}_{(V_h,L)}$ is given by (6);*
3. *Type-(V_h, H) bidder i mixes over $\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ according to CDF $G_{(V_h,H)}(\cdot)$ specified by (7) where $\bar{b}_{(V_h,H)}$ is given by (8).*

3.3 Revenue Non-Equivalence

Now we compare the seller's expected revenue from the SPA and the FPA. Because bidders bid their own private valuations in the SPA, the seller receives V_h if and only if both bidders have valuation type V_h (an event that occurs with probability p_h^2) and the seller receives V_l otherwise. Hence the seller's expected revenue from the SPA, denoted by R^{SPA}, is

$$R^{\text{SPA}} = \left(1 - p_h^2\right) V_l + p_h^2 V_h. \tag{19}$$

Since in the SPA a bidder obtains positive surplus $(V_h - V_l)$ only when her valuation is V_h and her opponent's valuation is V_l, an event that occurs with probability $p_h p_l$, each bidder's *ex ante* expected surplus from the SPA, denoted by M^{SPA}, is

$$M^{\text{SPA}} = p_h p_l (V_h - V_l), \tag{20}$$

In the unique equilibrium of the FPA characterized in Proposition 1, the object is always efficiently allocated. Thus, the expected social welfare is $p_l^2 V_l + \left(1 - p_l^2\right) V_h$. In equilibrium, bidders with valuation type V_l obtains zero expected surplus; and type-(V_h, L) and type-(V_h, H) bidders respectively obtain expected surplus $K_{(V_h,L)}$ and $K_{(V_h,H)}$ as described by (10) and (13). The *ex ante* probabilities that bidder i is of type (V_h, L) and (V_h, H) are, respectively, $\Pr[t_i = (V_h, L)] = p_h [p_h (1 - q) + p_l q]$ and $\Pr[t_i = (V_h, H)] = p_h [p_h q + p_l (1 - q)]$. Thus, the *ex ante* expected surplus of each bidder from the FPA, denoted by M^{FPA}, is

$$\begin{aligned} M^{\text{FPA}}(q) &= \Pr[t_i = (V_h, L)] K_{(V_h,L)} + \Pr[t_i = (V_h, H)] K_{(V_h,H)} \\ &= \frac{p_h q (1 - q) + p_l q}{p_h (1 - q)^2 + p_l q} p_h p_l (V_h - V_l). \end{aligned}$$

We have the following observations. First, M^{FPA} depends on q and M^{SPA} is independent of q. The intuition is simply that bidders strategically use their information about opponent's valuation only in the FPA. Second, $M^{\text{FPA}}(q) > M^{\text{SPA}}$ for all $q \in (1/2, 1)$ and $M^{\text{FPA}}(1/2) = M^{\text{FPA}}(1) = M^{\text{SPA}}$. That is, a bidder's expected surplus is strictly higher in the FPA than that in the SPA except for the completely uninformative and completely informative signal cases. When $q = 1/2$, the signals are completely uninformative, and bidders would simply disregard their information type. We can see from Lemma 1 that the probability densities of $G_{(V_h,L)}$ and $G_{(V_h,H)}$ can be smoothly pasted at $\bar{b}_{(V_h,L)}$ when $q = 1/2$, which implies that effectively, when $q = 1/2$, bidders of valuation type V_h are simply playing a mixed strategy on the whole support of $\left[V_l, \bar{b}_{(V_h,H)}\right]$. When $q = 1$, the FPA becomes a complete information auction, and it is well known that it is revenue equivalent to the SPA.

The seller's expected revenue in the FPA, denoted by R^{FPA}, is simply the difference between the expected social welfare and the sum of the bidders' expected surplus. That is,

$$\begin{aligned} R^{\text{FPA}}(q) &= \left[p_l^2 V_l + \left(1 - p_l^2\right) V_h\right] - 2M^{\text{FPA}}(q) \\ &= \left(1 - p_h^2\right) V_l + p_h^2 V_h - \frac{2p_h^2 p_l (2q-1)(1-q)}{p_h (1-q)^2 + p_l q} (V_h - V_l) \\ &= R^{\text{SPA}} - \frac{2p_h^2 p_l (2q-1)(1-q)}{p_h (1-q)^2 + p_l q} (V_h - V_l). \end{aligned} \tag{21}$$

The following proposition summarizes the comparison between $R^{\text{FPA}}(q)$ and R^{SPA} :

Proposition 2 ***(Revenue Non-Equivalence)*** *For any $q \in (1/2, 1)$, $R^{\text{FPA}}(q) < R^{\text{SPA}}$; and $R^{\text{FPA}}(1/2) = R^{\text{FPA}}(1) = R^{\text{SPA}}$; moreover, $R^{\text{FPA}}(q)$ has a unique minimizer.*

That $R^{\text{FPA}}(q)$ has a unique minimizer in q follows from simple algebra. Figure 1 depicts the seller's expected revenues as a function of $q \in [1/2, 1]$ from the two auction mechanisms for an example where $p_h = 0.75, V_l = 0, V_h = 1$.

Now we explain why the standard revenue equivalence theorem (see, for example, Mas-Colell *et. al.*, p. 890) breaks down in our multidimensional auction environment. An important condition for the revenue equivalence theorem is that the equilibrium winning probability for every possible bidder type must be equal in the two auction mechanisms. In our setting, however, this condition is violated. For example, the probability of winning is the same bidder types (V_h, L) and (V_h, H) in the SPA because both bid V_h in equilibrium. In the FPA, however, the probability of winning is strictly higher for type-(V_h, H) bidder because, as characterized in Proposition 1, the support of the equilibrium mixed strategy for type-(V_h, H) bidder is strictly higher than that of type-(V_h, L) bidder.

We also note that in this two-valuation example (since $p_m = 0$), the SPA generates a higher expected seller revenue than the FPA despite the fact that both auction mechanisms are allocatively

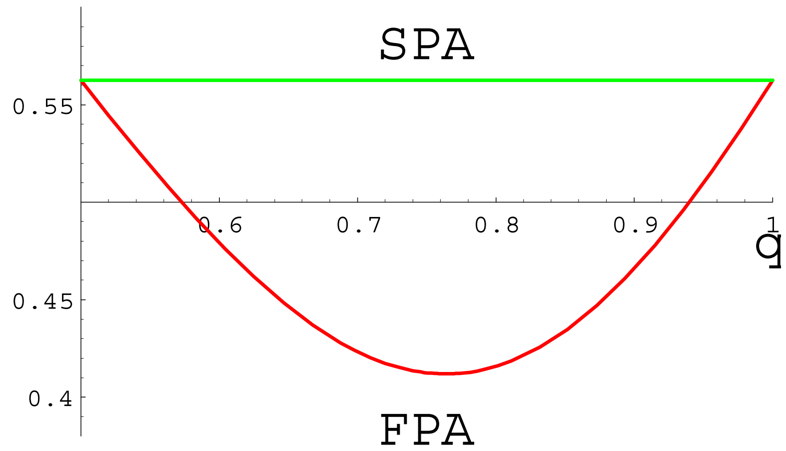

Figure 1: Seller's expected revenues in the SPA and FPA: $p_h = 0.75, V_h = 1, V_l = 0$.

efficient. This is similar to the one-dimensional APV auctions: the objects are allocated efficiently in both the first and second-price APV auctions, but the SPA generates higher expected revenue for the seller (Milgrom and Weber 1982).

4 Efficiency

While the SPA is always allocatively efficient in equilibrium, we argue in this section that the allocative efficiency of the FPA in Section 3 is an artifact of the two-valuation example. When we make all three valuations occur with positive probability, i.e., $p_k > 0$ for $k \in \{l, m, h\}$, the unique symmetric equilibrium of the FPA may be allocatively inefficient. Allocative inefficiency may arise in the FPA if a type-(V_m, H) bidder may infer that her opponent is mostly likely of valuation type V_h and hence bid more aggressively than a type-(V_h, L) bidder, who perceives her opponent to be weak and is willing to sacrifice the probability of winning to exchange for a bigger surplus when winning against an opponent with valuation V_l. The subtle point is that this intuition works only if the following conditions are met: (1) Type-(V_m, H) bidder's posterior belief about her opponent puts a small weight on (V_h, L), and big weight on (V_h, H). This requires that q_m be sufficiently small and $\Pr(v_j = V_h | s_i = H)$ be sufficiently large; (2). Type-(V_h, L) bidder's posterior belief about her opponent puts a big weight on V_l. This requires that q_l be sufficiently large; (3). V_h can not be too large relative to V_m since otherwise, type-(V_h, L) bidder is not willing to lower her probability of winning by bidding conservatively.

