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COWLES FOUNDATION DISCUSSION PAPER NO. 1126R

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INCOMPLETE DERIVATIVE MARKETS AND PORTFOLIO INSURANCE

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Revised June 1997

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Abstract

We present necessary and sufficient conditions on the asset span of incomplete derivative markets for insuring marketed portfolios. If the asset span is finite dimensional there exists a polynomial-time algorithm for deciding if every marketed portfolio is insurable, moreover this algorithm computes the minimum cost insurance portfolio.

In addition, we extend the Cox–Leland characterization of optimal portfolio insurance in complete derivative markets to asset spans of incomplete derivative markets where every marketed portfolio is insurable.

1 Portfolio Insurance

In Arrow's *The Role of Securities in the Optimal Allocation of Risk-bearing*, asset markets are complete and the state space is finite. Arrow assumes the existence of marketed securities e_j that pay one unit if state j occurs and zero otherwise. This family of assets, subsequently called Arrow securities, is a basis for the asset span, the linear vector space spanned by the marketed securities. In fact, it is the unique (up to a normalization) basis where every limited liability asset has a unique representation as a nonnegative linear combination of basis vectors.

Ross, in *Options and Efficiency*, also considers marketed subspaces spanned by a finite number of securities, where the state space is finite. In his paper, markets may be incomplete, i.e., the dimension of the asset span can be less than the number of states, and he does not assume that Arrow securities are marketed. By augmenting

*The authors are pleased to acknowledge comments of Peter Bossaerts, and Philippe Henrotte. The research of C. D. Aliprantis was partially supported by the 1995 PENED Program of the Ministry of Industry, Energy and Technology of Greece and by the NATO Collaborative Research Grant #941059. Roko Aliprantis also expresses his deep appreciation for the hospitality provided by the Department of Economics and the Center for Analytic Economics at Cornell University and the Division of Humanities and Social Sciences of the California Institute of Technology where parts of this paper were written during his sabbatical leave (January–June, 1996).

†This is a revision of CFDP 1126, entitled "Hedging with Derivatives in Incomplete Markets," May 1996.

the asset span with (European) put options on the marketed securities, Ross is able to complete the market for derivative assets, assets whose payoffs depend only on the payoffs of the underlying marketed securities. Such assets are called (generalized) options by Ross. His construction implies that markets for derivatives are complete if and only if put options on every marketed security are in the asset span.

Investors often wish to insure themselves against the value of their portfolios falling below a certain value tomorrow. One means of acquiring portfolio insurance as a hedge against falling stock prices is to hold in conjunction with the risky asset s, a put option on s. If the striking price of the put option is k, then the portfolio consisting of the asset and the put will have value at least k tomorrow. How can a rational investor hedge the value of her portfolio against the uncertainty of tomorrow's stock prices if not every put option is marketed?

This is the question we address in this paper. To that end, we restate Ross's main result: *Formally, options markets are complete if and only if the asset span is a Riesz subspace of the Riesz space of all state-contingent claims.*

Riesz spaces (vector lattices) arise naturally as models of derivatives where the lattice operation of max ($\vee$) in conjunction with the standard vector space operations of forming marketed portfolios is used to represent (European) put options. Recalling the canonical example: the state space Ω is the set of possible prices tomorrow of a risky asset s; the stock's payoff tomorrow is its price, hence s is represented as the identify function on the state space; the riskless asset b is represented as the constant function $\mathbf{1}$ on the state space. In this model, a put option on s with striking price k is the state-contingent claim $\max\{kb - s, 0\} = (kb - s) \vee 0$, denoted $(kb - s)^+$.

If X is the Riesz space of state-contingent claims on some state space Ω, then X^+, the positive cone of X, is the family of limited liability assets. This cone induces a partial order $\geq$ on X, where $x \geq y$ if and only if $x - y \in X^+$, i.e., $x(\omega) \geq y(\omega)$ for every state of the world ω. Under this ordering of payoff dominance, $x \vee y$ is the least upper bound (l.u.b.) of $z \in X$ such that $x \leq z$ and $y \leq z$. If M is a linear subspace of X and $x, y \in M$ implies $x \vee y \in M$, then M is said to be Riesz subspace of the Riesz space X. The set $\mathcal{L}_V(M)$ of derivative assets defined on M, is the smallest Riesz subspace of X containing M (see Brown and Ross [5], Theorem 1]. Derivative markets are complete if $M = \mathcal{L}_V(M)$ and incomplete otherwise. If the state space is finite and $b \in M$, then derivatives markets are incomplete if and only if there exists a marketed asset s and a striking price k such that $(kb - s)^+ \notin M$, i.e., portfolio insurance is not attainable for all portfolios by purchasing marketed puts.

