A Computational Analysis of the Core of a Trading Economy With Three Competitive Equilibria and a Finite Number of Traders

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A Computational Analysis of the Core of a Trading Economy with Three Competitive Equilibria and a Finite Number of Traders*

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Abstract

In this paper we examine the structure of the core of a trading economy with three competitive equilibria as the number of traders (N) is varied. We also examine the sensitivity of the multiplicity of equilibria and of the core to variations in individual initial endowments. Computational results show that the core first splits into two pieces at $N = 5$ and then splits a second time into three pieces at $N = 12$. Both of these splits occur not at a point but as a contiguous gap. As N is increased further, the core shrinks by $N = 600$ with essentially only the 3 competitive equilibria remaining. We find that the speed of convergence of the core toward the three competitive equilibria is not uniform. Initially, for small N, it is not of the order $1/N$ but when N is large, the convergence rate is approximately of the order $1/N$. Small variations in the initial individual endowments along the price rays to the competitive equilibria make the respective competitive equilibrium (CE) unique and once a CE becomes unique, it remains so for all allocations on the price ray. Sensitivity analysis of the core reveals that in the large part of the endowment space where the competitive equilibrium is unique, the core either converges to the single CE or it splits into two segments, one of which converges to the CE and the other disappears.
Keywords: Core, Multiple competitive equilibria, Speed of convergence, Sensitivity Analysis.
JEL Classification: C62, C71.

1 Introduction

In a brief note Shapley and Shubik (1977) presented an example of a two-commodity, two-trader type smooth trading economy with multiple equilibria. Figure 1 shows a graphical representation of this economy in an Edgeworth box diagram. The example is robust in the sense that its qualitative features would survive small perturbations to the model parameters. In this paper we extend the example to perform a sensitivity analysis on the core as the number of traders is varied. We also examine the sensitivity of the multiplicity of equilibria and of the core to variations in individual initial endowments. In particular, we address the following questions: (a)

*The MATLAB programs used for performing the computations in this paper are available from the authors upon request.

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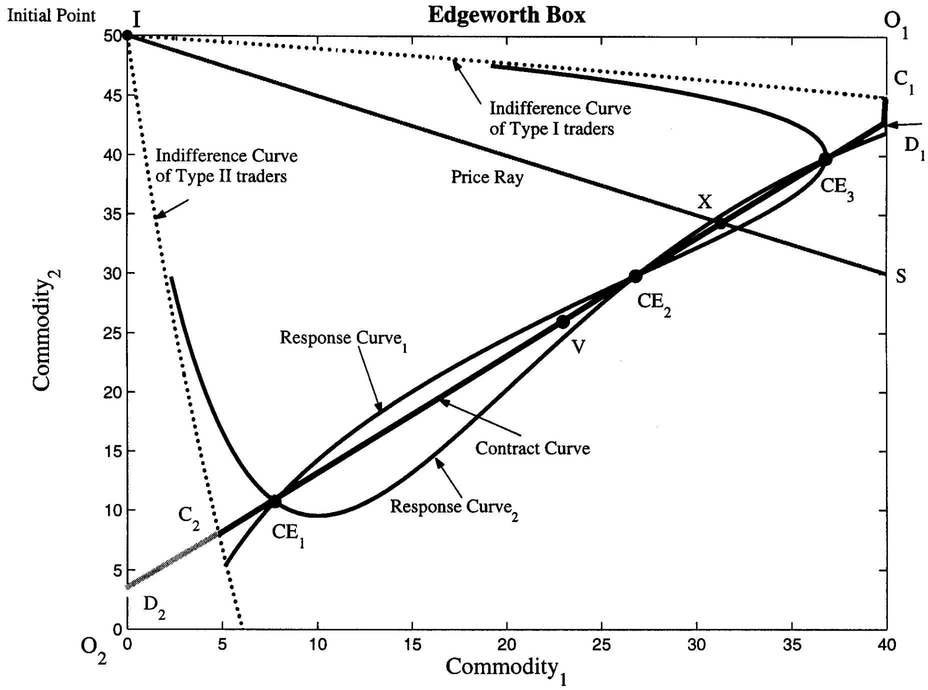

Figure 1. An Edgeworth box diagram with three competitive equilibria: *I* represents the initial allocations to both trader types. O_1 and O_2 are the origins for type I and type II traders respectively. CE_1, CE_2 and CE_3 are the three competitive equilibrium points and *V* is the value solution. IC_1 and IC_2 are the zero gain indifference curves. $O_1D_1C_1D_2C_2O_2$ is the Pareto set which consists of the locus of the points of tangency of the indifference curves of the two trader types (D_1D_{12}) and maximal points along the perimeter of the Edgeworth box (O_1D_1 and O_2D_2). The portion of the Pareto set, C_1C_2, which lies between the zero gain indifference curves is the contract curve. It includes a short piece D_1C_1 along the boundary of the box. C_1 and C_2 represent the worst contracts for type I and type II traders respectively. The response curve represents the optimal allocation of the trader type for an exogeneously specified exchange ratio (or price).

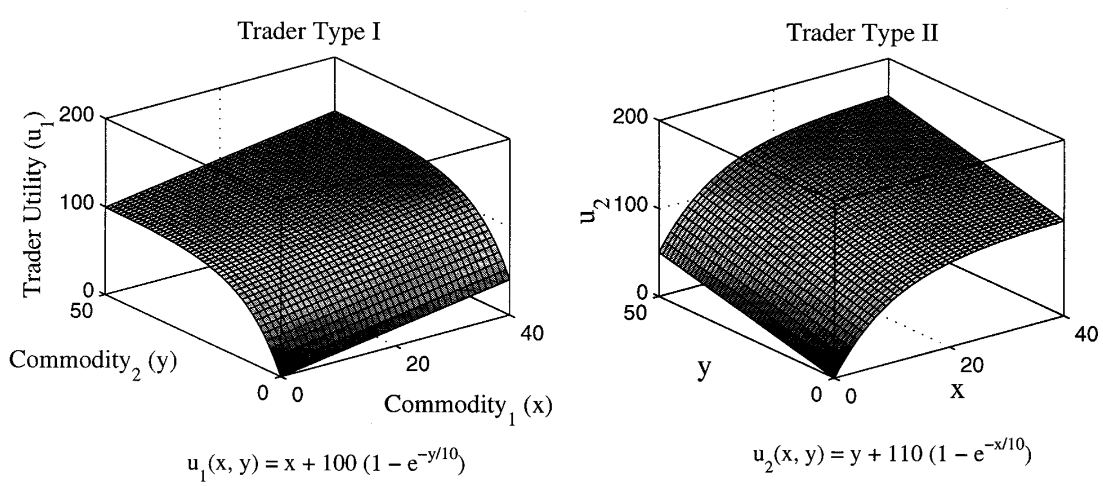

Figure 2. Utility functions of the two trader types: The utility functions are concave, smooth C^{∞}) and each has an additively separable linear term.

Core Split: for what values of N (the number of traders of each type) does the core split? Does the core split at a single point or is a gap formed? Does the core split into more segments than there are competitive equlibria (CEs)? (b) *Speed of convergence:* Do the segments of the core containing the CEs converge at different rates to their respective CE? (c) *Robustness:* How are the number of CEs influenced as the initial endowments of traders are varied while preserving the total endowment? Using analytical methods, these types of questions can only be approached at a high level of generality and do not permit us to examine the fine structure of specific cases. However, by adopting a computational approach we are able to "visualize" the fine structure of the specific example discussed in Shapley and Shubik (1977).

2 A Trading Economy with Three Competitive Equilibria

Consider a two-commodity[1], two-trader type exchange economy with $2N$ agents where there are N agents of each type. In Figure 1 we represent this economy graphically in an Edgeworth box. We assume that all traders of one type have identical preferences and identical endowments (initial allocations) at the beginning. Let $\mathcal{T}$ represent the set of all $2N$ traders in the economy and let $\mathcal{T}_1$ and $\mathcal{T}_2$ represent the set of traders of types I and II respectively. All traders of a given type have identical preferences which can be represented by the following utility functions (see Figure 2 for a plot of the two utility functions):

$$u_1(x,y) = x + 100(1 - e^{-y/10})$$

$$u_2(x,y) = y + 110(1 - e^{-x/10})$$

Both the utility functions are not only concave and smooth (C^∞) but also additively separable, with one good entering linearly in each case. It is well known that the competitive equilibrium is unique if the *same* good is linear and separable in all utility functions, provided only that this good is in sufficient supply and the preference sets are smooth (C^1) and strictly convex, as they are here. The Shapley–Shubik (1977) example shows that this transferable utility or welfare maximizing approach to uniqueness does not allow even a modest tinkering with the hypotheses.

The Pareto set for this economy is the locus of the points of tangency of the indifference curves of the two trader types and it is given by:

$$y = x + 50 - 10\log(110) = x + 2.995$$

This is represented as $O_1C_1D_1C_2D_2O_2$ in Figure 1. It includes the maximal points along the perimeter of the Edgeworth box (O_1D_1 and O_2D_2). The portion of the Pareto set, C_1C_2, which lies between the zero gain indifference curves is the contract curve. It includes a short piece D_1C_1 along the boundary of the box. C_1 and C_2 represent the worst contracts for type I and type II traders respectively.