Arguments similar to those in Section 3 can be used to establish that, first, in any symmetric equilibrium of the FPA, bidders with valuation V_l must bid V_l in pure strategy regardless of their information type; second, other types of bidders must bid in mixed strategies with contiguous and non-overlapping supports; third, the support of type-(V_m, L) bidder's mixed strategy must be lower than that of type-(V_m, H); the support of type-(V_h, L) bidder's mixed strategy must be lower than that of type-(V_h, H) ; the support of type-(V_m, H) bidder's mixed strategy must be lower than that of type-(V_h, H), and the support of type-(V_m, L) bidder's mixed strategy must be lower than that of type-(V_h, L) . Thus the symmetric equilibrium of the FPA in this section takes only two possible forms depending on the order of the mixed strategy supports of type-(V_m, H) and type-(V_h, L) bidders. A symmetric equilibrium is *efficient* if the equilibrium mixed strategy support of type-(V_m, H) bidder is lower than that of type-(V_h, L) bidder; and it is *inefficient* if the equilibrium mixed strategy support of type-(V_h, L) bidder is lower than that of type-(V_m, H) bidder. We first show the following result:

Proposition 3 ***(Efficient and Inefficient Equilibria Can not Coexist in the FPA)*** *Any auction environment $e \in \mathcal{E}$ can not simultaneously have both an efficient and an inefficient symmetric equilibrium in the FPA.*

Proof. Suppose to the contrary that there is an auction environment that admits both types of symmetric equilibrium in the FPA. First, since the support of type-(V_m, L) bidders must be lower than those of type-(V_m, H) , (V_h, L) , and (V_h, H) bidders in both equilibria, the upper limit of type-(V_m, L) bidders' mixed strategies in both equilibria must be the same, which we denote by $\bar{b}_{(V_m,L)}$.

Let $\bar{b}^{\text{eff}}_{(V_h,L)}$ be the upper limit of the mixed strategy support of type-(V_h, L) bidder in the efficient equilibrium and let $\bar{b}^{\text{ineff}}_{(V_m,H)}$ be the upper limit of the mixed strategy support of type-(V_m, H) bidder in the inefficient equilibrium. We then consider two possible cases:

Case 1: $\bar{b}^{\text{eff}}_{(V_h,L)} \geq \bar{b}^{\text{ineff}}_{(V_m,H)}$. This case is illustrated in Figure 2. Since in the inefficient equilibrium type-(V_h, L) bidder is indifferent between any bids in $\left[\bar{b}_{(V_m,L)}, \bar{b}^{\text{ineff}}_{(V_h,L)}\right]$, her expected surplus in the inefficient equilibrium is the same as that when she bids $\bar{b}_{(V_m,L)}$ (recall our tie-breaking rule), which is simply:

$$Z_1 = \left(V_h - \bar{b}_{(V_m,L)}\right)\left\{\frac{p_l q_l}{\sum_{k\in\{l,m,h\}} p_k q_k} + \frac{p_m q_m (1-q_h)}{\sum_{k\in\{l,m,h\}} p_k q_k}\right\}, \tag{22}$$

where the term in the bracket is the expected probability of winning against bidders with V_l valuation and type-(V_m, L) bidders. Given that her opponent follows the prescribed strategy in the

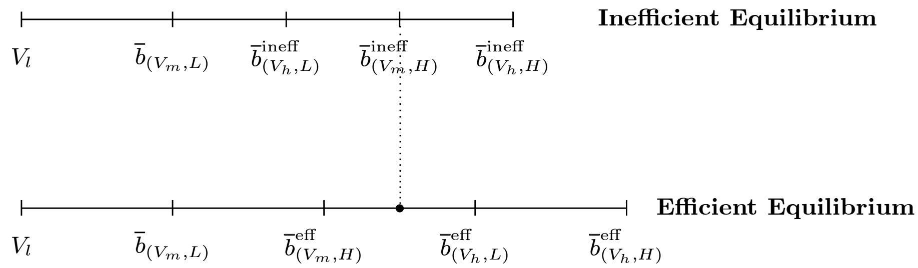

Figure 2: Case 1 in the Proof of Proposition 3.

inefficient equilibrium, her expected payoff from deviating to a bid of $\bar{b}^{\text{ineff}}_{(V_m,H)}$ is

$$Z_2 = \left(V_h - \bar{b}^{\text{ineff}}_{(V_m,H)}\right)\left\{\frac{p_l q_l}{\sum_{k\in\{l,m,h\}} p_k q_k} + \frac{p_m q_m}{\sum_{k\in\{l,m,h\}} p_k q_k} + \frac{p_h q_h\,(1-q_h)}{\sum_{k\in\{l,m,h\}} p_k q_k}\right\}, \tag{23}$$

where the term in the bracket is the expected probability of winning against bidders with V_l and V_m valuations and type-(V_h, L) bidders. By the requirement of the inefficient equilibrium, we have $Z_1 > Z_2$.[6] Moreover, since in this case $V_h > \bar{b}^{\text{eff}}_{(V_h,L)} \geq \bar{b}^{\text{ineff}}_{(V_m,H)}$ by assumption, we immediately have:

$$Z_2 \geq \left(V_h - \bar{b}^{\text{eff}}_{(V_h,L)}\right)\left\{\frac{p_l q_l}{\sum_{k\in\{l,m,h\}} p_k q_k} + \frac{p_m q_m}{\sum_{k\in\{l,m,h\}} p_k q_k} + \frac{p_h q_h\,(1-q_h)}{\sum_{k\in\{l,m,h\}} p_k q_k}\right\} \tag{24}$$

But the right hand side of inequality (24) is exactly type-(V_h, L) bidder's expected surplus in the efficient equilibrium, which by the definition of the efficient equilibrium is required to be larger than Z_1 [as given by expression (22)]. This is so because Z_1 is also type-(V_h, L) bidder's expected surplus from deviating to a bid of $\bar{b}_{(V_m,L)}$ in the efficient equilibrium. Thus we have $Z_2 \geq Z_1$, which is a contradiction to our earlier conclusion that $Z_1 > Z_2$.

Case 2: $\bar{b}^{\text{eff}}_{(V_h,L)} < \bar{b}^{\text{ineff}}_{(V_m,H)}$. A contradiction can be derived for type-(V_m, H) bidder using arguments analogous to Case 1. ■

Our next result shows that, in contrast to the SPA, the FPA may be allocatively inefficient.

Proposition 4 ***(Inefficiency of First-Price Auction)*** *There exists an open set of auction environments in $\mathcal{E}$ in which the unique symmetric equilibrium of the FPA is inefficient.*

[6] The strict, rather than the usual weak, inequality, is valid because, tedious algebra shows that the deviation surplus function is strictly decreasing in the whole interval $\left[\bar{b}^{\text{ineff}}_{(V_h,L)}, \bar{b}^{\text{ineff}}_{(V_m,H)}\right]$ in order to be consistent with the inefficient equilibrium.

Proof. The proof of this Proposition is completely constructive, and we proceed by providing an explicit auction environment with an inefficient symmetric equilibrium. Proposition 3 then guarantees that it does not admit any efficient symmetric equilibrium. That there is an open set of auction environments in $\mathcal{E}$ with inefficient equilibrium in the FPA follows from continuity.

Consider an example of the model presented in Section 2 as follows:

- $V_l = 0, V_m = 1$ and $V_h = 2$. We can set $V_l = 0$ and $V_m = 1$ by normalization and scaling with no loss of generality. For the inefficient equilibrium described below to exist, V_h can not be too high relative to V_m.

- $p_l = p_m = p_h = 1/3$. That is, ex ante bidders' valuations of the object take on the three values with equal probability. This assumption is purely for computational ease in Bayesian updating;

- $q_l = 0.9, q_m = 0.1, q_h = 0.05$. As we described in the beginning of this section, the intuition for inefficient equilibrium requires that q_l be big, q_m to be small, and q_h even smaller.

As in the example in Section 3, we know that bidders with valuation type V_l will bid V_l regardless of their information types. Now we show that the following mixed strategies for the other types of bidders constitute the unique symmetric equilibrium:

- Type-(V_m, L) bidders bid according to a mixed strategy on the support $\left[V_l, \bar{b}_{(V_m,L)}\right]$ with CDF $G_{(V_m,L)}(\cdot)$ where

$$\begin{aligned} \bar{b}_{(V_m,L)} &= \frac{q_m^2 V_m + q_l V_l}{q_l + q_m^2}, \\ G_{(V_m,L)}(b) &= \frac{q_l}{q_m^2}\left(\frac{b - V_l}{V_m - b}\right); \end{aligned}$$

- Type-(V_h, L) bidders bid according to a mixed strategy on the support $\left[\bar{b}_{(V_m,L)}, \bar{b}_{(V_h,L)}\right]$ with CDF $G_{(V_h,L)}(\cdot)$ where

$$\begin{aligned} \bar{b}_{(V_h,L)} &= \frac{[q_l + q_m q_h]\,\bar{b}_{(V_m,L)} + q_h^2 V_h}{q_l + q_m q_h + q_h^2}, \\ G_{(V_h,L)}(b) &= \frac{q_l + q_m q_h}{q_h^2}\frac{b - \bar{b}_{(V_m,L)}}{V_h - b}; \end{aligned}$$

- Type-(V_m, H) bidders bid according to a mixed strategy on the support $\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_m,H)}\right]$ with