The natural state space for investors holding portfolios is the family of possible equilibrium prices of marketed portfolios tomorrow. Of course, in equilibrium only arbitrage-free prices are possible. Market prices are arbitrage-free if they assign positive value to every positive portfolio, where a portfolio is positive if it has positive value under every arbitrage-free price system. This self-referential definition, due to Henrotte [10], is not circular if asset markets are defined by a payoff matrix where the entries (possible prices for marketed assets tomorrow) are assumed known, as in the binomial pricing model of Cox, Ross and Rubenstein [6]. An equivalent assumption is the specification of a cone in the space of portfolios, i.e., the cone of positive portfolios. In this case, the dual cone in price space is the cone of arbitrage-free prices.

We can now state our main result: *The market for portfolio insurance is complete if and only if the space of portfolios is a Riesz space of price-contingent contracts, i.e., functions on the state space of arbitrage-free price vectors.* The important notion of price-contingent contracts is due to Kurz [11].

This proposition which is independent of the dimension or the topology of the space of portfolios is an immediate consequence of the remarkable **Choquet–Kendall Theorem:** If V is a vector space ordered by a generating cone V^+ with a base B, then V is a Riesz space if and only if B is a "simplex." (See Peressini [14] for a discussion.)

If the market for portfolio insurance is complete then we denote a put on a portfolio θ with striking price k as $(kb-\theta)^{++}$, where b is the portfolio which assigns value 1 to every arbitrage-free price vector and $(kb-\theta)^{++}$ is the point-wise maximum of the portfolio $kb-\theta$ and 0 over the state-space of arbitrage-free price vectors. Then the portfolio $\theta+(kb-\theta)^{++}$ not only has value at least k under every arbitrage-free price vector but also it is the minimum cost insurance portfolio, i.e., if $\theta+y$ has value at least k tomorrow then $(kb-\theta)^{++}$ costs no more than y.

Since these two properties characterize put option in markets for state contingent claims as considered by Ross [15], how do asset spans with complete markets for portfolio insurance, *termed lattice subspaces in this paper*, differ from complete derivative markets, i.e., Riesz subspaces?

Derivative markets are complete if and only if the space of portfolios is a Riesz subspace of the Riesz space of price-contingent contracts. Portfolio insurance markets are complete if and only if the space of portfolios is a Riesz space of price-contingent contracts (but not necessarily a Riesz subspace of price-contingent contracts).

Of course, in general, M, the asset span, need not be a lattice-subspace. In addition to our main result, we present a number of sufficient conditions for M to be a lattice-subspace. The most striking such condition is that if the marketed subspace is spanned by two linearly independent limited liability assets, say a stock and a bond, then M is always a lattice-subspace.

If the asset span is finite-dimensional, then (as shown by Choquet and Kendall) a necessary and sufficient condition for M to be a lattice-subspace is the existence of a Yudin basis for M, i.e., a basis of limited liability assets such that every marketed limited liability asset has a unique representation as a nonnegative linear combination of basis vectors. The notion of Yudin basis is a generalization to incomplete asset markets of a basis of Arrow securities for complete securities markets. Moreover, if the state space is finite then Abramovich, Aliprantis and Polyrakis present an algorithm for deciding if the asset span M has a Yudin basis and computing if it exists.

We present another algorithm for deciding and computing the Yudin basis. Another consequence of the Choquet–Kendall theorem for ℓ-dimensional asset spans, equivalent to the existence of a Yudin basis, is that the set of normalized arbitrage-free price vectors is a ℓ-dimensional simplex in the ℓ-dimensional linear space of asset prices. An affinely independent basis for this simplex is called a family of *fundamental states* in Abramovich, Aliprantis and Polyrakis [1].

Hence to decide if the asset span is a lattice subspace, we need only decide if the set of normalized arbitrage-free price vectors is an ℓ-dimensional simplex. The existence of a polynomial-time decision procedure for this problem is given by a theorem of Edmonds, Lovász and Pulleybank [8].

