Liquidity, Default and Crashes:
Endogenous Contracts in General Equilibrium

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Abstract

The possibility of default limits available liquidity. If the potential default draws nearer, a liquidity crisis may ensue, causing a crash in asset prices, even if the probability of default barely changes, and even if no defaults subsequently materialize.

Introducing default and limited collateral into general equilibrium theory (GE) allows for a theory of endogenous contracts, including endogenous margin requirements on loans. This in turn allows GE to explain liquidity and liquidity crises in equilibrium. A formal definition of liquidity is presented.

When new information raises the probability and shortens the horizon over which a fixed income asset may default, its drop in price may be much greater than its objective drop in value for two reasons: the drop in value reduces the relative wealth of its natural buyers and also endogenously raises the margin required for its purchase. The liquidity premium rises, and there may be spillovers in which other assets crash in price even though their probability of default did not change.

Keywords: Liquidity, default, collateral, crashes, general equilibrium, contracts, spillover, liquidity premium

JEL Classification: D4, D5, D8, D41, D52, D81, D82

1 Default and Endogenous Contracts

Standard general equilibrium theory is unable to answer the question: which contracts are traded in equilibrium? I argue that introducing default into general equilibrium makes room for a competitive theory of endogenous contracts, and that in such a model, liquidity and liquidity crises can be explained.

Let $\mathcal{C}$ be the set of marketed contracts, and let $\mathcal{C}^*$ be the set of contracts that are positively traded by at least one agent in equilibrium. A contract in $\mathcal{C} \backslash \mathcal{C}^*$ is priced by the market, but untraded. If there are far fewer promises in $\mathcal{C}^*$, then we can say that the forces of supply and demand select the set of traded promises.

When there is the possibility of default, promises must be augmented by contract provisions which give the seller the incentive to deliver what he promised. These generally take one of two forms, punishment or collateral. It would seem to be far more daunting a task for competitive equilibrium theory to explain the terms of the loan contracts, as well as their promises and prices. Given a fixed promise, there are many attendant terms, such as how much collateral needs to be put up, what should the penalty for default be, what is the maximum allowable sales, and so on. It would seem that instead of one equation matching supply and demand and one endogenous price, as in conventional general equilibrium theory, there is now a whole host of new endogenous variables representing contract terms, but the same single market clearing equation for each promise. Equilibrium looks to be underdetermined.

The answer to the puzzle is to let each specification of contract terms $c \in C$ define another market, and therefore another market clearing price. The contract terms themselves are not endogenous variables like prices, which get set by equilibrium at one determinate value. Instead they are parameters that help to define the different markets. But equilibrium can set their values just as well. Equilibrium will choose determinate levels of trade q_c in each market $c \in C$. And if, for example, $q_c = 0$ for all $c \neq c^*$, then we can say that the forces of supply and demand have determined the contract terms c^*. This possibility is often obscured by the economist's preoccupation with price.

The public, and unfortunately the Federal Reserve, also share the economists' preoccupation with price. Every day the newspapers print the interest rates, and the Federal Reserve monitors them closely and systematically. But it might happen that the contract terms c^* attending most new loans dramatically change, while interest rates stay put. (This would imply that the prices for loans at the old terms had also dramatically shifted, but the newspapers do not print the prices of loans that are hardly transacted.) A change in c^* may be a more important harbinger of a liquidity crisis than a change in interest rates.

Scarce collateral provides a compelling explanation for endogenous contracts. Simply put, the quantity of desired promises exceeds the value of the available collateral, and so the forces of supply and demand (operating through margin requirements) will ration the set of positively traded contracts. As real conditions and expectations change, these margin requirements will need to change in order to maintain equilibrium. These margin changes will in turn have real effects, necessitating further adjustments in margins, and occasionally creating an *equilibrium* cascade into crisis.

The mechanisms by which scarce collateral and punishment ration contracts are similar. Both make the marginal utility of buying less than the marginal disutility of selling, and thus create the real possibility that an inessential contract will not be traded. With a positive probability of actual default, the buyer of a promise usually receives less than the seller delivers. For example, if a seller partially defaults and serves time in jail as punishment, he delivers both goods and jail time, while the buyer of the promise only receives the goods. Similarly, a provision of the contract might be that the seller is forced to put up collateral, that is to buy and hold some durable good that he otherwise might not want, or to hold cash reserves that he would otherwise spend. The seller of the promise delivers goods to the buyer of the promise, but he also delivers the disutility of making an inconvenient transaction with a third party. The marginal utility of buying a promise may thus be less than the marginal disutility of selling the promise.

When the price of a promise is greater than the marginal utility of buying it and is less than the marginal disutility of selling it for every agent, the promise will not be traded at all. Standard general equilibrium theory cannot explain which assets are traded because it cannot explain which assets are *not* traded. General equilibrium with default does.[1]

Together with Dubey and Shubik in [4], and with Dubey in [5] and [6], I built a theory of endogenous punishment and endogenous insurance. In [7], and also with Zame [8], I constructed a model of endogenous collateral levels. In this paper I build on the latter work [7] and [8], reinterpreting collateral levels in terms of liquidity and explaining how shifts in equilibrium collateral levels (margin requirements) can cause equilibrium crises.

2 Liquidity Crises

In 1994 and again in 1998, fixed income markets, especially derivatives and mortgage derivatives, suffered terrible liquidity crises which at the time seemed to threaten the stability of the whole financial system. Though we shall see that economists have had trouble precisely defining liquidity, the general features of the liquidity crises can be succinctly described. In both episodes one saw:

(1) A price crash in defaultable assets, especially for the riskiest assets, but *without* a commensurate increase in subsequent defaults.

(2) These effects spilled over many markets, such as high risk corporate bonds and mortgages, even though the risks of default are probably not correlated between the markets.

(3) A huge income loss for the most adventurous buyers (e.g., hedge funds purchasing derivatives).

(4) An increase in the spread between more "liquid" and less "liquid" securities (like off-the-run Treasuries and on-the-run Treasuries), even though the assets had the same probability of default. Thus default spreads and liquidity spreads both increased.

(5) The margin requirements on borrowing were raised.

(6) Borrowing decreased.

Another crucial observation is that the crises did not seem to be driven by changes in the riskless interest rate. In 1994 Treasury interest rates were rising before the crisis, while in 1998 they were falling. Moreover, when the margin requirements on borrowing were raised, the interest rate charged remained virtually the same.

[1] Moreover, it is not necessarily the default, nor even the probability of default, but the potential for default which puts the wedge between buying and selling utilities. Even if it is known that the default will not occur, given the contract provisions, these provisions may so onerous as to choke off trade in the contract.

3 Collateral and Margin

A contract j is defined in this paper by a promise A_j made by the seller and the collateral C_j required to back the promise. A classic example might be the promise of $100,000 backed by a house as collateral, which is called a mortgage.

The price $\pi_j = \pi(A_j, C_j)$ is the amount the buyer must pay the seller to obtain the contract. A contract i making the same promise $A_i = A_j$, but with a different collateral requirement, $C_i \neq C_j$, is a different contract, and may sell for a different price $\pi_i \neq \pi_j$. If C_i is a car, then the same promise $A_i = A_j$ is now called a car loan.

To avoid adverse selection problems, which are not my concern in this paper, I suppose that collateral is completely observable, so that one seller's collateral C_j is as good as any other seller's C_j. Furthermore, I suppose that every seller always delivers the minimum of the promise and the value of the collateral. Every sale of contract j is thus identical.

The collateral C_j will be taken to be a vector of durable, marketed goods. Since C_j is marketed, in equilibrium one can always observe the value of the collateral $p(C_j)$, as well as the price of the contract $\pi(A_j, C_j)$.

Notice that contracts have an important homogeneity property: if $\lambda > 0$, the contracts $(A_j, \lambda C_j)$ and $(\frac{1}{\lambda}A_j, C_j)$ are essentially identical. The first will sell for a price λ times bigger than the second, so that per dollar spent, the contracts are absolutely identical.

Notice also that in equilibrium we will always have $\pi_j = \pi(A_j, C_j) \leq p(C_j)$, since by assumption the payoff from the contract will never exceed the value of the collateral.

The margin on a contract (A_j, C_j) in equilibrium is defined as

$$m_j = \frac{p(C_j) - \pi(A_j, C_j)}{p(C_j)}.$$

The margin m_j will be positive for essentially three reasons. First, the collateral may provide utility before the promises come due, boosting the price of the collateral above the price of the promise. Second, there may be a mismatch between future collateral values and the promises, so that in some states, the collateral is worth more than the promises. Third, to the extent the mismatch is variable, risk averse lenders might prefer higher margins m_j to higher interest rates (i.e., to lower prices π_j).

We shall see that sometimes we can associate with each collateral a single loan. In that case we can think of the margin requirement as pertaining to the collateral. Each collateral asset ℓ might then have a margin requirement m_ℓ. For example, houses generally are bought with 20% cash and the rest borrowed. We shall see in later sections how this margin requirement is determined endogenously in equilibrium.

4 Liquidity

Liquidity is an elusive concept in economics. Sometimes it is used to refer to the volume of trade in a particular market, sometimes it means the average time needed to sell, sometimes it means the bid/ask spread in the market, sometimes it means the price function relating the change in price to the change in quantity orders, sometimes it refers to the spread between two assets with the same promises (such as the spread between on-the-run and off-the-run Treasuries).

Some of these definitions seem to require a non-competitive view of the world, since they presume that trades are not instantly transacted at one price. Yet some of the other definitions apply in competitive markets. It is evident that economists do not all have the same notion in mind when they speak of liquidity. However, every definition of liquidity identifies a wedge between buying and selling in each market separately.

By contrast, I offer a definition of the "liquidity of the system" that depends on the interactions of agents between markets. I identify the liquidity cost $m_\ell p_\ell$ of buying an asset $\ell \in L$ as its margin requirement m_ℓ multiplied by its price p_ℓ. The liquidity cost of a bundle of assets $x \in \mathbb{R}^L_+$ is thus $\sum_{\ell \in L} m_\ell p_\ell x_\ell$.

Each agent h must choose a bundle $x^h \in \mathbb{R}^L_+$ subject to two constraints. First, the value of his total expenditures must not exceed his wealth W^h, and second, the liquidity cost of his expenditures must not exceed his liquid wealth $\tilde{W}^h$:

$$\sum_{\ell \in 1}^{L} p_\ell x_\ell \le W^h$$

$$\sum_{\ell = 1}^{L} m_\ell p_\ell x_\ell \le \tilde{W}^h.$$

In Section 12, I offer a definition of liquid wealth $\tilde{W}^h$, after discussing a series of examples in the intervening sections. However, we can see already what the point will be, before we come to the definition of $\tilde{W}^h$. Let x^h be the bundle agent h would choose if he only faced the first constraint. Then $[\sum_{\ell=1}^{L} m_\ell p_\ell x_\ell - \tilde{W}^h]^+$ measures how much the liquidity constraint deflects his intentions.

