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Abstract

One set of n objects of type I, another set of n objects of type II, and an amount M of money is to be completely allocated among n agents in such a way that each agent gets one object of each type with some amount of money. We propose a new solution concept to this problem called a perfectly fair allocation. It is a refinement of the concept of fair allocation. An appealing and interesting property of this concept is that every perfectly fair allocation is Pareto optimal. It is also shown that a perfectly fair allocation is envy free and gives each agent what he likes best, and that a fair allocation need not be perfectly fair. Furthermore, we give a necessary and sufficient condition for the existence of a perfectly fair allocation. Precisely, we show that there exists a perfectly fair allocation if and only if the valuation matrix is an optimality preserved matrix. Optimality preserved matrices are a class of new and interesting matrices. An extension of the model is also discussed.

Keywords: Perfectly fair allocation, indivisibility, discrete optimization, multi-person decision, existence theorem, optimality preserved matrix

JEL Classification: D3, D31, D6, D61, D63, D7, D74

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1 Introduction

The subject of this paper is the distribution of a collection of objects (such as houses and cars) and an amount of money among a group of people. It is concerned with fairness, equity, justice, and efficiency of such distributions. These problems arise naturally in many situations, and are both difficult and controversial. Recall that given an allocation, we say agent i envies agent j if agent i prefers the bundle of agent j to his or her own. An allocation is *envy free* or *fair* if no agent envies any other. As it has been noted, the concept of fairness may not exactly correspond to the everyday notion of fairness. In fact, how we define equity, fairness, and justice has been, and remains a most provocative question in the course of mankind's endless quest for equity, fairness, and justice. The goal of this paper is to propose a new solution concept to a class of fair allocation problems and to investigate what conditions can ensure the existence of such a solution which is both fair and Pareto optimal.

The study of fair division problem can date back at least to Steinhaus (1948). But most of the literature has evovled from Foley (1967) in which the concept of envy free allocation is precisely formulated. A major defect of this concept is that an envy free allocation may not be efficient (i.e., Pareto optimal). Various criteria on equity and justice are discussed in Rawls (1971). Furthermore, in Varian (1974) a general formulation of fair division of divisible goods is given. He proved the existence of an envy free and efficient allocation by imposing certain conditions on the model.

The fair allocation problem of indivisible objects is investigated by Svensson (1983), and further studied by Maskin (1987), Alkan, Demange and Gale (1991), Su (1999), and Yang (1998). In these papers it is shown that in an economy if each agent consumes only one indivisible object and there is a divisible good (say money), then the set of envy free and efficient allocations is not empty under certain mild conditions. In these models a fundamental assumption in common is that each agent has no use for more than one indivisible object. As noted by Svensson (1983) this assumption leads to a nice conclusion that an envy free allocation must also be efficient. Unfortunately, this property does not automatically carry into more general situations where agents are allowed to consume more than one indivisible object. Sun and Yang (2000) have recently developed a more general model in which there are no restrictions on the agents' consumption of indivisible objects. A sufficient condition is introduced for the existence of an envy free and efficient allocation. On the other hand, an algorithmic procedure is proposed by Klijn (2000) to find an envy free allocation in a setting where agents have quasi-linear utilities in money and there are the same number of agents as objects. Furthermore, Alkan et al. (1991) and Tadenuma and Thomson (1991) have given two different sets of criteria for selecting desirable envy free allocations when there exist multiple envy free allocations.

In this paper we consider the following problem: One set of n objects of type I, another set of n objects of type II, and an amount M of money, are to be completely allocated among n agents in such a way that each agent gets one object of each type with some amount of money. We propose a new solution concept to this problem called a perfectly fair allocation. It is a refinement of the concept of fair allocation. An appealing and interesting property of this concept is that every perfectly fair allocation is Pareto optimal. It is also shown that a perfectly fair allocation is envy free and gives each agent what he likes best, and that a fair allocation need not be perfectly fair. Furthermore, we give a necessary and sufficient condition for the existence of a perfectly fair allocation. To be more precise, we show that there exists a perfectly fair allocation if and only if the valuation matrix is an optimality preserved matrix. We stress that optimality preserved matrices are a class of new and interesting matrices and might be worth being studied in their own right. An extension of the model is also discussed.

The rest of the paper is organized as follows. In Section 2 basic concepts are introduced, and the formal model is defined. In Section 3 several existence theorems are established. Finally in Section 4 an extension of the basic model is discussed and existence results are derived.

2 The Model of Perfectly Fair Allocation

We first introduce some notation. Let I_k be the set of first k positive integers and $\mathbb{R}^k$ the k-dimensional Euclidean space.

Our model consists of a finite number (n) of agents, denoted by I_n, the same number of indivisible objects of type I, denoted by $\mathcal{O}_1$, the same number of indivisible objects of type II, denoted by $\mathcal{O}_2$, and a fixed amount of money, denoted by M. One might think of $\mathcal{O}_1$ and $\mathcal{O}_2$ as the collections of houses and cars, respectively. Here M can be any real number. If M is negative, this will be the case in cost sharing problems. Here money will be treated as a perfectly divisible good. It is assumed that each agent demands or consumes exactly one of the indivisible objects of each type and a certain amount of money. The preference relation of each agent $i \in I_n$ can be represented by a utility function $u_i : \mathcal{O}_1 \times \mathcal{O}_2 \times \mathbb{R} \mapsto \mathbb{R}$. Throughtout the paper it will be assumed that $u_i(h, c, m)$ is a nondecreasing and continuous function in money (i.e., in m).

