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The KPSS Test with Seasonal Dummies

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Abstract

It is shown that the KPSS test for stationarity may be applied without change to regressions with seasonal dummies. In particular, the limit distribution of the KPSS statistic is the same under both the null and alternative hypotheses whether or not seasonal dummies are used.

Keywords: KPSS test, seasonal dummies, stationarity test, unit root.

JEL Classification Number: C32 Time Series Models.

1 Introduction

Kwaitkowski et al (1992) – hereafter, KPSS(1992) – proposed an LM test of the null hypothesis that a series is level stationary or trend stationary, gave a limit theory for the test under the null and analyzed asymptotic power under the alternative hypothesis of difference stationarity. The KPSS test is now widely used in empirical work to test trend stationarity and works as a complement to standard unit root tests in analyzing the properties of time series data.

In the same manner as most unit root limit theory, the asymptotic distribution of the KPSS test depends on whether the data has been filtered by a preliminary regression. In particular, if a mean, linear trend or higher order trend is extracted then the asymptotic distribution of the test statistic changes and critical values need to be adjusted accordingly. When economic time series are observed at weekly, monthly and quarterly frequencies, they commonly exhibit seasonal effects. These seasonal effects are often dealt with by the use of regression dummies, especially when the series is

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considered to be otherwise stationary or trend stationary. A recent overview of this approach to removing deterministic seasonality is given in Ghysels and Osborn (2001, chapter 2). When testing trend stationarity against difference stationary alternatives with such data, seasonal dummies will usually be retained in the regression and applied researchers are interested in whether the use of seasonal dummies affects the limit theory of the KPSS test and requires new critical values, just like other data filters. Similar questions arise in cointegrating regression applications, where the KPSS test has natural extensions in testing for cointegration (e.g. Harris and Inder, 1994)

The present note investigates the validity of the KPSS test in such contexts. It is shown that the limit theory under the null and the alternative of the KPSS test is invariant to the presence of seasonal dummy variables in the regression[1]. The corresponding result also holds for seasonal-frequency KPSS tests (e.g. Busetti and Harvey, 2000; Lyhagen, 2000) and under local alternatives, as recently demonstrated in independent related work by Taylor (2002). Taylor applied spectral independence arguments that have been used in the seasonal unit root literature, whereas we use here a scaled partial sum approach, which may make the arguments more accessible to empirical researchers. We note here, although it is not pursued in what follows, that similar invariance properties apply to the limit theory of the KPSS test in an FM cointegrating regression.

2 Preliminaries

To fix ideas, let y_t be a time series composed of a seasonal effect, a unit root process and a stationary error as follows

$$y_t = x_t'\beta + r_t + u_t, \quad t = 1, ..., n, \tag{1}$$

which we write in observation form as $y = X\beta + r + u$. For convenience in what follows, we assume that the seasonal effect is quarterly, that $t = 1$ corresponds to the first quarter and that $n = 4N$ for some integer N. No loss of generality is involved in these assumptions and it will be readily apparent how the results below extend to other cases. The components of x_t comprise quarterly seasonal dummies ($\delta_{st} = 1, 0$ depending on whether t falls in quarter s or not), so that $\beta' x_t = \sum_{s=1}^4 \beta_{0s}\delta_{st}$. Again, it will be clear in what follows how the analysis extends to the case of seasonal effects plus linear trend in which case $\beta' x_t = \sum_{s=1}^4 \beta_{0s}\delta_{st} + \beta_1 t$, or to the case of seasonal trend effects in which case $\beta' x_t = \sum_{s=1}^4 (\beta_{0s} + \beta_{1s}t)\,\delta_{st}$. The component u_t is weakly dependent with mean zero, autocovariogram $\gamma_h = E(u_t u_{t+h})$, and long run variance $\omega_u^2 > 0$, and r_t is a unit root process

$$r_t = r_{t-1} + v_t, \tag{2}$$

[1] Since we specify seasonal effects as in equation(1) below, the KPSS test would not be based on a maximal invariant if we did not extract seasonal dummies.

where v_t is weakly dependent with mean zero, variance σ_v^2 and long run variance $\omega_v^2 > 0$. It will be sufficient in what follows if the standardized partial sums $n^{-1/2}\sum_{j=1}^{[n\cdot]}(u_j, v_j)$ satisfy a functional CLT.