Theorem [Edmonds, Lovász and Pulleybank]. *There exists a polynomial-time algorithm that for any polytope, P, defined as the convex hull of a given finite set of vectors, determines the affine hull of P. Specifically the algorithm finds affinely independent vertices $v_0, v_1, ... v_\ell$ of P such that* $\mathrm{aff}(P) = \mathrm{aff}(\{v_0, v_1, ..., v_\ell\})$.

It is important to notice that the Yudin basis for the asset span is simply the family of portfolios dual to the affinely independent basis of the simplex, i.e., "fundamental states," of normalized arbitrage-free price vectors.

In addition to finite state-space models, we consider $C[0,1]$, the canonical model of a space of state-contingent claims over an infinite-dimensional state space (see Brown and Ross [5], Theorem 1, p. 9) for a characterization of asset spaces isomorphic to $C[0,1]$. In their model the riskless asset b is the constant function $\mathbf{1}$, the risky asset s is the identity function $i_d(t) = t$ and the asset span M is the vector subspace of $C[0,1]$ spanned by $\{b, s\}$. In Example 3.1, we compute the Yudin basis for the space of marketed portfolios, the space of arbitrage-free prices and the asset span. In this model, the space of derivative assets, i.e., the Riesz subspace of $C[0,1]$ generated by M, is spanned by the stock, the bond and put options on the stock $(kb-s)^+$, a result originally due to Cox and Rubenstein [7].

Is it rational, i.e., optimal, for investors to purchase portfolio insurance?

If derivative markets are dynamically complete and the risky asset follows a geometric Brownian motion then Cox and Leland prove (we quote from Grossman and Vila [9]) "for a non-decreasing concave utility of final wealth ... an optimal policy with a nonnegativity constraint on wealth is a combination of the policy that would be optimal without the constraint (for a smaller initial wealth) and a policy that duplicates a European put on the final value of the unconstrained policy."

Grossman and Vila give an elementary proof of the Cox–Leland characterization of optimal portfolio insurance for a two-period, finite state space investment model where derivative markets are complete. Using the argument of Grossman and Vila, we extend the Cox–Leland characterization to incomplete derivative markets where the asset span is a lattice subspace.

It will be convenient to carry out our proof in the space of portfolios, which by assumption is a finite dimensional Riesz space Θ, with positive cone Θ^+.

The investor has an initial wealth w_0 and a smooth, concave and monotone (indirect) utility function over marketed portfolios, denoted $V(\theta)$. $V(\theta) = \int_{p\in\Delta} u(\theta \cdot p)dp$, where $u : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing, strictly concave and C^2. If asset prices in the first period are denoted as q, then the unconstrained optimal investment problem is:

$$\left.\begin{array}{l}\max_{\theta\geq 0} V(\theta) \\ \text{s.t. } q\cdot\theta = w_0\end{array}\right\} \Pi(w_0)$$

Let $\theta(w_0)$ denote the unique optimal solution to $\Pi(w_0)$. If $(kb-\theta(w_0))^{++}$ is a put on $\theta(w_0)$ with striking price k, where $b(p)=1$ for all $p\in\Delta$, then $P_k(w_0) = q\cdot(kb-\theta(w_0))^{++}$, the price in period 0 of the put option.

Now consider the constrained investment problem:

$$\left.\begin{array}{l}\max_{\theta\geq 0} V(\theta) \\ \text{s.t. } q\cdot\theta = w_0 \text{ and} \\ \min_{p\in\Delta}\theta\cdot p \geq k\end{array}\right\} \Pi_k(w_0)$$

If this problem is feasible, i.e., $(q\cdot e)k\leq w_0$, where $e=(1,1,...,1)$ and $q\cdot e=1$ then it has a unique optimal solution $\theta_k(w_0)$.

Following Grossman and Vila, our argument has two parts. First for any w_0 s.t. $(q\cdot e)k\leq w_0$ there exists $\bar{w}_0$ s.t. $P_k(\bar{w}_0)+\bar{w}_0=w_0$. This follows from the Intermediate Value theorem, since $P_k(w)+w$ is a continuous function of w which goes to infinity as w goes to infinity and $P_k(0)+0=(q\cdot e)k\leq w_0$.

