# Multifractal Products of Cylindrical Pulses

**By**

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**January 2001**

**COWLES FOUNDATION DISCUSSION PAPER NO. 1287**

**COWLES FOUNDATION FOR RESEARCH IN ECONOMICS**
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# Multifractal products of cylindrical pulses

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August 4, 2000

**Abstract**

A new class of random multiplicative and statistically self-similar measures is defined on $\mathbb{R}$. It is the limit of measure-valued martingales constructed by multiplying random functions attached to the points of a statistically self-similar Poisson point process in a strip of the plane. Several fundamental problems are solved, including the non-degeneracy and the distribution of the limit measure, $\mu$; the finitness of the (positive and negative) moments of the total mass of $\mu$ restricted to bounded intervals.

Compared to the familiar canonical multifractals generated by multiplicative cascades, the new measures and their multifractal analysis exhibit strikingly novel features which are discussed in detail.

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**Key Words.** Random measures, Multifractal analysis, Continuous time martingales, Statistically self-similar Poisson point processes.

**AMS Classification.** 28A80, 60G18, 60G44, 60G55, 60G57.

## 1 Introduction, and Discussion with Examples.

### 1.1 Introduction.

This paper deals with a new class of random multifractal measures to be called "multifractal products of cylindrical pulses", MPCP. A broader class, "multifractals products of pulses", MPP, was introduced in [Ma9] together with a number of heuristic arguments and conjectures, which this paper will prove for MPCP. This introduction will define the MPCP and contrast them on several accounts with the familiar "canonical cascade multifractals", CCM, introduced in [Ma4,Ma5]. Measures obtained by either process will be denoted by $\mu$.

CCM involve a prescribed $b$-adic grid of intervals of $[0, 1]$. This basis $b$ is artificial and was motivated by an extraneous reason: to allow the conjectures in [Ma4,Ma5] to be proven [K1,P,KP] and much extended since ([K4], [Ho-Wa], [Fa], [O1], [Mol], [Ar-Pa], [B1,B2]).

The key virtue of the MPP is that they involve no $b$-adic grid. Neither do the limit lognormal multifractals introduced in [Ma2] and mentioned later in the discussion. Neither do the "fractal sums of pulses" introduced in [Ma7], which inspired the present study. The absence of grid creates serious mathematical complications, as will be seen. However, it brings in a great increase of realism and versatility which is very valuable for the applications. Those improvements are due to several novelties, essential to a varying degree, that this section will discuss. It will be noted that an irreducible part of the common role of the basis will be played by a constant $\delta$, called "density", which is a more general formal replacement for $1/\log b$.

Denote by $S = \{(t_h, \lambda_h)\}$ a Poisson point process in the strip $\{0 < \lambda < 1\}$ of the plane, with the intensity

$$\Lambda(dtd\lambda) = \frac{\delta}{2}\frac{dtd\lambda}{\lambda^2}.$$

The "cylindrical pulses" investigated in this paper are a denumerable family of functions $P_h(t)$, each of which is identically 1 outside of an interval $[t_h - \lambda_h, t_h + \lambda_h]$ called "trema", and identically equal to a weight $W_h$ within $[t_h - \lambda_h, t_h + \lambda_h]$, where the $W_h$'s are copies of a positive integrable random variable $W$, independent of one another and independent of $S$. We shall write $V = E(W)$.

One defines the approximating measures $\mu_\varepsilon$, $0 < \varepsilon \leq 1$, with density with respect to the Lebesgue measure $\ell$ given by

$$\frac{d\mu_\varepsilon}{d\ell}(t) = \varepsilon^{\delta(V-1)} \prod_{(t_h,\lambda_h)\in S,\, \lambda_h \geq \varepsilon} P_h(t).$$

The product of the pulses, $\mu$, is defined as the measure obtained on the whole real line as the weak limit (on compact subsets) of the approximating measures $\mu_\varepsilon$.

The familiar CCM are also defined as products of cylindrical pulses, but on $[0,1]$ rather than $\mathbb{R}$ and with the deterministic rather than random set $S = \{(\frac{k+1/2}{b^n}, \frac{1}{2b^n})$; integer $n$ and $k$; $n \geq 1$, $0 \leq k < b^n\}$ $(b \geq 2)$. The countable family of approximating measures $(\mu_n)_{n\geq 1}$ is then given by

$$\frac{d\mu_n}{d\ell}(t) = V^{-n} \prod_{(t_h,\lambda_h)\in S,\, \lambda_h \geq \frac{1}{2b^n}} P_h(t).$$

The normalizing factors insure that one deals with a measure-valued martingale. They are respectively $\varepsilon^{\delta(V-1)}$ and, writing $b^{-n} = \varepsilon$, $\varepsilon^{\delta \log V}$.

Each pulse is represented by an address point in the "address space" $H$. The three parts of Figure 1 show three sets that will be important in the sequel: in the center, for a given pulse, the set of values of $t$ that it "rules"; to the left, for a given value of $t$, the set of pulses $t$ -is "ruled by"; and to the right, for two given values $t$ and $t'$, the set of pulses $t$ and $t'$ are ruled by, either singly or together.

In this paper, a first open literature publication of the contents of [Ma9] is preceded by a discussion on the properties and the relevance of MPCP by B. Mandelbrot. Next, J. Barral proves and much strengthenes the conjectures in [Ma9] on conditions for the non-degeneracy of $\mu$, on the finiteness of the moments of pieces of $\mu$, and on the multifractal analysis of $\mu$ via the function $\tau_{\mathrm{MPCP}}$ written below in the discussion. Several proofs begin with a sequence of non obvious reductions, that make it possible to use arguments or approach developped for CCM. The geometry of the construction is statistically self-similar but very complex compared with the tree structure of the CCM.

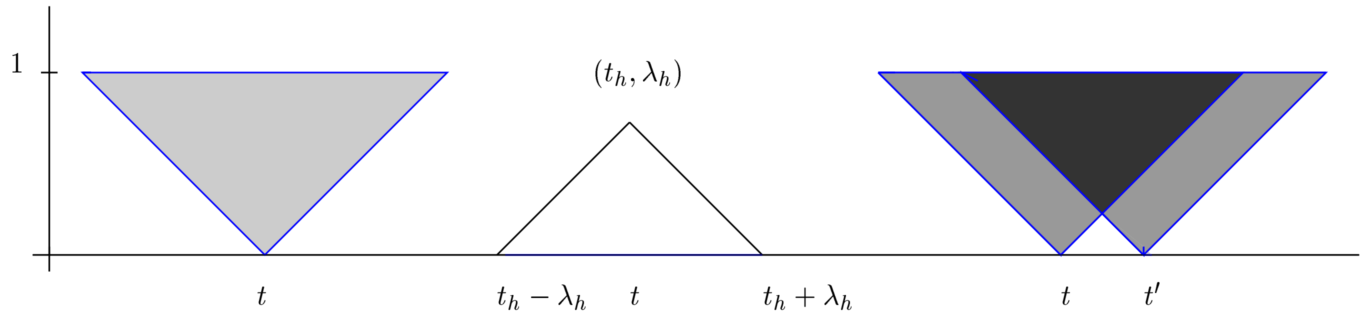


**Figure 1** .

Section 2 tightens up the construction of the measure $\mu$ and describes a self-similar property. Section 3 studies the non-degeneracy of the process, that is conditions under which $\mu$ is positive with positive probability, and the existence of finite moments for pieces of $\mu$. Section 4 treats the multifractal analysis of $\mu$ by performing with probability 1, the whole multifractal spectrum (as in [B2] for CCM), and not only each of its points with probability 1. Section 5 contains basic lemmas used in sections 1 to 4.

We end this section with the following discussion by B. Mandelbrot.

### 1.2 Discussion with Examples.

**Motivations.** The cascades behind CCM are not part of physical reality, but an artificial device made up to simplify definitions and proofs. Their self-similarity properties are restricted to reduction ratios of the form $b^{-n}$, with integers $b$ and $n$.

It is good to recall why [Ma4,Ma5] introduced the terms "microcanonical" (often replaced by "conservative") and "canonical". These terms are a reminder of two physical ensembles in the Gibbs statistical theory; canonical is less constrained statistically than microcanonical. The Gibbs theory then continues by introducing "grand canonical ensembles" which are made of a Poisson distributed number of canonical ensembles, therefore are infinitely divisible [Ma1].

The move from CCM to MPP loosens statistical constraints in the further spirit of grand canonical ensembles. Let us show how. The definition of the CCM approximating measures can be restated as follows. Let $W(t)$ be a function of positive $t$ that is constant in the intervals between successive integers and whose values in different intervals are statistically independent and with the distribution of $W$. Then

$$\frac{d\mu_n}{d\ell}(t) = V^{-n} \prod_{0<m\leq n} W_m(b^m t),$$

where the functions $W_m(t)$ are statistically independent and distributed as $W(t)$. Similarly, the corresponding approximating measures of the limit lognormal measures of [Ma2] is a product of statistically independent sinusoids. In his powerful advocacy of Fourier analysis, Norbert Wiener often pointed out to sinusoids as providing the proper base for the study of stationnary phenomena. But multifractals are not stationnary, either visually or in the usual mathematical sense (they are conditionally stationary sporadic functions, as defined in chapter 10 in [Ma10]). One response is to replace sinusoids by wavelets. The response of [Ma7] and the present paper is to use "pulses".

**Digression on a generalization.** The product $\prod W_n(b^n t)$ remains meaningful if the base ceases to be an integer. It is made more elegant and extended from $[0,1]$ to $\mathbb{R}$ if random phases $\varphi_n$ are introduced and the multipliers replaced by $W_n(b^n t + \varphi_n)$.

**The multifractal function $\tau(q)$.** The multifractal functions $\tau(q)$ and $f(\alpha)$ are familiar to the reader. A source of novelty is that $\tau(q)$ takes altogether different forms for CCM and MPCP. For the former, [Ma4,Ma5] obtained and [K1,P,KP] confirmed a now classical expression that is convenient by writing as

$$\tau_{\text{CCM}}(q) = -1 + q[1 + \log_b V] - \log_b E(W^q).$$

For the sake of symmetry with $\tau_{\text{MPCP}}$, it is best to use $\delta = 1/\log b$ and write

$$\tau_{\text{CCM}}(q) = -1 + q[1 + \delta \log V] - \delta \log E(W^q).$$

On the contrary, [Ma9] obtained, as this paper will confirm, the form

$$\tau_{\text{MPCP}}(q) = -1 + q[1 + \delta(V-1)] - \delta(E(W^q) - 1), \delta > 0. \tag{1}$$

**The role of $\tau'(1)$: condition of nondegeneracy and dimension of the non-degenerate "support".** Despite the change in the form of $\tau(q)$, the condition for non-degeneracy of $\mu$ remains $\tau'(1) > 0$. If so, $\tau'(1)$ is the Hausdorff-Besicovitch dimension of the "support" of the measure. For CCM, this is shown in [Ma4,P,KP]; for MPCP this is shown in sections 3 and 4 (however, a fine point concerning the converse remains open).

**Covariance and the role of $\tau(2)$.** Let $\varepsilon > 0$, denote by $\mu'_\varepsilon$ the density of the approximating measure $\mu_\varepsilon$ and consider two points $t'$ and $t''$ with $r = |t' - t''| > 2\varepsilon$. If $\mu$ is a non-degenerate MPCP, a measure of the dependence between $\mu$ at $t'$ and $t''$ is

$$E[\mu'_\varepsilon(t')\mu'_\varepsilon(t'')] = \varepsilon^{2\delta(V-1)} E\{[\Pi'_\varepsilon P_h(t')][\Pi_\varepsilon P_h(t'')]\},$$

where $\Pi'_\varepsilon$ and $\Pi''_\varepsilon$ are products of the pulses that rule $t'$ and $t''$.

Denote by $N_L$ and $N_R$ the numbers of pulses in $H \cap \{\lambda \geq \varepsilon\}$ that only affect $t'$ and $t''$. The pulses that rule only $t'$ or $t''$ but not both contribute $V^{N_L+N_R}$ in the product $\Pi'_\varepsilon \Pi''_\varepsilon$. The pulses in $H \cap \{\lambda \geq \varepsilon\}$ that affect both $t'$ and $t''$ contribute $[E(W^2)]^{N_0}$, where $N_0$ is their number.

Since $\varepsilon < r/2$, the subset $S(t', t'')$ of $H$ whose pulses rule both $t'$ and $t''$ does not depend on $\varepsilon$ (see Figure 1). Moreover it follows from elementary computations based on the construction (and helped by Figure 1) that $E(V^{N_L+N_R})E(V^{N_0})^2 = \varepsilon^{-2\delta(V-1)}$, $E(V^{N_0}) = e^{\Lambda(S(t',t''))(V-1)}$ and $E([E(W^2)]^{N_0})$
$= e^{\Lambda(S(t',t''))(E(W^2)-1)}$. Thus

$$E[\mu'_\varepsilon(t')\mu'_\varepsilon(t'')] = e^{\Lambda(S(t',t''))\{[E(W^2)-1]-2(V-1)\}}$$

which do not depend on $\varepsilon$. The correlation of $\mu$ at $t'$ and $t''$ is the limit as $\varepsilon \to 0$ of

$$\frac{E[\mu'_\varepsilon(t')\mu'_\varepsilon(t'')]}{[E(\mu'_\varepsilon(t'))]^2} - 1.$$

As $[E(\mu'_\varepsilon(t'))]^2 = 1$ and $\Lambda(S(t', t''))$ behaves like $-\delta \log{(r/2)}$ as $r \ll 1/2$, this correlation behaves like $r^{\tau(2)-1}$ as $r \ll 1/2$ ($\tau(2) - 1 < 0$ if $W$ is not the constant 1). For $r > 1/2$, the correlation vanishes.