CDF $G_{(V_m,H)}(\cdot)$ where

$$\begin{aligned}\bar{b}_{(V_m,H)} &= \frac{[1-q_l+(1-q_m)\,q_m+(1-q_h)\,q_m]\,\bar{b}_{(V_h,L)}+(1-q_m)^2\,V_m}{1-q_l+(1-q_m)\,q_m+(1-q_h)\,q_m+(1-q_m)^2},\\ G_{(V_m,H)}(b) &= \frac{1-q_l+(1-q_m)\,q_m+(1-q_h)\,q_m}{(1-q_m)^2}\frac{b-\bar{b}_{(V_h,L)}}{V_m-b};\end{aligned}$$

- Type-(V_h, H) bidders bid according to a mixed strategy on the support $\left[\bar{b}_{(V_m,H)}, \bar{b}_{(V_h,H)}\right]$ with CDF $G_{(V_h,H)}(\cdot)$ where

$$\begin{aligned}\bar{b}_{(V_h,H)} &= \frac{(1-q_h)^2\,V_h+\left[3-q_l-q_m-q_h-(1-q_h)^2\right]\bar{b}_{(V_m,H)}}{3-q_l-q_m-q_h},\\ G_{(V_h,H)}(b) &= \left[\frac{3-q_l-q_m-q_h-(1-q_h)^2}{(1-q_h)^2}\right]\frac{b-\bar{b}_{(V_m,H)}}{V_h-b}.\end{aligned}$$

Under the above parameterization,

$$\begin{aligned}\bar{b}_{(V_h,H)} &\approx 1.32531 > \bar{b}_{(V_m,H)} \approx 0.744012\\ &> \bar{b}_{(V_h,L)} \approx 0.0164684 > \bar{b}_{(V_m,L)} \approx 0.010989 > V_l = 0.\end{aligned}$$

To show that the above strategy profile constitutes an equilibrium, we need to demonstrate that, given that the opponent follows the postulated strategies, each type-(v, s) bidder, where $v \in \{V_m, V_h\}$ and $s \in \{L, H\}$, obtains a constant expected surplus from any bids in the support of the CDF $G_{(v,s)}(\cdot)$, which is in turn higher than the expected surplus from any other deviation bids. The details of the verifications are straightforward but arithmetically tedious, thus relegated to Appendix A. Figures 4-9 in Appendix A depict the expected surplus for each type of bidders from different bids, and it is clear that no type has incentive to deviate from the prescribed strategies.■

Our model has an interesting implication regarding the impact of more information on efficiency. The probability of the object being inefficiently allocated in an inefficient equilibrium is the probability that the two bidders' types are (V_m, H) and (V_h, L) respectively, which is given by

$$\begin{aligned}&2\Pr\{t_i=(V_m,H)\,,t_j=(V_h,L)\,,i\neq j\}\\ =\;&2p_m p_h\,(1-q_h)\,q_m.\end{aligned}$$

That is, the probability of inefficient allocation in the FPA is increasing in $1 - q_h$ locally in the set of auction environments with inefficient equilibrium. Recall that $1 - q_h$ is the probability of bidder i obtaining $s_i = H$ when her opponent's valuation $v_j = V_h$. Thus the higher $1 - q_h$ is, the more informative the signal H is about V_h, and also of course, the more informative the signal L is about V_l. Thus we have shown that:

Corollary 1 *The probability of inefficient allocation may be increasing in the informativeness of the signals.*

Finally, it can also be verified that the seller's expected revenue in the inefficient equilibrium in the FPA is again smaller than that in the SPA except for some knife-edge cases with measure zero.

The key observations of this section would clearly continue to hold in more general settings. For example, suppose that bidders' private values were independently drawn from a continuous distribution and each bidder observed a continuous signal, correlated with the value of the other bidder. In a continuous setting, efficiency would require that each bidder's strategy depend only on his valuation and not his signal. This would be impossible if the signal was informative. The probability of inefficiency would therefore always be non-monotonic, since we have inefficiency with intermediate informativeness of signals, but we have efficiency with either no information about others' values (efficiency is a well known property of the FPA with symmetric distributions and independent private values) or full information about others' values (there is efficiency in the FPA with complete information).

5 Equilibrium Existence

Up to now, we have assumed that bidders' information about the opponent's private valuation is of the same accuracy. In this section, we show that the existence of equilibrium in the FPA is contingent on this assumption in our model. For this purpose, we consider again the example we used in Section 3, with the exception that the accuracy of bidder i's signal regarding bidder j's valuation is $q_i \in [1/2, 1)$ and we let $q_1 > q_2$. Recall that $q_i = \Pr(s_i = L|v_j = V_l) = \Pr(s_i = H|v_j = V_h)$. Our main result in this section can be stated as follows.

Proposition 5 *If $p_m = 0$ and $1 \geq q_1 > q_2 \geq 1/2$, then generically the FPA does not admit any equilibrium.*

Proof. Using standard ε-deviation arguments, we can show that (1). bidders with valuation V_l must bid V_l in pure strategy in any equilibrium; (2). each bidder of type-(V_h, L) and type-(V_h, H) must bid in mixed strategies with bids higher than V_l; (3). the highest bid that may be submitted by each bidder must be the same; (4). there is no gap in the bids submitted in equilibrium. We denote the mixed strategy CDF of type-(V_h, L) and type-(V_h, H) bidder i by $G_{i(V_h,L)}$ and $G_{i(V_h,H)}$ respectively, where $i = 1, 2$.

Next, we show that for each bidder i, the supports of $G_{i(V_h,L)}$ and $G_{i(V_h,H)}$ can not overlap at more than one point. Without loss of generality, consider bidder 1. Let B_1 be the set of points in which the supports of $G_{i(V_h,L)}$ and $G_{i(V_h,H)}$ overlap. For any overlap bid $b \in B_1$, the following must be true:

$$\begin{aligned}(V_h - b)\left[\frac{p_l q_1}{p_l q_1 + p_h(1-q_1)} + \frac{p_h(1-q_1)(1-q_2)G_{2(V_h,L)}(b)}{p_l q_1 + p_h(1-q_1)} + \frac{p_h(1-q_1)q_2 G_{2(V_h,H)}(b)}{p_l q_1 + p_h(1-q_1)}\right] &= \tilde{K}_{1(V_h,L)} \\ (V_h - b)\left[\frac{p_l(1-q_1)}{p_l(1-q_1) + p_h q_1} + \frac{p_h q_1(1-q_2)G_{2(V_h,L)}(b)}{p_l(1-q_1) + p_h q_1} + \frac{p_h q_1 q_2 G_{2(V_h,H)}(b)}{p_l(1-q_1) + p_h q_1}\right] &= \tilde{K}_{1(V_h,H)}\end{aligned}$$

Similar to the arguments in step 2 of the proof of Lemma 3, the above system equations can hold for at most one value of b.

Therefore, we are left with four possible cases to consider depending on the order of the supports of type-(V_h, L) and (V_h, H) mixed strategies for each bidder. We will derive a contradiction for one of the cases, and the other cases can be dealt with analogously.

We consider the following case: The support of $G_{i(V_h,L)}$ is $\left[V_l, \bar{b}_{i(V_h,L)}\right]$ and the support of $G_{i(V_h,H)}$ is $[\bar{b}_{i(V_h,L)}, \bar{b}_{i(V_h,H)}]$. From discussions above, $\bar{b}_{1(V_h,H)} = \bar{b}_{2(V_h,H)} = \bar{b}_{(V_h,H)}$.

Step 1: Simple calculation shows that it must be the case that $\bar{b}_{1(V_h,L)} > \bar{b}_{2(V_h,L)}$.

Step 2: From the necessary indifference condition of type-(V_h, L) bidder 1 in the interval $\left[V_l, \bar{b}_{2(V_h,L)}\right]$, we can obtain $G_{2(V_h,L)}$:

$$\begin{aligned}&(V_h - b)\left[\frac{p_l q_1}{p_l q_1 + p_h(1-q_1)} + \frac{p_h(1-q_1)(1-q_2)G_{2(V_h,L)}(b)}{p_l q_1 + p_h(1-q_1)}\right] = (V_h - V_l)\frac{p_l q_1}{p_l q_1 + p_h(1-q_1)} \\ \Rightarrow\ & G_{2(V_h,L)}(b) = \frac{p_l q_1}{p_h(1-q_1)(1-q_2)}\frac{b - V_l}{V_h - b} \\ & \bar{b}_{2(V_h,L)} = \frac{p_h(1-q_1)(1-q_2)V_h + p_l q_1 V_l}{p_h(1-q_1)(1-q_2) + p_l q_1}\end{aligned}$$