I define the illiquidity of the system as

$$\frac{\sum_{h \in H} \left[\sum_{\ell=1}^{L} m_\ell p_\ell x^h_\ell - \tilde{W}^h \right]^+}{\sum_{h \in H} W^h},$$

where $x^+ \equiv \max\{x, 0\}$. The system is illiquid not simply because the aggregate liquid wealth $\sum_{h \in H} \tilde{W}^h$ is too low, but also because it could be distributed badly.

The system becomes more illiquid, *ceteris paribus*, if the margin requirements m_ℓ go up. Since m_ℓ are endogenous variables, we shall discuss reasons why they might rise. The system also becomes more illiquid if $\tilde{W}^h$ is redistributed from liquidity constrained households h to surplus liquidity households. In a dynamic economy, liquid wealth is endogenous, and we shall see precisely why such transfers might take place. Lastly, the system becomes more illiquid if $\sum_{h \in H} \tilde{W}^h$ falls.

5 Default and Collateral

The difficulty with promises is that they require some mechanism to make sure they are kept. This can take the form of penalties, administered by the courts, or collateral. As we mentioned at the outset, more and more often collateral has displaced penalties. In this paper I shall exclusively deal with collateral, by supposing that there is no penalty, legal or reputational, to defaulting. Of course, even collateral requires the courts to make sure the collateral changes hands in case of default.

The simplest kind of collateral is pawn shop collateral — valuable goods like watches or jewelry left with third parties (warehoused) for safekeeping. Financial markets have advanced as the number of goods that could function as collateral has increased, from watches and jewelry, to stocks and bonds. A further advance occurred when lenders (instead of warehouses) held collateral, like paintings, that afforded them utility. This required a more sophisticated court system, because the lender had to be obliged to return the collateral if the promise was kept. The biggest advance, however, was in allowing the borrower himself to continue to hold the collateral. This enabled houses, and later cars, to be used as collateral, which again is only possible because of a finely tuned court system that can enforce the confiscation of collateral.

More recently the complexity of collateral has taken several more giant steps forward. Pyramiding occurs when an agent A puts up collateral for his promise to B, and then B in turn uses A's promise to him, and hence in effect the same collateral, for a promise he makes to C, who in turn reuses the same collateral for a promise he makes to D. Mortgage passthrough securities offer a classic example of pyramiding. Pyramiding naturally gives rise to chain reactions, as a default by Mr. A ripples through, often all the way to D.

Still more complex is tranching, which arises when the same collateral backs several promises to different lenders. Needless to say, the various lenders will be concerned about whether their debts are adequately covered. Tranching usually involves a legal trust which is assigned the duty of dividing up the collateral among the different claims according to some contractual formula. Again collateralized mortgage obligations offer a classic example of tranching.

Every one of these innovations is designed to increase or to stretch the available collateral to cover as many promises as possible. We shall see later that active default is another way of stretching the available collateral.

For the formal analysis in this paper I shall avoid pyramiding and tranching. All collateral will by assumption be physical commodities. Collateral must be put up at the moment the promise is sold, even if the delivery is not scheduled for much later. Agents are not allowed to pledge their future endowment as collateral, because that would raise questions in the minds of lenders about whether the borrowers actually will have the endowments they pledged, and therefore it would once again destroy the anonymity of markets.

5.1 Contracts with Collateral

To each promise j we must formally associate levels of collateral. Any good can potentially serve as collateral, and there is no reason why the single promise j cannot be backed by a collection of goods. The bundle of goods that is required to be warehoused for contract j is denoted $C_j^W \in R_+^L$, the vector of goods that the lender is allowed to hold is denoted $C_j^L \in R_+^L$, and the vector of goods the borrower is obliged to hold is denoted $C_j^B \in R_+^L$. A contract j is defined by the promise it makes *and* the collateral backing it, $(A_j, C_j^W, C_j^L, C_j^B)$. It is quite possible that there will be many assets which make the same promises $A_j = A_{j'}$, but trade at different prices because their collateral levels are different $(C_j^W, C_j^L, C_j^B) \neq (C_{j'}^W, C_{j'}^L, C_{j'}^B)$. Similarly the two contracts might require exactly the same collaterals, but trade at different prices because their promises are different.

The price of contract j is denoted by π_j. A borrower sells contract j, in effect borrowing π_j, in return for which he promises to make deliveries according to A_j.

5.2 Production

Collateral is useful only to the extent that it is still worth something when the default occurs. Durability is a special case of production, so we introduce production into our model, and allow all goods to be durable, to varying degrees.

For ease of notation we shall suppose that production is of the fixed coefficient, constant returns to scale variety. One unit of commodity ℓ becomes a vector of commodities next period. A house may become a house that is one year older, wine may become a wine that is one year older, grapes may become wine one year later and so on. In these examples, one good became a different good the next period, but there is no reason not to permit one good to become several goods. By linearity, we can talk more succinctly about the transformation of a vector of goods $x \in \mathbb{R}_+^L$ into goods $f_s(x) \in \mathbb{R}_+^L$ for each $s \in S$.

The transformation of a commodity depends of course on how it is used. We suppose a bundle of goods $x \in \mathbb{R}_+^L$ is transformed into a vector $f_s^0(x) \in R_+^L$ in each state s if it is used for consumption (e.g., living in a house, or using a light bulb). If it is warehoused, then we assume that it becomes a vector $f_s^W(x) \in R_+^L$ in each state s. Likewise, if it is held as collateral by the lender it becomes a vector $f_s^L(x) \in R_+^L$ in each state s, while if it is held by the borrower it becomes the vector $f_s^B(x) \in R_+^L$ in each state s. The linear functions f^0, f^W, f^L, f^B summarize these different durabilities.

Observe that we have allowed for differential durability depending on the use to which the commodity is put. But we have not allowed the durability to be affected by the identity of the user. In this way the anonymity of markets is maintained, and our modeling problem becomes easier.

Given the collateral requirements (C_j^W, C_j^L, C_j^B) for each contract j, the security they provide in each state s is

$$p_s \cdot [f_s^W(C_j^W) + f_s^L(C_j^L) + f_s^B(C_j^B)].$$

The collateral is owned by the borrower but may be confiscated by the lender (actually by the courts on behalf of the lender) if the borrower does not make his promised deliveries. Since we have assumed that the borrower has nothing to lose but his collateral from walking away from his promise, it follows that the actual delivery by every agent h on asset j in state s will be:

$$D_{sj}^h = \min\{p_s \cdot A_s^j,\ p_s \cdot [f_s^W(C_j^W) + f_s^L(C_j^L) + f_s^B(C_j^B)]\}.$$

6 Collateral Equilibrium

We are now ready to put together the various elements of our model. An economy E is defined by a vector

$$E = ((u^h, e^h)_{h \in H}, (A^j, C_j^W, C_j^L, C_j^B)_{j \in J}, (f^0, f^W, f^L, f^B))$$

of agent utilities and endowments, asset promises and collateral levels, and the durability of goods kept by consumers, warehouses, lenders, and borrowers, respectively.

In keeping with the standard methodological approach of general equilibrium and perfect competition, we suppose that *in equilibrium* agents take the prices (p, π) of commodities and assets as given.

Our price taking hypothesis has the implication that agents have rational expectations about future prices, for these are taken as given as well. Agents in our model have perfect conditional foresight, in that they anticipate at time 0 what the prices p_s will be, depending on which state s prevails at time 1. Since they know the collateral that has been put up, and they know the production technology, they also understand in each state how much each asset will actually pay.

It might seem therefore that we could simply replace each asset promise A_j with an actual delivery vector, and thereby bypass the complications of collateral. But this is not possible, since whether an asset defaults or not in state s depends on whether the promise or the collateral is worth more. Since both are vectors, this cannot be known in advance until the prices $p_s \in R_+^L$ have been determined in equilibrium.

6.1 The Budget Set

Given the prices (p, π), each agent h decides what commodities to consume, x_0^h, and what commodities, x_W^h, to save in a warehouse. He also decides what contract purchases θ, and what contract sales φ he will make at time 0. Note that for every promise φ_j that he makes, he must put up the corresponding collateral $(C_j^W, C_j^L, C_j^B)\varphi_j$. The value of all his net trades at time 0 must be less than or equal to zero, that is, the agent cannot purchase anything without raising the money by selling something else (initial endowments of money are taken to be zero).

After the state of nature is realized in period 1, the agent must again decide on his net purchases of goods $(x_s - e_s^h - f_s^0(x_0) - f_s^W(x_W))$. Recall that the goods x_0 whose services he consumed at time zero may be durable, and still available (in the form $f^0(x_0)$) for consumption at time 1 in each state s. These net expenditures on goods can be financed out of sales of the collateral that the agent put up in period 0, and from the receipts from contracts j that he purchased at time 0, less the deliveries the agent makes on the contracts he sold at time 0. Putting all these transactions together, and noting again that the agent cannot buy anything without also selling something else of at least equal value, we derive the budget set for agent h:

$$B^h(p,\pi) = \{(x,\theta,\varphi) \in R_+^L \times \mathbb{R}_+^L \times R_+^{SL} \times R_+^J \times R_+^J :$$
$$p_0(x_0 + x_W - e_0^h) + \pi(\theta - \varphi) + p_0 \sum_{j\in J}(C_j^W + C_j^L + C_j^B)\varphi_j \leq 0 \text{ and for all } s \in S,$$
$$p_s(x_s - e_s^h - f_s^0(x_0) - f_s^W(x_W)) \leq \sum_{j\in J} \varphi_j p_s \cdot [f_s^W(C_j^W) + f_s^L(C_j^L) + f_s^B(C_j^B)] +$$
$$\sum_{j\in J}(\theta_j - \varphi_j) \min\{p_s \cdot A_s^j,\ p_s \cdot [f_s^W(C_j^W) + f_s^L(C_j^L) + f_s^B(C_j^B)]\}$$

6.2 Equilibrium

The economy $E = ((u^h, e^h)_{h\in H}, (A_j, C_j^W, C_j^L, C_j^B)_{j\in J}, (f^0, f^W, f^L, f^B))$ is in equilibrium at macro prices and individual choices $((p,\pi), (x^h, \theta^h, \varphi^h)_{h\in H})$ if supply equals demand in all the goods markets and asset markets, and if given the prices, the designated individual choices are optimal, that is if:

1) $\sum_{h\in H}(x_0^h + x_W^h - e_0^h + \sum_{j\in J}(C_j^W + C_j^L + C_j^B)\varphi_j^h) = 0,$

1′) $\sum_{h\in H}(x_s^h - e_s^h - f_s^0(x_0^h)) - \sum_{j\in J}\sum_{h\in H}\varphi_j^h[f_s^W(C_j^W) + f_s^L(C_j^L) + f_s^B(C_j^B)] = 0$

2) $\sum_{h\in H}(\theta^h - \varphi^h) = 0$

3) $(x^h, \theta^h, \varphi^h) \in B^h(p,\pi)$

4) $(x,\theta,\varphi) \in B^h(p,\pi) \Rightarrow u^h(x_0 + \sum_{j\in J}[C_j^B\varphi_j + C_j^L\theta_j], \bar{x})$
$\leq u^h(x_0^h + \sum_{j\in J}[C_j^B\varphi_j^h + C_j^L\theta_j^h], \bar{x}^h).$

We write $x^h = (x_0^h, \bar{x}^h)$, so consumption at time 0 is $x_0^h + \sum_{j\in J}[C_j^B\varphi_j^h + C_j^L\theta^h]$.