A formally identical expression of the correlation holds for CCM, but in that case the physical euclidean distance between $t'$ and $t''$ is replaced by the artificial ultrametric distance.

**Upper critical power $q_{\text{crit.pos}}$ and conditions under which it is finite.** For non-degenerate CCM and MPCP, $E[\mu([0,1])] < \infty$ and if $q > 1$, the condition of finiteness of $E[\mu([0,1])^q]$ is $\tau(q) > 0$. The critical power $q_{\text{crit.pos}}$ was introduced in [Ma4,Ma5] for a CCM as the supremum of $\{q \geq 0;\ E(\mu([0,1])^q) < \infty\}$. It is also defined for MPCP, and when the equation $\tau(q) = 0$ has a solution $> 1$, that solution is $q_{\text{crit.pos}}$.

*Conditions for finiteness of* $q_{\text{crit.pos}}$. These conditions brings out a third difference between MPCP and CCM, and a third source of novelty.

For a CCM, $q_{\text{crit.pos}} = \infty$ holds in all the elementary examples (binomial and multinomial), and all the cases when $W \leq bE(W)$ (a condition that necessarily holds in the conservative - as opposed to canonical - cascades). One has $q_{\text{crit.pos}} < \infty$ if and only if $P(\{W > bE(W)\}) > 0$. So the finiteness of $q_{\text{crit.pos}}$ depends on $b$ and on $W$ having a long tail. A finite $q_{\text{crit.pos}}$ is widely perceived as an anomaly associated, in terms of $f(\alpha)$, with the complication of negative Hölder-like components and negative dimensions (see [Ma8]). Indeed, the condition $\tau(q) = 0$ expresses that the tangent of $f(\alpha)$ whose slope is $q$ crosses the vertical axis of abscissa $\alpha = 0$ at the point of ordinate 0. This is well-known to be the case for $q = 1$. But for $q > 1$, this cannot be the case unless the graph of $f(\alpha)$ crosses into the lower left quadrant where $\alpha < 0$ and $f < 0$.

This behavior of $f(\alpha)$ and the fact that $q_{\text{crit.pos}} < \infty$ occur in the limit lognormal multifractals introduced in [Ma2] and made rigorous in [K2,K3]. But those fractals are not widely known. In any event, the deep importance of the case $q_{\text{crit.pos}} < \infty$ is not sufficiently widely appreciated and its frequent occurences in applications continue to be a source of surprise.

For MPCP, the contrary, a simple sufficient condition for $q_{\text{crit.pos}} < \infty$ is that $\max W > 1$. If so, the term $E(W^q)$ in $\tau_{\text{MPCP}}(q)$ does not vanish at $\infty$, implying $\lim_{q \to \infty} \tau_{\text{MPCP}}(q) = -\infty$.

**A guess.** Consider the following sequence of multifractal processes: non-random cascades, conservative cascades, "effectively conservative" cascades defined as having the same $\tau(q)$ as a non-random or conservative cascade, canonical cascades and MPCP. Aside from the "effectively conservative" cascades, each step from one to the next eliminates some constraints on randomness that simplified the theory but were arbitrary. As a result, the following tentative conclusion deserves careful attention. It may be that in further evolution of the models, the cases where $q_{\text{crit}} = \infty$ will increasingly become "anomalous" and the cases where $q_{\text{crit}} < \infty$ will increasingly become the norm.

*The concrete importance of* $q_{\text{crit.pos}} < \infty$ *and more generally of* $f(\alpha)$ *that is negative for some* $\alpha$ *(see [Ma8]).* In that case, a single sample of the process can only yield $f(\alpha)$ where it is positive. The negative $f(\alpha)$, which can only be obtained by "supersampling" characterize the level of randomness of the process. Therefore, if the above guess proves correct, random multifractals will prove to be typically highly random.

**Lower critical power.** $q_{\text{crit.neg}}$ **and conditions under which it is finite.** Both CCM and MPCP also involve a second critical power $q_{\text{crit.neg}} = \inf\{q;\ E[\mu([0,1]^q)] < \infty\}$, which depends on

$W$ and also the artificial base $b$ for CCM to the contrary, providing a fourth source of novelty: it will be shown in section 3 that for MPCP, $q_{\text{crit.neg}} = \inf\{q;\ \tau_{\text{MPCP}}(q) < \infty\} = \inf\{q;\ E(W^q) < \infty\}$; so $q_{\text{crit.neg}}$ depends only on $W$ and not on the conterpart of $b$ provided by $\delta$; to the contrary, for CCM [B1] obtained, $q_{\text{crit.neg}} = b \inf\{q;\ E(W^q) < \infty\}$.

Comment. Despite the symmetry between the definitions, the two critical power are extremely different in nature.

**The role of $E(W)$; CCM only depend on $W/E(W)$, while MPCP also depend on $E(W)$; this fact is a major source of versatility.** The MPCP exhibit a major fifth source of novelty that is revealed by writing $W = W_1 V$, where $W_1 = W/E(W)$, therefore $E(W_1) = 1$. For CCM, the normalization needed to define $\mu$ yields

$$\tau_{\text{CCM}}(q) = -1 + q - \delta \log E(W_1^q).$$

That is $V$ drops out and $\tau$ is independent of $V$. To the contrary,

$$\tau_{\text{MPCP}}(q) = -1 + q[1 + \delta(V-1)] - \delta[V^q E(W_1^q) - 1]$$

involves both $W_1$ and $V$explicitely and inseparably. So does the dimension

$$\tau'_{\text{MPCP}}(1) = 1 + \delta[(V-1) - V \log V - V E(W_1 \log W_1)].$$

So do $\tau(2)$ and the $q_{\text{crit.pos}}$. To the contrary, $q_{\text{crit.neg}}$ only involves $W_1$.

**Special case 1:** *pulses of non random height* $V$. They correspond to $W_1 \equiv 1$ yet in the MPP case suffice to generate an interesting random multifractal measure with a single parameter $V$. This measure has no counterpart in cascades. To pinpoint the origin of this novelty, recall the approximating measures $\mu_\varepsilon$ obtained by pulses of width $\geq \varepsilon$. For CCM, the number of pulses that affect $\mu_\varepsilon$ at a fixed $t$ is non random and independent of $t$. Therefore, when $W$ is non random, it degenerates to a constant that is eventually renormalized to 1. For MPCP, this number is a Poisson random variable and its randomness sufficies to create a non-degenerate process (it may, but need not, be useful).

Remark on a class of multidimentional Poisson random variables. Contrary to the Gaussian, the Poisson distribution has no intrinsic multivariable version. This process provides a "natural" candidate.

We do not recall seting mentioned previously.

In the case of two instants $t'$ and $t''$, the values of $\mu'_\varepsilon$ are of the form $\log \mu'(t') = P_L + P_0$ and $\log \mu'(t'') = P_0 + P_R$, where $P_L$, $P_0$, and $P_R$ are independent Poisson variables that correpond to the three areas to the right of Figure 1. The same expressions (with Poisson replaced by Gaussian) hold for positively correlated Gaussian variables.

**Special case 2:** $W$ *uniformly distributed between* 0 *and* $2V$. Fix $V > 0$ and assume that $W \in [0, 2V]$ and $W$ is uniformly distributed, that is

$$P_W(dx) = \mathbf{1}_{\{0 \leq W \leq 2V\}} \frac{dx}{2V}$$

Then for every $q > -1$,

$$E(W^q) = \frac{(2V)^q}{q+1}$$

$(V = E(W))$; $\tau_{\text{CCM}}$ and $\tau_{\text{MPCP}}$ are elementary functions and one can discuss explicitely the degeneracy of $\mu$ and the finiteness of the critical values of $q$.

**-The CCM case.** This case was studied in [Ma6] with the basis $b = 2$. For every basis $b \geq 2$, we have $P(W \leq bE(W)) = 1$ and, independently of $V$,

$$\tau_{\text{CCM}}(q) = \begin{cases} -\infty & \text{if} \quad q \leq -1 \\ -1 + q(1 - \log_b(2)) + \log_b(q+1) & \text{if} \quad q > -1 \end{cases}.$$

Thus $\lim_{q\to\infty} \tau_{\text{CCM}}(q) > 0$, so $\tau'_{\text{CCM}}(1) > 0$ hence $\mu$ is non-degenerate, and $q_{\text{crit.pos}} = \infty$. Moreover [B1] yields $q_{\text{crit.neg}} = -b$.

**-The MPCP case. When** $\max W \leq 1$**,** we find either $q_{\text{crit.pos}} < \infty$ or $q_{\text{crit.pos}} = \infty$ according to the value of $\delta$. In this case $E(W^q)$ vanishes at $\infty$ and

$$\tau'_{\text{MPCP}}(1) = 1 + \delta(V-1) - \delta E(W \log W)$$

with $E(W \log W) \leq 0$. There are two cases:

1) $1/\delta \geq 1 - V$, that is $1 + \delta(V-1) \geq 0$, and $\lim_{q\to\infty} \tau_{\text{MPCP}}(q) > 0$. Then $q_{\text{crit.pos}} = \infty$. Moreover such a $\delta$ yields always $\tau'_{\text{MPCP}}(1) > 0$ as can be seen on the expression of $\tau'_{\text{MPCP}}(1)$;

2) $1/\delta < 1 - V$, that is $1 + \delta(V-1) < 0$, and $\lim_{q\to\infty} \tau_{\text{MPCP}}(q) = -\infty$. So $q_{\text{crit.pos}} < \infty$. For such a $\delta$, the non-degeneracy holds if and only if $1 + \delta(V-1) - \delta E(W \log W) > 0$. This yields the following condition to be satisfied by $\delta$:

$$1 - V + E(W \log W) < 1/\delta < 1 - V.$$

Furthermore

$$\tau_{\text{MPCP}}(q) = \begin{cases} -\infty & \text{if} \quad q \leq -1 \\ -1 + q(1 + \delta(V-1)) - \delta(\frac{(2V)^q}{q+1} - 1) & \text{if} \quad q > -1 \end{cases}$$

and

$$\tau'_{\text{MPCP}}(1) = 1 + \delta(V-1) - \delta\theta(V)$$

with $\theta(V) = V\,[\log(2V) - 1/2]$.

<u>Summary of special case 2:</u>

If $\mu$ is non-degenerate, $q_{\text{crit.neg}} = -1$, a special case of the general rule.

If $V > 1/2$ then $\max W > 1$ and $q_{\text{crit.pos}} < \infty$ as long as $\tau'_{\text{MPCP}}(1) > 0$, that is $1/\delta > 1 - V + \theta(V)$.

If $0 < V \leq 1/2$ then $\max W \leq 1$, $\theta(V) < 0$ and
1) if $1/\delta \geq 1 - V$ then $\tau'_{\text{MPCP}}(1) > 0$ and $q_{\text{crit.pos}} = \infty$.

2) if $1 - V + \theta(V) < 1/\delta < 1 - V$ then $\tau'_{\text{MPCP}}(1) > 0$ and $q_{\text{crit.pos}} < \infty$. One can check that $1 - V + \theta(V)$ describes $[1/4, 1[$, so in this case $\delta$ must be in $]1, 4[$.

3) In all other cases, $\mu$ is degenerate.

**Special case 3:** *pulses with* $V = 1$. For them, $\tau'_{\text{MPCP}}(1)$ takes a form familiar from the CCM case.

**The general case $W = W_1 V$, with $P(\{W_1 = 1\}) < 1$ and $V > 0$:** observe that in the formula for the codimension $1 - \tau'(1)$, every term contains $V$. Therefore the codimension corresponding to $W_1 V$ is not the sum of the codimensions corresponding to $W_1$ and $V$ taken separatly. That is, the "typical behavior" of the intersection of "independent" sets is not applicable.

**Marginal distribution of density for the approximant measures.** The quantity $\log(d\mu_\varepsilon/d\ell)$ is, up to the constant $\delta(V-1)\log\varepsilon$, the sum of $N$ independent random variables of the form $\log W$, where $N$ is a Poisson random variable of expectation $-\delta\log\varepsilon$, independent of the $W$'s. When $W \equiv V$, $\log(d\mu_\varepsilon/d\ell)$ is a Poisson random variable. In all other cases, $\log(d\mu_\varepsilon/d\ell)$ is a very special infinitely divisible random variable. The early de Finetti theory, later generalized by Lévy (and Khinchine) (see [GKo] p 68), involved the sum of this very special variable and of a Gaussian.

The Gaussian term alone is the foundation of the "limit lognormal" multifractals (LLNM) introduced in [Ma2]. In the Gaussian context, the whole process is determined by its covariance. In the context of our pulses, it is not the case (as will be emphasized by the more general MPP refered to all the end of this section).

Scientific models are often compromises between the numbers of parameters, the ease of calculation and the quality of fit. Both LLNM and MPCP with $W \equiv$ a constant involve a single parameter. MPCP is far easier to calculate.