A feasible allocation is a 3-tuple of vectors $(\pi, \rho, z = (x, y))$ where $\pi = (\pi(1), \cdots, \pi(n))$ and $\rho = (\rho(1), \cdots, \rho(n))$ are the permutations of the elements in $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively, and where $\sum_{i=1}^{n}(x_i + y_i) = M$. Thus, at a feasible allocation, all objects and money will be completely distributed to the agents in a way that every agent gets exactly one indivisible object of each type and a certain amount of money. More precisely, each agent i receives a bundle of goods $(\pi(i), \rho(i), x_{\pi(i)} + y_{\rho(j)})$ consisting of object $\pi(i)$ of type I and object $\rho(i)$ of type II and the amount $x_{\pi(i)} + y_{\rho(i)}$ of money. If $x_{\pi(i)} + y_{\rho(i)} < 0$, then agent i pays others the amount $|x_{\pi(i)} + y_{\rho(i)}|$ of money.

Let $T = \{z = (x, y) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n}(x_j + y_j) = M\}$ be the $(2n-1)$-dimensional hyperplane and let $\Theta = \{\pi \mid \pi = (\pi(1), \cdots, \pi(n))$ a permutation of $I_n\}$. Thus a feasible allocation (π, ρ, z) is merely an element of $\Theta \times \Theta \times T$.

We can now introduce the major solution concept of the paper.

Definition 2.1 *A feasible allocation (π, ρ, z) is a perfectly fair allocation if it holds*

$$u_i(\pi(i), \rho(i), x_{\pi(i)} + y_{\rho(i)}) \geq u_i(\pi(j), \rho(k), x_{\pi(j)} + y_{\rho(k)}), \ \forall i, j, k \in I_n.$$

Recall that a feasiable allocation is fair or envy free if no agent prefers any other agent's bundle to his own. Clearly a perfectly fair allocation must be a fair allocation but the reverse is not true in general. Furthermore, a perfectly fair allocation gives each agent what he likes best. The concept of perfectly fair allocation can be also explained as follows. An auctioneer chooses a compensation scheme vector $z = (x, y) \in T$ for the pairs of objects in $\mathcal{O}_1 \times \mathcal{O}_2$ in such a way that every agent can pick up a pair of house and car with their compensation which he likes best without conflicting his interest with any other's. The following concept is a familiar one.

Definition 2.2 *A feasible allocation is efficient or Pareto optimal if there is no other feasible allocation which makes everyone at least as well as before and at least one agent strictly better off.*

The problem of the concept of fair allocation lies in the fact that it is not necessarily efficient. The following example indicates that a fair allocation indeed need not be efficient.

Example 1. Consider the case in which there are two agents 1, 2 and there are two houses $h1$, $h2$, and two cars $c1$, $c2$, and total money (say, dollar)

M is equal to zero. Both agents have quasi-linear utilities in money (i.e., $u_i(h, c, m) = \alpha(i, h, c) + m$, $i = 1, 2$) and the values of the agents for the different pairs of house and car are given in Table 1.

In this example when agent 1 gets house $h1$ and car $c2$ with \$1 and agent 2 gets house $h2$ and car $c1$ by paying \$1, this allocation is fair but not Pareto optimal, because another allocation in which agent 1 gets house $h2$ and car $c2$ by paying \$0.5 and agent 2 gets house $h1$ and car $c1$ with \$0.5 makes both agents strictly better off.

Table 1: The values of objects for both agents

$\alpha(1, h, c)$	$C1$	$C2$	$\alpha(2, h, c)$	$c1$	$c2$	
$h1$	2	3	$h1$	3	2	
$h2$	4	5	$h2$	4	4	

One of the most appealing and interesting properties of perfectly fair allocation is that it is also efficient as shown below.

Theorem 2.3 *Every perfectly fair allocation is Pareto optimal.*

Proof: Let (π, ρ, z) be a perfectly fair allocation. Then it follows that

$$u_i(\pi(i), \rho(i), x_{\pi(i)}+y_{\rho(i)}) \geq u_i(\pi(j), \rho(k), x_{\pi(j)}+y_{\rho(k)}), \forall i, j, k \in I_n; \quad (2.1)$$

Now suppose to the contrary that (π, ρ, z) is not efficient. Then there would exist a feasible allocation $(\bar{\pi}, \bar{\rho}, \bar{z})$ weakly preferred by all agents and strictly preferred by at least one agent. That is, it holds

$$u_i(\bar{\pi}(i), \bar{\rho}(i), \bar{x}_{\bar{\pi}(i)} + \bar{y}_{\bar{\rho}(i)}) \geq u_i(\pi(i), \rho(i), x_{\pi(i)} + y_{\rho(i)}), \forall i \in I_n; \quad (2.2)$$

and there is some $j \in I_n$ satisfying

$$u_j(\bar{\pi}(j), \bar{\rho}(j), \bar{x}_{\bar{\pi}(j)} + \bar{y}_{\bar{\rho}(j)}) > u_j(\pi(j), \rho(j), x_{\pi(j)} + y_{\rho(j)}). \quad (2.3)$$

Inequalities (2.1), (2.2) and (2.3) imply that for all $i \in I_n$,

$$u_i(\bar{\pi}(i), \bar{\rho}(i), \bar{x}_{\bar{\pi}(i)} + \bar{y}_{\bar{\rho}(i)}) \geq u_i(\bar{\pi}(i), \bar{\rho}(i), x_{\bar{\pi}(i)} + y_{\bar{\rho}(i)}),$$

and

$$u_j(\bar{\pi}(j), \bar{\rho}(j), \bar{x}_{\bar{\pi}(j)} + \bar{y}_{\bar{\rho}(j)}) > u_j(\bar{\pi}(j), \bar{\rho}(j), x_{\bar{\pi}(j)} + y_{\bar{\rho}(j)}).$$