7 Properties of Collateral Equilibrium

7.1 The Orderly Function of Markets

The agents we have described must anticipate not only what the prices will be in each state of nature, and not only what the contracts promise in each state of nature, but also what they will actually deliver in each state of nature. The hypothesis of agent rationality is therefore slightly more stringent in this model than in the conventional models of intertemporal perfect competition. Nevertheless, equilibrium always exists in this model, (under the assumptions made so far), yet in the standard model of general equilibrium with incomplete asset markets, equilibrium may not exist. The following theorem is taken from [8].

Theorem 1 (Geanakoplos, Zame) *Under the assumptions on endowments and utilities already specified, equilibrium must exist, no matter what the structure of contracts and collateral.*

If a contract contains no provision for collateral whatsoever, then of course everybody will rightly anticipate that it will deliver nothing, and its equilibrium price will be 0. Indeed the economy would function exactly the same way if it were not available at all. For assets with some nonzero collateral, agents will not be able to sell arbitrarily large quantities, because they will not be able to obtain the required collateral. This limiting factor helps to guarantee the existence of equilibrium.

7.2 Endogenous Assets

One of the major shortcomings of the standard general equilibrium model is that it leaves unexplained which assets are traded. Generically, all the assets exogenously allowed into the model will be traded. When default can only be avoided by collateral, the situation is different and much more interesting.

The crucial idea is that without the need for collateral, the marginal utility $\mu_j^h(B)$ to an agent h of buying the first unit of an asset j is almost exactly the same as the marginal utility loss $\mu_j^h(S)$ in selling the first unit of the asset; we can call both μ_j^h. Only by an incredible stroke of luck will it turn out that $\mu_j^h = \mu_j^{h'}$ for different agents h and h', and hence asset j will almost surely be traded in a GEI equilibrium. When collateral must be provided by the seller, the disutility of making a promise goes up, sometimes by as much as the consumption foregone by buying the collateral. If the required collateral is borrower held, and if it is something that agent h planned to hold anyway, then there is no extra utility loss from selling the first unit of asset j. But if agent h did not plan to hold the collateral for consumption, or if all that he intended to hold as consumption has already been allocated as collateral for other promises, then the loss in utility from selling even the first unit of asset j would be larger than the marginal utility from buying the first unit of asset j, $\mu_j^h(S) > \mu_j^h(B)$. It might well transpire that

$$\min_{h\in H} \mu_j^h(S) > \pi_j > \max_{h\in H} \mu_j^h(B)$$

and hence that asset j does not trade at all in equilibrium.

This situation can be most clearly seen when the value of the Arrow–Debreu promises in some state exceeds the salvage value of all the durable goods carried over into that state. It is then physically impossible to collateralize every socially useful promise up to the point that every delivery is guaranteed without exception. The market system, through its equilibrating mechanism, must find a way to ration the quantity of promises. This rationing is achieved by a scarcity of collateral. The resulting wedge between the buying marginal utility of each asset and the selling marginal utility of the asset, however, not only serves to limit the quantity of each promise, but more dramatically, it chokes off most promises altogether, so that the subset of assets that are actually traded is endogenous and potentially much smaller than the set of available assets.

Let $\mathcal{C} = \{(C^W, C^L, C^B) \in Z_+^L \times Z_+^L \times Z_+^L : C_\ell^i \leq 10^{100}\}$ be a finite set of (virtually) all potential collateral levels. Fix a promise $a \in \mathbb{R}_+^{SL}$. Consider the set $J = \mathcal{C}$ of all possible contracts with promise a and collateral levels $c \in \mathcal{C}$. In equilibrium all of these many contracts will be priced, but only a very few of them will actually be traded. The rest will not be observable in the marketplace, and therefore the appearance will be given of many missing markets. The untraded contracts will lie dormant not because their promises are irrelevant to spreading risk efficiently, but because the scarce collateral does not permit more trade.

7.2.1 Collateral and Default

It would be interesting to catalogue the rules by which the market implicitly chooses one promise over another, or one level of collateral over another. This issue is more fully developed in Geanakoplos–Zame [8], but let us note some things here. The easiest way of economizing on collateral is by allowing default in some states of nature. Moreover, if one vector of collaterals guarantees full delivery in every state of nature, there is no point in trading the same promise collateralized by greater levels of collateral. Finally, if a vector of promises is very different from the vector of its collateral values across the states of nature, the asset is not well drawn. In some states there will be too much collateral, and in others not enough. One might suspect that such an asset would also not be traded. The general principle is that the market chooses assets that are as efficient as possible, given the prices. We make this precise in the next section.

7.3 Constrained Efficiency

It is to be expected that an increase in available collateral, either through an improvement in the legal system (e.g., borrower held collateral), or through the increased durability of goods, will be welfare improving. But could it lower welfare in a pathological case? More subtly, we might wonder whether government intervention could improve the functioning of financial markets given a fixed level of available collateral. After all, the unavailability of collateral might create a wedge that prevents agents from trading the promises in J that would lead to a Pareto improving sharing of future risks. If the government transferred wealth to those agents unable to afford collateral, or subsidized some market to make it easier to get collateral, could the general welfare be improved? What if the government prohibited trade in assets with low collateral levels? The answer, surprisingly, is no, at least under some important restrictions.

Constrained Efficiency Theorem (Geanakoplos–Zame, 1998) *Each collateral equilibrium is Pareto efficient among the allocations which* (1) *are feasible and* (2) *given whatever period 0 decisions are assigned, respect each agent's budget set at every state s at time 1 at the old equilibrium prices, and* (3) *assume agents will deliver no more on their asset promises than they have to, namely the minimum of the promise and the value of the collateral put up at time* 0*, given the original prices.*

In particular, no matter how the government redistributes income in period 0, and taxes and subsidizes various markets at time 0, if it allows markets to clear on their own at time 1, then we can be sure that if the time 1 market clearing relative prices are the same as they were at the old equilibrium, then the new allocation cannot Pareto dominate the old equilibrium allocation. This will be illustrated in our examples.

7.4 Volatility I: Natural Buyers, the Marginal Buyer and the Distribution of Wealth

In any general economic equilibrium, the price of a good depends on the utilities of the agents and the distribution of wealth. If the agents who are fondest of the good are also relatively wealthy, the good's price will be particularly high. Any redistribution of wealth away from these "natural buyers" toward agents who like the good less will tend to lower the price of the good.

To a large extent, the value of durable goods depends on the expectations, and, when markets are incomplete, on the risk aversion of potential investors, as well as on intrinsic utility for the good. These multiple determinants of value make it quite likely that there will be wide divergences in the valuations different agents put on durable goods.

For example, farms in 1929 could be thought of as an investment, available to farmers and bankers, but to farmers there is a superior intrinsic value that made it sensible for them to own them and use them at the same time. Since the farmers did not have enough money to buy a farm outright, they typically borrowed money and used the farm as collateral. Similarly mortgage derivatives in the 1990s were worth much more to investors who had the technology and understanding to hedge them than they were to the average investor.

Needless to say, the value of many durable assets will be determined by the marginal utilities of those who like them the most. This is literally true if one cannot sell the asset short.

Since the 1929 stock market crash it has been widely argued that low margin requirements can increase the volatility of stock prices. The argument is usually of the following kind: when there is bad news about the stocks, margins are called and the agents who borrowed against the stocks are forced to put them on the market, which lowers their prices still further.

The trouble with this argument is that it does not quite go far enough. In general equilibrium theory, every asset and commodity is for sale at every moment. Hence the crucial step in which the borrowers are forced to put the collateral up for sale has by itself no bite. On the other hand the argument is exactly on the right track.

We argue that indeed using houses or stocks, or mortgage derivatives as collateral for loans (i.e., allowing them to be bought on margin) makes their prices more volatile. The reason is that those agents with the most optimistic view of the assets' future values, or simply the highest marginal utility for their services, will be enabled by buying on margin to hold a larger fraction of them than they could have afforded otherwise.

The initial price of those assets will be much higher than if they could not be used as collateral for two reasons: every agent can afford to pay more for them by promising future wealth, and second, the marginal buyer will tend to be somebody with a higher marginal utility for the asset than would otherwise be the case.

As a result of the margin purchases, the investment by the optimistic agents is greatly leveraged. When the asset rises in value, these agents do exceedingly well, and when the asset falls in price, these agents do exceedingly badly. Thus on bad news the stock price falls for two reasons: the news itself causes everyone to value it less, and this lower valuation causes a redistribution of wealth away from the optimists and toward the pessimists who did not buy on margin. The marginal buyer of the stock is therefore likely to be someone less optimistic than would have been the case had the stock not been purchased on margin, and the income redistribution not been so severe. Thus the fall in price is likely to be more severe than if the stock could not have been purchased on margin.[2]

7.5 Volatility and Incomplete Markets

The analysis here depends on the (endogenous) incompleteness of risk markets. When risk markets are incomplete, trade in assets and contracts might make the distribution of wealth very different across states. If asset prices are very sensitive to the distribution of wealth, this can lead to occasional, large changes in asset prices.

When risk markets are complete, trade in contracts will tend to make the distribution of wealth fairly constant across states, eliminating the wild swings possible with incomplete markets.

Scarce collateral endogenously limits the contract trade, forcing incomplete risk markets even when any contract could be written (but delivery not enforced).