Using LM principles, KPSS(1992) developed a test of the null hypothesis

$$H_0 : \sigma_v^2 = 0, \tag{3}$$

under which

$$y_t = x_t'\beta + u_t, \tag{4}$$

and the data is trend stationary after seasonal mean correction. Under the alternative $H_1 : \sigma_v^2 > 0$, y_t has a unit root with drift. Let $\widetilde{u}_t, t = 1, 2, ...n$ be the residuals from a least squares regression on (4) and let $\widetilde{\omega}_u^2$ be a consistent estimate of the long run variance ω_u^2 based on $\widetilde{u}_t$ and using lag kernel methods. KPSS used the Bartlett kernel giving

$$\widetilde{\omega}_u^2(\ell) = n^{-1}\sum_{t=1}^{n}\widetilde{u}_t^2 + 2n^{-1}\sum_{s=1}^{\ell}k(s,\ell)\sum_{t=s+1}^{n}\widetilde{u}_t\widetilde{u}_{t-s}, \tag{5}$$

with weight function $k(s,\ell) = 1 - \frac{s}{\ell+1}$ and with lag truncation parameter $\ell \to \infty$ as $n \to \infty$. We assume $\frac{1}{\ell} + \frac{\ell}{n^{1/2}} \to 0$ as $n \to \infty$ under both the null and the alternative.

Define the partial sum residual process $\widetilde{S}_t = \sum_{j=1}^{t}\widetilde{u}_j$, and then the KPSS test statistic is

$$LM = \frac{\frac{1}{n^2}\sum_{t=1}^{n}\widetilde{S}_{t-1}^2}{\widetilde{\omega}_u^2(\ell)}, \tag{6}$$

giving an upper tail test of H_0. Asymptotic critical values are supplied in KPSS(1992).

3 Results

To find the limit distribution of (6) we first consider the limit of the standardized partial sum process

$$\begin{aligned}\frac{\widetilde{S}_{[nr]}}{\sqrt{n}} &= \frac{1}{\sqrt{n}}\sum_{j=1}^{[nr]}\widetilde{u}_j = \frac{1}{\sqrt{n}}\sum_{j=1}^{[nr]}[u_j - x_j'(\widetilde{\beta}-\beta)] = \frac{1}{\sqrt{n}}\sum_{j=1}^{[nr]}u_j - \frac{1}{\sqrt{n}}\sum_{j=1}^{[nr]}x_j'(X'X)^{-1}(X'u) \\ &= \frac{1}{\sqrt{n}}\sum_{j=1}^{[nr]}u_j - \left(\frac{\sum_{j=1}^{[nr]}x_j'}{n}\right)\left(\frac{X'X}{n}\right)^{-1}\left(\frac{X'u}{\sqrt{n}}\right) \end{aligned} \tag{7}$$

By standard limit theory arguments (Phillips and Solo, 1992) we have $n^{-1/2}\sum_{j=1}^{[nr]}u_j \to_d B(r)$, where B is Brownian motion with variance ω_u^2. We also have $n^{-1}\sum_{j=1}^{[nr]}x_j' \to$

$(\frac{r}{4}, \frac{r}{4}, \frac{r}{4}, \frac{r}{4})$, $n^{-1}X'X \to (1/4)I_4$, and

$$\frac{1}{\sqrt{n}}X'u = \frac{1}{2}\frac{1}{\sqrt{\frac{n}{4}}}\begin{pmatrix} \sum_{j=1}^{\frac{n}{4}} u_{4j-3} \\ \sum_{j=1}^{\frac{n}{4}} u_{4j-2} \\ \sum_{j=1}^{\frac{n}{4}} u_{4j-1} \\ \sum_{j=1}^{\frac{n}{4}} u_{4j} \end{pmatrix} \to \frac{1}{2}\begin{pmatrix} B_{u.1}(1) \\ B_{u.2}(1) \\ B_{u.3}(1) \\ B_{u.4}(1) \end{pmatrix} \equiv \frac{1}{2}N\left(0, \begin{pmatrix} \omega_0 & \omega_1 & \omega_2 & \omega_3 \\ \omega_1 & \omega_0 & \omega_1 & \omega_2 \\ \omega_2 & \omega_1 & \omega_0 & \omega_1 \\ \omega_3 & \omega_2 & \omega_1 & \omega_0 \end{pmatrix}\right)$$

where $B_{u.i}(1) =_d N(0, \omega_0)$, $i = 1, ..., 4$, and where

$$\omega_{c=a-b} = \text{lrcov}\,(u_{4j-a}, u_{4j-b}) = \sum_{h=-\infty}^{\infty} \gamma_{4h+a-b}, \quad c = 0, ..., 3,$$

or, more specifically, $\omega_0 = \gamma_0 + 2\sum_{h=1}^{\infty} \gamma_{4h}$, $\omega_1 = \sum_{h=1}^{\infty} \gamma_{2h-1}$, $\omega_2 = 2\sum_{h=1}^{\infty} \gamma_{4h-2}$, and $\omega_3 = \sum_{h=1}^{\infty} \gamma_{2h-1} = \omega_1$. It follows that