Let $\theta^*=\theta(\bar{w}_0)+(kb-\theta(\bar{w}_0))^{++}$, then $q\cdot\theta^*=q\cdot\theta(\bar{w}_0)+q\cdot(kb-\theta(\bar{w}_0))^{++}=\bar{w}_0+P_k(\bar{w}_0)=w_0$ and $\theta^*(p)\geq k$ for all $p\in\Delta$, i.e., θ^* is feasible in $\Pi_k(\bar{w}_0)$. To complete the argument that $\theta^*=\theta_k(w_0)$, the optimal solution to $\Pi_k(w_0)$, we observe that:

(a) $DV(\theta(\bar{w}_0))\leq\lambda q$ for some $\lambda>0$

(b) $\partial^2 V/\partial\theta_i\partial\theta_j\leq 0$ for all i, j and all $\theta\in\Theta^+$

(c) $DV(\theta^*)=DV(\theta(\bar{w}_0))+R_1$, where
$R_1=\int_0^1 D^2V(\theta(\bar{w}_0)+t(kb-\theta(w_0))^{++})(kb-\theta(w_0))^{++}dt$

(d) $V(\theta)\leq V(\theta^*)+DV(\theta^*)\cdot(\theta-\theta^*)$ for all $\theta\in\Theta^+$.

It follows from (a), (b), and (c) that $DV(\theta^*)\leq\lambda q$. Call this implication (e), then the feasibility of θ^*, (d) and (e) imply that $V(\theta)\leq V(\theta^*)$ for all θ feasible in $\Pi_k(\bar{w}_0)$. This completes the proof.

2 Yudin Basis and Lattice-Subspaces

As mentioned in the previous section, this research utilizes the notions of Yudin basis and lattice-subspace. We shall discuss here briefly the basic properties of these concepts. For details and proofs we refer the reader to [1] and [3]. We follow the notation and terminology of the monograph [12].

An order relation $\geq$ on a vector space X is said to be *a linear order* if, in addition to being reflexive, antisymmetric and transitive, it is also compatible with the algebraic structure of X in the sense that $x \geq y$ implies:

a. $x + z \geq y + z$ for each z, and

b. $\alpha x \geq \alpha y$ for all $\alpha \geq 0$.

A vector space equipped with a linear order is called a *partially ordered vector space* or simply an *ordered vector space.* In a partially ordered vector space $(X, \geq)$ any vector satisfying $x \geq 0$ is known as a *positive vector* and the collection of all positive vectors $X^+ = \{x \in X : x \geq 0\}$ is referred to as the *positive cone* of X. A linear operator $T : X \to Y$ between two partially ordered vector spaces is *positive* if $Tx \geq 0$ for each $x \geq 0$ (i.e., if $T(X^+) \subseteq Y^+$).

A subset C of a vector space X is said to be a *cone* if:

1. $C + C \subseteq C$,
2. $\lambda C \subseteq C$ for each $\lambda \geq 0$, and
3. $C \cap (-C) = \{0\}$.

Notice that (1) and (2) guarantee that every cone is a convex set. An arbitrary cone C of a vector space X defines a linear order on X by letting $x \geq y$ if $x - y \in C$, in which case $X^+ = C$. On the other hand, if $(X, \geq)$ is an ordered vector space, then X^+ is a cone in the above sense. These show that the linear order relations and cones correspond in one-to-one fashion.

A partially ordered vector space X is said to be a *vector lattice* or a *Riesz space* if it is also a lattice. That is, a partially ordered vector space X is a vector lattice if for every pair of vectors $x, y \in X$ their *supremum* (least upper bound) and *infimum* (greatest lower bound) exist in X. Any cone of a vector space that makes it a Riesz space will be referred to as a *lattice cone.* As usual, the supremum and infimum of a pair of vectors x, y in a vector lattice are denoted by $x \vee y$ and $x \wedge y$, respectively. In a vector lattice, the elements $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$ are called the *positive part, negative part,* and *absolute value* of x. We always have the identities

$$x = x^+ - x^- \quad \text{and} \quad |x| = x^+ + x^-.$$

The necessity of augmenting the space of derivative assets, by marketing put options on marketed securities, to complete the options markets is a consequence of the results stated below.