**Critical density.** When $W$ is fixed, the condition $\tau'(1) = 0$ defines a critical density $\delta_{\text{crit}}(1)$ beyond which $\mu = 0$. For CCM, there is also a critical $\delta$, but a critical base is only defined when $\exp(1/\delta_{\text{crit}}(1))$ is an integer. There is also for each $q$ a critical density $\delta_{\text{crit}}(q)$ beyond which $E[\mu([0,1])^q] = \infty$. For $W_1 \equiv 1$ and $V < e$, the function $\delta_{\text{crit}}(1)$ is two-to-one, that is, the same criticality $\mu \equiv 0$ can be achieved by a small $V$ and a $V$ close to $e$.

**Generalization.** A paper that follows will study the following alternative construction: $W$ is a measurable mapping from $[0,1]$ to $\mathbb{R}_+^*$ and $P_h(t) = W(\frac{t-t_h+\lambda_h}{2\lambda_h})$ within $[t_h - \lambda_h, t_h + \lambda_h]$, $E(W^q)$ denotes $\int_{[0,1]} W^q(t)dt$, and under some condition we show that for the limit measure generated above, the associated multifractal function $\tau$ is again given by (1). Then a third construction combines the two previous types of pulses, by multiplying cylindrical pulses and pulses generated by a positive measurable function.

# 2 Definitions and Notations.

## 2.1 Construction of the limit measure.

Let $(\Omega, \mathcal{B}, P)$ be the probability space on which the random variables (r. v.) are defined in this paper.

Let $W > 0$ be an integrable r. v. and denote $E(W)$ by $V$; fix $\delta > 0$ and define on the strip $H = \{0 < \lambda < 1\}$ the positive measure

$$\Lambda(dtd\lambda) = \frac{\delta}{2}\frac{dtd\lambda}{\lambda^2}.$$

Let $\{B_k\}_{k\geq 1}$ be a partition of $H$ such that for all $k \geq 1$, $0 < \Lambda(B_k) < \infty$. For every $k \geq 1$, choose $(M_{k,n})_{n\geq 1}$ a sequence of $H$ valued r. v.'s with common law $\frac{\Lambda_{|B_k}}{\Lambda(B_k)}$, and $N_k$ a Poisson variable with parameter $\Lambda(B_k)$. Then choose $(W_{k,n})_{k,n\geq 1}$ a sequence of copies of $W$.

Assume that the r. v.'s $M_{k,n}$, $W_{k,n}$ and $N_k$, $k, n \geq 1$, are independent of one another.

Then, the associated Poisson point process with intensity $\Lambda$, is defined as $S = \{M_{k,n};\ 1 \leq k,\ 1 \leq n \leq N_k\}$.

For $M = M_{k,n} \in S$, define $W_M = W_{k,n}$, $I_M = [t_M - \lambda_M, t_M + \lambda_M]$ and define the *cylindrical pulse* associated to $M$ and $W_M$ as being

$$p_M : \mathbb{R} \to \mathbb{R}_+,\ t \mapsto 1 + (W_M - 1)\,\mathbf{1}_{I_M}(t).$$

For all $\varepsilon \in ]0, 1]$ and $t \in \mathbb{R}$, define the truncated cone $C_\varepsilon(t) = \{(t', \lambda) \in H;\ t - \lambda \leq t' \leq t + \lambda,\ \varepsilon \leq \lambda < 1\}$ and

$$Q_{C_\varepsilon(t)} = \prod_{M \in S \cap C_\varepsilon(t)} W_M.$$

Then for every $0 < \varepsilon \leq 1$, let $\mu_\varepsilon$ be the measure on $\mathbb{R}$ with density with respect to $\ell$, the Lebesgue measure, given by

$$\frac{d\mu_\varepsilon}{d\ell}(t) = Q_\varepsilon(t) = \varepsilon^{\delta(V-1)} \prod_{M \in S \cap \{\lambda \geq \varepsilon\}} p_M(t) = \varepsilon^{\delta(V-1)} Q_{C_\varepsilon(t)}.$$

and define $F_\varepsilon = \sigma\left(M, W_M;\ M \in S \cap \{\lambda \geq \varepsilon\}\right)$.

**Remark 1** The random variables defined in this paper do not depend on the choice of $\{B_k\}_{k\geq 1}$.
Because of the properties of a Poisson point process, if $H_1, \ldots, H_l$ are mutually disjoint subsets of $H$, the $\sigma$-algebras $\sigma\left(M, W_M;\ M \in S \cap H_i\right)$, $1 \leq i \leq l$, are mutually independent.

In all the text, weak convergence of measures on a locally compact Hausdorff set $K$ means weak*-convergence in the dual of $C(K)$, the space of real continuous functions on $K$.

**Theorem 1 (Existence of the limit measure)** *i) For every compact subset $K$ of $\mathbb{R}$, with probability one the measures $\mu_\varepsilon^K$, $0 < \varepsilon \leq 1$, obtained by restriction to $K$ of the measures $\mu_\varepsilon$ converge weakly to a measure $\mu^K$ as $\varepsilon \to 0$. Moreover, given $K$ and $t_0$ a point of $K$, with probability one, $\mu^K(\{t_0\}) = 0$.*

*ii) It follows from i) that with probability one, there is an unique measure $\mu^{\mathbb{R}}$ on $\mathbb{R}$ such that for all $n \in \mathbb{Z}$, $\mu^{\mathbb{R}}(\{n\}) = 0$ and the restriction of $\mu^{\mathbb{R}}$ to $[n, n+1]$ is $\mu^{[n,n+1]}$.*

**Proof.** *i)* By Lemma 2 in section 5, for every $t \in \mathbb{R}$, $(Q_{1/s}(t))_{s\geq 1}$is a non negative right-continuous martingale of mean 1 with respect to the filtration $\left(F_{1/s}\right)_{s\geq 1}$. Then the conclusion follows from an immediate extension to continuous time (right continuous) martingales of the general theory of [K3]. *ii)* It is a verification.

Because of the form of the intensity $\Lambda$, the measure $\mu^{\mathbb{R}}$ is statistically invariant by translations in the $t$-axis direction.

So, in the rest of the paper, we consider only the measure $\mu = \mu^{[0,1]}$.

### 2.2 Other definitions and a principle of self-similarity.

If $B$ is a Borel subset of $H$ and $\Lambda(B) < \infty$, define

$$Q_B = \prod_{M \in S \cap B} W_M.$$

If $I$ is a compact subinterval of $[0,1]$, then $|I|$ stands for its length and we define (see Figure 2)

$$\begin{aligned} T_I &= \{(t,\lambda) \in H;\ 0 < \lambda < |I|,\ \inf(I) - \lambda \le t \le \sup(I) + \lambda\}, \\ B_\gamma^I &= \{(t,\lambda) \in H;\ |I| \le \lambda < 1,\ t \in [\inf(I) + \gamma\lambda, \sup(I) + \gamma\lambda]\},\ \gamma \in \{-1,1\}, \\ T^I &= \{(t,\lambda) \in H;\ |I| \le \lambda < 1,\ t \in [\sup(I) - \lambda, \inf(I) + \lambda]\}, \end{aligned}$$

$B^I = B_{-1}^I \cup B_1^I$ and $f_I$ the affine transformation on $\mathbb{R}$ which maps $\inf(I)$ onto 0 and $\sup(I)$ onto 1.

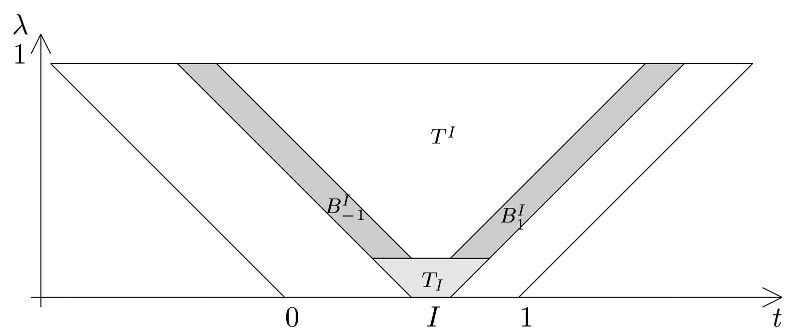


**Figure 2** : basic sets in $H$.

Then, with the notations of section 2.1, for all $0 < \varepsilon \le 1$ define $\mu_\varepsilon^I$ the measure on $I$ with density with respect to the Lebesgue measure given by

$$\frac{d\mu_\varepsilon^I}{d\ell}(t) = \varepsilon^{\delta(V-1)} \prod_{M \in S \cap \{\varepsilon|I| \le y < |I|\}} p_M(t) = \varepsilon^{\delta(V-1)}\, Q_{C_{\varepsilon|I|} \setminus C_{|I|}(x)}.$$

Theorem 2 examines the strong similarity between the measures $\mu_\varepsilon$ constructed on $[0,1]$ via the random weights distributed in $T_{[0,1]}$ in section 2.1 and the measures $\mu_\varepsilon^I$ on $I$ via the weights distributed on $T_I$. This relation is the object of the

**Theorem 2** *With probability one, for every non trivial compact subinterval $I$ of $[0,1]$ and for all $0 < \varepsilon \le |I|$, one has*

$$\begin{aligned} \mu_\varepsilon(I) &= |I|^{\delta(V-1)} \int_I Q_{C_{|I|}(t)} \mu^I_{\varepsilon/|I|}(dt) \\ &= |I|^{\delta(V-1)} Q_{T^I} \int_I Q_{B^I \cap C_{|I|}(t)}\ \mu^I_{\varepsilon/|I|}(dt). \end{aligned}$$

*Moreover, for any such interval and $0 < \varepsilon \le 1$, for all $f \in C(I)$, the r.v.'s $\int_I f(t)\, d\mu_\varepsilon^I(t)$ and $|I| \int_{[0,1]} f \circ f_I^{-1}(t)\, \mu_\varepsilon(dt)$ have the same distribution.*

*With probability one $(\mu_\varepsilon^I)_{0<\varepsilon\leq 1}$ converges weakly to a measure $\mu^I$ as $\varepsilon \to 0$, and for all $f \in C(I)$, the r.v.'s $\int_I f(t)\,\mu^I(dt)$ and $|I| \int_{[0,1]} f \circ f_I^{-1}(t)\,\mu(dt)$ have the same distribution.*

*In particular $\|\mu_\varepsilon^I\|$ (resp. $\|\mu^I\|$ ) has the same distribution as $|I|\,\|\mu_\varepsilon\|$ (resp. $|I|\,\|\mu\|$).*

*Moreover, $Q_{T^I}$ and the function $t \mapsto Q_{B^I \cap C_{|I|}(t)}$ are independent and they are also independent of the $\mu_\varepsilon^I$'s and $\mu^I$.*

**Proof**. It is a consequence of the definitions, Theorem 1, and Proposition 1.12 of [M2] which concludes that $\Lambda$ is invariant by the homotheties with apex in the $t$-axis and positive ratio which map $H$ into $H$, and also by translations in the $t$-axis direction.

Now define $Y = \|\mu\| = \|\mu^{[0,1]}\|$ and for all $s \geq 1$, $Y_s = \|\mu_{1/s}^{[0,1]}\|$. A simple computation using Lemma 2 shows that $(Y_s, F_{1/s})_{s\geq 1}$ is a right-continuous martingale of mean 1. If $I$ is a non trivial compact subinterval of $[0,1]$, define $Y_I = \frac{1}{|I|}\|\mu^I\|$ and for all $s \geq 1$, $Y_{s,I} = \frac{1}{|I|}\|\mu_{1/s}^I\|$.

The measure $\mu$ will be represented as the image of a measure on an homogeneous tree.

## 2.3 Measure on a tree associated to $\mu$.

Each time we fix an integer $b \geq 2$, for every integer $m \geq 0$ we denote by $A_m$ the set of finite words of length $m$ on the alphabet $\{0, \dots, b-1\}$ ($A_0 = \{\epsilon\}$). Then for $a \in A_m$, $|a| = m$ and we denote by $I_a$ the closed $b$-adic subinterval of $[0,1]$ naturally encoded by $a$.

We denote $\bigcup_{m=0}^\infty A_m$ by $A$ and $\{0, \dots, b-1\}^{\mathbb{N}}$ by $\partial A$; $A \cup \partial A$ is equipped with the concatenation operation and for every $a \in A$, $C_a$ denotes $a\partial A$, the cylinder generated by $a$; $\mathcal{A}$ denotes the $\sigma$-field generated by the $C_a$'s in $\partial A$.

We denote by $\pi$ the mapping from $\partial A$ to $[0,1]$ defined by $t = t_1 \dots t_i \dots \mapsto \sum_{i\geq 1} t_i/b^i$.

$\tilde{\ell}$ is the measure on $(\partial A, \mathcal{A})$ such that for all $a \in A$, $\tilde{\ell}(C_a) = b^{-|a|}$.

Now if $\rho$ is a non negative measure on $(\partial A, \mathcal{A})$, for $n \geq 1$ define $D_n.\rho$ the measure with density with respect to $\tilde{\ell}$ equal to

$$\frac{d(D_n.\rho)}{d\rho}(t) = D_n(t) = b^{-n\delta(V-1)} Q_{C_{b^{-n}}(\pi(t))}.$$

By the same arguments as for Theorem 1, with probability one, the sequence $(D_n.\rho)_{n\geq 1}$ converges weakly to a non-negative random measure $D.\rho$. Moreover by [K3], the operator $L : \rho \mapsto E(D.\rho)$ on non negative measures possesses the important property to be a projection (by definition if $f \in C(\partial T)$ then $\int_{\partial A} f(t)\,E(D.\rho)(dt) = E(\int_{\partial A} f(t)\,D.\rho(dt))$).