Since $u_i(j,k,\cdot)$, $i,j,k \in I_n$, are nondecreasing in money, we have that for all $i \in I_n$,

$$\bar{x}_{\bar{\pi}(i)} + \bar{y}_{\bar{\rho}(i)} \geq x_{\bar{\pi}(i)} + y_{\bar{\rho}(i)},$$

and

$$\bar{x}_{\bar{\pi}(j)} + \bar{y}_{\bar{\rho}(j)}) > x_{\bar{\pi}(j)} + y_{\bar{\rho}(j)}.$$

This implies that

$$M = \sum_{j=1}^{n} (\bar{x}_j + \bar{y}_j) > \sum_{j=1}^{n} (x_j + y_j) = M,$$

yielding a contradiction. Therefore, (π, ρ, z) must be efficient as well. □

In Example 1 there is a perfectly fair allocation, namely, agent 1 gets house $c2$ and car $c2$ by paying \$1 and agent 2 gets house $h1$ and car $c1$ with \$1. However, perfectly fair allocations may not always exist as shown in the following example.

Example 2. Consider the case in which there are two agents 1, 2 and there are two houses $h1$, $h2$, and two cars $c1$, $c2$, and total money (say, dollar) M is equal to zero. The values of the agents for the different pairs of house and car are given in Table 2, and utility functions are given by $u_i(h,c,m) = \alpha(i,h,c) + m$, $i = 1{,}2$.

In this example there is only one fair and efficient allocation, namely, agent 1 gets house $h1$ and car $c1$ and x\$ with $2 \leq x \leq 2.5$, and agent 2 gets house $h2$ and car $c2$ by paying x\$. Suppose that this allocation is perfectly fair. Then for agent 2, the following system of inequalities must have a solution.

$$\begin{array}{rcl} 5 + x_2 + y_2 & \geq & x_1 + y_1 \\ 5 + x_2 + y_2 & \geq & 4.5 + x_1 + y_2 \\ 5 + x_2 + y_2 & \geq & 4.5 + x_2 + y_1 \\ x_1 + y_1 & = & -(x_2 + y_2) \\ x_1 + y_1 & = & x \\ & 2 \leq x \leq & 2.5 \end{array}$$

It follows from the second and third inequalities that $x \leq 0.5$, yielding a contradiction to the sixth inequality. Thus there does not exist any perfectly fair allocation in this example.

Table 2: The values of objects for both agents

$\alpha(1,h,c)$	$c1$	$c2$	$\alpha(2,h,c)$	$c1$	$c2$	
$h1$	5	4.5	$h1$	0	4.5	
$h2$	4.5	9	$h2$	4.5	5	

In the next section we will establish several existence theorems for perfectly fair allocations in the case that agents have quasi-linear utilities in money. Relaxing the assumption of quasi-linearity in money still poses a difficult challenge to us.

3 Existence Theorems

Given an $n \times n \times n$ trimatrix $A = (\alpha(i,h,c))$, an assignment $(\pi, \rho) \in \Theta \times \Theta$ is an *optimal assignment* if $\sum_{i \in I_n} \alpha(i, \pi(i), \rho(i)) \geq \sum_{i \in I_n} \alpha(i, \tau(i), \gamma(i))$ for every $(\tau, \gamma) \in \Theta \times \Theta$. Similarly, given an $n \times n$ matrix $B = (\beta(i,o))$, we call an assignment $\pi \in \Theta$ an *optimal assignment* if $\sum_{i \in I_n} \beta(i, \pi(i)) \geq \sum_{i \in I_n} \beta(i, \tau(i))$ for every $\tau \in \Theta$.

When we restrict to the case where every agent has quasi-linear utilities in money, then the model described in Section 2 can be simply represented as $\mathcal{E} = ((\alpha(i,h,c)), n, M)$ where $(\alpha(i,h,c))$ is an $n \times n \times n$ trimatrix, n is the number of agents, and M is the total amount of money. Recall that $\alpha(i,h,c)$ is the value of a pair of house h and car c to agent i. We call $(\alpha(i,h,c))$ the *valuation matrix.* Furthermore, for a specific model where objects are only houses or cars, we will simply represent such a model by $\mathcal{E} = ((\beta(i,o)), n, M)$, where $(\beta(i,o))$ is an $n \times n$ matrix, n is the number of agents, and M is the total amount of money. $\beta(i,o)$ is the value of object o to agent i.

Recall the following duality theorem from linear programming, which has been used by Shapley and Shubik (1972), and Alkan et al. (1991) for related models.

Lemma 3.1 *Let $B = (\beta(i,o))$ be an $n \times n$ matrix. If $\pi \in \Theta$ is an optimal assignment, there exist two n-vectors v and w such that*

$$v_i + w_o \geq \beta(i,o), \ \forall i \in I_n, o \in \mathcal{O}_1$$

and

$$v_i + w_{\pi(i)} = \beta(i, \pi(i)), \ \forall i \in I_n.$$

Lemma 3.2 *Given a model $\mathcal{E} = ((\beta(i,o)), n, M)$, then there exists at least one optimal assignment with respect to the matrix $(\beta(i,o))$. For each optimal assignment π, there exists a distribution n-vector x of money M such that (π, x) is an efficient and fair allocation.*

Proof: The first statement is obvious, since there are only a finite number of assignments. The second statement can be seen as follows. Since π is an optimal assignment, it follows from Lemma 3.1 that there exists v and w such that

$$v_i + w_o \geq \beta(i,o), \ \forall i \in I_n, o \in \mathcal{O}_1$$

and

$$v_i + w_{\pi(i)} = \beta(i, \pi(i)), \ \forall i \in I_n.$$

From the above inequalities we obtain

$$\beta(i, \pi(i)) - w_{\pi(i)} \geq \beta(i,o) - w_o, \ \forall i \in I_n, o \in \mathcal{O}_1.$$