Step 3: The indifference condition for type-(V_h, L) bidder 2 requires that $G_{1(V_h,L)}(b)$ must satisfy, for $b \in \left[V_l, \bar{b}_{2(V_h,L)}\right]$,

$$(V_h - b)\left[\frac{p_l q_2}{p_l q_2 + p_h(1-q_2)} + \frac{p_h(1-q_2)(1-q_1)G_{1(V_h,L)}(b)}{p_l q_1 + p_h(1-q_2)}\right] = (V_h - V_l)\frac{p_l q_2}{p_l q_2 + p_h(1-q_2)},$$

from which we can obtain $G_{1(V_h,L)}(b)$ for $b \in (V_l, \bar{b}_{2(V_h,L)}]$ as

$$G_{1(V_h,L)}(b) = \frac{p_l q_2}{p_h(1-q_1)(1-q_2)}\frac{b - V_l}{V_h - b}.$$

Step 4: To obtain the $G_{1(V_h,L)}(b)$ for $b \in \left[\bar{b}_{2(V_h,L)}, \bar{b}_{1(V_h,L)}\right]$, we make use of the indifference condition of type-(V_h, H) bidder 2, which is given by

$$\begin{aligned}&(V_h - b)\left[\frac{p_l(1-q_2)}{p_l(1-q_2) + p_h q_2} + \frac{p_h q_2(1-q_1)}{p_l(1-q_2) + p_h q_2}G_{1(V_h,L)}(b)\right] \\ =\ & \left(V_h - \bar{b}_{2(V_h,L)}\right)\left[\frac{p_l(1-q_2)}{p_l(1-q_2) + p_h q_2} + \frac{p_h q_2(1-q_1)}{p_l(1-q_2) + p_h q_2}G_{1(V_h,L)}\left(\bar{b}_{2(V_h,L)}\right)\right]\end{aligned}$$

hence, for $b \in \left[\bar{b}_{2(V_h,L)}, \bar{b}_{1(V_h,L)}\right]$

$$G_{1(V_h,L)}(b) = \frac{p_l(1-q_2)\left[b - \bar{b}_{2(V_h,L)}\right] + p_h q_2 (1-q_1) G_{1(V_h,L)}\left(\bar{b}_{2(V_h,L)}\right)\left[V_h - \bar{b}_{2(V_h,L)}\right]}{p_h q_2 (1-q_1)(V_h - b)}.$$

Setting $G_{1(V_h,L)}(b) = 1$, we obtain

$$\bar{b}_{1(V_h,L)} = \frac{p_h q_2 (1-q_1)\left\{\left[1 - G_{1(V_h,L)}\left(\bar{b}_{2(V_h,L)}\right)\right] V_h + G_{1(V_h,L)}\left(\bar{b}_{2(V_h,L)}\right)\bar{b}_{2(V_h,L)}\right\} + p_l(1-q_2)\bar{b}_{2(V_h,L)}}{p_h q_2 (1-q_1) + p_l (1-q_2)}$$

Step 5: The indifference condition of type-(V_h, L) bidder 1 for the bids in the interval $\left[\bar{b}_{2(V_h,L)}, \bar{b}_{1(V_h,L)}\right]$ requires that $G_{2(V_h,H)}(b)$ for $b \in \left[\bar{b}_{2(V_h,L)}, \bar{b}_{1(V_h,L)}\right]$ must satisfy

$$\begin{aligned}
& (V_h - b)\left[\frac{p_l q_1}{p_l q_1 + p_h (1-q_1)} + \frac{p_h (1-q_1)(1-q_2)}{p_l q_1 + p_h (1-q_1)} + \frac{p_h (1-q_1) q_2}{p_l q_1 + p_h (1-q_1)} G_{2(V_h,H)}(b)\right] \\
= \; & (V_h - V_l)\frac{p_l q_1}{p_l q_1 + p_h (1-q_1)}
\end{aligned}$$

thus,

$$G_{2(V_h,H)}(b) = \frac{b - V_l}{V_h - b}\frac{p_l q_1}{p_h (1-q_1) q_2} - \frac{p_h (1-q_1)(1-q_2)}{p_h (1-q_1) q_2}$$

from which can obtain $G_{2(V_h,H)}\left(\bar{b}_{1(V_h,L)}\right)$.

Step 6: The indifference condition of type-(V_h, H) bidder 2 requires that $G_{1(V_h,H)}(b)$ satisfy

$$\begin{aligned}
& (V_h - b)\left[\frac{p_l (1-q_2)}{p_l (1-q_2) + p_h q_2} + \frac{p_h q_2 (1-q_1)}{p_l (1-q_2) + p_h q_2} + \frac{p_h q_2 q_1}{p_l (1-q_2) + p_h q_2} G_{1(V_h,H)}(b)\right] \\
= \; & \left(V_h - \bar{b}_{2(V_h,L)}\right)\left[\frac{p_l (1-q_2)}{p_l (1-q_2) + p_h q_2} + \frac{p_h q_2 (1-q_1)}{p_l (1-q_2) + p_h q_2} G_{1(V_h,L)}\left(\bar{b}_{2(V_h,L)}\right)\right]
\end{aligned}$$

which implies a value for $\bar{b}_{1(V_h,H)}$.

Step 7: Likewise, the indifference condition of type-(V_h, H) bidder 1 requires that $G_{2(V_h,H)}(b)$ satisfy, for $b \in \left[\bar{b}_{1(V_h,L)}, \bar{b}_{2(V_h,H)}\right]$,

$$\begin{aligned}
& (V_h - b)\left[\frac{p_l (1-q_1)}{p_l (1-q_1) + p_h q_1} + \frac{p_h q_1}{p_l (1-q_1) + p_h q_1} G_{2(V_h,H)}(b)\right] \\
= \; & \left(V_h - \bar{b}_{1(V_h,L)}\right)\left[\frac{p_l (1-q_1)}{p_l (1-q_1) + p_h q_1} + \frac{p_h q_1}{p_l (1-q_1) + p_h q_1} G_{2(V_h,H)}\left(\bar{b}_{1(V_h,L)}\right)\right]
\end{aligned}$$

which implies a value for $\bar{b}_{2(V_h,H)}$.

Step 8: Generically, $\bar{b}_{1(V_h,H)}$ and $\bar{b}_{2(V_h,H)}$ are not equal, which contradicts the equilibrium requirement by the standard ε-deviation argument. (see Figure 3 for a graphic illustration of the above steps) ■

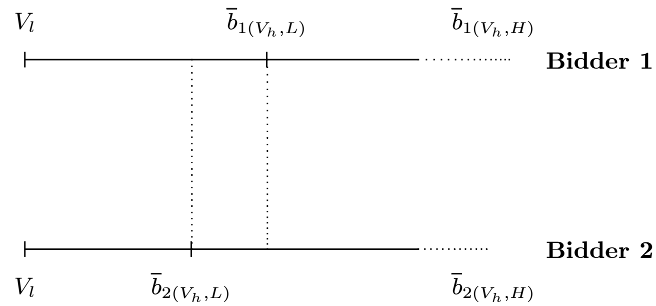

Figure 3: A Graphic Illustration of the Proof of Proposition 5.

Examples of non-existence of equilibrium with multidimensional types are also presented in Jackson (1999) in the context of auctions with both private and common value components, and the example of this section has a similar flavor.

The non-existence problem is surely not an artifact of the discrete type assumption. Consider again the case where bidders' private values were independently drawn from a continuous distribution and each bidder observed a continuous signal, correlated with the value of the other bidder. Even with strong assumptions of the signals (e.g., the monotone likelihood ratio property), if bidder 1 knows that bidder 2 is following a strategy that is monotonic in his valuation and his signal of bidder 1's valuation, bidder 1 will not have a best response that is monotonic (in the same sense). To see why, suppose that bidder 1 has a bimodal distribution on bidder 2's valuation, and thus on bidder 2's bid. Suppose that improvements in bidder 1's signal translate up bidder 1's beliefs about bidder 2's bids. For low values of the signal, it will be optimal for bidder 1 to bid such that he wins against both modal bids. However, as his signal improves, there will be a point where he will give up on winning against 2's high modal bid and his bid will jump down to just above the 2's low modal bid. Thus his bid will jump downwards as his signal improves. It is hard to think of a primitive assumption on the signal structure that will prevent this type of non-monotonicity. This monotonicity implies that the existence arguments such as those of Athey (2001) and Reny and Zamir (2002) will not help in this problem.

6 Discussion: Revenue and Information Acquisition

In this section, we first present two examples of multidimensional private value auctions in which the revenue ranking of the FPA and the SPA are reversed; and then discuss the incentives of information acquisition.

6.1 Revenue

We have observed that the SPA is efficient and the FPA is not in general. This may suggest the possibility that the SPA will generate more revenue (as suggested by the three valuation example of section 4). We have also seen an example where the seller's revenue in the SPA is higher than in the FPA, even though the there is efficient allocation of the object under both auctions (the two valuation example in section 3). What can be said in general about the revenue ranking?

There is an easy way to see that a general revenue ranking is not possible. For some special information structures, each bidder will know what private signal the other bidder has observed. This will be true if each bidder observes a partition of the other bidder's valuations. Now even though we start with a model that is completely symmetric across bidders, conditional on the observed signals, bidders are playing in an independent private values environment with asymmetric distributions. But from the work of Maskin and Riley (2000), we already know that revenue ranking may go either way. We can use this insight to construct the following example where revenue in the FPA is higher than in the SPA. Presumably, this revenue ranking would continue to hold in nearby models where private signals were not common knowledge among the players.