The conditions for a competitive allocation reduce by elementary calculus to the following transcendental equation:

$$x(1 + 11e^{x/10}) = 10\log(110)$$

[1] For example, bread and wine.

Table 1: Numerical Data for Figure 1

Point in the Figure	Holdings of Type I Traders	Exchange Ratio	Utility Payoff to Type I Traders	Utility Payoff to Type II Traders
Initial Allocation				
I	$(50, 0)$		40.00	50.00
Core Solution: endpoints of the core				
C_1	$(40, 44.89)$	0.13	40.00	152.88
C_2	$(4.83, 7.83)$	8.73	133.69	50.00
Competitive Solutions				
CE_1	$(7.74, 10.74)$	5.07	130.29	70.01
CE_2	$(26.83, 29.82)$	0.75	99.88	132.30
CE_3	$(36.78, 39.77)$	0.28	67.27	146.99
Value Solution				
V	$(23.00, 25.99)$	1.04	107.94	124.96

For holdings of type II traders, subtract the holdings of type I traders from $(40, 50)$.

Exchange ratio is the price of the first commodity in the unit of the second commodity.

This equation has three roots in the region of interest which determine the three competitive equilibrium solutions, CE_1, CE_2 and CE_3, as shown in Figure 1 and Table 1. The allocations represented by the competitive equilibrium points are optimal for both the trader types, given the exchange ratio (or the price). At these allocations, the response curves of the two trader types intersect. Of the three competitive equilibria, two are stable (CE_1 and CE_3) and the third one (CE_2) is unstable in the sense that raising the price of either good would create a positive excess demand for that good. In Table 1 the core and the value solutions of the two person trading game are given in order to suggest outcomes alternative to those of the competitive equilibrium (see Shapley and Shubik, 1969).

Initial Conditions: We assume that all traders of one type begin with an identical initial endowment of commodities: $(a, 0)$ for type I traders and $(0, b)$ for type II traders. For $a = 40, b = 50$, the initial utilities are $(u_1, u_2) = (40, 50)$.

3 The Computational Approach

The core of an N-times replicated exchange economy converges to the set of competitive equilibria as the number of agents become large (Debreu and Scarf, 1963). The core shrinks in this manner because for a given core allocation, a group of traders or an (m, n) coalition[2] may be able to redistribute their resources such that some (at least one) members of the coalition improve their utilities while no member's utility deteriorates. This means:

$$u_c(x_j, y_j) \geq u_c(x_i, y_i), \forall c \in \mathcal{T}_{\mathcal{C}}(N) \text{ and } \exists c \in \mathcal{T}_{\mathcal{C}}(N) \text{ s.t. } u_c(x_j, y_j) > u_c(x_i, y_i)$$

where (x_i, y_i) is the allocation prior to exchange, (x_j, y_j) is the allocation after the exchange and $\mathcal{T}_{\mathcal{C}}(N) \subset \mathcal{T}$ is the set of all traders in the coalition. This process of exchange is known as a recontract. An allocation X is said to be "blocked" if a profitable recontract exists for an (m, n)

[2] An (m, n) coalition is a subset of the set of traders $\mathcal{T}$, where there are m traders of type I and n traders of type II.

coalition.[3] The blockable set depends only upon the type-compression ratio (the relative number of players of types I and II) of the coalitions.

As N increases, more allocations in the core can be blocked, i.e., points on the core that cannot be recontracted decreases and the core shrinks. For a very large N, all points in the core can be blocked except the competitive equilibrium points. The core typically shrinks at a uniform rate with the rate of convergence of the order $1/N$ (Debreu, 1975; Shapley, 1975). When there is one competitive equilibrium, the core uniformly converges to the CE. With multiple equilibria, however, the core must split into several pieces at some value of N and then it must remain disconnected for all larger N. We are interested in identifying the split points of the core as the number of traders in the economy increases and the subsequent convergence of the segments.

In order to identify the points on the core that can be blocked as N increases, the core is first discretized. Then, each allocation in the core (say X) is examined sequentially to check if it can be blocked by an (m, n) coalition. The symmetric allocations achievable by an (m, n) coalition lie on the line (price ray IXS in Figure 3) joining the initial allocation point (represented by I) and the point on the core under investigation. If P and Q are the allocations of the two trader types along the price ray after an exchange, then the following relation must hold in order to satisfy the resource constraint and the equal treatment[4] property:

$$\frac{\mathrm{IP}}{\mathrm{IQ}} = \frac{n}{m}$$

At the competitive allocations, the two indifference curves and the price ray are tangent to each other and thus no profitable recontract is possible. However, if X is a non-competitive allocation, the price ray IXS is not tangent to the two indifference curves through X and cuts them again at A and B respectively. The m type I traders can improve by recontracting to an allocation on the line segment AX and the n type II traders can improve by moving to an allocation on the line segment BX. Segments AX and BX are both non-empty and hence, a profitable recontract can take place if the ratio m/n is such that feasible allocations for type I and type II traders (allocations P and Q respectively) lie on segments AX and BX respectively. As we move closer to the CEs, the line segments AX and BX become smaller and in the limit, at the CEs, both the segments converge to the competitive equilibrium point. When we are dealing with very large coalitions, the ratio $m/m \approx 1$ and by recontracting, allocations very close to X along the price ray (IXS) are attainable. Hence, for large N, as the size of the coalitions increase, the core contains little more than the CEs.

The following algorithm describes the computational procedure used for computing the core of the exchange economy for a given value of N:

An Algorithm for computing the Core: *Given $N = N_{\max}$, compute the core of the exchange economy.*
Let

$N_{\max}$ = maximum number of traders in the economy.

I = initial individual endowments to the traders.

$\mathcal{C}(N, I)$ = core of the economy with N traders of each type.

[3] For details of the blocking procedure, see Shubik (1984, pp. 315–317).

[4] All traders of one type must have the same allocation because they all start with the same initial endowments and they have identical preferences. Given convex preferences, the traders of one type can always engage in a mutually beneficial exchange if this condition does not hold.

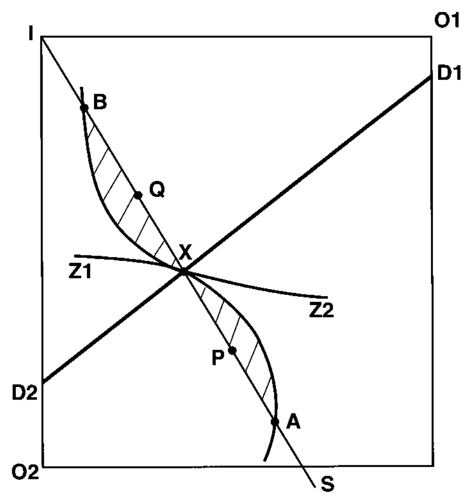

Figure 3. A schematic diagram illustrating the blocking of a non-competitive core allocation by a process of recontracting: *IXS* is the price ray and AXZ_1 and BXZ_2 are the two indifference curves through *X*. Since the two indifference curves are not tangent to the price ray, a profitable recontract is possible.

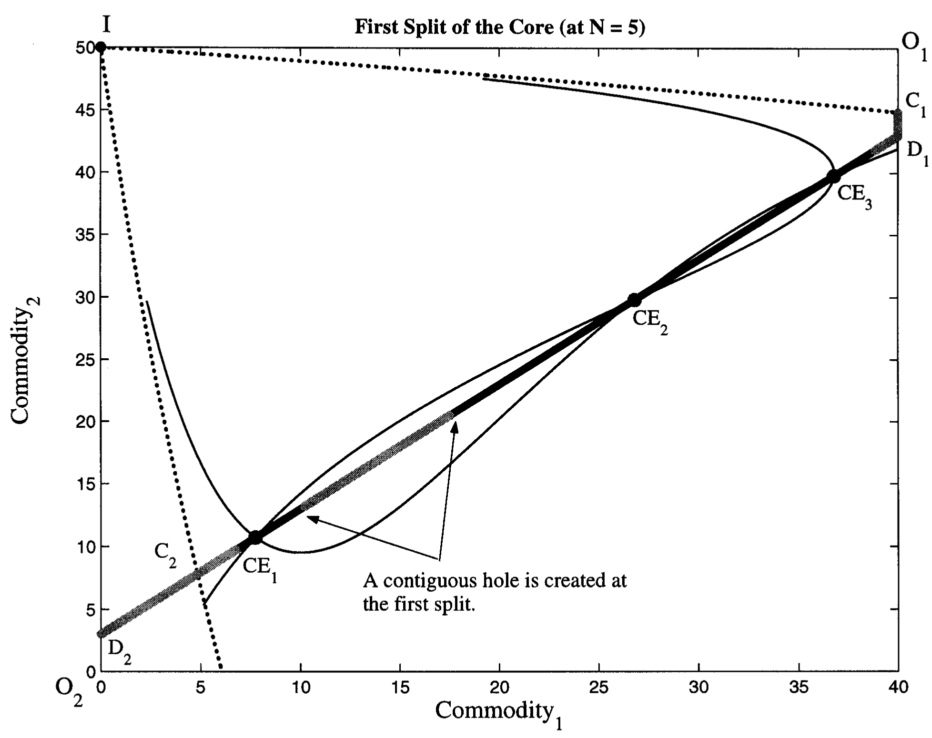

Figure 4. First split of the core. At N = 5, coalition (5,4) is able to dominate a continuous sequence of points in the core that lie between CE_1 and CE_2. The core splits into two pieces and a "gap" is created between CE_1 and CE_2. The points in the darker segments of $D_2C_2D_1C_1$ are inside the core while the points in the lighter segments can be blocked and hence they are no longer a part of the core. The co-ordinates of the end points of the contiguous hole are (10.20, 13.20) and (17.52, 20.52). Discretization parameters for the computation: δ_{cc} = 0.001 (1049 points on the core), δ_{pr} = [0.0001,0.0001f], where f = min(m/n, n/m).