Let $y_i = -w_i$, $\delta = (M - \sum_{i \in I_n} y_i)/n$, and $x_i = y_i + \delta$ for each $i \in I_n$. Define $x = (x_1, \cdots, x_n)$. Then we have

$$\beta(i, \pi(i)) + x_{\pi(i)} \geq \beta(i,o) + x_o, \ \forall i \in I_n, o \in \mathcal{O}_1$$

and

$$\sum_{i \in I_n} x_i = M.$$

Thus, (π, x) is an efficient and fair allocation. □

Using the same argument of the above lemma or Theorem 4.1 of Sun and Yang (2000), we have

Theorem 3.3 *Given a model $\mathcal{E} = ((\alpha(i,h,c)), n, M)$, then there exists at least one optimal assignment with respect to the matrix $(\alpha(i,h,c))$. For each optimal assignment (π, ρ), there exists a distribution $2n$-vector (x, y) of money M such that $(\pi, \rho, (x, y))$ is an efficient and fair allocation.*

As Example 2 indicates that perfectly fair allocations may not always exist, this motivates a natural question: Under what circumstance does a perfectly fair allocation exist? The remaining section is to present a necessary and sufficient condition for the existence of a perfectly fair allocation.

Condition 3.4 *The trimatrix* $(\alpha(i, h, c))$ *has the following property: For every* $i \in I_n$*, it holds*

$$\alpha(i, h1, c1) + \alpha(i, h2, c2) = \alpha(i, h1, c2) + \alpha(i, h2, c1), \\ \forall h1, h2 \in \mathcal{O}_1, c1, c2 \in \mathcal{O}_2.$$

Condition 3.5 *The trimatrix* $(\alpha(i, h, c))$ *has the following property: For every* $i \in I_n$*, there exist two* n *vectors* $H(i) = (H_1(i), \cdots, H_n(i))$ *and* $C(i) = (C_1(i), \cdots, C_n(i))$ *such that it holds*

$$\alpha(i, h, c) = H_h(i) + C_c(i), \ \forall h \in \mathcal{O}_1, c \in \mathcal{O}_2.$$

Lemma 3.6 *Conditions 3.4 and 3.5 are equivalent.*

Proof: Condition 3.5 clearly implies Condition 3.4. Now we prove that Condition 3.4 implies Condition 3.5. From Condition 3.4, we see that

$$\alpha(i, 1, 1) + \alpha(i, h, c) = \alpha(i, 1, c) + \alpha(i, h, 1) \qquad \text{for all } h \in \mathcal{O}_1 \text{ and } c \in \mathcal{O}_2.$$

Thus we obtain that

$$\alpha(i, h, c) - \alpha(i, h, 1) = \alpha(i, 1, c) - \alpha(i, 1, 1) \qquad \text{for all } h \in \mathcal{O}_1 \text{ and } c \in \mathcal{O}_2.$$

For each $h \in \mathcal{O}_1$ and $c \in \mathcal{O}_2$, let

$$H_h(i) = \alpha(i, h, 1), \quad \text{and} \quad C_c(i) = \alpha(i, 1, c) - \alpha(i, 1, 1).$$

Then we have that $\alpha(i, h, c) = H_h(i) + C_c(i)$ for all $h \in \mathcal{O}_1$ and $c \in \mathcal{O}_2$. That is, Condition 3.5 holds. □

Definition 3.7 *Given an* $n \times n \times n$ *trimatrix* $A = (\alpha(i, h, c))$ *and an assignment* $(\pi, \rho) \in \Theta \times \Theta$*, the following process is called an* $\mathcal{M}$*-transformation of* A *from* (π, ρ) *if each element* $\alpha(i, h, c)$ *except for* $\alpha(i, \pi(i), \rho(i))$*,* $i \in I_n$ *is added with a nonnegative number* $\delta(i, h, c)$ *so that the new trimatrix* $T = (\alpha(i, h, c) + \delta(i, h, c))$ *satisfies Condition 3.4.*

The trimatrix T above will be called an $\mathcal{M}$-matrix resulted from (π, ρ).

Definition 3.8 *An* $n \times n \times n$ *trimatrix* $(\alpha(i, h, c))$ *is an optimality preserved matrix if there exist an optimal assignment* $(\pi, \rho) \in \Theta \times \Theta$ *and an* $\mathcal{M}$*-transformation from* (π, ρ) *such that* (π, ρ) *is still an optimal assignment in the* $\mathcal{M}$*-matrix resulted from* (π, ρ)*.*

Obviously, a trimatrix satisfying Condition 3.4 is an optimality preserved matrix. We are now ready to introduce the main existence result of this paper which states a necessary and sufficient condition for the existence of a perfectly fair allocation.