Example 1 *Consider a private value auction with two bidders,* $i = 1, 2$*. Suppose that* v_1 *and* v_2 *are independent and both drawn from Uniform* $[0, 1]$*. Bidders also observe a noisy signal about their opponent's valuation. Suppose that the signal is generated as follows: for* $i \neq j$*,*

$$s_i = \begin{cases} L \text{ if } v_j \in \left[0, \frac{1}{2}\right] \\ H \text{ if } v_j \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

That is, a bidder observes a signal that tells her if her opponent's value is higher or lower than $1/2$*; and this information structure is common knowledge.*

In the equilibrium of the SPA for the auction environment described in Example 1, each bidder will bid their own private valuation. In the FPA, however, we have to consider three cases: (i) $v_i \in [0, 1/2]$ for $i = 1, 2$; (ii) $v_i \in (1/2, 1]$ for $i = 1, 2$; and (iii) $v_i \in [0, 1/2]$ and $v_j \in (1/2, 1]$ where $i \neq j$. In case (i), both bidders effectively compete in an auction environment in which it is common knowledge that the valuations are both drawn from Uniform $[0, 1/2]$ distributions. In case (ii), both bidders effectively compete in an auction environment in which it is common knowledge that the valuations are both drawn from Uniform $[1/2, 1]$ distributions. In case (iii), however, the bidders are asymmetric in their valuation distributions and it is common knowledge. Clearly the FPA and the SPA are revenue equivalent in case (i) and (ii) events. In case (iii) events, however, the bidder asymmetry breaks the revenue equivalence. It can be easily verified that case (iii) events satisfy the conditions for a theorem of Maskin and Riley (2000a, Proposition 4.3) which shows that the FPA would generate a higher expected revenue for the seller than the SPA under this type of asymmetry. Thus, we reach the conclusion that overall in this example, the seller's expected revenue is higher in the FPA than that in the SPA. It can also be numerically verified that in case (iii) events, the FPA may be allocatively inefficient. Thus we have an example that the FPA generates higher expected revenue for the seller than the SPA despite its possible allocative inefficiency relative to the SPA.

6.2 Information Acquisition

So far we have assumed that bidder's information about her opponent's valuation or valuation distribution is provided by nature without incurring any cost. In reality, of course, such information is be costly to acquire.[7] Now we argue that if bidders have to costly acquire such information, then different auction mechanisms provide vastly different incentives for such information acquisition. This, together with the difference in revenue and allocative efficiency between the FPA and the SPA we documented earlier, provides yet another reason for the auction designer to prefer one auction mechanism over another even in private value auction environments.

In the SPA, bidders do not strategically use information about their opponents' valuation, thus there is no incentives at all to acquire such information if it is costly. This observation is completely general for any private value auction environments. The lack of incentives to acquire information about one's opponents in the SPA is related to the fact that bidding one's private valuation is an *ex post* equilibrium in the SPA.

In the FPA, however, information about the opponent's valuation does have strategic consequences in the bidding, thus bidders do have incentives to acquire such information if the cost is sufficiently small. We illustrate such incentives using an extension of Example 1 above. Suppose that bidder i can, at a cost $c_i > 0$, purchase a signal about her opponent's valuation that reveals whether her opponent's valuation is below or above $1/2$. Assume that a bidder's signal purchase decision is observable to her opponent. Suppose that the timing of the game is as follows: first, bidders decide whether to purchase such a signal technology at cost c_i; second, nature draws private valuations from Uniform $[0, 1]$ for each bidder; third, a bidder observes whether her opponent's private valuation is below or above $1/2$ if and only if she purchased the signal technology; and finally, bidders compete for the object in the FPA.

The equilibrium bidding strategies in the FPA depend on the signal purchase decisions:

[7] Most of the existing literature in information acquisition in auctions are concerned with common value auctions (for example, Matthews 1984, and Persico 2000).

- If neither bidder purchased the signal technology in stage 1, then both bidders will play the symmetric FPA in which the opponent's valuation is drawn from Uniform $[0,1]$ distribution. Thus bidders bid $v/2$ and the expected surplus for each bidder is given by $\left(\int_0^1 v^2 dv\right)/2 = 1/6 \approx 0.16667$.

- If both bidders purchase the signal technology, then bidder i's ex ante expected surplus from the subsequent FPA is calculated as follows. (1). With probability $1/4$, both bidder valuations will be below $1/2$. In this case, bidders will bid $v/2$ in equilibrium and the expected surplus for each bidder is $\left(\int_0^{1/2} v^2 dv\right)/2 = 1/48$. (2). With probability $1/4$, both valuations will be above $1/2$. In this case, bidders will bid $v/2$ again in equilibrium and the expected surplus for each bidder is $\left(\int_{1/2}^1 v^2 dv\right)/2 = 7/48$. (3). With probability $1/4$, bidder i's valuation is below $1/2$ and bidder j's valuation is above $1/2$, where $j \neq i$. (4). With probability $1/4$, bidder i's valuation is above $1/2$ and bidder j's valuation is below $1/2$. The equilibria of the FPA in the events of case (3) and (4) can not be analytically solved, but numerical calculation shows that bidder i's expected surplus in case (3) and (4) are 0.01848 and 0.34808 respectively.[8] Thus bidder i's *ex ante* expected surplus if both bidders purchased the signal technology is

$$\frac{1/48 + 7/48 + 0.01848 + 0.34808}{4} \approx 0.13331.$$

- If bidder i does not purchases the signal technology but bidder j does, then bidder i and j's *ex ante* expected surplus from the subsequent FPA can be calculated as follows. (1). With probability $1/2$, bidder i's valuation is below $1/2$. In this case, bidder i's belief about j's valuation is Uniform $[0,1]$ while bidder j's belief about i's valuation is Uniform $[0,1/2]$ and this is common knowledge. Hence, bidder i's expected surplus is that of a "weak" bidder (in the terminology of Maskin and Riley 2000a) with Uniform $[0,1/2]$ valuation distribution against a "strong" bidder with Uniform $[0,1]$ distribution in the FPA, which can be analytically calculated to be approximately 0.0242334.[9] Likewise, bidder j (the "strong" bidder in this case)'s ex ante expected payoff is approximately 0.253449. (2). With probability $1/2$, bidder i's

[8] John Riley and Estelle Cantillon graciously provided various versions of BIDCOMP2 fortran codes that are used in calculating the bidders' ex ante expected payoffs in the asymmetric auctions.

[9] The unique equilibrium of a two-bidder asymmetric FPA with valuation distributions Uniform $[0,h_1]$ and Uniform $[0,h_2]$ respectively where $h_1 > 0, h_2 > 0$ and $h_1 \neq h_2$ are given by

$$b_1(v) = \frac{\sqrt{1+mv^2}-1}{mv}, \qquad b_2(v) = \frac{1-\sqrt{1-mv^2}}{mv}$$

where $m = \left(h_1^2 - h_2^2\right)/\left(h_1 h_2\right)^2$ is a constant. Appendix B provides an elementary derivation of the above equilibrium. See also Griesmer, Levitan and Shubik (1967) and Plum (1992).

bidder $i\diagdown$**bidder** j	No Purchase	Purchase
No Purchase	$0.16667, \quad 0.16667$	$0.12135, \quad 0.17878 - c_j$
Purchase	$0.17878 - c_i, \quad 0.12135$	$0.13331 - c_i, \quad 0.13331 - c_j$

Table 1: The expected payoff matrix.

valuation is above $1/2$. In this case, bidder i's belief about bidder j's valuation is Uniform $[0, 1]$ while bidder $j's$ belief about i's valuation is Uniform $[1/2, 1]$ and this is common knowledge. Hence, bidder i's expected surplus is that of a "strong" bidder with Uniform $[1/2, 1]$ valuation distribution against a "weak" bidder with Uniform $[0, 1]$ distribution in the FPA, which can be numerically calculated to be 0.218465. Likewise, bidder j (the "weak" bidder in this case)'s expected surplus is approximately 0.104104. Thus bidder i (the non-purchaser)'s *ex ante* expected payoff is approximately

$$\frac{0.0242334 + 0.218465}{2} = 0.12135;$$

and bidder j (the purchaser)'s *ex ante* expected payoff is approximately

$$\frac{0.253449 + 0.104104}{2} = 0.17878.$$

Table 1 lists the ex ante expected payoff matrix for the two bidders taking into account the information acquisition cost c_i and c_j. When c_i and c_j are sufficiently small, the unique equilibrium in the information acquisition stage is that both bidders purchase the signals. Both bidders are made worse off through two channels. First, they incur the information acquisition cost; second, in the subsequent FPA, they will be engaged in more fierce competition and the seller will be able to extract a higher revenue. The social welfare is also decreased for two reasons. First, the information acquisition cost is dissipative; second, the object will be allocated inefficiently with positive probability.