$\mathcal{L}(N, I)$ = a discrete representation of $\mathcal{C}(N, I)$.

$\mathcal{O}(N)$ = a set of all possible coalitions for a given N.

δ_{cc} = step size for discretizing the contract curve.

δ_{pr} = step size for discretizing the price ray.

Step 1: Choose an appropriate step size (for example, $\delta_{\mathrm{cc}} = 0.001$) and discretize the core which is a part of the Pareto set represented by $y = x + 2.995$. $\mathcal{L}(1, I)$ is a discrete representation of the core for $N = 1$.

Step 2: Choose N, the number of traders of each type in the economy. Start with N = 2.

Step 3: Generate all possible (m, n) coalitions for a given value of N. For example, for $N = 3$, the set of coaltions are $(1, 1), (1, 2)$ and $(2, 1)$.

Step 4: Sequentially consider each point (say X) in the set $\mathcal{L}(N-1, I)$ and check if any of the coalition from $\mathcal{O}(N)$ can block it. If the point X can be blocked, remove it from the set $\mathcal{L}(N-1, I)$.

Step 5: If $N \leq N_{\max}$, set $N = N + 1$ and go to *Step 2*. Otherwise, proceed to *Step 6*.

Step 6: The points in set $\mathcal{L}(N_{\max}, I)$ constitute the core of the exchange economy for $N = N_{\max}$. Note that $\mathcal{C}(N_{\max}, I) \approx \mathcal{L}(N_{\max}, I)$.

The details of the procedure and the parameters used for discretizing the core and the price rays are provided in Appendix A.1.

4 Computational Results

We compute the core of the economy for different values of N, starting at $N = 1$. At the beginning, as N is increased, the core is fully connected and shrinks from both the ends, though not uniformly. At $N = 5$, the core splits for the first time into two pieces between the first and the second competitive equilibria (CE_1 and CE_2). In fact, a contiguous section from the core (see Figure 4) disappears at $N = 5$. The blocking coalition that achieves this is $(5, 4)$. The co-ordinates of the end points of the contiguous gap[5] are $(10.20, 13.20)$ and $(17.52, 20.52)$.

In Figure 5 we provide some details of the gap created at the first split. The figure shows the allocations of the traders and the associated utility payoffs after recontracting from allocations inside the gap. When several profitable recontracts are possible, recontracting can be carried out in many different ways. In the figure we show the results from two possible recontracting methods: (a) allocations are chosen such that type I traders maximize their payoffs when multiple profitable recontracts are possible, (b) type II traders maximize their payoffs. Figure 5(i) shows that when type I traders maximize their payoffs during a recontract, the corresponding payoffs to type II

[5] We have discretized the core and we check for domination only at discrete locations on the core. So it is possible for the regions in between the chosen points to have allocations that are not blocked by the $(5, 4)$ coalition and the core may split into multiple pieces. However, even with a very small step size of 10^-5 which generates $104,739$ points on the core, we find that all consecutive points inside the first gap are blocked by the $(5, 4)$ coalition. The issue of whether a core *always* splits as a gap and into two pieces is an open question.

traders are also positive. However, when recontracting is performed such that type II traders maximize their payoffs (see Figure 5(ii)), the corresponding extra payoffs to type I traders are negligible (almost zero). Table 2 provides the numerical values for a few recontracts during the first split of the core.

With a further increase in the number of traders, a second split in the core occurs at $N = 12$ with a blocking coalition $(11, 12)$. Similar to the first split, this split also introduces a gap between the second and the third competitive equilibrium points though the size of this gap is smaller than the gap in the first split (see Figure 6). The co-ordinates of the end points of the contiguous gap are $(32.12, 35.12)$ and $(33.84, 36.84)$.

Table 2: Some details of the first split of the core. The first gap contains 184 points and the co-ordinates of the end points of the gap are $(10.20, 13.20)$ and $(17.52, 20.52)$ respectively. Numerical values for only 18 of these 184 points are shown in the table. For the data shown in the table, when multiple profitable recontracts are possible, recontracting is performed such that type I traders maximize their payoffs.

Blocking Coalition	**Core Allocations**	**Allocations after Recontracting**	**Core Utilities**	**Payoffs from Recontracting**
First 6 points from the gap				
(5,4)	(29.80,36.80),(10.20,13.20)	(30.30,34.99),(12.12,6.26)	(127.28,83.53)	(0.01,0.00)
(5,4)	(29.76,36.76),(10.24,13.24)	(30.25,35.00),(12.18,6.26)	(127.23,83.73)	(0.01,0.00)
(5,4)	(29.72,36.72),(10.28,13.28)	(30.20,35.00),(12.25,6.26)	(127.18,83.93)	(0.01,0.00)
(5,4)	(29.68,36.68),(10.32,13.32)	(30.16,34.99),(12.31,6.26)	(127.13,84.12)	(0.01,0.00)
(5,4)	(29.64,36.64),(10.36,13.36)	(30.11,34.99),(12.36,6.27)	(127.08,84.32)	(0.01,0.00)
(5,4)	(29.60,36.60),(10.40,13.40)	(30.06,34.98),(12.42,6.27)	(127.03,84.52)	(0.01,0.00)
6 points from the middle of the gap				
(5,4)	(27.00,34.00),(13.00,16.00)	(27.52,32.64),(15.60,9.20)	(123.66,96.02)	(0.03,0.06)
(5,4)	(26.96,33.96),(13.04,16.04)	(27.48,32.60),(15.64,9.25)	(123.61,96.18)	(0.03,0.06)
(5,4)	(26.92,33.92),(13.08,16.08)	(27.45,32.56),(15.69,9.30)	(123.56,96.34)	(0.03,0.06)
(5,4)	(26.88,33.88),(13.12,16.12)	(27.41,32.51),(15.74,9.36)	(123.50,96.49)	(0.03,0.06)
(5,4)	(26.84,33.84),(13.16,16.16)	(27.37,32.47),(15.78,9.41)	(123.45,96.65)	(0.04,0.06)
(5,4)	(26.80,33.80),(13.20,16.20)	(27.34,32.43),(15.83,9.46)	(123.40,96.81)	(0.04,0.06)
Last 6 points from the gap				
(5,4)	(22.68,29.68),(17.32,20.32)	(24.07,27.31),(19.92,15.86)	(117.54,110.85)	(0.01,0.00)
(5,4)	(22.64,29.64),(17.36,20.36)	(24.04,27.26),(19.95,15.93)	(117.48,110.97)	(0.01,0.00)
(5,4)	(22.60,29.60),(17.40,20.40)	(24.01,27.20),(19.98,16.00)	(117.42,111.09)	(0.01,0.00)
(5,4)	(22.56,29.56),(17.44,20.44)	(23.99,27.15),(20.02,16.07)	(117.36,111.21)	(0.01,0.00)
(5,4)	(22.52,29.52),(17.48,20.48)	(23.96,27.09),(20.05,16.14)	(117.30,111.32)	(0.01,0.00)
(5,4)	(22.48,29.48),(17.52,20.52)	(23.94,27.04),(20.08,16.21)	(117.24,111.44)	(0.01,0.00)

Discretization parameters for the computation: $\delta_{\text{cc}} = 0.001$ (1049 points on the core), $\delta_{\text{pr}} = [0.0001, 0.0001f]$, where $f = \min(m/n, n/m)$.