Theorem 3.9 *Given a model $\mathcal{E} = ((\alpha(i,h,c)), n, M)$, there exists a perfectly fair allocation if and only if the valuation trimatrix $(\alpha(i,h,c))$ is an optimality preserved matrix.*

Proof: Since $(\alpha(i,h,c))$ is an optimality preserved matrix, then there exist an optimal assignment $(\pi,\rho) \in \Theta\times\Theta$ and an $\mathcal{M}$-transformation from (π,ρ) such that (π,ρ) is still an optimal assignment in the $\mathcal{M}$-matrix resulted from (π,ρ). Let $T = (\bar{\alpha}(i,h,c))$ be the $n\times n\times n$ $\mathcal{M}$-matrix resulted from (π,ρ). So we have

$$\bar{\alpha}(i,h,c) \geq \alpha(i,h,c)$$

$$\bar{\alpha}(i,\pi(i),\rho(i)) = \alpha(i,\pi(i),\rho(i))$$

for all $i \in I_n$, $(h,c) \in \mathcal{O}_1 \times \mathcal{O}_2$, and

$$\sum_{i\in I_n} \bar{\alpha}(i,\pi(i),\rho(i)) \geq \sum_{i\in I_n} \bar{\alpha}(i,\tau(i),\gamma(i)), \forall(\tau,\gamma) \in \Theta\times\Theta. \tag{3.4}$$

Since T satisfies Condition 3.4, then there exist two n-vectors $H(i)$ and $C(i)$ for each $i \in I_n$ so that $\bar{\alpha}(i,h,c) = H_h(i) + C_c(i)$ for every $h \in \mathcal{O}_1$, $c \in \mathcal{O}_2$. Then we can rewrite equation (3.4) as

$$\sum_{i\in I_n}(H_i(\pi(i))+C_i(\tau(i)) \geq \sum_{i\in I_n}(H_i(\tau(i))+C_i(\gamma(i))), \forall(\tau,\gamma) \in \Theta\times\Theta. \tag{3.5}$$

It follows from equation (3.5) that

$$\begin{aligned} \sum_{i\in I_n} H_i(\pi(i)) &\geq \sum_{i\in I_n} H_i(\tau(i)) \\ \sum_{i\in I_n} C_i(\rho(i)) &\geq \sum_{i\in I_n} C_i(\gamma(i)) \end{aligned}$$

for all $(\tau,\gamma) \in \Theta\times\Theta$. By Lemma 3.2 there exist two n-vectors x and y such that $\sum_{i\in I_n} x_i = M/2$, $\sum_{i\in I_n} y_i = M/2$, and

$$\begin{aligned} H_i(\pi(i)) + x_{\pi(i)} &\geq H_i(j) + x_j \\ C_i(\rho(i)) + y_{\rho(i)} &\geq C_i(l) + y_l \end{aligned}$$

for all $i, j, l \in I_n$. It follows that

$$\begin{aligned}\alpha(i,\pi(i),\rho(i)) + x_{\pi(i)} + y_{\rho(i)} &= \bar{\alpha}(i,\pi(i),\rho(i)) + x_{\pi(i)} + y_{\rho(i)} \\ &= H_i(\pi(i)) + C_i(\rho(i)) + x_{\pi(i)} + y_{\rho(i)} \\ &\geq H_i(j) + x_j + C_i(l) + y_l \\ &= \bar{\alpha}(i,j,l) + x_j + y_l \\ &\geq \alpha(i,j,l) + x_j + y_l\end{aligned}$$

for all $i, j, l \in I_n$. Thus $(\pi, \rho, (x, y))$ is a perfectly fair allocation.

Now suppose that $(\pi, \rho, (x, y))$ is a perfectly fair allocation. Then it holds that

$$\alpha(i,\pi(i),\rho(i)) + x_{\pi(i)} + y_{\rho(i)} \geq \alpha(i,h,c) + x_h + y_c$$

for all $i \in I_n$, $h \in \mathcal{O}_1$, $c \in \mathcal{O}_2$. It is readily seen that (π, ρ) is an optimal assignment with respect to $(\alpha(i,h,c))$. Let $A_i = \alpha(i,\pi(i),\rho(i)) + x_{\pi(i)} + y_{\rho(i)}$ for each $i \in I_n$. Let $d_i(h,c) = A_i - \alpha(i,h,c) - x_h - y_c$ for every $h \in \mathcal{O}_1$, $c \in \mathcal{O}_2$. Clearly, $d_i(h,c) \geq 0$. Furthermore, $d_i(\pi(i),\rho(i)) = 0$ for all $i \in I_n$. Let $H_h(i) = A_i - x_h$ and $C_c(i) = -y_c$. Now define $\bar{\alpha}(i,h,c) = \alpha(i,h,c) + d_i(h,c)$. Clearly $\bar{\alpha}(i,h,c) = H_h(i) + C_c(i)$ and $\bar{\alpha}(i,\pi(i),\rho(i)) = \alpha(i,\pi(i),\rho(i))$ for all $i \in I_n$. Thus $(\bar{\alpha}(i,h,c)$ satisfies Condition 3.4. Furthermore, for any $(\tau,\gamma) \in \Theta \times \Theta$, we have

$$\begin{aligned}\sum_{i\in I_n} \bar{\alpha}(i,\pi(i),\rho(i)) &= \sum_{i\in I_n} \alpha(i,\pi(i),\rho(i)) \\ &= \sum_{i\in I_n} (A_i - x_{\pi(i)} - y_{\rho(i)}) \\ &= \sum_{i\in I_n} (\bar{\alpha}(i,\tau(i),\gamma(i)) + x_{\tau(i)} + y_{\gamma(i)} - x_{\pi(i)} - y_{\rho(i)}) \\ &= \sum_{i\in I_n} \bar{\alpha}(i,\tau(i),\gamma(i)) - \sum_{i\in I_n} x_{\pi(i)} \\ &\quad - \sum_{i\in I_n} y_{\rho(i)} + \sum_{i\in I_n} x_{\tau(i)} + \sum_{i\in I_n} y_{\gamma(i)} \\ &= \sum_{i\in I_n} \bar{\alpha}(i,\tau(i),\gamma(i)).\end{aligned}$$

This means that $(\alpha(i,h,c))$ is an optimality preserved matrix. This completes the proof. □

One can easily verify that the matrix $(\alpha(i,h,c))$ in Example 1 is an optimality preserved matrix and thus there exists a perfectly fair allocation, whereas the matrix $(\alpha(i,h,c))$ in Example 2 is not an optimality preserved matrix and therefore there is no perfectly fair allocation in the example.