This example also illustrates the possibility that a decrease in the cost of information acquisition may increase allocative inefficiency in the first price auctions. Imagine that initially the information acquisition cost c_i are sufficiently high that in equilibrium neither bidder purchases the signal technology. Thus we know that the subsequent FPA is allocatively efficient. However, as c_i is sufficiently low, both bidders will purchase information in equilibrium and the subsequent FPA is allocatively inefficient with positive probability.

7 Conclusion

This paper presents examples of two-bidder private value auctions in which each bidder observes her own private valuation as well as noisy signals about her opponent's private valuation. This departs from the one-dimensional symmetric IPV paradigm and relaxes the assumption that bidder's belief regarding her opponents is common knowledge. We partially characterize the equilibrium of the FPA when each bidder's signal about her opponent's valuation is drawn from the same distribution, and show that the revenue-equivalence between standard auctions fails. Our examples demonstrate that, first, the revenue ranking between the FPA and the SPA is ambiguous; second, the equilibrium allocation of the object could be inefficient in the FPA but is always efficient in the SPA, but the revenue and allocative efficiency may not coincide: an inefficient FPA may generate a higher expected revenue for the seller; but it is also possible that the seller's expected revenue is higher in the SPA even when the object is efficiently allocated in both auctions. We also show that the equilibrium existence of the FPA may be problematic in multidimensional type environments. Finally, we illustrate that different auction mechanisms provide different incentives for bidders to acquire cost information about opponents' private valuations. We also provide examples that the allocative inefficiency in the FPA may increase as the signal becomes more informative; and the allocative inefficiency may increase in the FPA as the information acquisition costs are decreased. While the results in our paper are derived in examples, we have explained how the underlying intuitions are general.

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Appendix A: Details of the Inefficient Equilibrium.

In this appendix, we provide details of the inefficient equilibrium in the proof of Proposition 4. As described in the text, bidders with valuation V_l will bid V_l in pure strategy in any equilibrium. In an inefficient symmetric equilibrium, type-(V_m, L) bidder bids in mixed strategies on the support $(V_l, \bar{b}_{(V_m,L)}]$ according to CDF $G_{(V_m,L)}$, type-(V_h, L) bidder bids according to $G_{(V_h,L)}$ on the support $\left[\bar{b}_{(V_m,L)}, \bar{b}_{(V_h,L)}\right]$, type-$(V_m, H)$ bidder bids according to $G_{(V_m,H)}$ on the support $\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_m,H)}\right]$, and finally type-$(V_h, H)$ bidder bids according to $G_{(V_h,H)}$ on the support $\left[\bar{b}_{(V_m,H)}, \bar{b}_{(V_h,H)}\right]$ where $V_l < \bar{b}_{(V_m,L)} < \bar{b}_{(V_h,L)} < \bar{b}_{(V_m,H)} < \bar{b}_{(V_h,H)}$.

A1. Deriving the Candidate Inefficient Equilibrium Strategies.

We first derive the four CDFs and four thresholds from the bidders' indifference conditions. Suppose that bidder 2 plays according to the above prescribed strategies.

Type-(V_m, L) bidder 1. Type-(V_m, L) bidder 1's indifference condition on the support $\left[V_l, \bar{b}_{(V_m,L)}\right]$ is:

$$(V_m - b)\left[\frac{p_l q_l}{\sum p_k q_k} + \frac{p_m q_m}{\sum p_k q_k} \times q_m G_{(V_m,H)}(b)\right] = (V_m - V_l)\frac{p_l q_l}{\sum p_k q_k}.$$

Thus,

$$\begin{aligned} G_{(V_m,H)}(b) &= \frac{p_l q_l}{p_m q_m^2}\left(\frac{b - V_l}{V_m - b}\right), \\ \bar{b}_{(V_m,L)} &= \frac{p_l q_l V_l + p_m q_m^2 V_m}{p_l q_l + p_m q_m^2}. \end{aligned}$$

Type-(V_h, L) bidder 1. Type-(V_h, L) bidder 1's indifference condition on the support $\left[\bar{b}_{(V_m,L)}, \bar{b}_{(V_h,L)}\right]$ is:

$$\begin{aligned} &(V_h - b)\left[\frac{p_l q_l}{\sum p_k q_k} + \frac{p_m q_m}{\sum p_k q_k} q_h + \frac{p_h q_h}{\sum p_k q_k} q_h G_{(V_h,L)}(b)\right] \\ = \ &\left(V_h - \bar{b}_{(V_m,L)}\right)\left(\frac{p_l q_l}{\sum p_k q_k} + \frac{p_m q_m}{\sum p_k q_k} q_h\right). \end{aligned}$$

Thus,

$$\begin{aligned} G_{(V_h,L)}(b) &= \frac{p_l q_l + p_m q_m q_h}{p_h q_h^2}\frac{b - \bar{b}_{(V_m,L)}}{V_h - b}, \\ \bar{b}_{(V_h,L)} &= \frac{p_h q_h^2 V_h + (p_l q_l + p_m q_m q_h)\,\bar{b}_{(V_m,L)}}{p_h q_h^2 + p_l q_l + p_m q_m q_h}. \end{aligned}$$

Type-(V_m,H) bidder 1. Type-(V_m,H) bidder 1's indifference condition on the support $\left[\bar{b}_{(V_h,L)},\bar{b}_{(V_m,H)}\right]$ is:

$$\begin{aligned}&(V_m-b)\left[\frac{p_l(1-q_l)}{\sum p_k(1-q_k)}+\frac{p_m(1-q_m)q_m}{\sum p_k(1-q_k)}+\frac{p_h(1-q_h)q_m}{\sum p_k(1-q_k)}+\frac{p_m(1-q_m)^2}{\sum p_k(1-q_k)}G_{(V_m,H)}(b)\right]\\ =\ &\left(V_m-\bar{b}_{(V_h,L)}\right)\left[\frac{p_l(1-q_l)}{\sum p_k(1-q_k)}+\frac{p_m(1-q_m)q_m}{\sum p_k(1-q_k)}+\frac{p_h(1-q_h)q_m}{\sum p_k(1-q_k)}\right].\end{aligned}$$

Thus,

$$\begin{aligned}G_{(V_m,H)}(b)&=\frac{p_l(1-q_l)+p_m(1-q_m)q_m+p_h(1-q_h)q_m}{p_m(1-q_m)^2}\frac{b-\bar{b}_{(V_h,L)}}{V_m-b},\\ \bar{b}_{(V_m,H)}&=\frac{[p_l(1-q_l)+p_m(1-q_m)q_m+p_h(1-q_h)q_m]\,\bar{b}_{(V_h,L)}+p_m(1-q_m)^2V_m}{p_l(1-q_l)+p_m(1-q_m)q_m+p_h(1-q_h)q_m+p_m(1-q_m)^2}.\end{aligned}$$

Type-(V_h,H) bidder 1. Type-(V_h,H) bidder 1's indifference condition on the support $\left[\bar{b}_{(V_m,H)},\bar{b}_{(V_h,H)}\right]$ is:

$$\begin{aligned}&(V_h-b)\left[\frac{p_l(1-q_l)}{\sum p_k(1-q_k)}+\frac{p_m(1-q_m)}{\sum p_k(1-q_k)}+\frac{p_h(1-q_h)q_h}{\sum p_k(1-q_k)}+\frac{p_h(1-q_h)^2}{\sum p_k(1-q_k)}G_{(V_h,H)}(b)\right]\\ =\ &\left(V_h-\bar{b}_{(V_m,H)}\right)\left[\frac{p_l(1-q_l)}{\sum p_k(1-q_k)}+\frac{p_m(1-q_m)}{\sum p_k(1-q_k)}+\frac{p_h(1-q_h)q_h}{\sum p_k(1-q_k)}\right].\end{aligned}$$

Thus,

$$\begin{aligned}G_{(V_h,H)}(b)&=\left[\frac{p_l(1-q_l)+p_m(1-q_m)+p_h(1-q_h)q_h}{p_h(1-q_h)^2}\right]\frac{b-\bar{b}_{(V_m,H)}}{V_h-b},\\ \bar{b}_{(V_h,H)}&=\frac{p_h(1-q_h)^2V_h+[p_l(1-q_l)+p_m(1-q_m)+p_h(1-q_h)q_h]\,\bar{b}_{(V_m,H)}}{p_h(1-q_h)^2+p_l(1-q_l)+p_m(1-q_m)+p_h(1-q_h)q_h}.\end{aligned}$$

Clearly, we can obtain the corresponding expressions in the proof of Proposition 4 when we set $p_l=p_m=p_h=1/3$.

A2. Deriving the Expected Surplus from Deviations.

Now we derive the expressions of different types of bidders' expected surplus from bids in different regions. In order to verify that the prescribed strategies indeed constitute an inefficient equilibrium, each bidder type's expected surplus should be maximized in the support of its prescribe strategy.