We define $\tilde{\mu} = D.\tilde{\ell}$ and $\tilde{\mu}_n = D_n.\tilde{\ell}$ for all $n \geq 1$. By construction $\mu = \tilde{\mu} \circ \pi^{-1}$ and $\mu_{b^{-n}} = \tilde{\mu}_n \circ \pi^{-1}$ for $n \geq 1$.

We end this section by three relations, (2), (3), (4) that will prove to be fundamental.

By Theorem 2 for all $n > m > 1$

$$Y_{b^n} = \sum_{a \in A_m} \mu_{b^{-n}}(I_a) = b^{-m\delta(V-1)} \sum_{a \in A_m} Q_{T^{I_a}} \int_{I_a} Q_{B^{I_a} \cap C_{b^{-m}}(t)}\, \mu_{b^{m-n}}^{I_a}(dt) \qquad (2)$$

and for all $m \geq 1$ and $a \in A_m$

$$\tilde{\mu}(C_a) = b^{-m\delta(V-1)} Q_{T^{I_a}} \int_{I_a} Q_{B^{I_a} \cap C_{b^{-m}}(t)} \; \mu^{I_a}(dt) \tag{3}$$

(indeed $\tilde{\mu}(C_a) = \lim_{n\to\infty} \tilde{\mu}_n(C_a)$ since the space $\partial A$ is totaly disconnected and for all $n \geq 1$ and $a \in T$, $\tilde{\mu}_n(C_a) = \mu_{b^{-n}}(I_a)$). So for all $m \geq 1$,

$$Y = \sum_{a \in A_m} \tilde{\mu}(C_a) = b^{-m\delta(V-1)} \sum_{a \in A_m} Q_{T^{I_a}} \int_{I_a} Q_{B^{I_a} \cap C_{b^{-m}}(t)} \; \mu^{I_a}(dt). \tag{4}$$

# 3 Non-degeneracy of the process; Moments.

Define $\tau(q) = -1 + q\,(1 + \delta(V-1)) - \delta(E(W^q) - 1) \in \mathbb{R} \cup \{-\infty\}$ for $q \in \mathbb{R}$ (note that $\tau$ is concave and finite on $[0,1]$). Recall that $Y = \|\mu\|$.

**Theorem 3 (Non degeneracy)** *i) If $\tau'(1^-) > 0$ then $P(\mu \neq 0) = 1$ and $E(Y) = 1$.*

*ii) If $P(\mu \neq 0) > 0$ then $P(\mu \neq 0) = 1$, $E(Y) = 1$ and $\tau'(1^-) \geq 0$. If moreover $E((1 + W)|\log W|^{2+\gamma}) < \infty$ for some $\gamma > 0$ then $\tau'(1^-) > 0$.*

**Theorem 4 (Moments of positive orders)** *Let $h > 1$.*
*i) If $\tau(h) > 0$ then $0 < E(Y^h) < \infty$. ii) If $0 < E(Y^h) < \infty$ then $\tau(h) \geq 0$.*

**Theorem 5 (Moments of negative orders)** *When $Y$ is not degenerate and $a > 0$, $E(Y^{-a}) < \infty$ if and only if $E(W^{-a}) < \infty$.*

**Remark 2** The alternatives for non-degeneracy and moments of positive orders are the same as for CCM. But we have not yet obtained the necessity of the sufficient conditions excepted for the non-degeneracy under the additional condition of Theorem 3*ii*), and for moments in the case where $h = 2$. [Ma9] conjectures that the equivalence holds in each case.

**Proof of Theorem 3**. Fix an integer $b \geq 2$.

*i*) Define $c = E(Y)$ $(\leq 1)$. By the invariance properties of $\Lambda$ (proof of Theorem 2) and the relation (4), $E(\tilde{\mu}(C_a)) = cb^{-|a|} = c\tilde{\ell}(C_a)$ for every $a \in A$. Thus with the notations of section 2.3, as $\tilde{\mu} = D.\tilde{\ell}$, we have $L(\tilde{\ell}) = c\tilde{\ell}$. So as the operator $L$ is a projection, we have $c^2 = c$.

Moreover as $W > 0$, we see on (4) by the independences between r. v.'s that $\{Y = 0\}$ is a tail event; this, together with the relation $c^2 = c$, implies that we only have to prove that $P(Y > 0) > 0$ to establish *i*). For, the approach in [KP] to study the non-degeneracy of multiplicative cascades can be adapted, only after some (delicate) operations on the equation (2).

Fix $n > m > 1$ two integers. By Lemma C of [KP], if $h < 1$ is large enough, our expression (2) yields

$$Y_{b^n}^h \geq \sum_{a \in A_m} \mu_{b^{-n}}^h(I_a) - (1-h) \sum_{a \neq a' \in A_m} \mu_{b^{-n}}^{\frac{h}{2}}(I_a)\mu_{b^{-n}}^{\frac{h}{2}}(I_{a'}).$$

Moreover Theorem 2 and the Jensen inequality yield $\mu_{b^n}^h(I_a) \geq f_{a,n,m}(h)$ with

$$f_{a,n,m}(h) = b^{-mh\delta(V-1)} Y^{h-1}_{b^{n-m},I_a} b^{-m(h-1)} \int_{I_a} Q^h_{C_{b^{-m}}(t)}\ \mu^{I_a}_{b^{m-n}}(dt)$$

So by writing $\frac{E(Y^h_{b^n}) - \sum_{a\in A_m} f_{a,n,m}(h)}{h-1} \le \sum_{a\neq a'\in A_m} E\left(\mu^{\frac{h}{2}}_{b^{-n}}(I_a)\mu^{\frac{h}{2}}_{b^{-n}}(I_{a'})\right)$ and letting $h$ tend to 0, by using the fact that $E(Y_{b^n}) = \sum_{a\in A_m} f_{a,n,m}(1) = 1$ and Lemma 5$i$) one obtains

$$\begin{aligned} & m\ \log(b)\,\tau'(1^-) + E(Y_{b^n}\ \log\ Y_{b^n}) \ - E(Y_{b^{n-m}}\log Y_{b^{n-m}}) \\ \le & \sum_{a\neq a'\in A_m} E(\mu^{\frac{1}{2}}_{b^{-n}}(I_a)\mu^{\frac{1}{2}}_{b^{-n}}(I_{a'})). \end{aligned}$$

As $(Y_{b^n})_{n\ge 1}$ is a martingale, $E(Y_{b^n}\ \log\ Y_{b^n}) - E(Y_{b^{n-m}}\ \log\ Y_{b^{n-m}})$ is non negative so $m\ \log(b)\,\tau'(1^-) \le \sum_{a\neq a'\in A_m} E\left(\mu^{\frac{1}{2}}_{b^{-n}}(I_a)\mu^{\frac{1}{2}}_{b^{-n}}(I_{a'})\right)$.

Now a geometrical remark shows that given $a \in A_m$, there are at least two and at most four $a' \neq a$ in $A_m$ such that $T_{I_a} \cap T_{I_{a'}} \neq \emptyset$, and then $E(Y^{\frac{1}{2}}_{b^{n-m},I_a} Y^{\frac{1}{2}}_{b^{n-m},I_{a'}}) \le 1$ by the Cauchy-Schwarz inequality; if $a' \in A_m$ and $T_{I_a} \cap T_{I_{a'}} = \emptyset$ then $Y_{b^{n-m},I_a}$ and $Y_{b^{n-m},I_{a'}}$ are independent copies of $Y_{b^{n-m}}$, so $E(Y^{\frac{1}{2}}_{b^{n-m},I_a} Y^{\frac{1}{2}}_{b^{n-m},I_{a'}}) = (E(Y^{\frac{1}{2}}_{b^{n-m}}))^2$. This with Lemma 5$ii$) yield $C > 0$, independent of $m$ such that

$$\sum_{a\neq a'\in A_m} E\left(\mu^{\frac{1}{2}}_{b^{-n}}(I_a)\mu^{\frac{1}{2}}_{b^{-n}}(I_{a'})\right) \le 4b^m.Cb^{-m} + b^{2m}.Cb^{-m}(E(Y^{\frac{1}{2}}_{b^{n-m}}))^2.$$

So $m\ \log(b)\,\tau'(1^-) - 4C \le Cb^m(E(Y^{\frac{1}{2}}_{b^{n-m}}))^2$ and as $\tau'(1^-) > 0$, we can choose $m$ to have $m\ \log(b)\,\tau'(1^-) - 4C > 0$, which yields $C' > 0$ such that for all $n > 1$, $E(Y^{\frac{1}{2}}_{b^n}) \ge C'$. As $(Y^{\frac{1}{2}}_{b^n})_{n\ge 1}$ is uniformly integrable since $E(Y_{b^n}) = 1$, we have $E(Y^{\frac{1}{2}}) > 0$ and so $P(Y > 0) > 0$.

$ii$) That $P(\mu \neq 0) > 0$ implies $P(\mu \neq 0) = 1 = E(Y)$ is proved in $i$).

Fix $h \in ]0,1[$. For all $m > 1$, one has $Y^h \le \sum_{a\in A_m} \tilde{\mu}^h(C_a)$ by (4) and by Lemma 4.$ii$), there exists $C > 0$ such that $E(Y^h) \le C.b^{-m\tau(h)}E(Y^h)$ for all $m > 1$. Since $\tau(1) = 0$, if $Y$ is not degenerate, it is necessary to have $\tau(h) \le 0$ in a left neighbourhood of 1, so $\tau'(1^-) \ge 0$.

Now we assume that $E((1+W)|\log W|^{2+\gamma}) < \infty$ and $\tau'(1^-) = 0$ and show that $P(Y = 0) = 1$:

For every $n \ge 1$, define $\bar{P}_n$ the probability measure on $(\Omega, F_{b^{-n}})$ with density with respect to $P$ equal to $Y_{b^n}$, and let $\bar{P}$ be the Kolmogorov extension of the $\bar{P}_n$'s on $(\Omega, F_\infty = \bigcup_{n\ge 1} F_{b^{-n}})$. By Theorem 2.5.20 of [D-CDu], $Y_{b^n} = \frac{d\bar{P}_n}{dP}$ converges $\frac{1}{2}(P+\bar{P})$-almost surely (a. s.) to a r. v. $Y_\infty$ in $\mathbb{R}_+ \cup \{\infty\}$ and if $\bar{P}(Y_\infty = \infty) = 1$ then $P(Y_\infty = 0) = 1$. As $Y = Y_\infty$ $P$-a. s., it is so enough to show that $\bar{P}(\limsup_{n\to\infty} Y_{b^n} = \infty) = 1$ to have the conclusion.

We adapt the approach used in [Wa-Wi] for multiplicative cascades (here again this requires new ideas, see Lemma 6): for every $t \in [0,1]$ and $n \ge 1$, define the measure $P_{t,n}$ on $F_{b^n}$ by

$$\frac{dP_{t,n}}{dP}(\omega) = Q_{b^{-n}}(t)(\omega).$$

By Lemma 2 $(Q_{b^{-n}}(t), F_{b^{-n}})_{n\geq 1}$ is a martingale with mean one, so $(P_{t,n})_{n\geq 1}$ is a consistent family of probability measures. Let $P_t$ denote the Kolmogorov extension of the $P_{t,n}$ to $F_\infty$. Then for every $n \geq 1$ define on $(\Omega\times[0,1], F_{b^{-n}}\otimes\mathcal{B}([0,1]))$ the probability measure $\mathcal{Q}_n(d\omega\times dt) = P_{t,n}(d\omega)\ell(dt)$, and define $\mathcal{Q}$ on $(\Omega\times[0,1], F_\infty\otimes\mathcal{B}([0,1]))$, the Kolmogorov extension of $(\mathcal{Q}_n)_{n\geq 1}$.

Let $\pi_\Omega$ be the first coordinate projection map on $\Omega\times[0,1]$. By construction, for every $n\geq 1$, $\bar{P}_n = \mathcal{Q}_n\circ\pi_\Omega^{-1}$ and so $\bar{P} = \mathcal{Q}\circ\pi_\Omega^{-1}$. Moreover $\mathcal{Q}(d\omega\times dt) = P_t(d\omega)\ell(dt)$, and by Lemma 6, $P_t(\limsup_{n\to\infty} Y_{b^n} = \infty) = 1$ for every $t\in[0,1]$. So $\bar{P}(\limsup_{n\to\infty} Y_{b^n} = \infty) = 1$.

**Proof of Theorem 4**. *i)* It is enough to show that $(Y_{b^n})_{n\geq 1}$ is bounded in $L^h$ norm for some integer $b\geq 2$. Fix $b = 3$ and $m > 1$. After some reductions we use computations similar to those of [KP] (Th. 2).

Number the intervals $I_a$, $a\in A_m$, as they follow one another from 0 on the real line, and write $\{I_a;\, a\in A_m\} = \{J_i;\, 0\leq i < 3^m\}$. Then, for $i\in\{0,1,2\}$ and $n > m$ define

$$Z_{i,n} = \sum_{0\leq 3k+i<3^m} \mu_{3^{-n}}(J_{3k+i}).$$

By the invariance properties of the intensity $\Lambda$, the $Z_{i,n}$'s are identically distributed, so $E(Y_{3^n}^h) \leq 3^h E(Z_{0,n}^h)$.