7.6 Volatility II: Asset Values and Margin Requirements

Even without any change in the marginal utility of buyers, changes in margin requirements can dramatically affect asset values. Durable assets can provide dividends for years into the future, vastly exceeding the quantity of consumption goods in the present. But if the buyers of the assets cannot spend by borrowing against their

[2] Instead of imagining that the shock and income redistribution causes the assets to become partly owned by less enthusiastic buyers, which we called the marginal buyer effect, we could imagine instead that the original buyers themselves became less enthusiastic as their diminished wealth (and inevitable diminished consumption) lowered the asset's marginal utility relative to the marginal utility of consumption.

future wealth, the price of the assets might remain quite low simply because the "liquidity constrained" buyers cannot call on enough financial resources. A toughening of margin requirements can thus cause asset prices to tumble.

7.7 Why Margin Requirements Get Tougher

Regulators can enforce high margins. We saw already that if returns become more volatile, or lenders become more risk averse, margin requirements will likely stiffen. Furthermore, if worries about adverse selection or moral hazard increase, margin requirements will toughen. All these factors probably contributed to the crises of 1994 and 1998. But here we are seeking reasons stemming purely from the logic of collateral equilibrium which could explain why adverse shocks might lead to tighter margins, and thus a multiplier effect on margins.

One possibility is that when lenders lose money to some defaulting borrowers, their decreased wealth makes them more risk averse. They might then demand higher margins and this may lead to a fall in asset prices.

One important question is whether a fall in asset prices themselves will lead to higher margins. The answer depends on what caused the fall in price. Very often bad news for asset prices will lead to a reduction in margin requirements.

For example, if asset prices decline because of an income shock to the natural buyers, lenders may demand less onerous margins because they feel asset prices have less far to fall.

If asset values follow a geometric random walk, then after an adverse shock prices may be lower, but the standard deviation of outcomes is also scaled down, so the margin requirement (which is a ratio) may very well hold constant.

A productivity shock that raises the probability of (the same) bad outcome will tend to lower asset prices, but also to *ease* margins. For example, suppose that asset Y could produce 1 with probability b or $R < 1$ with probability $1 - b$. If the natural buyers were risk neutral and believed in b, the asset would sell for $p_Y = b1 + (1-b)R$, provided that buyers had access to enough cash. If the lenders were infinitely risk averse, it is not unreasonable to guess that lenders will lend at most R against one unit of Y as collateral. The margin requirement would then be

$$m = \frac{p_Y - R}{p_Y} = \frac{b1 + (1-b)R - R}{b1 + (1-b)R} = \frac{b(1-R)}{b(1-R) + R} = \frac{1}{1 + \dfrac{R}{b(1-R)}}.$$

It is clear that if the probability of a bad event increases, b goes down, and m *decreases*, if $0 < R < 1$. The reason is that the drop Δ in p_Y causes a percentage drop Δ/p_Y in the price of Y, but a bigger percentage drop, $\Delta/(p_Y - R)$, in the required downpayment.

On the other hand, in the same situation, a productivity shock that lowers R, keeping b fixed, dramatically *raises* the margin requirement, as can be seen in the formula above.

Bad news about assets typically does not take the form that *if* default occurs, the recovery will be less. Typically the bad news suggests that default is more likely, and sooner. We shall see that the combination of more likely and sooner can lead to higher margins (even though more likely by itself often leads to lower margins).

We must rigorously investigate how the margin is set. In the last paragraph, we described utilities for which it seemed plausible that the margin would be set high enough to eliminate default. Let us now describe different utilities such that the same asset could be purchased with no money down. For example, if the natural buyers of Y get a utility boost simply from holding Y, and if all agents are risk neutral and agree that the probability of 1 is b, then the margin requirement will be zero, independent of R. The relevant loan will promise 1 in both states, but because of default it will deliver what Y delivers, namely 1 or R. Thus the price of the loan will be the same as the price of Y, and buyers will finance the entire purchase price, putting no money down.

The properties of collateral equilibrium are illustrated in the next two sections via two examples worked out in detail to see how equilibrium determines a unique collateral level.

8 A Simple Example

We begin with a simple example in which there are two goods X and Y in each state $s = 0, 1, 2$. X is a storable consumption good, like tobacco, and Y is an investment good which delivers 1 unit of X when things go well in state $s = 1$, and a smaller amount $R < 1$ in state $s = 2$. Y is reminiscent of a securitized mortgage or asset-backed bond in which there are normal payments in state $s = 1$ and default with recovery R in state $s = 2$.

We suppose that there are optimists who think that state 1 is very likely and pessimists who do not. The price of Y (in terms of X) at time 0 will naturally be somewhere between 1 and R, reflecting the average opinion about the probability of the good state. At that price, the optimists would like to buy Y from the pessimists, but they do not have the cash. They would gladly borrow the money, but they must put up Y as collateral for their loans. There will be a menu of loans, some with low required collateral (low margin), but high interest rates, and other contracts with low interest rates but high margin requirements. Will only one contract be traded in equilibrium, thus determining both the interest rate and the margin requirement? If so, will it be the socially efficient contract? Let us be precise.

Let there be two states $s = 1$ and $s = 2$, in addition to $s = 0$, and let agent $h \in H \subset [0, 1]$ assign probability h to $s = 1$ and probability $1 - h$ to $s = 2$. Agents with h near 1 are optimists, agents with h near 0 are pessimists.

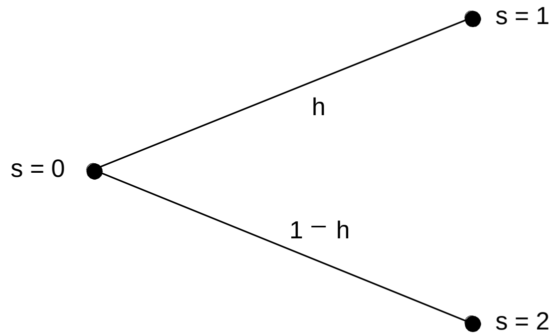

Figure 1

Suppose that each unit of X gives 1 unit of consumption utility in each state and that Y gives no utility of consumption:

$$u^h(x_0, y_0, x_1, y_1, x_2, y_2) = x_0 + hx_1 + (1-h)x_2$$

Suppose that each agent h has an endowment of e units of good X and one unit of good Y in state $s = 0$, and nothing otherwise:

$$e^h = (e^h_{0x}, e^h_{0y}, e^h_{1x}, e^h_{1y}, e^h_{2x}, e^h_{2y}) = (e, 1, 0, 0, 0, 0).$$

Suppose that X is perfectly durable if warehoused and extinguished if consumed (like tobacco). Suppose that 1 unit of Y gives 1 unit of X in state $s = 1$ and $R < 1$ units of X in $s = 2$.

We can write this formally as

$$\begin{aligned} f^0_s((x,y)) &= f^L_s((x,y)) = f^B_s((x,y)) = (0,0),\ s = 1,2 \\ f^W_s((x,y)) &= (x+y, 0),\ s = 1 \\ f^W_s((x,y)) &= (x+Ry, 0),\ s = 2. \end{aligned}$$

We suppose that every contract j promises 1 unit of X in each state $s = 1, 2$:

$$A^j_s = (1, 0),\ s = 1, 2,\ j \in J.$$

The collateral required by contract j is j units of good Y in a warehouse:

$$\begin{aligned} C^L_j &= C^B_j(0,0),\ j \in J \\ C^W_j &= (0, j),\ j \in J. \end{aligned}$$

Buying one unit of Y on margin via contract j in state 0 means selling $1/j$ units of contract j for π_j/j, then paying $p_{0Y} - \pi_j/j$ cash margin plus the borrowed π_j/j for the one unit of Y.

For convenience we take a continuum of agents $H = [0, a]$ and assets $J = [0, 10^{100}]$. (The definition of equilibrium must then be modified in the obvious way, replacing the sum $\sum_h$ by the integral $\int dh$ and restricting each agent to trade a finite number of contracts.) The parameter a will control the number of optimists. We proceed to compute equilibrium.

The first (and perhaps most important) property of equilibrium is indeed that only one contract will be traded. In fact, it is the contract with $j^* = 1/R$, guaranteeing that full delivery is just barely made in state $s = 2$ (and made with ease in $s = 1$). Let us temporarily take this claim on faith and construct the equilibrium, verifying the claim at the end.

Some agent $b \in (0, a)$ will be indifferent to buying or selling Y at time 0. Because of the linear utilities, we guess that agents $h > b$ will buy all they can afford of Y (after selling all their X and borrowing to the max), and agents $h < b$ will sell all they have of Y, lend (buy contract j^*), and consume X. We choose X as numeraire, fixing $p_{sx} = 1 \; \forall s = 0, 1, 2$. Since there is no default, the price $\pi_{j^*} = 1$, and the interest rate is 0. The total money spent on purchases of Y will be the X endowments of agents $h \in (b, a]$, totalling $e(a - b)$, plus the money they can borrow, which is R on each unit of Y they own, plus R on each unit of Y they buy. Total net sales of Y are the b units of agents $h \in [0, b)$, giving a price in equilibrium of

$$p_{0Y} = \frac{e(a - b) + aR}{b}. \tag{1}$$

A buyer on margin of Y must put down $p_{0Y} - R = [e(a - b) + aR)]/b - R = [(a - b) + (e + R)]/b$ of his own money, getting a payoff of $1 - R$ in state 1 and 0 in state 2. Since $h = b$ is indifferent to buying on margin, $[(a - b)/b](e + R) = b(1 - R)$, or $b^2(1 - R) + b(e + R) - a(1 + R) = 0$, or

$$b = \frac{-(e + R) + \sqrt{(e + R)^2 + 4a(e + R)(1 - R)}}{2(1 - R)}. \tag{2}$$

Notice that agent b is also indifferent to buying Y directly from cash, without borrowing, so

$$p_{0Y} = b1 + (1 - b)R. \tag{3}$$

The price of Y is given by the marginal utilities of the *marginal buyer* b.

Notice that buying Y on margin via contract j^* costs on net $b(1 - R)$ and pays $1 - R$ in state 1 and 0 in state 2.

Thus for $h > b$, $x_0^h = 0$, $y_0^h = 0$, $C_Y^W \varphi_{j^*}^h = 1 + \frac{b}{a-b} = \frac{a}{a-b}$, $\varphi_{j^*}^h = R\frac{a}{a-b}$, $x_1^h = (1 - R)\frac{a}{a-b}$, $x_2^h = 0$, and all other choice variables equal zero. For $h < b$, $x_0^h = e + \frac{(a-b)}{b}e$, $y_0^h = 0$, $\theta_{j^*}^h = R\frac{a}{b}$, $x_1^h = R\frac{a}{b} = x_2^h$, and all other choice variables 0. One can easily check that supply equals demand, and that each agent is balancing his budget, using the definition of p_{0Y}.