We continue to increase N and observe that the two gaps in the core continue to widen, though at different rates. The speed of convergence of the core towards CE_1 is the fastest and it is slowest towards CE_2. In Figure 7, the structure of the disappearing core can be "visualized" quite clearly. Each horizontal line in the figure represents the core for a given N. The top line shows the core at $N = 1$ and the bottom line represents the core at $N = 75$. Moving from the top

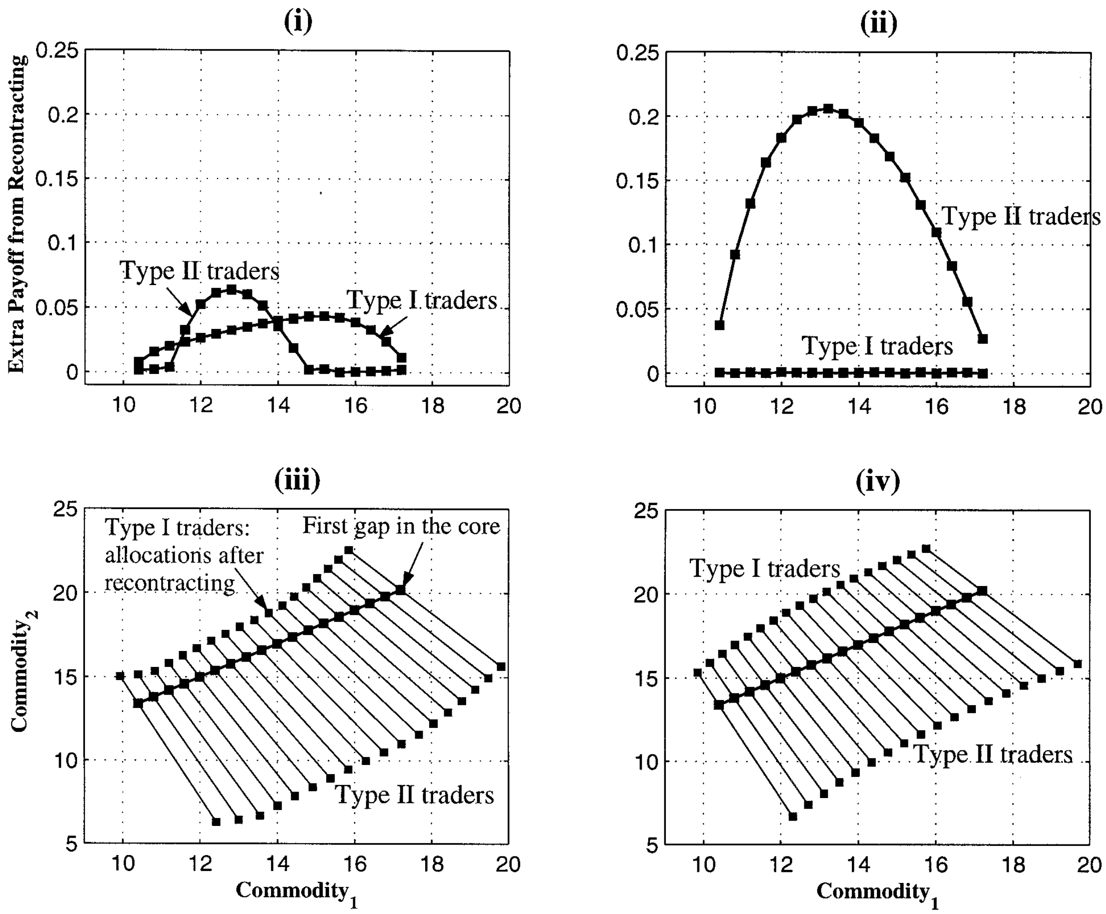

Figure 5. Second split of the core: At N = 12, with coalitions (11,12) and (12,11), the core splits again between the second and the third competitive equilibrium points. The second split in the core also occurs as a gap though the size of this gap is smaller than the gap in the first split. The co-ordinates of the end points of the contiguous hole are (32.60,35.60) and (33.40, 36.40). Discretization parameters for the computation: δ_{cc} = 0.001 (1049 points on the core), δ_{pr} = [0.0001,0.0001f], where f = min(m/n, n/m).

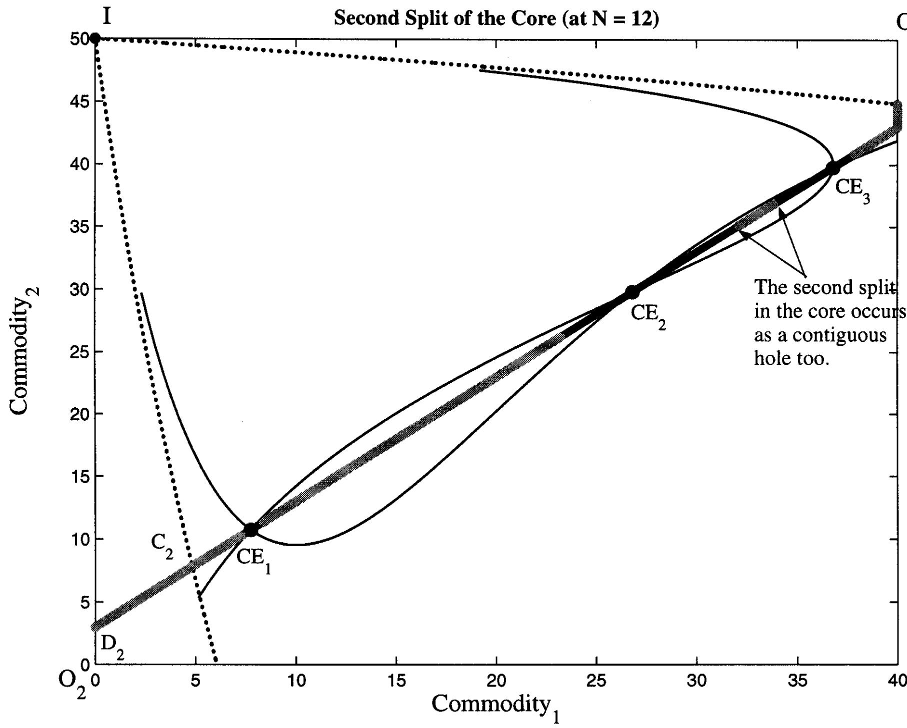

Figure 6. Some details of the first split of the core: (i-ii) Payoffs to the two trader types from recontracting. When multiple profitable recontracts are possible, recontracting can be carried out in many different ways. In case (i), when multiple recontracts are possible, recontracting is performed such that type I traders maximize their payoffs while in (ii) type II traders maximize their payoffs. **(iii-iv)** The allocations to the trader types after recontracting from points inside the "gap" created at the first split of the core. Discretization parameters for the computation: δ_{cc} = 0.01 (106 points on the core), δ_{pr} = [0.001,0.001*f*], where *f* = min(*m*/*n*, *n*/*m*). A coarse discretization is intentionally chosen for this illustration so that the payoffs from recontracting and the corresponding allocations can be seen clearly.

to the bottom, we can observe the emergence of the core gaps and their subsequent spread as N increases. Figure 8 shows the rates of convergence of the core toward the three equlibrium points in the utility space. The distance between the utilities of the trader types at the end points of the core piece around the CE is used to measure the speed of convergence.[6] By $N = 125$, the core has converged completely towards CE_1 for the chosen level of discretization. Only two points remain around this CE.[7] At $N = 150$, convergence towards CE_3 is complete as well. The convergence towards CE_2 is very slow and the core does not converge completely until $N = 600$. These results indicate that initially, for smaller values of N, the rate of convergence is slower than N but for larger N, the rate of convergence is close to $1/N$.

4.1 Sensitivity of Competitive Equilibria to Changes in Initial Allocations

Our example illustrates three competetive equilibria and multiple intersections of the bid and offer curves. The three CEs and the curves are based on the initial allocations of $(50, 0)$ to type I traders and $(0, 40)$ to type II traders. Conserving the total amount of both goods, we illustrate how the equilibria change with the change in the distribution of the initial endowments. It is well known that all points on the Pareto surface are CEs for some distribution of initial endowments. Furthermore, there is a neighborhood around every CE for which that CE is unique. The intuition behind this observation can be seen immediately from the observation that if the initial resources of the traders are selected at a point on the (individually rational segment) Pareto surface that point is a CE with no trade but a shadow price is given which supports the CE.

In order to analyze the sensitivity of the CEs, we note that for each trader type, a response curve, which is a locus of the optimal allocations of the trader for various prices (or exchange ratios), can be defined. For a given price level, the intersection of the best response curve with the price ray determines the optimal allocation for the trader. In the presence of multiple traders (or trader types in type economies), the response curves of the trader types may intersect at a common point on the price ray which gives us the competitive equilibrium allocation. In our example, with the initial allocations of $(40, 0)$ and $(0, 50)$ to the two trader types, the response curves intersect at three locations which correspond to the three competitive equilibrium solutions. However, when the initial allocations are perturbed, the best response functions change shape and they no longer intersect at three points. Figure 9 shows the sensitivity of the three competitive equilibrium solutions to changes in the initial allocations. We first consider our sensitivity analysis along the price ray from I to CE_2. All points on this ray have at least CE_2 as a competitive equilibrium. Generically the number of CEs in an exchange economy is odd as has been established by Harsanyi (1973). Thus as we move from $[(50, 0), (40, 0)]$ towards CE_2 we expect that at some endowment level, the competitive equilibrium becomes unique and is at CE_2. Similarly, if we go from the initial endowment in the direction of CE_1 or CE_3, we expect the multiplicity of equilibria to disappear.