Let $\tilde{h}$ be the integer such that $\tilde{h} < h \leq \tilde{h}+1$ and use the subadditivity of $x\mapsto x^{h/(\tilde{h}+1)}$ on $\mathbb{R}_+$ to write

$$Z_{0,n}^h \leq [\sum_{0\leq 3k+i<3^m} \mu_{3^{-n}}^{h/(\tilde{h}+1)}(J_{3k})]^{\tilde{h}+1}.$$

Then

$$E(Z_{0,n}^h) \leq \sum_{0\leq k<3^{m-1}} E(\mu_{3^{-n}}^h(J_{3k})) + \sum \alpha_{j_0\dots j_{3^{m-1}-1}} E(\prod_{0\leq k<3^{m-1}} \mu_{3^{-n}}^{j_k\frac{h}{\tilde{h}+1}}(J_{3k}))$$

where in the last sum the $j_i$'s are $\leq \tilde{h}$, $j_0+\cdots+j_{3^{m-1}-1} = \tilde{h}+1$, $j_i\geq 0$ and $\sum\alpha_{j_0\dots j_{3^{m-1}-1}} = 3^{(m-1)(\tilde{h}+1)} - 3^{m-1} = C_{m,\tilde{h}}$. With the notations of Lemma 4

$$\prod_{0\leq k<3^{m-1}} \mu_{3^{-n}}^{j_k\frac{h}{\tilde{h}+1}}(J_{3k}) \leq \prod_{0\leq k<3^{m-1}} (w_1(J_{3k}))^{j_k\frac{h}{\tilde{h}+1}} \prod_{0\leq k<3^{m-1}} Y_{3^{n-m},J_{3k}}^{j_k\frac{h}{\tilde{h}+1}}.$$

By construction, for all $0 < k \neq k' < 3^{m-1}$, $T_{J_{3k}}\cap T_{J_{3k'}} = \emptyset$, so the $Y_{3^{n-m},J_{3k}}$'s are i.i.d. They are also independent of $\prod_{0\leq k<3^{m-1}}(w_1(J_{3k}))^{j_k\frac{h}{\tilde{h}+1}}$, where the $w_1(J_{3k})$'s have the same distribution. So by applying the generalized Hölder inequality to $\prod_{0\leq k<3^{m-1}}(w(J_{3k},1))^{j_k\frac{h}{\tilde{h}+1}}$, Lemma 4*i)a)* to the $w_1(J_{3k})$'s and by bounding $E(Y_{3^{n-m},J_{3k}}^{j_k\frac{h}{\tilde{h}+1}})$ by $E(Y_{3^{n-m}}^{\tilde{h}})^{\frac{j_k h}{\tilde{h}(\tilde{h}+1)}}$ since $\frac{j_k h}{\tilde{h}(\tilde{h}+1)} \leq 1$, we obtain $C_h > 0$ (independent of $m$) such that

$$E(\prod_{0\leq k<3^{m-1}} \mu_{3^{-n}}^{j_k\frac{h}{\tilde{h}+1}}(J_{3k})) \leq C_h 3^{-m(\tau(h)+1)} E(Y_{3^{n-m}}^{\tilde{h}})^{h/\tilde{h}}.$$

Moreover Lemma 4$i)b)$ gives $E(\sum_{0\leq k<3^{m-1}}\mu^h_{3^{-n}}(J_{3k}))\leq 3^{-1}C_h 3^{-m\tau(h)}E(Y^h_{3^{n-m}})$. Thus

$$E(Y^h_{3^n})\leq 3^{h-1}C_h 3^{-m\tau(h)}E(Y^h_{3^{n-m}})+3^h C_{m,\tilde{h}}C_h 3^{-m(\tau(h)+1)}E(Y^{\tilde{h}}_{3^{n-m}})^{h/\tilde{h}}$$

and since $E(Y^h_{3^{n-m}})\leq E(Y^h_{3^n})$ ($(Y^h_{3^n})_{n\geq 1}$ is a submartingale) and $\tau(h)>0$, we can choose $m$ such that $3^{h-1}C_h 3^{-m\tau(h)}<1$ and

$$E(Y^h_{3^n})(1-3^{h-1}C_h 3^{-m\tau(h)})\leq 3^h C_{m,\tilde{h}}C_h 3^{-m(\tau(h)+1)}\sup_{n\geq 1}E(Y^{\tilde{h}}_{3^n})^{h/\tilde{h}}.$$

As $\sup_{n\geq 1}E(Y_{3^n})=1<\infty$, we have the conclusion for $h\in]1,2]$ ($\tilde{h}=1$); then, as $\tau(1)=0$ and $\tau$ is concave, if $h>2$ and $\tau(h)>0$ then $\tau(p)>0$ for all integer $p\in[2,h]$, so $\sup_{n\geq 1}E(Y^h_{3^n})<\infty$ by induction on $\tilde{h}$.

$ii)$ Fix $b$ an integer $\geq 2$ and $m\geq 1$. Letting $n$ tend to $\infty$ in Lemma 4$i)c)$ yields $E(Y^h)\geq b^{-m\tau(h)}e^{\frac{1}{2}(1-b^{-m})(C(V-E(W^h))}E(Y^h)$. So $b^{-m\tau(h)}$ must be bounded.

**Proof of Theorem 5**. If $E(W^{-a})<\infty$ then $E(Y^{-a})<\infty$: by (4) written with $b=4$ and $m=1$

$$Y\geq B_0 Y_{I_0}+B_3 Y_{I_3}$$

where for $i\in\{0,3\}$, $B_i=4^{-\delta(V-1)-1}Q_{T^{I_i}}m_{B^{I_i},I_i}$(with the notations of Lemma 3). $B_0$ and $B_3$ have the same distribution, the random variables $Y_{I_0}$ and $Y_{I_3}$ are independent copies of $Y$ (because $T_{I_0}\cap T_{I_3}=\emptyset$) and they are independent of $B_0$ and $B_3$. So the previous inequality yields for $t\geq 0$

$$E(e^{-tY})=\phi(t)\leq E[\phi(B_0 t)\phi(B_3 t)]. \tag{5}$$

Moreover $E(B_0^{-a})<\infty$ (use Lemma 1 and 3). Then (5) makes possible to use the approach of [Mol] for multiplicative cascades, and $E(Y^{-a})<\infty$.

If $E(Y^{-a})<\infty$ then $E(W^{-a})<\infty$: we use again the notations of Lemma 3, (4) with $b=2$ and $m=1$ and write

$$\begin{aligned} Y &\leq 2^{-\delta(V-1)}(Q_{T^{I_0}}M_{B^{I_0},I_0}2^{-1}Y_{I_0}+Q_{T^{I_1}}M_{B^{I_1},I_1}2^{-1}Y_{I_1})\\ &\leq 2^{-\delta(V-1)}Q_{T^{I_0}\cap T^{I_1}}[Q_{T^{I_0}\setminus T^{I_1}}M_{B^{I_0},I_0}+Q_{T^{I_1}\setminus T^{I_0}}M_{B^{I_1},I_1}](Y_{I_0}+Y_{I_1})\end{aligned}$$

with $Q_{T^{I_0}\cap T^{I_1}}$ independent of $[Q_{T^{I_0}\setminus T^{I_1}}M_{B^{I_0},I_0}+Q_{T^{I_1}\setminus T^{I_0}}M_{B^{I_1},I_1}]$ and these two r.v.'s independent of $Y_{I_0}+Y_{I_1}$. So $E(Y^{-a})<\infty$ yields $E[(Q_{T^{I_0}\cap T^{I_1}})^{-a}]<\infty$ and as $\Lambda(T^{I_0}\cap T^{I_1})>0$, Lemma 1$i)$ gives the conclusion.

# 4 Multifractal analysis of $\mu$.

For $t\in[0,1]$ and $r>0$, denote $[0,1]\cap[t-\frac{r}{2},t+\frac{r}{2}]$ by $I_r(t)$.

Given $f$ a mapping from $\mathbb{R}$ to $\mathbb{R}\cup\{-\infty\}$, one defines its Legendre-transform $f^*$ from $\mathbb{R}$ to $\mathbb{R}\cup\{-\infty,\infty\}$ by $f^*:\alpha\mapsto\inf_{q\in\mathbb{R}}\alpha q-f(q)$.

For $\alpha>0$, define

$$E_\alpha = \{t \in [0,1];\ \lim_{r\to 0^+} \frac{\log\ \mu(I_r(t))}{\log\ r} = \alpha\}.$$

Here, by the multifractal analysis of $\mu$, we mean the computation, on a largest as possible interval, of the mapping $\alpha \mapsto \dim_H\ E_\alpha$, where $\dim_H$ denotes the Hausdorff dimension.

Our study has points in common with the one made in [B2] for generalized CCM. However, as announced in the introduction, the geometry of $\mu$ does not depend on a particular $b$-ary tree. Hence the logarithmic density, in the definition of the $E_\alpha$'s is not expressed via $b$-adic intervals converging to $\{t\}$, but via the intervals centered at $t$; this gives rise to a different geometrical approach. Moreover, it is easily deduced from our study that for every integer $b \geq 2$, the multifractal analysis of $\mu$ in the sense of [B2] is deduced from the one of $\mu$ in the sense considered here.

The following result is an immediate consequence of Theorem 7 and 8 and Proposition 1 (recall that $\tau$ is defined in section 3):

**Theorem 6 (Multifractal analysis)** *Assume that $\tau$ is finite on an interval $J$ containing a neighbourhood of $[0,1]$, and that $\tau'(1) > 0$.*

*Define $J' = \{q \in Int(J);\ \tau'(q)q - \tau(q) > 0\}$, $I' = \{\tau'(q);\ q \in J'\}$, $\alpha_{\inf} = \inf(I')$ and $\alpha_{\sup} = \sup(I')$ ($[0,1] \subset J'$, $I' \subset ]0,\infty[$, $\alpha_{\inf} > 0$).*

*Then with probability one:*

*i) for all $\alpha \in I'$, $\dim_H\ E_\alpha = \tau^*(\alpha)$;*

*ii) If $\tau^*(\alpha_{\inf}) = 0$ then for all $\alpha \in ]0, \alpha_{\inf}[$, $E_\alpha = \emptyset$. If $\alpha_{\sup} < \infty$ and $\tau^*(\alpha_{\sup}) = 0$ then for all $\alpha \in ]\alpha_{\sup}, \infty[$, $E_\alpha = \emptyset$.*

### 4.1 Lower bounds for dimensions

For $q \in J'$, let $\mu_q$ be the measure constructed on $[0,1]$ as $\mu$, but this time with the random variables $W_{q,M} = W_M^q$. A verification shows that the condition $\tau'(q)q - \tau(q) > 0$ is equivalent to the sufficient condition for the non-degeneracy of $\mu_q$ obtained in Theorem 3.

**Theorem 7** *i) With probability one the measures $\mu_q$, $q \in J'$, are defined simultaneously and have $[0,1]$ as support.*

*ii)a) With probability one, for every $q \in J'$, for $\mu_q$-almost every $t \in [0,1]$,*

$$\liminf_{r\to 0} \frac{\log\ \mu_q(I_r(t))}{\log\ r} \geq \tau^*(\tau'(q)) \text{ and } \lim_{r\to 0} \frac{\log\ \mu(I_r(t))}{\log\ r} = \tau'(q),$$

*so for every $\alpha \in I'$, $\dim_H\ E_\alpha \geq \tau^*(\alpha)$.*

*b) The smallest Hausdorff dimension of a Borel set carrying a piece of $\mu$ is $\tau'(1)$.*

**Proof.** $i)$ It is a direct consequence of Lemma 7$i)$ since $\mu_q = \tilde{\mu}_q \circ \pi^{-1}$.

$ii)$ Assume the results on the logarithmic densities. Then, by a Billinglsey Lemma ([Bil], pp 136-145) and the result on $\liminf_{r\to 0} \frac{\log\ \mu_q(I_r(t))}{\log\ r}$, a. s. for every $q \in J'$ the smallest Hausdorff dimension of a Borel set carrying a piece of $\mu_q$ is $\tau^*(\tau'(q))$. Moreover by the result on $\lim_{r\to 0} \frac{\log\ \mu(I_r(t))}{\log\ r}$, a. s. for every $q \in J'$ the measure $\mu_q$ is carried by $E_{\tau'(q)}$. So a. s. for every $\alpha \in I'$, $\dim_H\ E_\alpha \geq \tau^*(\alpha)$.

The case $q = 1$, that is $\mu_q = \mu$, yields $b$) (one can show that it is not necessary that $E(W^{-a}) < \infty$ for some $a > 0$).

The result on $\liminf_{r\to 0} \frac{\log \mu_q(I_r(t))}{\log r}$ is a consequence of Lemma 7$ii$). Indeed, fix an integer $b \geq 2$ and for $\varepsilon > 0$, $q \in J'$ and $n \geq 1$ define

$$F_{q,n,\varepsilon} = \{t \in [0,1];\ \frac{\log \mu_q(I_{b^{-n}}(t))}{\log b^{-n}} \leq \tau^*(\tau'(q)) - \varepsilon\}.$$

It is enough to show the assertion: $(\mathcal{A}_1)$ for every compact subinterval $K$ of $J'$ and $\varepsilon > 0$, a. s. for every $q \in K$, $\sum_{n\geq 1} \mu_q(F_{q,n,\varepsilon}) < \infty$.