__**Type-(V_m, L) bidder 1.**__ Type-(V_m, L) bidder 1's expected surplus from bid b, denoted by $M_{(V_m,L)}$, is:

$$M_{(V_m,L)}(b) = \begin{cases} (V_m - V_l)\frac{p_l q_l}{\sum p_k q_k}, & \text{if } b \in \left[V_l, \bar{b}_{(V_m,L)}\right] \\ \frac{V_m - b}{\sum p_k q_k}\left[p_l q_l + p_m q_m^2 + p_h q_h q_m G_{(V_h,L)}(b)\right], & \text{if } b \in \left[\bar{b}_{(V_m,L)}, \bar{b}_{(V_h,L)}\right] \\ \frac{V_m - b}{\sum p_k q_k}\left[p_l q_l + p_m q_m^2 + p_h q_h q_m \right. \\ \left. + p_m q_m (1-q_m) G_{(V_m,H)}(b)\right], & \text{if } b \in \left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_m,H)}\right] \\ \frac{V_m - b}{\sum p_k q_k}\left[p_l q_l + p_m q_m + p_h q_h q_m \right. \\ \left. + p_h q_h (1-q_h) G_{(V_h,H)}(b)\right], & \text{if } b \in \left[\bar{b}_{(V_m,H)}, \bar{b}_{(V_h,H)}\right] \end{cases}$$

__**Type-(V_h, L) bidder 1.**__ Type-(V_h, L) bidder 1's expected surplus from bid b, denoted by $M_{(V_h,L)}$, is:

$$M_{(V_h,L)}(b) = \begin{cases} \frac{V_h - b}{\sum p_k q_k}\left[p_l q_l + p_m q_m q_h G_{(V_m,L)}(b)\right], & \text{if } b \in \left[V_l, \bar{b}_{(V_m,L)}\right] \\ \frac{V_h - \bar{b}_{(V_m,L)}}{\sum p_k q_k}\left(p_l q_l + p_m q_m q_h\right), & \text{if } b \in \left[\bar{b}_{(V_m,L)}, \bar{b}_{(V_h,L)}\right] \\ \frac{V_h - b}{\sum p_k q_k}\left[p_l q_l + p_m q_m q_h + p_h q_h^2 \right. \\ \left. + p_m q_m (1-q_h) G_{(V_m,H)}(b)\right], & \text{if } b \in \left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_m,H)}\right] \\ \frac{V_h - b}{\sum p_k q_k}\left[p_l q_l + p_m q_m + p_h q_h^2 \right. \\ \left. + p_h q_h (1-q_h) G_{(V_h,H)}(b)\right], & \text{if } b \in \left[\bar{b}_{(V_m,H)}, \bar{b}_{(V_h,H)}\right] \end{cases}$$

__**Type-(V_m, H) bidder 1.**__ Type-(V_m, H) bidder 1's expected surplus from bid b, denoted by $M_{(V_m,H)}$, is:

$$M_{(V_m,H)}(b) = \begin{cases} \frac{V_m - b}{\sum p_k (1-q_k)}\left[p_l (1-q_l) + p_m (1-q_m) q_m G_{(V_m,L)}(b)\right], & \text{if } b \in \left[V_l, \bar{b}_{(V_m,L)}\right] \\ \frac{V_m - b}{\sum p_k (1-q_k)}\left[p_l (1-q_l) + p_m (1-q_m) q_m \right. \\ \left. + p_h (1-q_h) q_m G_{(V_h,L)}(b)\right], & \text{if } b \in \left[\bar{b}_{(V_m,L)}, \bar{b}_{(V_h,L)}\right] \\ \frac{V_m - \bar{b}_{(V_h,L)}}{\sum p_k (1-q_k)}\left[p_l (1-q_l) + p_m (1-q_m) q_m + p_h (1-q_h) q_m\right], & \text{if } b \in \left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_m,H)}\right] \\ \frac{V_m - b}{\sum p_k (1-q_k)}\left[p_l (1-q_l) + p_m (1-q_m) + p_h (1-q_h) q_m \right. \\ \left. + p_h (1-q_h)(1-q_m) G_{(V_h,H)}(b)\right], & \text{if } b \in \left[\bar{b}_{(V_m,H)}, \bar{b}_{(V_h,H)}\right] \end{cases}$$

Type-(V_h, H) bidder 1. Type-(V_h, H) bidder 1's expected surplus from bid b, denoted by $M_{(V_h,H)}$, is:

$$M_{(V_h,H)}(b) = \begin{cases} \frac{V_h - b}{\sum p_k(1-q_k)} \left[p_l(1-q_l) + p_m(1-q_m)\, q_h G_{(V_m,L)}(b)\right], & \text{if } b \in \left[V_l, \bar{b}_{(V_m,L)}\right] \\ \frac{V_h - b}{\sum p_k(1-q_k)} \left[p_l(1-q_l) + p_m(1-q_m)\, q_h \right. \\ \qquad \left. + p_h(1-q_h)\, q_h G_{(V_h,L)}(b)\right], & \text{if } b \in \left[\bar{b}_{(V_m,L)}, \bar{b}_{(V_h,L)}\right] \\ \frac{V_h - b}{\sum p_k(1-q_k)} \left[p_l(1-q_l) + p_m(1-q_m)\, q_h + p_h(1-q_h)\, q_h \right. \\ \qquad \left. + p_m(1-q_m)(1-q_h)\, G_{(V_m,H)}(b)\right], & \text{if } b \in \left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_m,H)}\right] \\ \frac{V_h - \bar{b}_{(V_m,H)}}{\sum p_k(1-q_k)} \left[p_l(1-q_l) + p_m(1-q_m) + p_h(1-q_h)\, q_h\right], & \text{if } b \in \left[\bar{b}_{(V_m,H)}, \bar{b}_{(V_h,H)}\right]. \end{cases}$$

A3. The Graphs for the Example in the Proof of Proposition 4.

We collect the graphs for the example in the proof of Proposition 4, i.e., we let $V_l = 0, V_m = 1, V_h = 2$, and let $p_k = 1/3$ for $k \in \{l, m, h\}$ and $q_l = 0.9, q_m = 0.1, q_h = 0.05$. The following graphs show the expected surplus functions for the different types of bidders to show that the candidate inefficient equilibrium derived in **A1** indeed constitute an equilibrium.

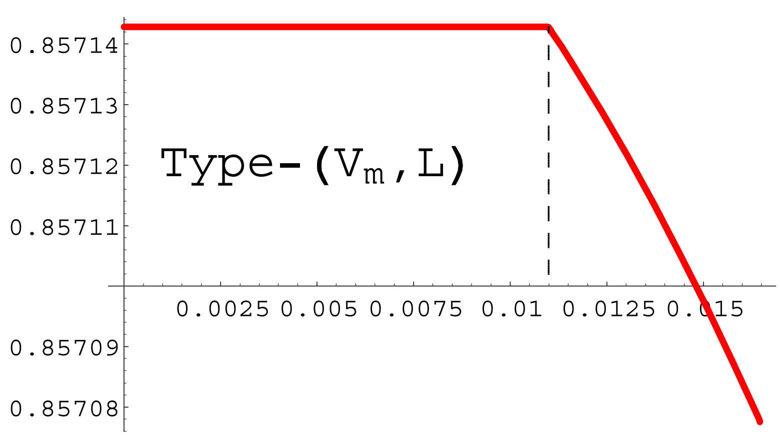

Figure 4: Type-(V_m, L) bidder's expected surplus when bidding $b \in \left[V_l, \bar{b}_{(V_h,L)}\right]$.

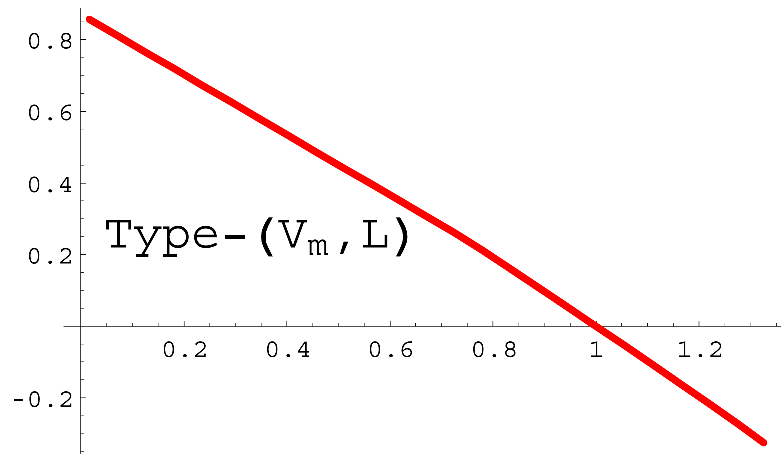

Figure 5: Type-(V_m, L) bidder's expected surplus when bidding $b \in \left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$.