To finish the description of the equilibrium, we must describe all the other prices, and show that the agent actions are optimal. In particular, we must check that no agent wants to buy or sell (lend or borrow) any contract j with collateral level $C_j \neq C_{j^*}$. Clearly $p_{0X} = p_{1X} = p_{2X} = 1$, and $p_{1Y} = p_{2Y} = 0$.

We already said that for collateral level $C_{j^*} = (0, j^*) = (0, 1/R)$, $\pi_{j^*} = 1$. In general, we set

$$\pi_j = b \min\{1, j\} + (1 - b) \min\{1, jR\} \tag{4}$$

equal to the marginal utility of agent b. For $j > j^*$, collateral levels are wasteful, since then the collateral more than covers the loan. Thus $\pi_j = 1$ for all $j > j^*$. Nobody has any reason to lend (buy) via contract $j > j^*$, since he gets the same price and return as with contract j^*. Similarly, nobody would sell (borrow via) $j > j^*$, since the price is the same on j as j^*, and the collateral terms are more onerous.

We now turn to contracts $j < j^*$. These contracts involve default, but they demand higher interest (lower price for the same promise). In effect they pay less in state 2 but more in state 1 than asset j^*. This is bad for optimistic borrowers $h > b$ and also bad for pessimistic lenders $h < b$, since these assets deliver more in the event borrowers think will happen and lenders think will not happen. If anything, cautious optimists with h barely bigger than b might want to lend via contract j. But lending requires money, and they would rather spend all their free liquidity on Y_0. We now make this argument rigorous.

A buyer of contract j receives $D_1^j = \min\{1, j\}$ in state 1 and $D_2^j = \min\{1, jR\} = jR < D_1^j$ in state 2. A seller of contract j must also buy the collateral consisting of j units of Y. On net in state s he receives $-D_s^j + jf_{s1}(0,1)$. In state 1 this is $-\min\{1, j\} + j1 \geq 0$, and in state 2 this is $-\min\{1, jR\} + jR = 0$. Notice that in both cases the payoff is at least as high in state 1 as in state 2. All prices are determined linearly by taking expectations with respect to $(b, 1-b)$. Agents $h < b$ will therefore regard each payoff as too expensive, or at best, as break even. To see that agents $h > b$ do not wish to trade either side of contracts $j \neq j^*$, observe that their budget set is included in $B \equiv \{(x_0, x_1, x_2) : x_0 + bx_1 + (1-b)x_2 = e + b1 + (1-b)R\}$. Every asset and contract trades at a price equal to its contingent X payoffs, valued at price $(1, b, 1-b)$. The collateral requirements make trades more difficult, reducing the real budget set strictly inside B. In B, agents $h > b$ clearly would take $x_1 = [e + b + (1-b)R]/b$, $x_0 = x_2 = 0$, that is, they would spend all their wealth in state 1. But, as we saw, that is exactly what they are able to do via margin borrowing on contract j^*. Therefore they have no incentive to trade any other contract $j \neq j^*$.

Below are equilibria for various values of the exogenous parameters (R, a, e).

R	0	0.1	0.2	0	0.2	0	0.2
a	1	1	1	0.75	0.75	1	1
e	1	1	1	1	1	0.75	0.75
b	0.618034	0.652091	0.686141	0.5	0.549038	0.568729	0.647233
p_{0Y}	0.618034	0.686882	0.748913	0.5	0.63923	0.568729	0.717786
m	1	0.854415	0.732946	1	0.687124	1	0.721366
x_{0H}	0	0	0	0	0	0	0
x_{1H}	2.618034	2.586882	2.548913	3	2.985641	2.318729	2.267786
x_{2H}	0	0	0	0	0	0	0
x_{0L}	1.618034	1.533529	1.457427	1.5	1.366025	1.318729	1.158779
x_{1L}	0	0.153353	0.291485	0	0.273205	0	0.309008
x_{2L}	0	0.153353	0.291485	0	0.273205	0	0.309008

8.1 The Marginal Buyer

A striking property of the example is that the prices of the asset Y and all the contracts $j \in J$ are a set by the marginal utilities of a particular marginal buyer $b \in H$.

8.2 Endogenous Margin Requirement

We saw that equilibrium endogenously sets the maximum loan backed by 1 unit of Y at R, thus ruling out default. The margin requirement is then

$$m \equiv \frac{p_{0Y} - \frac{1}{j^*}\pi_{j^*}}{p_{0Y}} = 1 - \frac{R}{p_{0Y}};\ (1-m) = \frac{R}{p_{0Y}} \tag{5}$$

where $p_{0Y} = b1 + (1-b)R$.

8.3 Margin Feedback Effects

In the table we see that a decrease in R leads to a decline in p_{0Y}. This is natural, since with lower R, Y has lower expected payoff. The interesting point is that p_{0Y} falls by *more* than the expected output of Y, calculated with respect to the probabilities of the marginal buyer b in the old equilibrium. For example, when $(R, a, e) = (.2, 1, 1)$, $p_{0Y} = b1 + (1-b).2 \approx .69(1) + .31(.2) \approx .75$. When R falls to .1, expected output at the old b falls to $.69(1) + (.31)(.1) \approx .72$. But actual p_{Y0} falls to .69, as can be seen in the table. Thus p_{0Y} falls by twice as much as would be expected from its drop in expected output.

Of course, the reason for this large fall in p_{0Y} is that b falls. As R falls, the formula (2) shows that b must fall.[3] By (3), p_{0Y} falls for two reasons. It falls because with lower R, the expected payoff from Y falls, computed with respect to the old probability b. But p_{0Y} falls again because the new marginal buyer is less optimistic, $\tilde{b} < b$, and with $\tilde{b}$ replacing b, p_{0Y} would fall even with the old R.

The reason for the drop in b is the sharp increase in margin requirements. With Y much less expensive, one would expect b to rise. The only possible explanation is that the equilibrium margin requirements have gone way up, which we can confirm analytically. Using the margin requirement m in (5), we can write $R = (1-m)p_{0Y}$. Plugging that into the RHS of (1) we get

$$p_{0Y} = \frac{e(a-b)}{b - a(1-m)} = \frac{1}{-1 + \dfrac{am}{a-b}}. \tag{6}$$

Since b and p_{0Y} fall when R falls, it follows from (6) that m must rise as R falls.

Thus a fall in R has a direct effect on p_{0Y}, because it lowers expected output, but it also has an indirect effect on p_{0Y} by raising margin requirements. And the indirect effect can be as large as the direct effect.

[3] To see this analytically, consider the equation $f(b, R) = b^2(1-R) + b(e+R) - a(e+R)$, and recall that $b(R)$ is defined so that $f(b(R), R) = 0$. Clearly $\partial f/\partial b > 0$, and $\partial f/\partial R = -b^2 + b - a < 0$. If R falls, b must fall to restore $f(b(R), R) = 0$.

To put the matter in different words, an asymmetrically perceived decrease in the productivity and safety of the asset Y leads to an even greater fall in its price, because it also makes it harder to borrow, and markets become less liquid.

By contrast, consider the effect of a decrease in liquid wealth e. This also reduces the value of Y. A drop in $(R, a, e) = (.2, 1, 1)$ to $(.2, 1, .75)$ causes p_{0Y} to drop from .75 to .72, and b to drop from .69 to .65. But the drop in liquidity is partly ameliorated by a decrease in margin requirements, from $m = .73$ to $m = .72$.

Similarly, a fall in the number of optimistic buyers a naturally leads to a drop in p_{0Y} and in b. As (R, a, e) falls from $(.2, 1, 1)$ to $(.2, .75, 1)$, p_{0Y} falls from .75 to .64. But m also falls from .73 to .69, partly damping what would have been a worse fall in p_{0Y}.

Thus we see that certain kinds of shocks tend to reduce asset prices, but in a damped way because they also lower margin requirements. Shocks that reduce value less for buyers than for sellers lower price by more than they lower expected value to the original marginal buyer, because they also tend to raise the margin requirement, making a less optimistic buyer the marginal buyer, giving a second reason for prices to fall.

8.4 Endogenous Default

We saw in the example that the equilibrium margin requirements were set so that there would be no default. But that is not necessarily the case. Consider a variant of the last example in which there are three states, with payoffs of Y and agent-dependent probabilities given below:

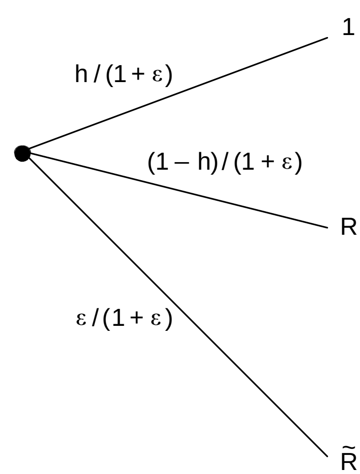

Figure 2

Note that all agents agree on the probability of $s = 3$. It is easy to check that for any $\tilde{R} < R$, in equilibrium, *only* asset $j^* = 1/R$ will be traded, exactly as before. If $\tilde{R} < R$, then there will be defaults in state 3. Rather than adjusting the collateral level to maintain zero default, equilibrium will adjust the price of all the loans to compensate lenders for the higher expected loss from default. In the new equilibrium, the price of Y and every contract j again is calculated according to the probabilities of the new marginal trader $\tilde{b}$:

$$\tilde{p}_{0Y} = \frac{\tilde{b}}{1+\varepsilon}1 + \frac{1-\tilde{b}}{1+\varepsilon}R + \frac{\varepsilon}{1+\varepsilon}\tilde{R}$$
$$\tilde{\pi}_j = \frac{\tilde{b}}{1+\varepsilon}\min\{1, j\} + \frac{1-\tilde{b}}{1+\varepsilon}\min\{1, jR\} + \frac{\varepsilon}{1+\varepsilon}\min\{1, j\tilde{R}\}.$$

If the new equilibrium with $\varepsilon > 0$ had the same marginal buyer as before, when $\varepsilon = 0$, then the new price $\tilde{p}_{0Y}$ would be less than the old p_{0Y} by the expected loss in output $[\varepsilon/(1+\varepsilon)](p_{0Y} - \tilde{R})$. The fall in $R\pi_{j^*}$ would, however, only be $[1/(1+\varepsilon)](R - \tilde{R})$, which is smaller. Hence agents would need less cash after borrowing to buy Y_0. This drives the price of Y_0 up, or equivalently it drives the marginal buyer $\tilde{b} > b$. (A countervailing force is that agents $h > b$ can borrow less on the Y they begin by owning. For $\tilde{R}$ near R, this is a less important effect.) Thus the equilibrium price $\tilde{p}_{Y0}$ falls *less* than the expected drop in output at the old b. Far from a feedback, news of a potential default, if universally agreed upon in probability, lowers asset prices by less than the direct effect.