¿From the covering $\bigcup_{t\in F_{q,n,\varepsilon}} I_{b^{-n}}(t)$ of $F_{q,n,\varepsilon}$ one can extract a finite subcovering with the property that it can be divided itself in two finite union of intervals with at most one point in common, $\bigcup_i J_i$ and $\bigcup_j J'_j$, each of them being covered by at most two adjascent $b$-adic intervals of length $b^{-n}$. Moreover by definition of $F_{q,n,\varepsilon}$, $1 \leq \mu_q^\eta(I) b^{n\eta(\tau^*(\tau'(q))-\varepsilon)}$ for $I \in \{J_i; J'_j\}$ and $\eta > 0$. So as $\mu_q(F_{q,n,\varepsilon}) \leq \sum_i \mu_q(J_i) + \sum_j \mu_q(J'_j)$, we have

$$\mu_q(F_{q,n,\varepsilon}) \leq \sum_i \mu_q^{1+\eta}(J_i) b^{n\eta(\tau^*(\tau'(q))-\varepsilon)} + \sum_j \mu_q^{1+\eta}(J'_j) b^{n\eta(\tau^*(\tau'(q))-\varepsilon)}.$$

For every $I \in \{J_i, J'_j; i, j\}$ we have $I \subset I_a \cup I_{a'}$ for some $a$ and $a' \in A_n$ and so $\mu_q^{1+\eta}(I) \leq 2^\eta(\mu_q^{1+\eta}(I_a) + \mu_q^{1+\eta}(I_{a'}))$. Hence, if $\eta \leq 1$

$$\mu(F_{q,n,\varepsilon}) \leq 8 \sum_{a\in A_n} \mu_q^{1+\eta}(I_a) b^{n\eta(\tau^*(\tau'(q))-\varepsilon)}.$$

But the relation $\mu_q = \tilde{\mu}_q \circ \pi^{-1}$ and the fact that a. s. $\tilde{\mu}_q$ as no atoms by Lemma 7$ii$) $(\tau^*(\tau'(q)) > 0)$ imply that a. s. $\mu_q(I_a) = \tilde{\mu}_q(C_a)$ for every $a \in A$. So the proof of Lemma 7$ii$) yields $(\mathcal{A}_1)$.

Now we indicate how to obtain that a. s. for every $q \in J'$, for $\mu_q$-almost every $t \in [0,1]$,

$$\liminf_{r\to 0} \frac{\log \mu(I_r(t))}{\log r} \geq \tau'(q) \text{ and } \limsup_{r\to 0} \frac{\log \mu(I_r(t))}{\log r} \leq \tau'(q).$$

We define the sets $F^{-1}_{q,n,\varepsilon} = \{t \in [0,1];\ \frac{\log \mu(I_{b^{-n}}(t))}{-n \log b} \geq \tau'(q) + \varepsilon\}$ and $F^1_{q,n,\varepsilon} = \{t \in [0,1];\ \frac{\log \mu(I_{b^{-n}}(t))}{-n \log b} \leq \tau'(q) - \varepsilon\}$, and we have to show the assertion: $(\mathcal{A}_2)$ a. s. for every $q \in J'$, $\sum_{n\geq 1} \mu_q(F^{-1}_{q,n,\varepsilon}) + \mu_q(F^1_{q,n,\varepsilon}) < \infty$.

The sets $F^{-1}_{q,n,\varepsilon}$ and $F^1_{q,n,\varepsilon}$ admit the same kind of covering that $F_{q,n,\varepsilon}$, but now if $\eta > 0$, $\gamma \in \{-1, 1\}$ and $I \in \{J_i, J'_j; i, j\}$ (the covering of $F^\gamma_{q,n,\varepsilon}$) then $1 \leq \mu^{\gamma\eta}(I) b^{n\gamma\eta(\tau'(q)-\gamma\varepsilon)}$ and so

$$\mu_q(F^\gamma_{q,n,\varepsilon}) \leq \sum_i \mu_q(J_i)\mu^{\gamma\eta}(J_i) b^{n\gamma\eta(\tau'(q)-\gamma\varepsilon)} + \sum_j \mu_q(J'_j)\mu^{\gamma\eta}(J'_j) b^{n\gamma\eta(\tau'(q)-\gamma\varepsilon)}.$$

By using the fact that for every $I \in \{J_i, J'_j; i, j\}$ we have $I \subset I_a \cup I_{a'}$ where $a$ and $a'$ are in $A_n$ and $I_a$ and $I_{a'}$ have one point in common, by defining $S_a = \{c \in A_{|a|};\ I_a \cap I_c \neq \emptyset\}$ $(\#S_a \leq 3)$ we obtain for $\eta \in ]0,1[$

$$\mu_q(F^1_{q,n,\varepsilon}) \leq 4b^{n\eta(\tau'(q)-\varepsilon)} \sum_{a\in A_n} \mu_q(I_a) \sum_{c\in S_a} \mu^\eta(I_c);$$

since $I$ contains an interval $I_{a''}$ with $a" \in A_{n+1}$, and $I_{a''} \subset I_a$ or $I_{a''} \subset I_{a'}$, by writing $\bar{S}_{a''}$ for the set of the $c$'s $\in A_{n+1}$ such that the father $\bar{c}$ of $c$ in $A_n$ is such that $I_{\bar{c}} \cap I_{\bar{a}} \neq \emptyset$ ($\#\bar{S}_{a''} \leq 6$), we obtain

$$\mu_q(F_{q,n,\varepsilon}^{-1}) \leq 2b^{-n\eta(\tau'(q)+\varepsilon)} \sum_{a" \in A_{n+1}} \mu^{-\eta}(I_a) \sum_{c \in \bar{S}_{a''}} \mu_q(I_c).$$

($\mathcal{A}_2$) comes from computations very similar to those of the proof of Lemma 7$ii$), using the fact that $\frac{\Lambda(T^{I_a})}{\Lambda(T^{I_a} \cap T^{I_c})}$, $c \in S_a$, and $\frac{\Lambda(T^{I_{a''}})}{\Lambda(T^{I_{a''}} \cap T^{I_c})}$, $c \in \bar{S}_{a''}$, tend to 1 as $|a|$ and $|a''|$ tend to $\infty$, and for $\eta$ small enough $E(Y^{-\eta}) < \infty$.

### 4.2 Upper bounds for dimensions

Proposition 1 is standard and deduced from [BMP] and [O2] (for another multifractal formalism eliminating a prescribed grid, see [R]).

**Proposition 1** *Let $b$ be an integer $\geq 2$. For $(q,t) \in \mathbb{R}^2$, define*
$C_b(q,t) = \limsup_{n\to\infty} C_{b,n}(q,t) = \sum_{a \in A_n} \mu^q(I_a)|I_a|^t$ *and*
$C(q,t) = \lim_{\delta\to 0} \inf\{\sum_{i\geq 1} \mu^q(I_i)|I_i|^t; [0,1] \subset \bigcup_{i\geq 1} I_{r_i}(t_i), t_i \in [0,1], |r_i| \leq \delta\}$.

*i) For all $q \in \mathbb{R}$, $\varphi_b(q) = \inf\{t \in \mathbb{R};\ C_b(q,t) = 0\}$ and $\varphi(q) = \inf\{t \in \mathbb{R};\ C(q,t) = 0\}$ are defined, the function $\varphi_b$ is convex and $\varphi \leq \varphi_b$.*

*ii) Fix $\alpha > 0$. If $(-\varphi)^*(\alpha) \geq 0$ then $\dim_H E_\alpha \leq (-\varphi)^*(\alpha)$ else $E_\alpha = \emptyset$.*

Then

**Theorem 8** *With probability one*

*i) $\varphi(q) \leq -\tau(q)$ for every $q \in J$ such that $E(Y^q) < \infty$.*

*ii) $(-\varphi)^*(\alpha) \leq \tau^*(\alpha)$ for every $\alpha > 0$ such that $\alpha = \tau'(q)$ with $q \in J$ and $E(Y^q) < \infty$.*

*iii) $\dim_H E_\alpha \leq \tau^*(\alpha)$ for all $\alpha \in I'$.*

**Proof.** $i$) Fix $(q,t) \in J \times \mathbb{R}$. Lemma 4$ii$)b) can be applied with $\mu$ instead of $\tilde{\mu}$ since now we know that $\mu$ has no atoms. Hence there exists $C_q > 0$ such that for all $n \geq 1$

$$E(C_{b,n}(q,t)) \leq C_q.b^{n(-\tau(q)-t)}.E(Y^q). \tag{6}$$

By (6) if $E(Y^q) < \infty$ and $t > -\tau(q)$ then a. s. $\sum_{n\geq 1} C_{b,n}(q,t) < \infty$ and so $C_b(q,t) = 0$ and $\varphi_b(q) \leq t$. Thus if $q \in J$ and $E(Y^q) < \infty$ then a. s. $\varphi_b(q) \leq -\tau(q)$. As $\varphi_b$ and $\tau$ are continuous, this holds a. s. for every $q \in J$ such that $E(Y^q) < \infty$. We conclude with Propoition 1.
$ii$) By definition of the Legendre transform.
$iii$) Use Proposition 1 and $ii$) after proving that $E(Y^q) < \infty$ for every $q \in J'$: if $q < 0$ it is the case by Theorem 5. If $0 \leq q \leq 1$, it is true by definition of $Y$. If $q > 1$, it is insured by Theorem 4 and the fact that by the concavity of $\tau$, $\tau(1) = 0$ and $\tau'(q)q - \tau(q) > 0$ imply $\tau(q) > 0$.

## 5 Basic Lemmas.

**Lemma 1** *Fix $B$ is a Borel subset of $H$ such that $\Lambda(B) < \infty$, $q \in \mathbb{R}$ and $\beta > 0$. Then (in $\mathbb{R} \cup \{-\infty, \infty\}$)*

*i)* $E(Q_B^q) = e^{\Lambda(B)(E(W^q)-1)}$;

*ii)* $E(Q_B^q \log Q_B) = \Lambda(B) E(W^q \log W) e^{\Lambda(B)(E(W^q)-1)}$;

*iii)* $E(Q_B^q |\log Q_B|) \leq \Lambda(B) E(W^q |\log W|) e^{\Lambda(B)(E(W^q)-1)}$;

*iv) Denote by $\bar{\beta}$ the integer such that $\bar{\beta} \leq \beta < \bar{\beta} + 1$. There exists a constant $C_\beta > 0$, independent of $B$ such that*

$$E(Q_B |\log Q_B|^\beta) \leq C_\beta (1 + \Lambda(B))^{\bar{\beta}+3} (V)^{\bar{\beta}+2} E(W |\log W|^\beta) e^{\Lambda(B)(V-1)}.$$

**Proof.** We prove only *iv)* because the other points are obtained by similar computations: conditionally to $\#S \cap B = k \geq 1$, $Q_B = \prod_{i=1}^k W_i$ where the $W_i$'s are independent copies of $W$. So

$$\begin{aligned} E(Q_B |\log Q_B|^\beta | \#S \cap B &= k) \leq E(\prod_{i=1}^k W_i [\sum_{i=1}^k |\log W_i|]^\beta) \\ &\leq k^{\beta+1} E(W |\log W|^\beta)(V)^{k-1}. \end{aligned}$$

Thus as $P(\#S \cap B = k) = e^{-\Lambda(B)} \frac{(\Lambda(B))^k}{k!}$, we have

$$E(Q_B |\log Q_B|^\beta) \leq E(W |\log W|^\beta) \sum_{k \geq 1} e^{-\Lambda(B)} \frac{(\Lambda(B))^k}{k!} k^{\bar{\beta}+2} (V)^{k-1}$$

and standard estimates give the conclusion.

**Lemma 2** *Fix $t \in \mathbb{R}$. For every $s \geq 1$, $\Lambda(C_{1/s}(t)) = C \log s$, and $(Q_{1/s}(t))_{s \geq 1}$ is a right continuous martingale with respect to $(F_{1/s})_{s \geq 1}$, with expectation 1.*

**Proof**. The right continuity is a consequence of the definition and the martingale property comes from the independences between the copies of $W$ and Lemma 1*i)* applied with $B = C_{1/s}(t) \backslash C_{1/s'}(x)$ for $1 \leq s' \leq s$.

Now if $B \subset H$ and $I$ is a nontrivial compact subinterval of $[0, 1]$ we define

$$m_{B,I} = \inf_{u \in I} Q_{B \cap C_{|I|}(u)} \text{ and } M_{B,I} = \sup_{u \in I} Q_{B \cap C_{|I|}(u)}.$$

**Lemma 3** *i) For every non trivial compact subinterval $I$ of $[0, 1]$*

*a)* $\Lambda(T^I) = C(\log \frac{1}{|I|} - \frac{1}{2}(1 - |I|))$; *b)* $\Lambda(B_I) = C(1 - |I|)$.