To make the reader more acquainted with optimality preserved matrices, we give one more example.

Example 3. Consider the case in which there are two agents 1, 2 and there are two houses $h1$, $h2$, and two cars $c1$, $c2$, and total money (say, dollar) M. The values of the agents for the different pairs of house and car are given in Table 3.

Table 3: The values of objects for both agents

$\alpha(1,h,c)$	$c1$	$c2$	$\alpha(2,h,c)$	$c1$	$c2$	
$h1$	4	$\underline{5}$	$h1$	3	4.5	
$h2$	4	0	$h2$	$\underline{5}$	5	

The matrix $(\alpha(i,h,c))$ is an optimality preserved matrix. This can be seen from the optimal assignment $((1,2),(2,1))$ which is underlined in the tables 3 and 4. The transformation operations are indicated in Table 4.

Table 4: The changed values of objects for both agents

$\alpha(1,h,c)$	$c1$	$c2$	$\alpha(2,h,c)$	$c1$	$c2$	
$h1$	4	$\underline{5}$	$h1$	$3+1.5$	4.5	
$h2$	4	$0+5$	$h2$	$\underline{5}$	5	

Up to this point, as the reader may have noticed, the class of optimality preserved matrices is fairly large and includes the matrices resulting from separable and additive value functions as its special subclass.

4 An Extension

In this section we consider an extension of the previous model. Suppose there are m different types of objects. There are n objects of each type, denoted by $\mathcal{O}_j$, $j \in I_m$. For example, one might think of $\mathcal{O}_1$ as the collection of houses, of $\mathcal{O}_2$ as cars, of $\mathcal{O}_3$ as trucks, and so on. The utility function of each agent is defined as $u_i : \mathcal{O}_1 \times \mathcal{O}_2 \times \cdots \times \mathcal{O}_m \times \mathbb{R} \mapsto \mathbb{R}$ which is assumed to be a nondecreasing and continuous function in money. Then we can extend the definition of perfectly fair allocation as follows.

Definition 4.1 *An allocation* $(\pi^1, \cdots, \pi^m, x^1, \cdots, x^m)$ *is a perfectly fair allocation if it holds that* $\pi^j \in \Theta$, $j \in I_m$, $\sum_{i \in I_n} \sum_{j \in I_m} x_i^j = M$, *and*

$$u_i(\pi^1(i), \cdots, \pi^m(i), x^1_{\pi^1(i)} + \cdots + x^m_{\pi^m(i)}) \geq u_i(h_1, \cdots, h_m, x^1_{h_1} + \cdots + x^m_{h_m}) \quad \forall i \in I_n,\ h_j \in \mathcal{O}_j, j \in I_m.$$

One can show that every perfectly fair allocation is also Pareto optimal. To obtain an existence result, once again we will focus our attention on the case where every agent has quasi-linear utilities in money. In this case we can represent the model by $\mathcal{E} = ((\alpha(i, h_1, h_2, \cdots, h_m)), n, M)$ where the matrix $(\alpha(i, h_1, \cdots, h_m))$ is an n^{m+1}-matrix, n is the number of agents and M is the total amount of money. Each entry $\alpha(i, h_1, \cdots, h_m)$ represents the value of the combination of objects h_1, h_2, $\cdots$, h_m to agent i.

Conditions 3.4 and 3.5 can be appropriately modified as follows:

Condition 4.2 *The* n^{m+1}*-matrix* $(\alpha(i, h_1, \cdots, h_m))$ *has the following property: For every* $i \in I_n$ *and* $j, k \in I_m$ *with* $1 \leq j < k \leq m$*, it holds*

$$\begin{aligned}
&\alpha(i, h_1, \cdots, h_j, \cdots, h_k, \cdots, h_m) + \alpha(i, h_1, \cdots, h_j', \cdots, h_k', \cdots, h_m) \\
&= \alpha(i, h_1, \cdots, h_j, \cdots, h_k', \cdots, h_m) + \alpha(i, h_1, \cdots, h_j', \cdots, h_k, \cdots, h_m), \\
&\forall\ (h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m\ \ h_j' \in \mathcal{O}_j,\ h_k' \in \mathcal{O}_k
\end{aligned}$$

Condition 4.3 *The* n^{m+1}*-matrix* $(\alpha(i, h_1, \cdots, h_m))$ *has the following property: For every* $i \in I_n$ *and* $j \in I_m$*, there exists an* n*-vector* $H_i(j) = (H_i(j, 1), \cdots, H_i(j, n))$ *such that it holds*

$$\alpha(i, h_1, \cdots, h_m) = \sum_{j \in I_m} H_i(j, h_j), \quad \forall\ (h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m.$$

Lemma 4.4 *Conditions 4.2 and 4.3 are equivalent.*

Proof: That Condition 4.3 implies Condition 4.2 is obvious. Now we prove that Condition 4.2 implies Condition 4.3 by induction. We have proved the case of $m = 2$ in Section 3. Suppose that the case of $m - 1$ is true. Now let us prove the case of m. It follows from Condition 4.2 and the assumption that for every $i \in I_n$, $j \in I_m \setminus \{m\}$, and each fixed $h_m \in \mathcal{O}_m$ there exists an n-vector $H_i'(j, h_m) = (H_i'(j, 1, h_m), \cdots, H_i'(j, n, h_m))$ such that

$$\alpha(i, h_1, \cdots, h_{m-1}, h_m) = \sum_{j=1}^{m-1} H_i'(j, h_j, h_m), \quad \forall (h_1, \cdots, h_{m-1}) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_{m-1}.$$