$$\begin{aligned} \left(\frac{\sum_{j=1}^{[nr]} x_j'}{n}\right)\left(\frac{X'X}{n}\right)^{-1}\left(\frac{X'u}{\sqrt{n}}\right) &\to_d (\frac{r}{4}, \frac{r}{4}, \frac{r}{4}, \frac{r}{4})\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}\frac{1}{2}\begin{pmatrix} B_{u.1}(1) \\ B_{u.2}(1) \\ B_{u.3}(1) \\ B_{u.4}(1) \end{pmatrix} \\ &= \frac{1}{2}r[B_1(1) + B_2(1) + B_3(1) + B_4(1)] = rB(1), \end{aligned}$$

and then $n^{-1/2}\widetilde{S}_{[nr]} \to_d B(r) - rB(1)$, giving

$$\eta = n^{-2}\sum_{t=1}^{n} \widetilde{S}_{t-1}^2 \to_d \omega^2 \int_0^1 V(r)^2 dr, \tag{8}$$

where $V(r)$ is a standard Brownian bridge process. For any consistent estimate $\widetilde{\omega}_u^2(\ell)$ of ω^2, it now follows that $LM \to_d \omega^2 \int_0^1 V(r)^2 dr$, so that the limit distribution of the KPSS is the same under the null when seasonal dummy extraction is performed as it is for the standard case where deseasonalization is not required.

The same conclusion holds under the alternative. In particular, when $\sigma_v^2 > 0$, a functional law applies to r_t, giving

$$n^{-1/2} r_{[sn]} = n^{-1/2} \sum_{j=1}^{[sn]} v_j \to_d B_v(s) \tag{9}$$

for $s \in [0, 1]$ and B_v Brownian motion with variance ω_v^2. Then

$$\begin{aligned} n^{-3/2}\widetilde{S}_{[an]} &= n^{-3/2}\sum_{j=1}^{[an]} \left\{r_j + u_j - x_j'(X'X)^{-1}X'(u + r)\right\} \\ &= n^{-1}\sum_{j=1}^{[an]} n^{-1/2} r_j - n^{-1}[\frac{[an]}{n}, \frac{[an]}{n}, \frac{[an]}{n}, \frac{[an]}{n}](n^{-1/2}\begin{pmatrix} \sum_{j=1}^{\frac{n}{4}} r_{4j-3} \\ \sum_{j=1}^{\frac{n}{4}} r_{4j-2} \\ \sum_{j=1}^{\frac{n}{4}} r_{4j-1} \\ \sum_{j=1}^{\frac{n}{4}} r_{4j} \end{pmatrix})] + o_p(1) \end{aligned}$$

$$= \quad n^{-1}\sum_{j=1}^{[an]} n^{-1/2} r_j - n^{-1}(\frac{[an]}{n})(n^{-1/2}\sum_{j=1}^{n} r_j)$$

$$\to_d \int_0^a B_v(s)ds - a\int_0^1 B_v(s)ds = \int_0^a \underline{B}_v(s)ds,$$

where $\underline{B}_v(s)$ is the demeaned Wiener process $\underline{B}_v(s) = B_v(s) - \int_0^1 B_v(b)db$. Thus,

$$n^{-4}\sum_{t=1}^{n} \widetilde{S}_{t-1}^2 = n^{-1}\sum_{t=1}^{n}(n^{-3/2}\widetilde{S}_{t-1})^2 \to \int_0^1 \left(\int_0^a \underline{B}_v(s)ds\right)^2 da, \qquad (10)$$

so that $n^{-2}\sum_{t=1}^{n} \widetilde{S}_{t-1}^2$ is $O_p(n^2)$. Just as in KPSS(1992),

$$(n\ell)^{-1}\widetilde{\omega}_u^2(\ell) \to K\int_0^1 \underline{B}_v(s)^2 ds, \qquad (11)$$

provided $n^{-1/2}\ell \to 0$ as $n \to \infty$. The constant K is defined by $K = \int_{-1}^1 k(s)ds$, where $k(s)$ represents the weighting function used in $\widetilde{\omega}_u^2(\ell)$. It follows that $LM = O_p(n/\ell)$ as $n \to \infty$, and the test is consistent. Moreover, (6) and (10) imply that

$$(\ell/n)LM \to_d \int_0^1 \left(\int_0^a \underline{B}_v(s)ds\right)^2 da / \int_0^1 \underline{B}_v(s)^2 ds,$$

giving the same limit distribution under the alternative hypothesis for the scaled LM statistic as in the case where there is no deseasonalization.

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