Investigating the three rays we find uniqueness starts at perturbations shown in Table 3. In Figures 9(i-iv), the initial allocations are chosen at various points along the line (price ray) joining I and the second competitive equilibrium (CE_2). At $[(44.10, 4.44), (5.90, 45.56)]$, CE_2 becomes unique. When the initial allocations are chosen on the line joining I and CE_1, CE_1

[6] If (u_{11}, u_{12}) and (u_{21}, u_{22}) are the utilities at the end points of a core piece, the convergence metric, $d = \sqrt{(u_{11} - u_{21})^2 + (u_{12} - u_{22})^2}$. Convergence of the core towards a CE is complete if $d = 0$.

[7] It is certainly possible for the points around the CEs to disappear but we did not observe this phenomenon even with a very fine discretization of the contract curve ($\delta_{cc} = 10^-5$) and very large coalitions ($(1000, 999)$ and $(999, 1000)$).

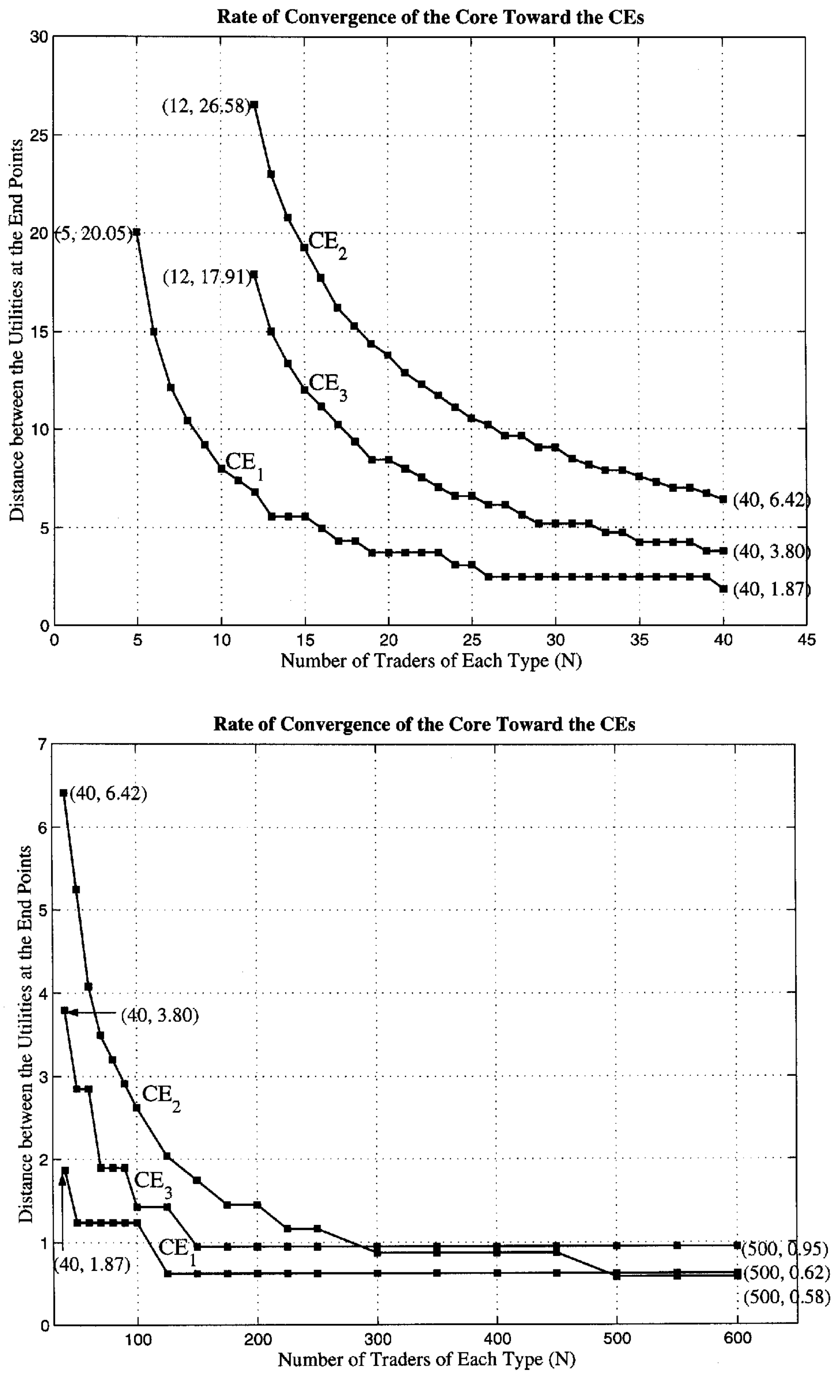

Figure 8. Speed of convergence of the core (in utility space) toward the CEs: The core converges toward the three competitive equilibrium points at different speeds. Initially, for smaller values of N, the rate of convergence is not of the order $1/N$ but for larger N, the rate of convergence is close to $1/N$.

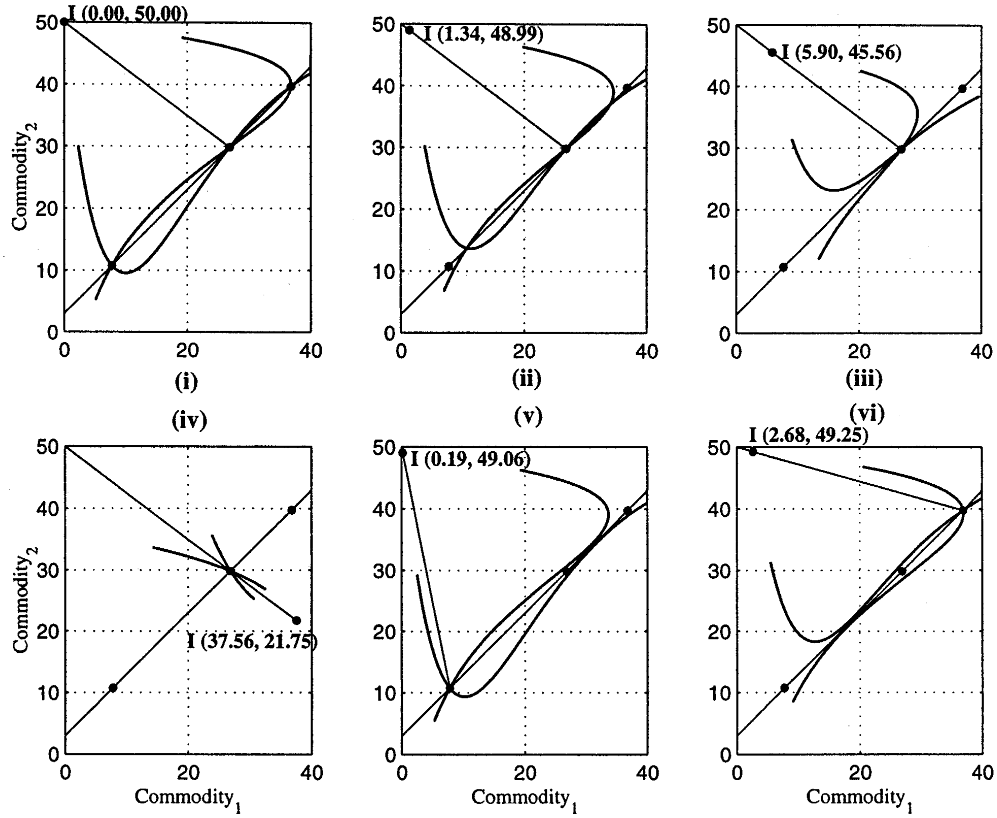

Figure 9. Influence of changes in the initial individual allocations on the number of competitive equilibria: (i) Original configuration: the three competitive equilibria and the response curves when the initial allocation is [(40,0), (0,50)]. **(ii)** The initial individual endowments are perturbed slightly and moved along the line (price ray) joining *I* and the second competitive equilibrium (CE_2): multiple equilibria are still present though both CE_1 and CE_3 have moved inward. **(iii)** Further perturbation along *I*-CE_2 to (5.90, 45.56): CE_2 becomes unique. **(iv)** Uniqueness of CE_2 is maintained everywhere beyond point (5.90,45.56) on the price ray, even after crossing the contract curve. **(v)** Initial allocations are chosen on the line joining *I* and CE_1: CE_1 becomes unique at (0.19, 49.06) with a very small perturbation to the initial allocations. **(vi)** Initial allocations are chosen on the line joining *I* and CE_3: CE_3 becomes unique at (2.68, 49.25).