This can be verified by noting that the economy with parameters $(a, e, R, \varepsilon, \tilde{R})$ has the same equilibrium marginal buyer as the old economy with parameters $(a, \tilde{e}, R)$, where $\tilde{e} = e + b^2(1-R)/(a-b) - (R - \tilde{R})$.

8.5 Efficiency vs. Constrained Efficiency

Collateral equilibrium is clearly not Pareto efficient. In our example, agents $h < b$ end up consuming $R(a/b)$ units in states $s = 1$ and $s = 2$. In particular, agent $h = 0$, who attaches probability zero to $s = 1$, consumes $R(a/b) > 0$ in state 1, if $R > 0$. It would be better if he could sell some of his $s = 1$ consumption to agent $h = b - \varepsilon$ in exchange for some $s = 2$ consumption.

When $R = 0$, agents $h > b$ consume nothing in state 2, and agents $h < b$ consume nothing in state 1. But still the collateral equilibrium is inefficient, because agents $h < b$ consume $1 + [(a-b)/b]$ units at time 0. Again agents $h = 0$ and $h = b - \varepsilon$ could *both* be made better off if they could trade some x_0 for some x_2.

We now compute the Arrow–Debreu prices $(1, b^*, 1 - b^*)$. They must induce all agents $h \in (b^*, a]$ to consume only in state 1, and all agents $h \in [0, b^*)$ to consume only in state 2, for some b^*. Since aggregate output in state 1 is $ea + a$, and in state 2 it is $ea + aR$, we conclude that $[I(a - b^*)]/b^* = ea + a$ and $Ib^*/(1 - b^*) = ea + aR$, where $I = b^*(e+1) + (1-b^*)(e+R)$ is the wealth at prices b^* of every agent $h \in [0, a]$. It follows from some algebra that

$$b^* = \frac{-(1+a)(e+R) + \sqrt{(1+a)^2(e+R)^2 + 4a(e+R)(1-R)}}{2(1-R)} < b. \qquad (7)$$

We see therefore that in collateral equilibrium it is possible for asset prices to be much higher than in Arrow–Debreu equilibrium. This of course is confirmed by simulation. When $R = 0$ and $e = a = 1$, $b^* = p^*_{0Y} = .414 < .61 = p_{0Y}$. When $R = 1/2$ and $e = a = 1$, $p^*_{0Y} = .46 < .62 = p_{0Y}$ and so on.

Since collateral equilibrium gives rise to asset prices that are too high ($p_{0Y} > p^*_{0Y}$), one is tempted to think that government intervention to impose high margin requirements would be beneficial. It is particularly tempting when there are defaults, as in the variant of the example considered in Section 8.4. But by the constrained efficiency theorem in Geanakoplos–Zame, no allocation achievable with the collateral enforcement mechanism for delivery could do better.

9 Crashes

We turn now to a dynamic context in which we can find feedbacks from wealth redistribution and margin changes at the same time. Imagine a multiperiod model, with the agent specific probabilities and payoffs from asset Y indicated below. It is now convenient to label agent's h's opinion of the probability of up by $g(h)$.

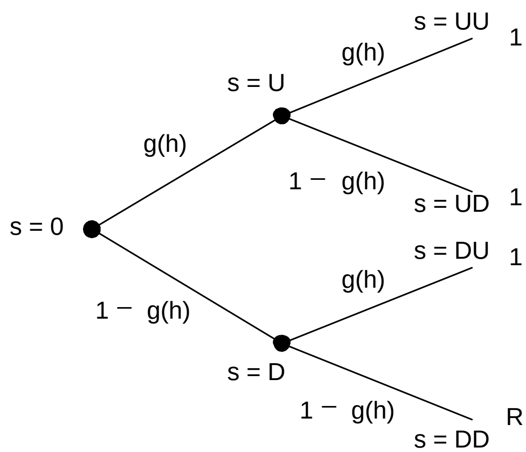

Figure 3

The tree roughly corresponds to the possibility of default getting closer, as well as more probable. An asset Y can pay off 1 or default and pay off $R < 1$. Each period there is either good news or bad news, independently drawn. The asset Y defaults only if there are two bad signals. After the first bad signal, the probability of default rises, and the horizon over which there may be a default shortens. (The rest of the tree, for example where there is one good signal and two bad signals, is compressed for simplicity into the simple three-stage tree above.)

Take the case where $g(h) = 1 - (1-h)^2$. At time 0, agent h attaches probability $(1-h)^4$ of eventual default in asset Y. If he gets bad news, $s = D$, then his probability rises to $(1-h)^2$ in the next period. For an optimist with h near 1, this may be hardly any change at all.

Each node or state s in the tree is defined by its history of U and D's. The node sD means the node where the move D occurred after the history s. Similarly for sU.

Again let there be two goods X and Y in each state, where X is like cigarettes and Y is like a tobacco plant that will produce only in the last period. Y produces one cigarette, unless two independent events go bad, in which case it produces only $R < 1$.

Endowments are 1 unit of X and Y at $s = 0$, and 0 otherwise:

$$e^h_{0X} = e^h_{0Y} = 1,\ e^h_{sX} = e^h_{sY} = 0\ \forall s \neq 0,\ \forall h \in H.$$

As before, X is durable but extinguishable by production, and Y is durable until its final output of X is produced. Let $f_s(z_1, z_2)$, denote the output in state s from the inputs (z_1, z_2) of X and Y in the unique state s^* preceding s. Then for $s \neq 0$, consumption destroys the good:

$$f^0_s(z_1, z_2) = f^B_s(z_1, z_2) = f^L_s(z_1, z_2) = 0$$

Warehousing on the other hand produces

$$\begin{aligned} f^W_U(z_1, z_2) &= f^W_D(z_1, z_2) = (z_1, z_2) \\ f^W_{UU}(z_1, z_2) &= f^W_{UD}(z_1, z_2) = f^W_{DU}(z_1, z_2) = (z_1 + z_2, 0) \\ f^W_{DD}(z_1, z_2) &= (z_1 + Rz_2, 0). \end{aligned}$$

Utility as before is given by the expected consumption of x,

$$U^h(x, x_W, y) = x_0 + g(h)x_U + (1-g(h))x_D + g^2(h)x_{UU} + g(h)(1-g(h))[x_{UD} + x_{DU}] + (1-g(h))^2 x_{DD}.$$

We assume now that $H = [0, \alpha]$.

In each state s^* there is a contract $j \in J$ that promises 1 unit of X in each successive state s and requires j units of Y as collateral at time s^*. We write

$$\begin{aligned} A_{sj} &= (1, 0)\ \forall s \neq 0 \\ C_{s^*j} &= (0, j)\ \forall s \neq 0. \end{aligned}$$

Prices as before are given by p_{sX}, p_{sY} $\forall s$ and π_{sj} for all states s and all $j \in J$. It is understood that $e^h_{0^*} = 0$ and that $\pi_{sj} = 0$ for the terminal states $s \in \{UU, UD, DU, DD\}$.

The budget set for each agent h is given by exactly the same equations as before, but for every state s separately. It is understood that the output $f_s(z_1, z_2)$ belongs to the owner of the input (z_1, z_2) at state s^*.

Let us now compute equilibrium. It is clear that in state U we will have $p_{UY} = 1 = \pi_{Uj^*}$ where $j^* = 1$. Let us guess again that all contract trade takes place at $s = D$ via the contract j_D where $j_D = 1/R$, and that all contract trade takes place at $s = 0$ via the contract j_0, where $j_0 = 1/p_{DY}$.

Following this guess we further suppose that at time 0, all agents $h \in (a, \alpha]$ borrow to the max and buy up all the Y. In the likely state $s = U$ they get rich. In the (for them) unlikely state D they lose everything. The rest of the agents $h \in [0, a)$ sell Y and lend at $s = 0$. Thus in state $s = D$ agents $h \in [0, a)$ begin with endowments α/a of both X and Y. Of these, agents $h \in (b, a)$ will borrow to the max to buy Y in state D, and agents $h \in [0, b)$ will sell Y and lend to them, as in our first example.

If our guess is right, then the price p_{DY} will crash far below p_{0Y} for three reasons. First, every agent believes that D is bad news, and so by each agent's reckoning, the expected output of Y is lower. Second, the optimistic agents at $s = 0$ leverage, by borrowing to the hilt, and so they suffer a huge wealth setback at $s = D$, creating a feedback on prices p_{DY}, as we saw in the last section. (The elimination of the top echelon of optimists reduces the price at $s = D$.) Third, the margin requirement increases.

Computing equilibrium is similar to the simple example from Section 9, but with one wrinkle.

Agent a is the marginal buyer at $s = 0$. But at state $s = D$ he is much more optimistic than the marginal buyer b. Therefore he anticipates that \$1 is worth much more than \$1 worth of consumption of x_D. Indeed, it is worth $g(a)/g(b)$ times as much. The reason is exactly as we saw in Section 8. Agent a can buy y_D on the margin, paying $g(b)\delta$ at time D to get δ in state DU, which gives expected utility $g(a)\delta$. It follows that he should not consume x_0, but rather save it, then consume it if $s = U$, but if $s = D$ use the X to buy Y on margin. The marginal utility to a of x_0 is therefore $g(a)1 + (1 - g(a))[g(a)/g(b)]$.

The marginal utility to agent h from buying Y at $s = 0$ and holding it to the end is

$$MU_Y^h \equiv [1 - (1 - g(h))^2]1 + (1 - g(h))^2 R.$$

Thus we must have

$$\frac{[1 - (1 - g(a))^2]1 + (1 - g(a))^2 R}{p_{0Y}} = g(a)1 + (1 - g(a))\frac{g(a)}{g(b)}1. \tag{8}$$

Agents $h \in (a, \alpha]$ will buy Y on margin, spending in total $(\alpha - a) + \alpha p_{DY}$, and agents $h \in [0, a)$ will sell Y, giving

$$p_{0Y} = \frac{(\alpha - a) + \alpha p_{DY}}{a}. \tag{9}$$

But as before,

$$p_{DY} = g(b)1 + (1 - g(b))R \tag{10}$$

and

$$p_{DY} = \frac{(a - b) + aR}{b}. \tag{11}$$

Combining (10) and (11) gives

$$a = \frac{b[(1 + g(b)) + (1 - g(b))R]}{1 + R}. \tag{12}$$

These five equations can be solved simultaneously via a simple algorithm. Choose b arbitrarily. From the last equation compute a. Then compute p_{DY} and then p_{0Y}. Finally, check that equation (8) holds. If not, iterate.