Note that: for each fixed $j \in I_m$, $(H_i'(j, h_j, h_m))$ can be looked as a trimatrix for all $i \in I_n$, $h_j \in \mathcal{O}_j$, and $h_m \in \mathcal{O}_m$. Recall that from Condition 4.2 we have: for every $j(\neq m) \in I_m$

$$\begin{aligned}&\alpha(i, h_1, \cdots, h_j, \cdots, h_m) + \alpha(i, h_1, \cdots, h_j', \cdots, h_m')\\&= \alpha(i, h_1, \cdots, h_j, \cdots, h_m') + \alpha(i, h_1, \cdots, h_j', \cdots, h_m).\end{aligned}$$

This implies that

$$H_i'(j, h_j, h_m) + H_i'(j, h_j', h_m') = H_i'(j, h_j, h_m') + H_i'(j, h_j', h_m),$$

for all $h_j, h_j' \in \mathcal{O}_j$, and $h_m, h_m' \in \mathcal{O}_m$. Then by Lemma 3.6, we see that for each $j(\neq m) \in I_m$ there exist two n-vectors $H_i(j)$ and $H_i'(j)$ such that $H_i'(j, h_j, h_m) = H_i(j, h_j) + H_i'(j, h_m)$ for all $h_j \in \mathcal{O}_j$ and $h_m \in \mathcal{O}_m$. Define $H_i(m) = \sum_{j=1}^{m-1} H_i'(j)$. Then we obtain that

$$\alpha(i, h_1, \cdots, h_m) = \sum_{j \in I_m} H_i(j, h_j), \quad \forall\, (h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m.$$

This says that Condition 4.3 is true for the case of m. $\square$

Definition 4.5 *Given an n^{m+1}-matrix $A = (\alpha(i, h_1, \cdots, h_m))$ and an assignment $(\pi^1, \cdots, \pi^m) \in \Theta \times \cdots \times \Theta$, the following process is called an $\mathcal{M}$-transformation of A from $(\pi^1, \cdots, \pi^m)$ if each element $\alpha(i, h_1, \cdots, h_m)$ except for $\alpha(i, \pi_1(i), \cdots, \pi_m(i))$, $i \in I_n$ is added with a nonnegative number $\delta(i, h_1, \cdots, h_m)$ so that the new n^{m+1}-matrix $T = (\alpha(i, h_1, \cdots, h_m) + \delta(i, h_1, \cdots, h_m))$ satisfies Condition 4.2.*

The n^{m+1}-matrix T above will be called an $\mathcal{M}$-matrix resulted from $(\pi^1, \cdots, \pi^m)$.

Definition 4.6 *An n^{m+1}-matrix $(\alpha(i, h_1, \cdots, h_m))$ is an optimality preserved matrix if there exist an optimal assignment $(\pi^1, \cdots, \pi^m) \in \Theta \times \cdots \times \Theta$ and an $\mathcal{M}$-transformation from $(\pi^1, \cdots, \pi^m)$ such that $(\pi^1, \cdots, \pi^m)$ is still an optimal assignment in the $\mathcal{M}$-matrix resulted from $(\pi^1, \cdots, \pi^m)$.*

Clearly, an n^{m+1}-matrix satisfying Condition 4.2 is an optimality preserved matrix.

Having these preparations, we can now establish the following existence theorem on this more general model. Here we render a complete proof, which we believe will provide some additional insight into the problem, although some part of the proof is similar to that given in Theorem 3.9.

Theorem 4.7 *Given a model $\mathcal{E} = ((\alpha(i, h_1, \cdots, h_m)), n, M)$, there exists a perfectly fair allocation if and only if the valuation n^{m+1}-matrix $(\alpha(i, h_1, \cdots, h_m))$ is an optimality preserved matrix.*

Proof: Since $(\alpha(i, h_1, \cdots, h_m))$ is an optimality preserved matrix, then there exist an optimal assignment $(\pi^1, \cdots, \pi^m) \in \Theta \times \cdots \times \Theta$ and an $\mathcal{M}$-transformation from $(\pi^1, \cdots, \pi^m)$ such that $(\pi^1, \cdots, \pi^m)$ is still an optimal assignment in the $\mathcal{M}$-matrix resulted from $(\pi^1, \cdots, \pi^m)$. Let $T = (\bar{\alpha}(i, h_1, \cdots, h_m))$ be the n^{m+1} $\mathcal{M}$-matrix resulted from $(\pi^1, \cdots, \pi^m)$. So we have

$$\bar{\alpha}(i, h_1, \cdots, h_m) \geq \alpha(i, h_1, \cdots, h_m)$$
$$\bar{\alpha}(i, \pi^1(i), \cdots, \pi^m(i)) = \alpha(i, \pi^1(i), \cdots, \pi^m(i))$$

for all $i \in I_n$, $(h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m$, and

$$\sum_{i \in I_n} \bar{\alpha}(i, \pi^1(i), \cdots, \pi^m(i)) \geq \sum_{i \in I_n} \bar{\alpha}(i, \tau^1(i), \cdots, \tau^m(i)), \tag{4.6}$$

for all $(\tau^1, \cdots, \tau^m) \in \Theta \times \cdots \times \Theta$. Since T satisfies Condition 4.2, then for every $i \in I_n$ and $j \in I_m$ there exists an n-vector $H_i(j)$ so that $\bar{\alpha}(i, h_1, \cdots, h_m) = \sum_{j \in I_m} H_i(j, h_j)$ for every $(h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m$. Then we can rewrite equation (4.6) as