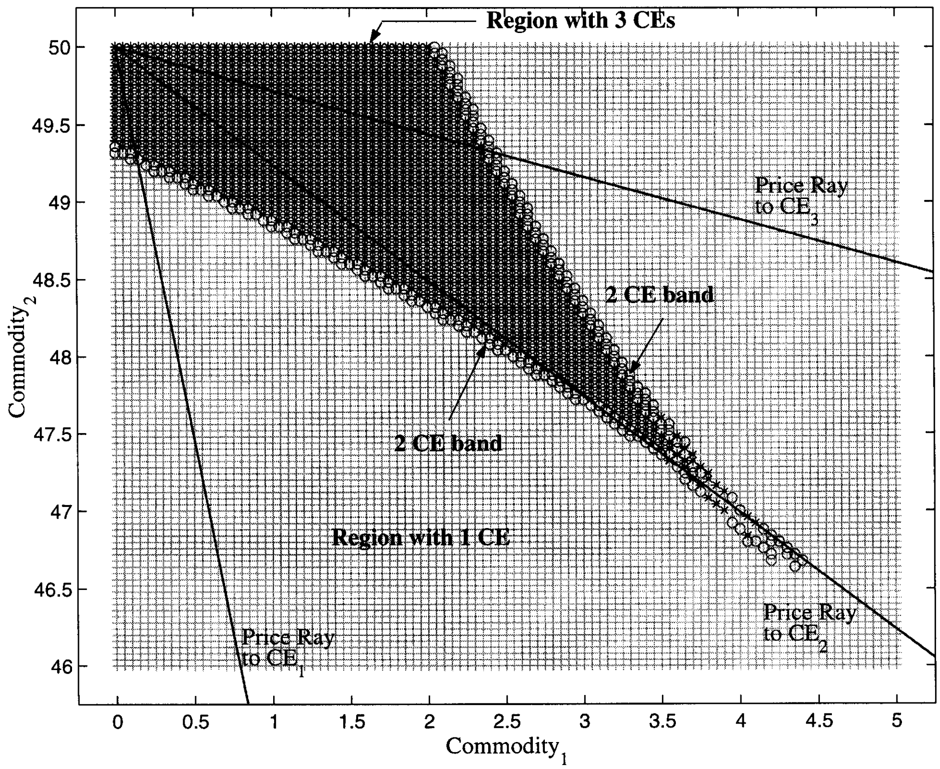

Figure 10. Identifying the region in the Edgeworth box where multiple competitive equilibria exist: The initial individual allocations are chosen in a small rectangular region at the top-left corner of the Edgeworth box. We discretize the rectangular region using step sizes of (0.05, 0.04) which gives a total of 10201 points (101×101) inside the rectangle. For each grid point, the number of competitive equilibria is computed and a color is assigned to the point depending upon the number of the CEs. The following color scheme is used: blue stars (the darker shade) for 3 CEs, red circles for 2 CEs and green plus marks (the lighter shade) for 1 CE. The 2-CE region is a narrow band around the 3-CE region. Some of the points in the 2-CE band are due to the computational error introduced by the combined effects of the discretization procedure and the interpolation technique used to identify the points of intersection of the response curves. See Appendix A.1 for details of the computational procedure.

becomes unique at $[(49.81, 0.84), (0.19, 49.06)]$ with a very small perturbation to the initial allocations. Finally, moving along the price ray I-CE_3, we find that CE_3 becomes unique at $[(47.32, 0.75), (2.68, 49.25)]$. Once a CE becomes unique for a certain initial allocation chosen along the price ray, it remains unique for all allocations on the price ray, even after crossing the contract curve (for an example, see Figure 9(iv)).

Extending this sensitivity analysis, we investigate the changes in the number of equilibria as the initial individual endowments are perturbed to a small rectangular region (say R) around the original initial endowment point I. The number of points of intersection of the two response curves determine the number of competitive equilibria. We find that there is only a very small region (see Figure 10) around I where 3 CEs exist. We also find 2 CEs in a narrow band around the 3-CE region. However, some of the points in the 2-CE band are due to the computational error introduced by the combined effects of the discretization procedure and the interpolation technique used to identify the points of intersection of the response curves. At all locations outside the rectangular region R the competitive equilbrium is unique.

Table 3: Initial individual endowments for uniqueness of the competitive equilibrium

Equilibrium Point	Holdings of Type I Traders	Holdings of Type II Traders
CE_1	$(49.81, 0.84)$	$(0.19, 49.16)$
CE_2	$(44.10, 4.44)$	$(5.90, 45.56)$
CE_3	$(47.32, 0.75)$	$(2.68, 49.25)$

4.2 Sensitivity of the Core to Changes in Initial Allocations

Similar to the sensitivity analysis of the competitive equilibria, we investigate the sensitivity of the core as the initial individual endowments are perturbed along the lines joining I and the three competitive equilibrium points, CE_1, CE_2 and CE_3. We observe that even when the initial conditions lie in the part of the endowment space where the CE is unique, the core does not always shrink uniformly from the two end-points as N is increased. The core may first split into two pieces and as N is increased further, the core segment containing the CE converges towards the CE while the other part disappears. Figures 11 and 13 show the core splitting phenomenon as the initial endowments are perturbed along the price rays I-CE_1 and I-CE_3 respectively. In Figure 11, moving the initial allocations to $[(49.61, 1.96), (0.39, 48.04)]$ on I-CE_1 we find that the core splits into two pieces at $N = 4$. The split occurs as a contiguous gap where the end-points of the gap are $(10.64, 13.64)$ and $(19.32, 22.32)$. The blocking coalition in this case is $(4, 3)$. With a further increase in N, the two segments of the core converge further and at $N = 8$, the core segment containing CE_1 converges towards the CE while the other part disappears completely. A similar core convergence diagram is shown in Figure 13 where the core converges towards CE_3. When the initial condition lie in the part of the endowment space where CE_2 is unique, the core shrinks uniformly from the two end-points as N is increased and there is no split in the core. This is shown in Figure 12. The sensitivity analysis of the core also indicates that as we move inward along the price rays toward the contract curve, the core splits at smaller values of N and in addition, the rate of convergence of the core towards a CE is faster.

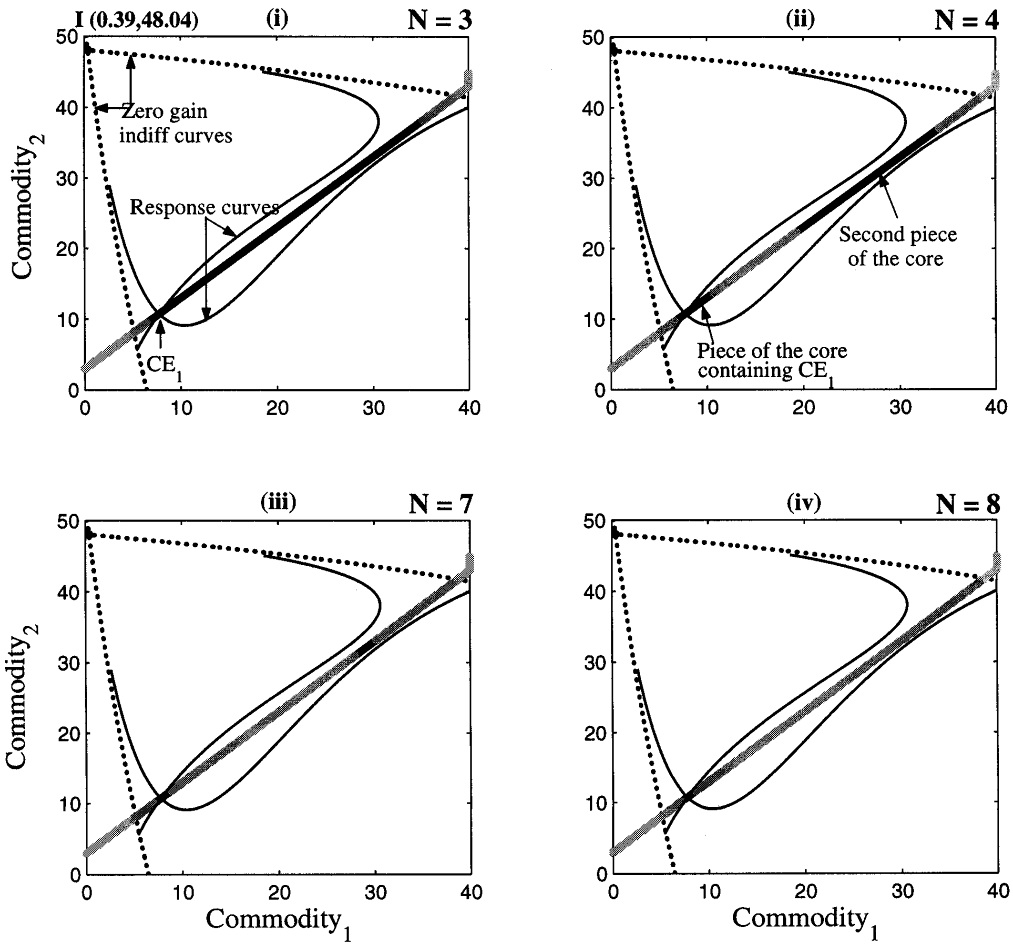

Figure 11. Convergence of the core when the initial individual endowments are such that CE_1 is the unique competitive equilibrium: The initial individual endowment is at [(49.61, 1.96), (0.39, 48.04)]. **(i)** At N = 3, the core is a continuous unit. **(ii)** At N = 4, the core splits for the first time into two pieces. The blocking coalition (4, 3) is able to dominate a continuous sequence of points in the core and creates a gap. The co-ordinates of the end points of the contiguous hole are (10.64, 13.64) and (19.32, 22.32). **(iii)** By N = 7, both pieces of the core has converged and only a very small part of both of them remain. **(iv)** At N = 8, the part of the core containing CE_1 converges towards it while the segment not containing CE_1 disappears completely. Discretization parameters for the computation: δ_{cc} = 0.001 (1049 points on the core), δ_{pr} = [0.0001,0.0001f], where f = min(m/n, n/m).