The following table describes the equilibrium for three different values of R, given $g(h) \equiv 1 - (1 - h)^2$, and $\alpha = 1$.

R	0	0.1	0.2
b	0.5012	0.55055	0.60022
$1-g(b)$	0.24880144	0.202005	0.159824
$g(b)$	0.75119856	0.797995	0.840176
p_{DY}	0.75119856	0.818195	0.872141
a	0.877700718	0.910007	0.936414
$1-a$	0.122299282	0.89993	0.063586
p_{0Y}	0.995211492	0.998002	0.999267
$1-g(a)$	0.014957114	0.008099	0.004043
$g(a)$	0.985042886	0.991901	0.995957
$(1-g(a))^2$	0.000223715	6.56E-05	1.63E-05
$g(a)/g(b)$	1.31129496	1.242992	1.185415
E_aY/p_{0Y}	1.004656074	1.001968	1.00075
$m0$	0.245134929	0.180146	0.127194
mD	1	0.87778	0.770679

9.1 What Caused the Crash? Feedback

Consider the case $R = .2$. In state D, the asset price p_{DY} crashes, falling from a price of $p_{0Y} = .9993$, to a price $p_{DY} = .8721$. Three factors explain this change. First, the probability of default increased from $(1-h)^4$ to $(1-h)^2$ for each agent h. For the marginal buyer $a = .9364$, this represents an increase from virtually zero to .0040, still a negligible number. The drop in expected output from Y is thus about .003, which itself is negligible, compared to the drop in price of $(.9993 - .8721) = .1272$.

Second, the drop in value of the price destroyed the wealth of the most optimistic buyers, effectively eliminating the purchasing power of every agent $h > a = .9364$. We can see what effect the disappearance of these agents would have *ceteris paribus*, by recomputing equilibrium in the two-period model with $\alpha = .9364$. The result is listed below.

α	0.9364136
R	0.2
b	0.563
$1-g(b)$	0.190969
$g(b)$	0.809031
p_{DY}	0.8472248
a	0.866656302
$1-a$	0.133343698
p_{0Y}	0.995908206
$1-g(a)$	0.017780542
$g(a)$	0.982219458
$(1-g(a))^2$	0.000316148
$g(a)/g(b)$	1.214069001
E_aY/p_{0Y}	1.003806263
$m0$	0.149335291
mD	0.763935144

We see that there is almost no effect on equilibrium prices from eliminating the 7% most optimistic buyers. *Ceteris paribus*, p_{0Y} drops from .9993 to .9959.

Third, the time of default gets closer, and the margin requirement jumps from 12.7% to 77%. We can compute the effect this change would have itself by returning to our one-period model, but with $g(h) = 1-(1-h)^4$, which is the probability each agent h attaches to no-default in the two-period model The result is listed below.

R	0.2
a	1
e	1
b	0.60585
$1-g(b)$	0.024134934
$g(b)$	0.975865066
p_{0Y}	0.980692052
m	0.796062383

We see again that the effect of changing the margin requirement from 12.7% to 79.6% (as well as bringing the possibility of default nearer) reduces price p_{0Y} from .9993 to .9807, again close to negligible.

The conclusion I draw is that the price crash in the example is not due to any one factor, but is due to the reinforcement each brings to the others.

9.2 Why Did the Margin Increase?

The margin requirement increased on Y as the crash grew nearer because everyone expected the rate of information flow to increase, and because agents continued to disagree in their forecasts. The pieces of information D and DD are completely symmetric, since the probability of bad news is $1-g(h)$ in both cases, and the two events are independent. However, from D, the significance of the information to be revealed at the next step is huge. It resolves whether Y is worth 1 or R, while at $s = 0$, the next step will resolve whether Y is worth 1 or p_{DY}. Putting it another way, the variance of the price of Y one period after $s = D$ is much higher than the variance of Y one period after $s = 0$. Since agents continued to disagree about the probability of future news, the higher volatility must result in higher margins.

9.3 Liquidity and Differences of Opinion

The size of the crash depends on how far b is from a, and on how fast $g(h)$ changes as h changes. With b near a, $g(b)$ is near $g(b)$ and b's valuation of Y is not much different from a's. But as b moves away from a, this difference accelerates, given the functional form $g(h) = 1 - (1 - h)^2$. Had we made $g(h)$ a constant, so there were no differences of opinion, there would have been no crash.

With $g(h)$ a constant, there is a deep reservoir of potential buyers of the asset at the same price. With $1 - g(h)$ very convex, this pool erodes at an accelerating pace, so that twice the bad news does more than twice the damage. Hence the power of multiple factors in the crash, when each alone makes little difference.

This appears to give us a different perspective on liquidity, closer to one of the conventional definitions. In that definition, liquidity is defined as the sensitivity of the reaction function of the price when an agent tries to sell more. I noted earlier that this definition seems to contradict perfect competition, but we can simply amend it to mean the elasticity of price change as the total supply of an asset changes. It would appear from the foregoing that we might describe a market as illiquid and vulnerable to crashes if changes in the supply of the asset dramatically affect its price.

This definition does not, however, capture what is going on. *Doubling* the supply of the asset (which is equivalent to reducing every agent's endowment of X by 50%) would change equilibrium p_{0Y} from .9993 to .9920, a negligible change. (See next table.) It is interesting that after the doubling, the economy becomes much more vulnerable to the shock D, because then price drops from .9920 to .7746.

α	1
R	0.2
b	0.46915
$1-g(b)$	0.281801723
$g(b)$	0.718198278
p_{DY}	0.774558622
a	0.854227396
$1-a$	0.145772604
p_{0Y}	0.992060109
$1-g(a)$	0.021249652
$g(a)$	0.978759348
$(1-g(a))^2$	0.000451548
$g(a)/g(b)$	1.362785708
E_aY/p_{0Y}	1.00770907
m_0	0.219188192
m_D	0.741788427

9.4 Profits after the Crash and Cautious Speculators

Before leaving the crash example, it is instructive to reconsider why it is difficult to imagine a crash in a rational expectations world. One would think that if the crash is foreseen, then nobody would want to hold the asset before the crash. Or better, that investors would hold their capital, waiting to buy after the crash. After the crash optimistic investors could make a far greater return than they can before the crash. Investor $a = .9364$ can see that he could make an expected return of 18% $(g(a)/g(b))$ above the riskless rate starting at $s = D$. Why don't investors wait to invest until after the crash (thereby eliminating the crash)?

In fact, a group of investors do wait. As $s = 0$, investor $h = a$ calculates the expected output of Y per dollar at 1.00075. Unleveraged, he anticipates a .075% return on his money, above the riskless rate, from investing in Y. He is risk neutral, yet he holds off investing in Y. Why? Because he foresees that if he keeps his money in liquid X, he can earn an 18% return $(g(a)/g(b))$ on his money above the riskless rate, after leverage, if state D should occur. There is a whole group of agents $h \in (\underline{a}, a)$ who regard Y_0 as a profitable investment, but who choose instead to sell it in order to stay liquid in X in anticipation of the crash. The probability of the crash is so low, however, that not many investors bother to prepare themselves this way, and so the crash still occurs.

10 The Liquidity Spread

Consider two assets which are identical except that only the first can be used as collateral. Will their prices be the same? To some extent this situation prevails with on-the-run and off-the-run Treasuries. The percentage of off-the-run Treasuries that are used as collateral is much smaller than the on-the-run Treasuries, and they sell for a lower price.

We can see in our simple example why this should be so. Suppose a fraction f of each agent's Y is painted blue, and can be used as collateral, while the remaining fraction $(1-f)$ is painted red and cannot. What will their equilibrium prices be? If the price p^* of blue is the same as the price p of red, then all h above the marginal buyer b will spend all their money on blue (since they strictly prefer Y to X, and leveraging is the way to get as much Y as possible). All the agents $h < b$ will sell Y (since they strictly prefer X to Y.) Thus there will be no buyers for red Y, and markets will fail to clear. It follows that $p^* > p$. A moment's thought shows that in equilibrium households $h \in [0, \alpha]$ will split into three pieces. The most optimistic $h \in (a, \alpha]$ will leverage and buy blue Y. Agent a will be indifferent to buying blue on margin at the high price, and red at the low price. Agents $h \in (b, a)$ will buy only the red Y, selling their blue. Agents $h \in [0, b)$ will sell all their Y. Agent b is indifferent between buying red Y and holding X.

More precisely, we can find equilibrium by solving the equations

$$1b + (1-b)R = p \tag{13}$$

$$\frac{e(a-b) + p^* f(a-b)}{(1-f)(\alpha - (a-b))} = p \tag{14}$$

$$\frac{e(\alpha - a) + p(1-f)(\alpha - a) + f\alpha R}{fa} = p^* \tag{15}$$

$$\frac{a(1-R)}{p^* - R} = \frac{a1 + (1-a)R}{p}. \tag{16}$$

Equation (13) asserts that agent b is indifferent between red Y and X. Equation (14) says that agents $h \in (b, a)$ take all their cash, plus the money they get selling off their blue Y, and spend it all on red Y. Everyone else sells their red Y. Equation (15) says that agents $h \in (a, \alpha]$ take all their cash, plus all the money they get selling their red Y plus all the money they can borrow in the blue Y, and use it to buy all the blue Y that is sold by agents $h \in [0, a)$. Finally, equation (16) insures that for agent a, the marginal utility of \$1 in blue Y is equal to the marginal utility of \$1 in red Y.

The following table gives equilibrium for various values of f, fixing $\alpha = 1$, $R = .2$, and $e = 1$.

f	0.4	0.5	.06
α	1	1	1
e	1	1	1
R	0.2	0.2	0.2
a	0.873007	0.841774	0.810657
b	0.627968	0.636558	0.645501
p	0.702374	0.709246	0.7164
p^*	0.746014	0.746839	0.747544

The equilibrium equations sharpen our intuition about why the price of blue Y and red Y differ, despite the fact that they are perfect substitutes. The buyers of blue Y and red Y can be disjoint sets. Y bought on the margin gives extreme payoffs $(1-R,0)$ that are not collinear with the payoffs $(1,R)$ from buying Y with cash.

One can see from the table that as f declines, the total value of Y falls, the spread between red and blue Y increases, and both blue Y and red Y fall in value. The fact that the total value of Y falls is obvious. Y is harder to purchase if its liquidity is lower.