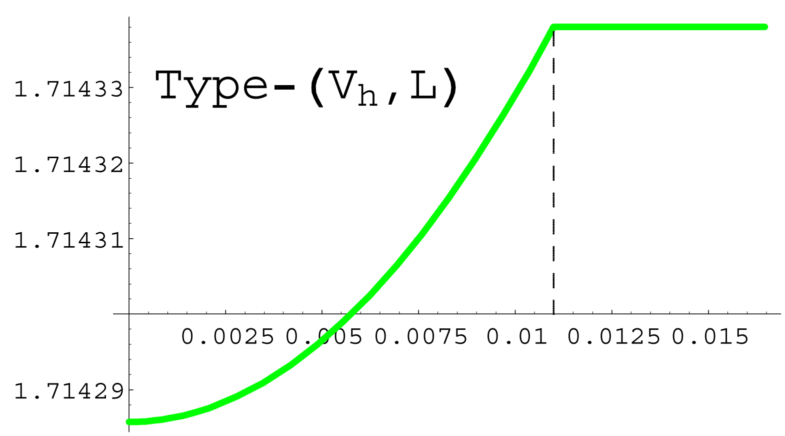

Figure 6: Type-(V_h, L) bidder's expected surplus when bidding $b \in \left[V_l, \bar{b}_{(V_h,L)}\right]$.

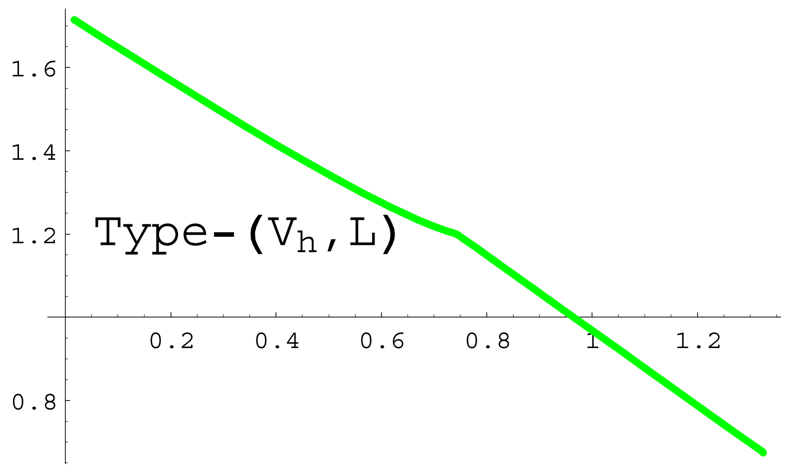

Figure 7: Type-(V_h, L) bidder's expected surplus when bidding $b \in \left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$.

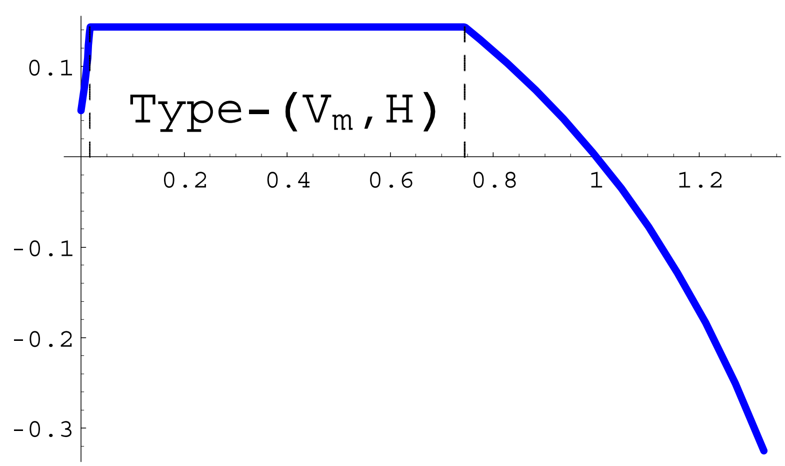

Figure 8: Type-(V_m, H) bidder's expected surplus when bidding $b \in \left[V_l, \bar{b}_{(V_h,H)}\right]$.

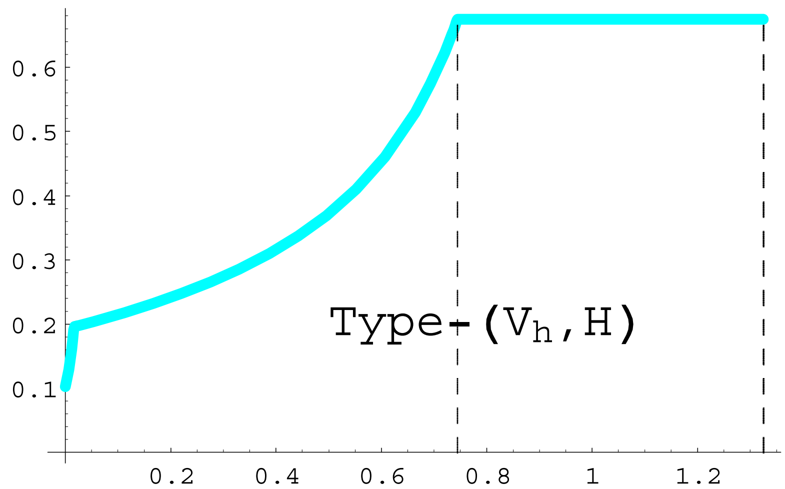

Figure 9: Type-(V_h, H) bidder's expected surplus when bidding $b \in \left[V_l, \bar{b}_{(V_h,H)}\right]$.

Appendix B.

In this appendix, we provide a detailed and elementary derivation of the equilibrium of the asymmetric FPA auction discussed in footnote (9).

Suppose that bidder 1 and $2'$s valuations are drawn independently from Uniform $[0, h_1]$ and $[0, h_2]$ respectively where $h_1 \neq h_2$. Let $b_i : [0, h_i] \to R_+$ be i's bidding function; and $\phi_i \equiv b_i^{-1}$ bidder i's inverse bid function. Given $\phi_2(\cdot)$, bidder 1 with valuation v solves

$$\max_{b \geq 0} \frac{(v-b)\,\phi_2(b)}{h_2}.$$

The first order condition is $\phi_2(b) = (v-b)\,\phi_2'(b)$, i.e.,

$$\phi_2'(b) = \frac{\phi_2(b)}{\phi_1(b) - b}. \tag{25}$$

Similarly, bidder 2's first order condition implies

$$\phi_1'(b) = \frac{\phi_1(b)}{\phi_2(b) - b}. \tag{26}$$

Consider the change-of-variable $k_i(b) \equiv \phi_i(b) - b$. We immediately have $k_i'(b) = \phi_i'(b) - 1$, which can be used to rewrite the system of differential equations (25) and (26) as

$$\left[1 + k_2'(b)\right] k_1(b) = k_2(b) + b, \tag{27}$$

$$\left[1 + k_1'(b)\right] k_2(b) = k_1(b) + b. \tag{28}$$

Summing up the two equations, we obtain $k_2'(b)\,k_1(b) + k_1'(b)\,k_2(b) = 2b$. That is, $d\left[k_1(b)\,k_2(b)\right]/db = 2b$. Together with the initial condition that $k_1(0) = k_2(0) = 0$, we have

$$k_1(b)\,k_2(b) = b^2.$$

Substitute $k_2(b) = b^2/k_1(b)$ into (28), we obtain

$$k_1'(b) = \frac{\left[k_1(b) + b\right] k_1(b)}{b^2} - 1. \tag{29}$$

This homogeneous differential equation can be solved as follows. Define $u_1(b) \equiv k_1(b)/b$. Hence

$$u_1'(b) = \frac{k_1(b) - b k_1'(b)}{b^2}.$$

The differential equation (29) can be rewritten as the following separable differential equation:

$$\frac{u_1'(b)}{u_1(b)^2 - 1} = \frac{1}{b},$$

which as a unique solution

$$u_1(b) = \frac{1+mb^2}{1-mb^2},$$

where m is a constant. Therefore,

$$k_1(b) = \frac{b\left(1+mb^2\right)}{1-mb^2},$$

and

$$k_2(b) = \frac{b^2}{k_1(b)} = \frac{b\left(1-mb^2\right)}{1+mb^2}.$$

Hence,

$$\begin{aligned}\phi_1(b) &= k_1(b) + b = \frac{2b}{1-mb^2},\\ \phi_2(b) &= k_2(b) + b = \frac{2b}{1+mb^2}.\end{aligned}$$

Solving for $\phi_i^{-1} \equiv b_i$, we obtain

$$\begin{aligned}b_1(v) &= \frac{\sqrt{1+mv^2}-1}{mv},\\ b_2(v) &= \frac{1-\sqrt{1-mv^2}}{mv}.\end{aligned}$$

To determine the constant m, we note that $b_1(h_1) = b_2(h_2)$ in equilibrium (for an argument, see Maskin and Riley 2000 for example). Solving for m from the equation

$$b_1(h_1) = \frac{\sqrt{1+mh_1^2}-1}{mh_1} = b_2(h_2) = \frac{1-\sqrt{1-mh_2^2}}{mh_2^2},$$

we obtain a unique solution that

$$m = \frac{h_1^2 - h_2^2}{(h_1 h_2)^2}.$$