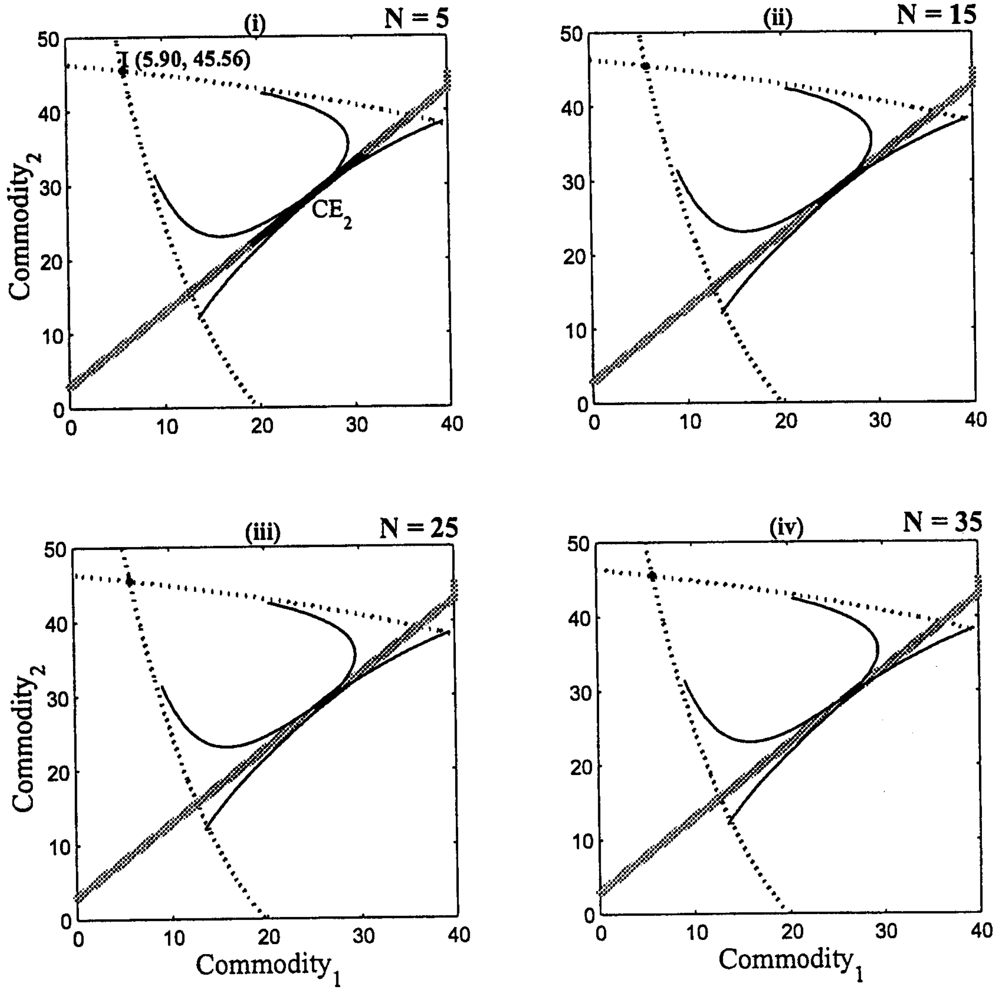

Figure 12. Convergence of the core when the initial individual endowments are such that CE_2 is the unique competitive equilibrium: The initial individual endowment is at [(44.10, 4.44), 5.90, 5.56)]. As *N* is increased, the core converges uniformly towards CE_2 from the two end-points.

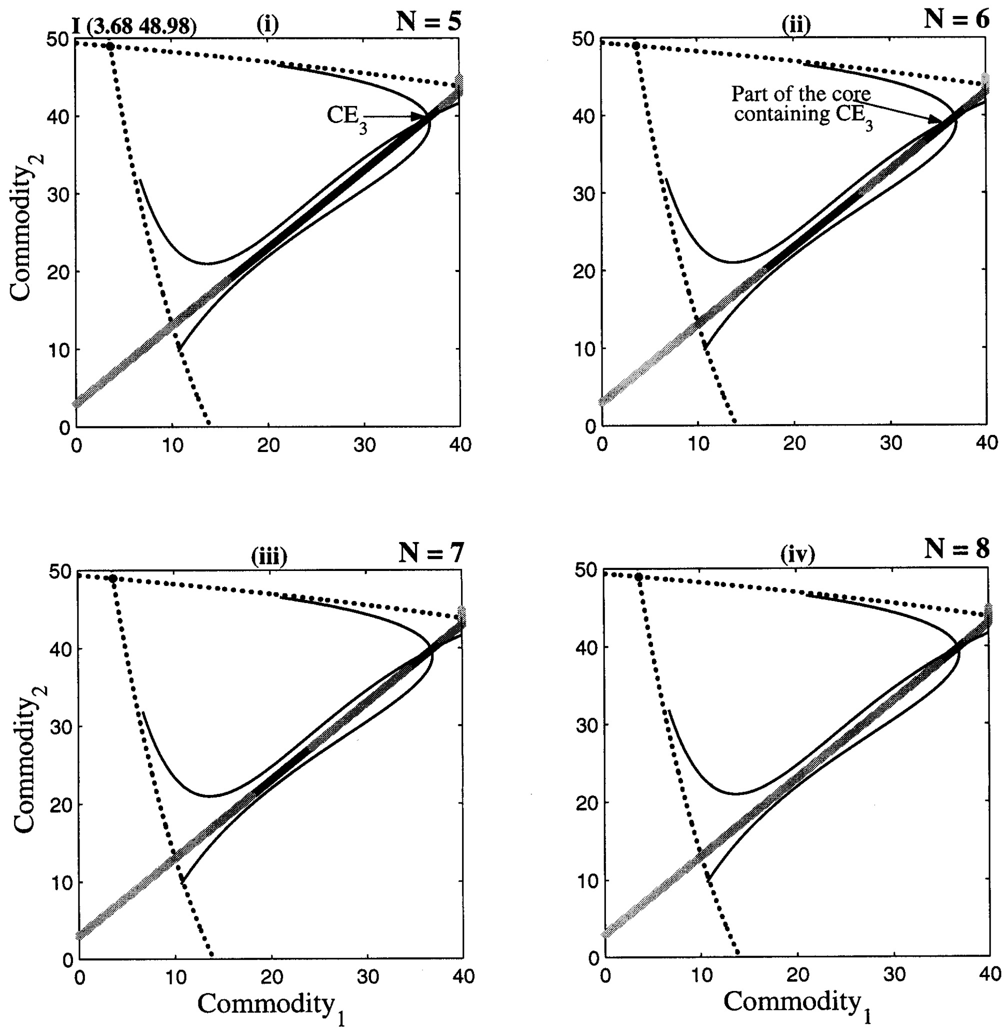

Figure 13. Convergence of the core when the initial individual endowments are such that CE_3 is the unique competitive equilibrium: The initial individual endowment is at [(46.32, 1.02), (3.68, 48.98)]. As N is increased, the core splits into two segments, one of which converges to CE_3 while the other disappears completely.

5 Dynamics: Virtual and Real

The broader focus of our research is on understanding the *dynamics* of adjustment in an exchange economy with a large number of traders. For example, we want to investigate if there is any significant relationship between the binary search models of trade and the core? We also want to consider other strategic and non-strategic trading behaviors. Each type of trading behavior specification defines a complete dynamic process model and we want to investigate if these models have any natural "attractors" (point attractors such as the competitive equilibrium points or regions on or around the Pareto surface)? If they do, can we identify the domains of attraction of the attractor points (or sets)? In general, the question "is there a dynamic process which starts at the initial point I and converges to one of CE_1, CE_2 or CE_3" is not as well defined as it might appear to be. In particular, are we concerned with a dynamics in which after every move the initial endowments are renormalized but price may have changed or are we concerned with a full dynamics where there may be increments of trading at each step until a final equilibrium is reached. The first process has a virtual trade in goods until a final price is reached. The tâtonnement procedure formerly used in the French stock-market was of this form. The second process, however, takes into account the fact that the competitive equilibria might themselves change as the trading process evolves. In a related paper we enlarge on these comments on dynamics and illustrate why path-dependent dynamics may not lead to the static competitive equilibria.