*ii) Fix $\beta > 0$. There exists $C_\beta > 0$ such that for every non trivial compact subinterval $I$ of $[0, 1]$ and $t \in I$,*

$$E\left[ Q_{|I|}(t) \left| \log |I|^{-1} \int_I Q_{B^I \cap C_{|I|}(u)} du \right|^\beta \right] \leq C_\beta.$$

*iii) For every non trivial compact subinterval $I$ of* $[0,1]$*:*

*a)* $E(M_{B^I,I}) \leq e^{\delta(E(\max(1,W))-1)}$*;*

*b) for* $q \in \mathbb{R}$ *define* $\gamma_I(q) = \mathbf{1}_{\{q<0\}}\, m^q_{B^I,I} + \mathbf{1}_{\{q\geq 0\}}\, M^q_{B^I,I}$*. For every compact subinterval $K$ of* $\mathbb{R}$*,*

$$E(\sup_{q\in K} \gamma_I(q)) \leq e^{\delta(E(\max(1,W^{\inf(K)}+W^{\sup(K)}))-1)}.$$

**Proof.** *i)* The verification is left to the reader.

*ii)* Define

$$T_1 = \left|\log\, |I|^{-1} \int_I Q_{B^I\cap C_{|I|}(t)\cap C_{|I|}(u)} Q_{(B^I\setminus C_{|I|}(t))\cap C_{|I|}(u)} du\right|^\beta.$$

One checks that $T_1 \leq 2^\beta \sum_{\varepsilon\in\{-1,1\}} (T_{2,\varepsilon} + T_{3,\varepsilon})$ where
$T_{2,\varepsilon} = |\log\, m_{B^I_\varepsilon\cap C_{|I|}(t)}|^\beta + |\log\, M_{B^I_\varepsilon\cap C_{|I|}(t)}|^\beta$ and
$T_{3,\varepsilon} = |\log\, m_{B^I_\varepsilon\setminus C_{|I|}(t)}|^\beta + |\log\, M_{B^I_\varepsilon\setminus C_{|I|}(t)}|^\beta$. So as

$$\begin{aligned} T &= Q_{|I|}(t) \left|\log\, |I|^{-1}\int_I Q_{B^I\cap C_{|I|}(u)} du\right|^\beta \\ &= |I|^{\delta(V-1)} Q_{C_{|I|}(t)\setminus B^I} Q_{B^I\cap C_{|I|}(t)} T_1, \end{aligned}$$

by taking account of the independences between variables and by the identity $|I|^{\delta(V-1)}E(Q_{C_{|I|}(t)}) = 1$ we obtain

$$E(T) \leq 2^\beta [\frac{1}{E(Q_{B^I})} E(Q_{B^I\cap C_{|I|}(t)}(T_{2,-1}+T_{2,1})) + E(T_{3,-1}+T_{3,1})]$$

where $(E(Q_{B^I}))^{-1} = e^{-\delta(1-|I|)(V-1)}$ is bounded independently of $I$ by *i)a)*.

It remains to show that $E(Q_{B^I\cap C_{|I|}(t)} T_{2,\varepsilon})$ and $E(T_{3,\varepsilon})$ are bounded independently of $I$ and $t\in I$ for $\varepsilon \in \{-1,1\}$.

We estimate $E(Q_{B^I\cap C_{|I|}(t)} T_{2,-1})$: conditionally to $\#S\cap B^I\cap C_{|I|}(t) = k \geq 1$, $S\cap B^I\cap C_{|I|}(t) = \{N_1,\dots N_k\}$, and conditionally to $\#S\cap B^I_{-1}\cap C_{|I|}(t) = 1 \leq l \leq k$ (if $k$ or $l=0$ then $T_{2,-1}=0$), we can assume that $N_1,\dots N_l \in B^I_{-1}$ and $t_{N_1} \leq \dots \leq t_{N_l}$. Then for every $u\in I$ we have $Q_{B^I_{-1}\cap C_{|I|}(t)\cap C_{|I|}(u)} \in \{\prod_{i=1}^j W_{N_i};\, 0\leq j\leq l\}$, so $T_{2,-1} \leq 2k^\beta \sum_{i=1}^k |\log\, W_{N_i}|^\beta$.

Thus for $\varepsilon\in\{-1,1\}$ and $k\geq 1$

$$\begin{aligned} E(Q_{B^I\cap C_{|I|}(t)} T_{2,\varepsilon} | \#S\cap B^I\cap C_{|I|}(t) &= k) \leq 2k^\beta (\prod_{j=1}^k W_{N_j}) \sum_{j=1}^k |\log\, W_{N_j}|^\beta \\ &\leq 2k^{\beta+1} E(W|\log\, W|^\beta)(V)^{k-1}. \end{aligned}$$

Similarly we obtain $E(T_{3,\varepsilon}|\#S\cap B^I\setminus C_{|I|}(t) = k)) \leq 2k^{\beta+1}E(|\log\, W|^\beta)$, and conclude as for Lemma 1*iv)* and by using the fact that $\Lambda(B^I\setminus C_{|I|}(t))$ and $\Lambda(B^I\cap C_{|I|}(t))$ are bounded independently of $I$ and $t\in I$.

*iii)a)* We obtain $E(M_{B^I,I}) \leq (e^{\Lambda(B^I_1)(E(\max(1,W))-1)})^2$ by using computations similar to those used in *ii)* and the fact that $M_{B^I,I} \leq M_{B^I_{-1},I}.M_{B^I_1,I}$ and $E(M_{B^I_{-1},I}.M_{B^I_1,I}) = (E(M_{B^I_1,I}))^2$.

*iii)b)* The verification is left to the reader.

**Lemma 4** *Fix $b$ an integer $\geq 2$ and $q \in \mathbb{R}$ such that $E(W^q) < \infty$. There exists $C_q = C_q(W) > 0$ such that for $n > m \geq 1$*

*i)a) $\mu^q_{b^{-n}}(I_a) \leq w_q(I_a) Y^q_{b^{n-m},I_a}$ with $w_q(I_a) = b^{-mq[1+\delta(V-1)]} Q^q_{T^{I_a}} \gamma_{I_a}(q)$ and $E(w_q(I_a)) \leq C_q b^{-m(\tau(q)+1)}$ for every $a \in A_m$;*

*b) $\sum_{a\in A_m} E(\mu^q_{b^{-n}}(I_a)) \leq C_q b^{-m\tau(q)} E(Y^q_{b^{n-m}})$;*

*c) if $q \geq 1$ then $E(Y^q_{b^n}) \geq b^{-m\tau(h)} e^{\frac{1}{2}(1-b^{-m})(\delta(V-E(W^q))} E(Y^q_{b^{n-m}})$.*

*ii)a) $\tilde{\mu}^q(C_a) \leq w_q(I_a) Y^q_{I_a}$ for every $a \in A_m$;*

*b) $\sum_{a\in A_m} E(\tilde{\mu}^q(C_a)) \leq C_q b^{-m\tau(q)} E(Y^q)$.*

We leave the proof to the reader which will use Theorem 2, Lemma 1 and 3, and for $i)c)$ the superadditivity of $x \geq 0 \mapsto x^q$ in (2) and the Jensen inequality in $E([\int_{I_a} Q_{B^{I_a}\cap C_{b^{-m}}(t)} \mu^{I_a}_{b^{m-n}}(dt)]^q | \cup_{0<\varepsilon\leq b^{-n}} F_\varepsilon)$.

**Lemma 5** *With the notations of the proof of Theorem 3i)*

*i) $E(f'_{a,n,m}(1^-)) = b^{-m}\left(-m\log(b)\,\tau'(1^-) + E(Y_{b^{n-m}} \log Y_{b^{n-m}})\right)$.*

*ii) There exists $C$ independent of $m$ such that for $a$ and $a'$ in $A_m$*

$$E\left(\mu^{\frac{1}{2}}_{b^{-n}}(I_a)\mu^{\frac{1}{2}}_{b^{-n}}(I_{a'})\right) \leq C\,.b^{-m} E(Y^{\frac{1}{2}}_{b^{n-m},I_a} Y^{\frac{1}{2}}_{b^{n-m},I_{a'}}).$$

**Proof.** $i)$ Differentiate $f_{a,n,m}$ at $1^-$ yields $E(f'_{a,n,m}(1^-)) = T_1 + T_2 + T_3$ with $T_1 = -m\log(b)[\delta(V-1)]E(f_{a,n,m}(1)) = -b^{-m} m \log(b)\delta(V-1)$,

$$\begin{aligned} T_2 &= b^{-m\delta(V-1)} E(\int_{I_a} E(Q_{C_{b^{-m}}(t)} \log(Q_{C_{b^{-m}}(t)}))\mu_{b^{m-n},I_a}(dt)) \\ &= b^{-m} m \log(b) \delta E(W\log W) \end{aligned}$$

by Lemma 1$ii)$, and

$$\begin{aligned} T_3 &= b^{-m\delta(V-1)} E([\log(Y_{b^{n-m},I_a}) - m\log(b)] \int_{I_a} E(Q_{C_{b^{-m}}(t)})\mu_{b^{m-n},I_a}(dt)) \\ &= b^{-m}[E(Y_{b^{n-m}} \log Y_{b^{n-m}}) - m\log(b)]. \end{aligned}$$

As $\tau'(1) = 1 + \delta(V-1) - \delta E(W\log W)$, we have the conclusion.

$ii)$ Consequence of Lemma 4$i)a)$ and the Cauchy-Schwarz inequality.

**Lemma 6** *Under the assumptions made in the proof of Theorem 3ii) for every $t \in [0,1]$, $P_t(\limsup_{n\to\infty} Y_{b^n} = \infty) = 1$.*

**Proof.** For $n \geq 1$, denote by $I_n(t)$ the $b$-adic subinterval of $[0,1]$ of the $n^{\text{th}}$ generation which contains $t$. One has $Y_{b^n} = \|\mu_{b^{-n}}\| \geq \mu_{b^{-n}}(I_n(t))$ so it is enough to show that $P_t(\limsup_{n\to\infty} \mu_{b^{-n}}(I_n(t)) = \infty) = 1$.

Define $R_{1,n}(t) = -\log Q_{C_{b^{-n}}(t)\setminus T^{I_n(t)}}$ and

$$R_{2,n}(t) = \log b^n \int_{I_n(t)} Q_{B^{I_n(t)}\cap C_{b^{-n}}(u)} du.$$

We have

$$\log\,\mu_n(I_n(t)) = \log\,Q_{b^{-n}}(t)\ - n\log{(b)} + R_{1,n}(t) + R_{2,n}(t)$$

so the conclusion will be immediate after showing that

1) $P_t(\limsup_{n\to\infty}\frac{\log\,Q_{b-n}(t)\,-n\log{(b)}}{(n\log\log n)^{1/2}}>0)=1$: for $k\geq 1$ define $X_k(\omega)=\log{(Q_{b^{-k}}(t)/Q_{b^{-(k-1)}}(t))}-\log{(b)}$. By construction the $X_k$'s are i.i.d. with respect to $P_t$ and by Lemma $1i)ii)iv)(q=1,\,\beta=2)$ applied with $B=C_{b^{-1}}(t)$

$$E_{P_t}(X_k) = E_{P_t}(X_1) = E(Q_{b^{-1}}(t)\log\,Q_{b^{-1}}(t)) - \log{(b)} = -\log(b)\tau'(1^-) = 0$$

and $E_{P_t}(X_k^2)<\infty$. Moreover $E_{P_t}(X_k^2)>0$, otherwise $P(Q_{C_{b^{-1}}(t)}=1)=1$, so $W=1$ a. s. and this contradicts $\tau'(1^-)=0$. So the result is a consequence of the law of the iterated logarithm.

2) $P_t(\lim_{n\to\infty}\frac{|R_{1,n}(t)|+|R_{2,n}(t)|}{(n\log\log n)^{1/2}}=0)=1$: it is easily seen that it is enough to prove that for $i\in\{1,2\}$, $\sup_{n\geq 1}E_{P_t}(|R_{i,n}(t)|^{2+\gamma})<\infty$.

$$\begin{aligned}E_{P_t}(|R_{1,n}(t)|^{2+\gamma}) &= E(Q_{b^{-n}}(t)|\log\,Q_{C_{b^{-n}}(t)\setminus T^{I_n(t)}}|^{2+\gamma})\\ &= b^{-n\delta(V-1)}E(Q_{T^{I_n(t)}})\\ &\qquad .E(Q_{C_{b^{-n}}(t)\setminus T^{I_n(t)}}|\log\,Q_{C_{b^{-n}}(t)\setminus T^{I_n(t)}}|^{2+\gamma})\end{aligned}$$

and $E_{P_t}(|R_{2,n}(t)|^{2+\gamma})=E(Q_{b^{-n}}(t)|\log\,b^n\int_{I_n(t)}Q_{B^{I_n(t)}\cap C_{b^{-n}}(u)}du|^{2+\gamma})$.

So by Lemma $1i)$ and $1iv)$ applied respectively with $B=T^{I_n(t)}$ and $B=C_{b^{-n}}(t)\setminus T^{I_n(t)}$ and Lemma $3ii)$, these expectations are uniformly bounded over $\mathbb{N}^*$.

Now we are under the assumptions and notations of Theorem 6.

Fix an integer $b\geq 2$ and for $q\in J'$ let $\tilde{\mu}_q$ be the measure constructed as $\tilde{\mu}$ on $\partial A$, but with the $W_{q,M}$'s: $\tilde{\mu}_q$ is almost surely the weak limit of $(\tilde{\mu}_{q,n})_{n\geq 1}$ with

$$\frac{d\tilde{\mu}_{q,n}}{d\tilde{\ell}}(t) = b^{-n\delta(E(W^q)-1)}\,Q^q_{C_{b^{-n}}(\pi(t))}.$$

The total mass of $\tilde{\mu}_q$ is denoted by $Y_q$ and for every $a\in A$, $Y_{q,I_a}$ denotes $b^{|a|}\|\tilde{\mu}_q^{I_a}\|$ and is a copie of $Y_q$.