The fact that blue Y is more valuable than its perfect substitute, red Y, just because it can be used as collateral, is of extreme importance, as is the principle that this spread gets wider as the general liquidity in the economy falls. This liquidity spread widening is one of the hallmarks of a liquidity crisis. In our example, spread widening is inevitable because the supply of blue Y went down and the supply of red Y went up. The only curiosity is that the price of blue Y went down. This is an accidental artifact of our parameters, coming from the fact that as p declines the liquid wealth of the superoptimists $h \in (a,\alpha]$, who are sellers of red Y, declines, thereby reducing their purchasing power for blue Y.

A subtler proposition is that when one asset Y becomes less liquid, say because margin requirements are raised on it, then the spread between liquid and less liquid assets that are unrelated to Y also tends to increase. We consider such questions in the next section.

11 Spillovers

Since the collapse of Long Term Capital Management in 1998, it has become clear that many assets are much more correlated in times of (liquidity) crisis than they are otherwise. Our simple example of Section 8 can be extended to show some reasons why.

Consider the situation where there are two assets Y and Z, and suppose that the margin requirement on Y is increased, say because R falls. Why should we expect the price of Z to fall?

At least three reasons some to mind. First, the same optimistic buyers might hold Y and Z. A negative shock to their wealth, or to their liquidity, will reduce their demand for all normal goods. Second, a decline in their liquidity will give them the incentive to shift into more liquid assets; if Z has relatively high margin requirements, and there is another comparable asset Z' with easier margin requirements, they will demand less Z. Finally, the equilibrium margin requirement may rise on Z, as a result of decreased recovery R on Y. We take these three explanations in turn.

11.1 Correlated Output

At first glance it would seem that if two assets had very similar returns, then they would be close substitutes. If R fell for Y, impairing its value, we might expect investors to switch to Z, possibly raising its value. But this substitution effect can easily be swamped by an income effect. If Y and Z are closely correlated, it is likely that optimists about Y are also optimistic about Z. The fall in R causes an income shock to its buyers, which impairs their ability to buy Z.

When R falls, we saw that the price of Y falls for two reasons. First, because the expected output goes down, and second because the new marginal buyer is a more pessimistic fellow. If Y and Z are very correlated, then a more pessimistic buyer for Y will be more pessimistic about Z, and so the price of Z should fall as well.

We can see this in the example from the last section. Holding the fraction of blue Y fixed at .5, and lowering R on both blue Y and $Z =$ red Y reduces the price of both by more than expected output decreases, as can be seen from the following table.

α	1	1
f	0.5	0.5
e	1	1
R	0.3	0.3
a	0.85873	0.841774
b	0.646957	0.636558
p	0.752871	0.709246
p^*	0.802224	0.746839

When R falls from .3 to .2, both prices p and p^* fall by more than the expected output of Y and Z (calculated with respect to either the possibilities $(a, 1-a)$ or $(b, 1-b)$). The gap between p^* and p narrows from .050 to .037.

In the example there is no substitution effect. Agents either prefer to buy expensive Y on the margin, or they prefer to buy cheaper Z. A change in the margin requirement simply reduces the amount of Y that can be bought on margin, but does not by itself induce an agent to switch. If we had three states and a more complicated example we could have had agents holding both Y and Z, and then adjusting the proportions of each. Then the gap might have narrowed more.

A similar example in which Y and Z are correlated but not identical is the following. Let Y pay 1 or R, as usual. Let Z pay 1 or 0. It is easy to see that the equilibrium is the same as it would be with one asset $W = Y + Z$. Lowering R for W will reduce p_W and make the marginal buyer b more pessimistic. But that lowers the price of both Y and Z.

11.2 Independent Outputs, Correlated Opinions

It is perfectly possible for each agent h to think that the returns from Y and Z are independent, yet for optimists about Y to be optimistic about Z. For example, we could imagine four states of nature giving payoffs from Y and Z as follows: $(1, 1)$, $(1, R)$, $(R, 1)$, (R, R). Each household h might regard their probabilities as $(h^2, h(1-h), (1-h)h, (1-h)^2)$, respectively. Thus everybody might agree that defaults by the Russian government and the American homeowners are independent. Yet many hedge funds might have been optimistic about both, and thus simultaneously invested in Russian debt and mortgages.

In our example, every agent is risk neutral, so equilibrium is exactly the same for the independent case as for the perfectly correlated case just given. In the example of 11.1, a decrease in R for Russian debt will lower American mortgage prices.

11.3 Cross-Collateralization and the Margin Requirement

Many lenders cross collateralize their loans. Thus if the same promise (say of \$1) is taken out by a borrower using C_1 as collateral, and another promise is made by the same borrower using C_2 as collateral, then the lender is paid in each state s

$$\min\{2, f_s^W(C_1) + f_s^W(C_2)\}$$

where $f_s^W(\ )$ is the value of the collateral in state s.

Consider the situation in the example in Section 11.2 in which assets Y and Z had independent payoffs. Total value of $Y + Z$ in the four states would then be $(2, 1 + R, 1 + R, 2R)$. It should be evident that if lenders could count on borrowers taking out an equal amount of Y-backed loans as Z-backed loans, then they would loan $1 + R$ for each collateral of $Y + Z$. But the margin requirement is then only $[2p - (1 + R)]/2p = 1 - (1 + R)/2p$, which is less than the margin requirement for Z alone, $(p - R)/p = 1 - R/p$. Thus cross-collateralization often leads to more generous loan terms.

If Y disappears, say because Russian debt collapsed, then lenders will be lending against only Z collateral, and thus margin requirements may rise on mortgages.

11.4 Rational Expectations and Liquidity Risk

We have assumed in our examples that agents may differ in their probability assessment of exogenous agents (U or D or UU), but that they all understand completely the endogenous implications of each event. In reality, of course, agents do not have identical opinions about endogenous variables. In particular, there are probably wide disparities in the probability assessments of a liquidity crisis. An optimist about liquidity crises would then be optimistic about all kinds of assets that crash in liquidity crises. He might therefore be led to hold all of them. But if enough liquidity optimists do this, then they create precisely the conditions we have been describing that lead to spillovers in a liquidity crisis.

12 Two More Causes of Liquidity Crises

There are other explanations of liquidity crises that our examples suggest, but that we shall not pursue. The first is that when lenders cross-collateralize, but leave it to the borrower to choose the proportions of collateral, there is a moral hazard problem. Desperate hedge funds facing collapse might be tempted to gamble, thus holding a less hedged portfolio, e.g., not balancing Y with Z. Anticipating this, lenders might raise margin requirements, thus causing the collapse they feared.

Second, I took the possibility of default (the state in which output is $R < 1$) to be exogenous, and looked for endogenous liquidity crashes. But in reality there is a long chain of interlocking loans and the probability of a cascade of defaults is endogenous, and also an effect of liquidity, rather than just a cause.

13 An Abstract Definition of Liquidity

In Section 4 I promised to give a formal definition of liquidity, and in particular of liquid wealth. I do so now in a very abstract, simple model, which fits our previous examples.

Consider an economy in which agents have endowments $e^h \in \mathbb{R}^L_+$ in the first period, and state-contingent endowments $\tilde{e}^h \in \mathbb{R}^S_+$ in the second period. The goods $\ell \in \mathbb{R}^L_+$ represent tangible commodities and assets and promises. Each of them (potentially) produces a vector of goods in the second period according to the linear production function f. Thus $x \in \mathbb{R}^L_+$ in period 1 gives rise to consumption $\tilde{e}^h + f(x)$ in period 2 for agent h. The consumption set of agent h is

$$X^h = \{x \in \mathbb{R}^L : \tilde{e}^h + f(x) \geq 0\}.$$

Utility $u^h : \mathbb{R}^L_+ \to \mathbb{R}$ is derived from the utility of consumption in both periods.

Each good ℓ in period one gives rise to liquid wealth if it is sold. But if it is sold short, the short sale gives rise to income but not liquid wealth. Every good requires a fraction of its purchase to be from liquid wealth. Given prices $p \in \mathbb{R}^L_{++}$ and liquidity requirements $m \in [0,1]^L$, define the budget set of agent h by

$$B^h(p, m, e^h) = \{x \in X^h : p \cdot x \leq p \cdot e^h \text{ and } \sum_{\ell=1}^{L} m_\ell p_\ell x_\ell^+ \leq p \cdot e^h\},$$

where $x_\ell^+ \equiv \max\{x_\ell, 0\}$ and $X^h \subset \mathbb{R}^L$ is the consumption set of h. Notice that B^h includes a Walrasian constraint $p \cdot x \leq p \cdot e^h$, as well as a liquidity constraint. The liquidity constraint may be binding if some $x_\ell < 0$. In that case $p \cdot x^+$ could be much greater than $p \cdot e^h$.

When $e_\ell^h = 0$, ℓ is a promise, and $x_\ell^h < 0$ means h is selling $-x_\ell^h$ units of the promise ℓ.

It would seem that liquid wealth $\tilde{W}^h$ is the same as wealth, $W^h = p \cdot e^h$. But $p \cdot e^h$ is not a measure of wealth, since by choosing some $x_\ell < 0$, agent h can spend much more than $p \cdot e^h$ on the rest of his goods. If there were some $\bar{x}^h$ with $f(\bar{x}^h) = \tilde{e}^h$, then wealth would be more accurately described as $p \cdot e^h + \sum_{\ell=1}^{L} p_\ell \bar{x}_\ell^h$, much greater than $\tilde{W}^h$.

A commodity ℓ with $m_\ell = 1$ is completely liquid, because selling one unit always raises liquidity by p_ℓ. If $m < 1$, then selling one unit of ℓ raises liquidity by $(1-m)p_\ell$.

Imagine a commodity ℓ, say a house, with margin $m_\ell = 20\%$. A buyer could sell a promise $x_j < 0$, say $-p_j x_j = .80 p_\ell$, and also sell enough of a completely liquid good ℓ' to raise $.20 p_\ell$, and then buy the house. It is easy to rewrite all our examples in this abstract form.

Raising liquid wealth while leaving the economy unchanged could be achieved in this abstract model by transforming future endowment into a current physical asset.

Add another commodity $\ell+1$, and replace one agent's e^h with $(e^h, 1)$ and replace his $\tilde{e}^h$ with $\tilde{e}^h - f(0,1)$.

Thus liquidity could also be studied via an abstract general equilibrium model with two budget constraints. It might be interesting in such a model to find sufficient conditions to guarantee that if liquid wealth declines, while leaving the economy unchanged, then in equilibrium the spread between the prices of low margin assets and high margin assets increases.

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