A Appendix: Details of the Computational Procedure

A.1 Discretization Procedure

The core and the price rays must be discretized in order to compute the core of the economy for a given value of $N(> 1)$. The following procedure is used to generate points along any given line:

1. Choose a step size for discretizing the line: $\delta = 0.001$.
2. Generate a set of points (call it α) between 0 and 1 using the step size of δ:

$$\alpha = [0 : \delta : 1]$$

   The total number of points generated inside the $(0, 1)$ interval is $N_l = 1 + 1/\delta$.

3. The points on the line are obtained using the known co-ordinates of the end points of the line, $[x(1), y(1)]$ and $[x(N_l), y(N_l)]$:

$$\begin{aligned} x(k) &= x(1) + \alpha(k)[x(N_l) - x(1)] \\ y(k) &= y(1) + \alpha(k)[y(N_l) - y(1)] \end{aligned}$$

   where $k = 2, 3, \ldots, (N_l - 1)$.

For discretizing the core, we use a step size of $\delta_{\mathrm{cc}} = 0.001$ which gives us a total of 1049 points. Note that 49 points are generated along the small vertical section (D_1C_1) of the contract curve. The price ray is discretized more finely using step sizes of $\delta_{\mathrm{pr}} = [0.0001, 0.0001f]$, where $f = \min(m/n, n/m)$. When $m \neq n$, the feasible allocations generated along the price ray do not coincide for the two trader types. A step size of 0.0001 is chosen for the faster moving points and the slower moving points are generated using a step size of $0.0001f$. The total number of allocations generated along a price ray depends upon its length which varies as the slope of the line changes.

A.2 Generating Feasible Allocations on the Price Rays

Consider an (m, n) coalition which consists of m type I traders and n type II traders. If the initial individual endowments of the two trader types are $(a, 0)$ and $(0, b)$ respectively, the total resources in this economy is (ma, nb). Figure 14 shows the feasible extreme allocations to both trader types. On one hand, at point T_1, type I traders divide the total resources equally among themselves and obtain $(a, bn/m)$ while type II traders get nothing. On the other hand, at point T_2, type II traders divide the total resources equally among themselves and obtain $(am/n, b)$ while type I traders get nothing. Intermediate feasible allocations are obtained by moving along lines O_2T_2 (type II allocations increase) and T_1O_1 (type I allocations decrease) in such a way that the ratio $T_1P/O_2Q = n/m$. The price ray $IPQS$ represents the exchange ratio that facilitates this exchange. Parallel shifts of lines O_1T_1 and O_2T_2 generate different feasible allocations on the given price ray.

A.3 Identifying the Multiple Equilibria Region in the Edgeworth Box

In this section we describe the computational procedure used for identifying the multiple equilibria region in the Edgeworth box and discuss some of the difficulties involved in the computations.

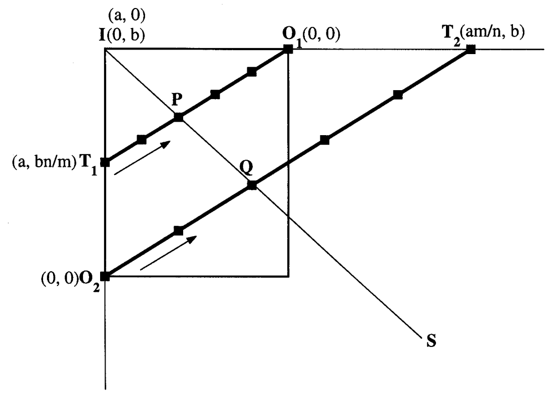

Figure 14. An illustration of how the feasible allocations are generated along the price rays for a given (*m, n*) coalition: In this figure, $m = 2$, $n = 1$, $a = 40$, $b = 50$. The allocations of type I traders are shown relative to the origin at O_1 while for type II traders the origin is at O_1.

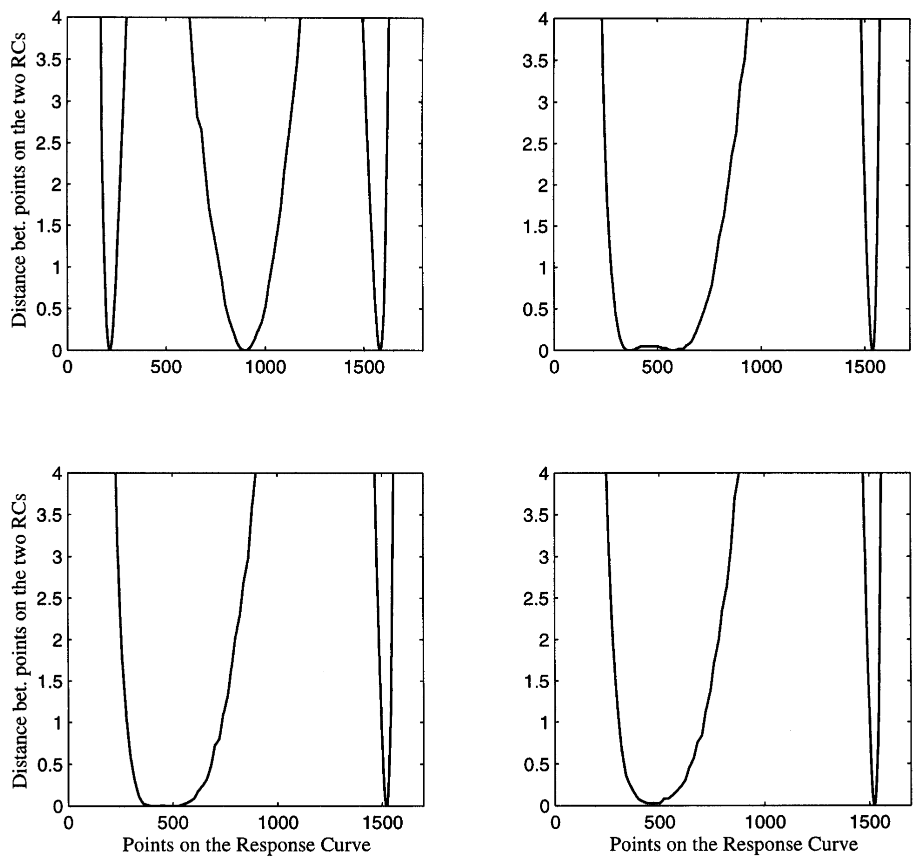

Figure 15. Typical distance functions: (i) Clearly there are three competitive equilibria and the computational procedure is able to identify them correctly. **(ii)** The intersection points have moved closer but three CEs can still be identified without any difficulty. **(iii)** In this situation it is not clear whether there are two CEs or three. The computational procedure identifies 2 CEs in cases such as these. **(iv)** There is only one CE and we are able to detect is correctly.

First of all, the core is discretized (using a step size of $\delta_{\mathrm{cc}} = 0.01$) and a discrete representation of the core (with 106 points) is obtained. Secondly, an initial individual allocation is chosen and price rays passing through this point and the points on the discretized core are drawn. Next, all price rays are discretized (using a step size $\delta_{\mathrm{pr}} = 0.001$) and the utilities of both trader types are computed at discrete locations on the price rays. Finally, for each trader type, the allocation with maximum utility is identified on each of the price rays. The collection of these optimal allocations for each trader type provides an approximate representation of their response curve.

After obtaining a discrete representation of the two response curves for a chosen initial condition, we identify the points of intersection of these two curves and determine the number of competitive equilibria (Number of CEs = number of intersections). A parameter ε_d is chosen to identify the points of intersection. Let $[(x_{1i}, y_{1i}), (x_{2i}, y_{2i})], i = 1, \ldots, n_{\mathrm{rc}}$ represent the points on the two response curves. Distance (d_i) between corresponding pairs of points on the two response curves are obtained and if $d_i < \varepsilon_d$, $[x_i(= x_{1i} = x_{1i}), y_i(= y_{1i} = y_{1i})]$ is an intersection point and represents a CE. The parameter ε_d must be chosen very carefully. If we choose a very small value of ε_d, we may miss the actual intersection points (due to the discretization of the core) and if we make ε_d large, we may identify spurious intersection points. Using a trial-and-error procedure, $\varepsilon_d = 0.005$ is chosen and used in the computations.

The core is discretized coarsely to keep the total time of computation within reasonable limits. With a step size of $\delta_{\mathrm{cc}} = 0.01$, evaluation of one initial condition takes 0.05 minute (3 seconds) on an 850 MHz, dual-processor, Pentium III computer. With 10201 initial conditions (101×101 points), the total computation time is approximately 510 minutes (≈ 8.5 hours). When the step size used for discretizing the core is reduced by half, the total computation time doubles and so a very fine discretization of the core is not feasible. A coarse core results in a coarser response curve and this leads to inaccuracies in the computation of the intersection points. We use a linear interpolation scheme to generate additional points between all consecutive pairs of points on the two response curves and thereby improve the accuracy of the computational procedure. However, it is quite difficult to avoid the error introduced by the discretization process altogether and some spurious CEs will nevertheless be identified. Figure 15 illustrates this phenomenon.