**Lemma 7** *i) With probability one, for all* $a\in A=\bigcup_{m\geq 0}A_m$, *the sequence of functions* $(q\mapsto\tilde{\mu}_{q,n}(C_a))_{n\geq 1}$ *converges uniformly on the compact subsets of* $J'$ *to* $q\mapsto\tilde{\mu}_q(C_a)$, *which is analytic and positive.*

*So the measures* $\tilde{\mu}_q$, $q\in J'$, *are defined with probability one simultaneously and have* $\partial A$ *as support.*

*ii) With probability one, for every* $q\in J'$, *for* $\tilde{\mu}_q$*-almost every* $t\in\partial A$ *(with* $t=t_1\dots t_n\dots$*)*

$$\lim_{n\to\infty}\frac{\log\,\tilde{\mu}_q(C_{t_1\dots t_n})}{-n\log\,b}\geq\tau^*(\tau'(q)).$$

**Proof**. *i*) Fix $a \in A$, $|a| = m$. In a deterministic complex neighbourhood of $J'$, for $n > m$, $q \mapsto \tilde{\mu}_{q,n}(C_a) = \sum_{a' \in A_{n-m}} \tilde{\mu}_{q,n}(C_{aa'})$ has the analytic extension

$$z \mapsto \psi_n(z) = \sum_{a' \in A_{n-m}} b^{-n\delta(E(W^z)-1)} \int_{I_{aa'}} Q^z_{C_{b^{-n}}(t)} \, dt.$$

Assume for instance the following assertion: $(\mathcal{A}_3)$ for every compact subinterval $K$ of $J'$, there exists $h > 1$, $c < 0$, $C > 0$ and a complex neighbourhood $U$ of $K$ such that for all $n \geq 1$, $\sup_{z \in U} E(|\psi_{n+1}(z) - \psi_n(z)|^h) \leq C b^{nc}$.

Then a similar using of the Cauchy formula than in [Bi] gives the uniform convergence of $\psi_n$ on the compact subsets of $U$, and so the one of $(q \mapsto \tilde{\mu}_{q,n}(C_a))_{n\geq 1}$ on the compact subsets of $J'$ to $q \mapsto \tilde{\mu}_q(C_a)$, which is analytic and non-negative. This happens simultaneously for all the $a$'s in $A$ since $A$ is countable, and so the measures $\tilde{\mu}_q$ are defined simultaneously. Moreover, the using of the Cauchy formula yields also for every compact subinterval $K$ of $J'$ an $h > 1$ such that $\sup_{q\in K} E(\|\tilde{\mu}_q\|^h) + \sup_{q \in K} E(|\frac{d}{dq}\|\tilde{\mu}_q\||^h) < \infty$.

To see that $q \mapsto \tilde{\mu}_q(C_a)$ is almost surely positive on $J'$ for every $a \in A$, that is the support of the $\tilde{\mu}_q$'s is $\partial A$, it is enough to show that it is the case on every non trivial compact subinterval $[\alpha, \beta]$ of $J'$. By Theorem 2 and equation (3), since $W > 0$, this is equivalent to show that $q \mapsto Y_q = \|\tilde{\mu}_q\|$ is positive on $[\alpha, \beta]$. For every compact subinterval $K$ of $[\alpha, \beta]$ and $a \in A$ denote by $N_K(a)$ the event $\{\exists\, q \in K; Y_{q,I_a} = 0\}$. By using (4) and choosing $m > 1$ and $(a, a') \in A_m^2$ such that $T_{I_a} \cap T_{I_{a'}} = \emptyset$, we have $N_K(\epsilon) \subset N_K(a) \cap N_K(a')$, where $N_K(a)$ and $N_K(a')$ are independent and with the same probability as $N_K(\epsilon)$. So $P(N_K(\epsilon)) \in \{0, 1\}$. Then if $P(N_{[\alpha,\beta]}(\epsilon)) = 1$, one constructs by dichotomy a decreasing sequence of intervals $K_n$ of length $\frac{\beta-\alpha}{2^n}$ such that $P(N_{K_n}(\epsilon)) = 1$. Then $P(N_{\{q_0\}}(\epsilon)) = 1$ where $\{q_0\} = \cap_{n\geq 1} K_n$, so $P(Y_{q_0} = 0) = 1$, in contradiction with the fact that $q_0 \in J'$.

Now we prove assertion $(\mathcal{A}_3)$: fix $K$ a compact subinterval of $J'$. For every $z$ in a complex neighbourhood of $K$
$\psi_{n+1}(z) - \psi_n(z) =$

$$\sum_{a' \in A_{n-m}} \int_{I_{aa'}} b^{-n\delta(E(W^z)-1)} Q^z_{C_{b^{-n}}(t)} [b^{-\delta(E(W^z)-1)} Q^z_{C_{b^{-(n+1)}}(t) \setminus C_{b^{-n}}(t)} - 1] \, dt.$$

We number from 0 to $b^{n-m} - 1$ the $b$-adic intervals of the $n^{\text{th}}$ generation that appear in the previous sum as they appear on the real line, we denote them by the $J_k$'s, $0 \leq k < b^{n-m}$, and for $t \in \cup_{k=0}^{b^{n-m}-1} J_k$ we denote $b^{-n\delta(E(W^z)-1)} Q^z_{C_{b^{-n}}(t)}$ by $u_n(z,t)$ and $[b^{-\delta(E(W^z)-1)} Q^z_{C_{b^{-(n+1)}}(t) \setminus C_{b^{-n}}(t)} - 1]$ by $v_n(z,t)$. Thus
$\psi_{n+1}(z) - \psi_n(z) = \int_{J_0} \Gamma_0(z,t) + \Gamma_1(z,t) + \Gamma_2(z,t) \, dt$ where for $i \in \{0, 1, 2\}$ and $t \in J_0$, $\Gamma_i(z,t) = \sum_{0 \leq 3k+i < b^{n-m}} u_n(z, t + \frac{3k+i}{b^{n-m}}) \; v_n(z, t + \frac{3k+i}{b^{n-m}})$, and for $h > 1$, $E(|\psi_{n+1}(z) - \psi_n(z)|^h)$

$$\leq 3^{h-1} |J_0|^{h-1} \int_{J_0} [E(|\Gamma_0(z,t)|^h) + E(|\Gamma_1(z,t)|^h) + E(|\Gamma_2(z,t)|^h)] \, dt.$$

For a fixed $t \in J_0$, in $\Gamma_i(z,t)$, the $v_n(z, t + \frac{3k+i}{b^{n-m}})$'s are mutually independent since the $T_{J_{3k+i}}$'s are pairwise disjoint, and by construction they are of mean 0, and are independent of the $u_n(z, t +$

$\frac{3k+i}{b^{n-m}}$)'s. So, if $1 < h \leq 2$, the complex version of a result by Von Bahr and Esseen used in [Bi] (see [Bi] Lemma 1) yields

$$E(|\Gamma_i(z,t)|^h) \leq 2^h \sum_{0\leq 3k+i<b^{n-m}} E(|u_n(z,t+\frac{3k+i}{b^{n-m}})|^h)E(|v_n(z,t+\frac{3k+i}{b^{n-m}})|^h).$$

By the invariance property of $\Lambda$, Lemma 1$i$) applied with $|W^z|$ instead of $W$ and $B = C_{b^{-(n+1)}}(t) \setminus C_{b^{-n}}(t)$ or $C_{b^{-n}}(t)$, and by defining $C_h = 12^h b^{-(m+1)(1-h)}$ we obtain finally

$$E(|\psi_{n+1}(z) - \psi_n(z)|^h) \leq C_h b^{(n+1)\{1-h(1+\delta[E(\Re(W^z))-1])+\delta[E(|W^z|^h)-1]\}}.$$

For $z = q \in J'$, $1 - h(1+\delta[E(W^q)-1]) + \delta[E(W^{qh})-1] = h\tau(q) - \tau(hq)$, so by the concavity of $\tau$ and the fact that on the compact $K$, $q \mapsto \tau'(q)q - \tau(q)$ is positive, we can choose $h \in ]1,2]$ to have $\sup_{q\in K} h\tau(q) - \tau(hq) < 0$ and then a complex neighbourhood $U$ of $K$ such that

$$c = \sup_{z\in U} 1 - h(1+\delta[E(\Re(W^z))-1]) + \delta[E(|W^z|^h)-1] < 0.$$

$ii$) For $q \in J', \varepsilon > 0$ and $n \geq 1$, define

$$E_{q,n,\varepsilon} = \{t \in \partial A; \frac{\log\, \tilde{\mu}_q(C_{t_1\dots t_n})}{-n\log\, b} \leq \tau^*(\tau'(q)) - \varepsilon\}.$$

It is enough to show that for every compact subinterval $K$ of $J'$ and $\varepsilon > 0$, a. s. for every $q \in K$, $\sum_{n\geq 1} \tilde{\mu}_q(E_{q,n,\varepsilon}) < \infty$.

Fix such a $K$ and $\varepsilon$. For every $\eta > 0$, $E_{q,n,\varepsilon}$ is by definition the union of the $C_a$'s, $a \in A_n$, such that $\tilde{\mu}_q^\eta(C_a) b^{n\eta(\tau^*(\tau'(q))-\varepsilon)} \geq 1$, so

$$\tilde{\mu}_q(E_{q,n,\varepsilon}) \leq \sum_{a\in A_n} \tilde{\mu}_q^{1+\eta}(C_a) b^{n\eta(\tau^*(\tau'(q))-\varepsilon)} \leq f_{n,\eta,\varepsilon}(q) \tag{7}$$

with (by using Lemma 4$ii$)$a$)) $f_{n,\eta,\varepsilon}(q)$

$$= b^{n\eta(\tau^*(\tau'(q))-\varepsilon)} \sum_{a\in A_n} [\sup_{q'\in K} \gamma_a((1+\eta)q')]\, b^{-n(1+\eta)(1+\delta(E(W^q)-1))} Q_{T^{I_a}}^{q(1+\eta)} Y_{q,I_a}^{1+\eta}.$$

We shall see that if $\eta$ is small enough, there exist $C_K > 0$ and $C'_K > 0$ such that for every $n \geq 1$, $\sup_{q\in K} E(f_{n,\eta,\varepsilon}(q)) \leq C_K b^{-nC'_K}$ and $\sup_{q\in K} E(|\frac{d}{dq} f_{n,\eta,\varepsilon}(q)|) < C_K n b^{-nC'_K}$. One checks that this implies that the series $\sum_{n\geq 1} f_{n,\eta,\varepsilon}(q)$ converge uniformly on $K$ and yields the conclusion by (7).

Define

$$b_n(q) = b^{n\eta[\tau^*(\tau'(q))-\varepsilon]-n(1+\eta)[1+\delta(E(W^q)-1)]}$$

and

$$\beta_a(q) = [\sup_{q'\in K} \gamma_a((1+\eta)q')] \left|\eta + \tau''(q) - (1+\eta)\delta[E(W^q \log W)]\right| \log b.$$

We have

$$\begin{aligned} |\frac{d}{dq} f_{n,\eta,\varepsilon}(q)| &\leq \sum_{a\in A_n} (1+\beta_a(q)) b_n(q) \\ &\times \left[ \begin{array}{c} (n + (1+\eta)|\log Q_{T^{I_a}}|)\, Q_{T^{I_a}}^{q(1+\eta)} Y_{q,I_a}^{1+\eta} \\ +(1+\eta) Q_{T^{I_a}}^{q(1+\eta)} |\frac{d}{dq} Y_{q,I_a}| Y_{q,I_a}^{\eta} \end{array} \right]. \end{aligned}$$

By a remark made in the proof of $i$), if $\eta$ is small enough, $\sup_{q\in K} E(Y_q^{1+\eta}) + \sup_{q\in K} E(|\frac{d}{dq} Y_q| Y_q^{\eta}) < \infty$ and by Lemma 1 and 3 $\sup_{q\in K} E(1+\beta_a(q))$, which do not depend on $a$, is finite. Thus by taking account of the independences, the invariance of $\Lambda$, and by using Lemma 1$i)iii)$ with $B = T^{I_a}$ and the fact that $\sup_{q\in K} E(W^{q(1+\eta)}|\log W|) < \infty$ if $\eta$ is small, we obtain for $\eta$ small enough a constant $C_K$ such that for every $n \geq 1$ and $q \in K$

$$E(|\frac{d}{dq} f_{n,\eta,\varepsilon}(q)|) \leq C_K n b^{n\{\eta[\tau^*(\tau'(q))-\varepsilon]-\tau((1+\eta)q)+(1+\eta)\tau(q)\}}$$

and a study of function shows that if $\eta$ is small enough then
$\sup_{q\in K} \eta[\tau^*(\tau'(q)) - \varepsilon] - \tau((1+\eta)q) + (1+\eta)\tau(q) < 0$.

We leave the simpler estimate of $\sup_{q\in K} E(f_{n,\eta,\varepsilon}(q))$ to the reader.

**Acknowlegments.** The authors thank Jacques Peyrière and Rudolf Riedi for valuable discussions.

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