$$\sum_{i \in I_n} \sum_{j \in I_m} H_i(j, \pi^j(i)) \geq \sum_{i \in I_n} \sum_{j \in I_m} H_i(j, \tau^j(i)), \tag{4.7}$$

for all $(\tau^1, \cdots, \tau^m) \in \Theta \times \cdots \times \Theta$. It follows from equation (4.7) that

$$\sum_{i \in I_n} H_i(j, \pi^j(i)) \geq \sum_{i \in I_n} H_i(j, \tau^j(i))$$

for every $j \in I_m$ and all $\tau^j \in \Theta$. By Lemma 3.2 for each $j \in I_m$ there exists an n-vector x^j such that $\sum_{h_j \in I_n} x^j_{h_j} = M/m$, and

$$H_i(j, \pi^j(i)) + x^j_{\pi^j(i)} \geq H_i(j, h_j) + x^j_{h_j}$$

for all $i \in I_n$, $j \in I_m$ and $h_j \in \mathcal{O}_j$. It follows that

$$\begin{aligned}
\alpha(i, \pi^1(i), \cdots, \pi^m(i)) + \textstyle\sum_{j \in I_m} x^j_{\pi^j(i)} &= \bar{\alpha}(i, \pi^1(i), \cdots, \pi^m(i)) + \textstyle\sum_{j \in I_m} x^j_{\pi^j(i)} \\
&= \textstyle\sum_{j \in I_m} H_i(j, \pi^j(i)) + \sum_{j \in I_m} x^j_{\pi^j(i)} \\
&= \textstyle\sum_{j \in I_m} (H_i(j, \pi^j(i)) + x^j_{\pi^j(i)}) \\
&\geq \textstyle\sum_{j \in I_m} (H_i(j, h_j) + x^j_{h_j}) \\
&= \bar{\alpha}(i, h_1, \cdots, h_m) + \textstyle\sum_{j \in I_m} x^j_{h_j} \\
&\geq \alpha(i, h_1, \cdots, h_m) + \textstyle\sum_{j \in I_m} x^j_{h_j}
\end{aligned}$$

for all $i \in I_n$ and $(h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m$. Thus $(\pi^1, \cdots, \pi^m, x^1, \cdots, x^m)$ is a perfectly fair allocation.

Now suppose that $(\pi^1, \cdots, \pi^m, x^1, \cdots, x^m)$ is a perfectly fair allocation. Then it holds that

$$\alpha(i, \pi^1(i), \cdots, \pi^m(i)) + \sum_{j \in I_m} x^j_{\pi^j(i)} \geq \alpha(i, h_1, \cdots, h_m) + \sum_{j \in I_m} x^j_{h_j}$$

for all $i \in I_n$ and $(h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m$. It is readily seen that $(\pi^1, \cdots, \pi^m)$ is an optimal assignment with respect to $(\alpha(i, h_1, \cdots, h_m))$. Let $A_i = \alpha(i, \pi^1(i), \cdots, \pi^m(i)) + \sum_{j \in I_m} x^j_{\pi^j(i)}$ for each $i \in I_n$. Let $d_i(h_1, \cdots, h_m) = A_i - \alpha(i, h_1, \cdots, h_m) - \sum_{j \in I_m} x^j_{h_j}$ for every $(h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m$. Clearly, $d_i(h_1, \cdots, h_m) \geq 0$. Furthermore, $d_i(\pi^1(i), \cdots, \pi^m(i)) = 0$ for all $i \in I_n$. Let $H_i(j) = -x^j$ for all $j \in I_m \backslash \{m\}$ and $H_i(m) = (A_i - x^m_i, \cdots, A_i - x^m_n)$. Now define $\bar{\alpha}(i, h_1, \cdots, h_m) = \alpha(i, h_1, \cdots, h_m) + d_i(h_1, \cdots, h_m)$. Clearly $\bar{\alpha}(i, h_1, \cdots, h_m) = \sum_{j \in I_m} H_i(j, h_j)$ and $\bar{\alpha}(i, \pi^1(i), \cdots, \pi^m(i)) = \alpha(i, \pi^1(i), \cdots, \pi^m(i))$ for all $i \in I_n$. Thus $(\bar{\alpha}(i, h_1, \cdots, h_m))$ satisfies Condition 4.2. Furthermore, for any $(\tau^1, \cdots, \tau^m) \in \Theta \times \cdots \times \Theta$, we have

$$\begin{aligned}
&\textstyle\sum_{i \in I_n} \bar{\alpha}(i, \pi^1(i), \cdots, \pi^m(i)) \\
&\quad = \textstyle\sum_{i \in I_n} \alpha(i, \pi^1(i), \cdots, \pi^m(i)) \\
&\quad = \textstyle\sum_{i \in I_n} (A_i - \sum_{j \in I_m} x^j_{\pi^j(i)}) \\
&\quad = \textstyle\sum_{i \in I_n} (\bar{\alpha}(i, \tau^1(i), \cdots, \tau^m(i)) + \sum_{j \in I_m} x^j_{\tau^j(i)} - \sum_{j \in I_m} x^j_{\pi^j(i)}) \\
&\quad = \textstyle\sum_{i \in I_n} \bar{\alpha}(i, \tau^1(i), \cdots, \tau^m(i)) + \sum_{i \in I_n} \sum_{j \in I_m} x^j_{\tau^j(i)} - \sum_{i \in I_n} \sum_{j \in I_m} x^j_{\pi^j(i)} \\
&\quad = \textstyle\sum_{i \in I_n} \bar{\alpha}(i, \tau^1(i), \cdots, \tau^m(i))
\end{aligned}$$

This means that $(\alpha(i, h_1, \cdots, h_m))$ is an optimality preserved matrix. This completes the proof. □

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