Theorem 2.1. *A partially ordered space X is a Riesz space if and only if x^+ exists for every $x \in X$.*

For a proof see [12], Theorem 1.5, p. 55. Ross's insight (in part) was to observe that in markets for state-contingent claims over a finite state space, a sufficient condition for completeness of the space of options is the marketing of $(kb - s)^+$ for each marketed asset s and every striking price k.

A vector subspace Y of a vector lattice X is said to be:

1. a *vector sublattice* if for each $x, y \in Y$ we have $x \vee y$ and $x \wedge y$ in Y; and

2. an *ideal* if $|y| \leq |x|$ and $x \in X$ imply $y \in Y$.

An ideal is always a vector sublattice but a vector sublattice need not be an ideal. A positive vector u in a vector lattice Y is said to be an *order unit* if the ideal it generates is all of Y, or equivalently, if for each $y \in Y$ there exists some $\lambda > 0$ such that $\lambda u \geq y$.

Definition 2.2. *A vector subspace Y of a partially ordered vector space X is said to be a lattice-subspace if Y under the induced ordering from X is a vector lattice in its own right. That is, Y is a lattice-subspace if for every $x, y \in Y$ the least upper bound of the set $\{x, y\}$ exists in Y when ordered by the cone $Y^+ = Y \cap X^+$.*

If Y is a lattice-subspace of X, then we shall denote the supremum and infimum of the set $\{x, y\} \subseteq Y$ in Y by $x \curlyvee y$ and $x \curlywedge y$, respectively. In particular, we have $x^{++} = x \curlyvee 0$.

If X is a Riesz space, then every Riesz subspace of X is automatically a lattice-subspace but a lattice-subspace need not be a Riesz subspace.

A normed space which is also a partially ordered vector space is a *partially ordered normed space.* A norm $||\cdot||$ on a vector lattice is said to be a *lattice norm* if $|x| \leq |y|$ implies $||x|| \leq ||y||$. A *normed vector lattice* is a vector lattice equipped with a lattice norm. A complete normed vector lattice is called a *Banach lattice.* A classical example of a Banach lattice is $C(\Omega)$, the vector space of all continuous real-functions on a compact topological space Ω, equipped with the sup norm. All Riesz spaces in this paper will be $C(\Omega)$-spaces, where the vector space and lattice operations are all defined pointwise and the constant function $\mathbf{1}$ is an order unit.

The notion of a Yudin basis extends the notion of Arrow securities for Euclidean spaces to finite dimensional lattice-subspaces.

Definition 2.3. *If M is a finite dimensional vector subspace of a partially ordered space X, then a Yudin basis for M is a basis $\{e_i\}_{i=1}^J$ such that every $e_i \in M^+ = M \cap X^+$ and every $x \in M^+$ has a unique representation of the form $x = \sum_{i=1}^J \lambda_i e_i$ where $\lambda_i \geq 0$.*

The fundamental connection between lattice-subspaces and Yudin bases is given next and is proved in [14], Proposition 1.5, p. 9.

Theorem 2.4 [Choquet–Kendall]. *A finite dimensional vector subspace M of a partially ordered vector space is a lattice-subspace if and only if it has a Yudin basis.*

When Yudin bases exist, they are essentially unique. For a proof of the next result (see [3], Lemma 8, p. 6).

Regarding lattice operations in a lattice-subspace we have the following important result.

Theorem 2.6. *Assume M is a lattice-subspace of the Riesz space X and $\{e_i\}_{i \in I}$ is a Yudin basis for M.*

If $x = \sum_{i\in I} \lambda_i e_i$ *and* $y = \sum_{i\in I} \mu_i e_i$ *are two elements in* M *then* $x \mathbin{\vee\!\!\!\vee} y = \sum_{i\in I}(\lambda_i \vee \mu_i)e_i$ *and* $x \mathbin{\wedge\!\!\!\wedge} y = \sum_{i\in I}(\lambda_i \wedge \mu_i)e_i$.

3 Two-Dimensional Lattice-Subspaces

In this section, we shall discuss the most important lattice-subspace in finance, the two-dimensional vector subspace of X, the space M of state-contingent claims, spanned by a bond b, the riskless asset, and a stock s, the risky asset. Both assets have limited liability, i.e., b and s are in X^+ and they are linearly independent. If M is the *asset span* of $\{b, s\}$, then the *portfolio space* for M is $\Theta = \mathbb{R}^2$. We shall denote the vectors of Θ by pairs $\theta = (\theta_b, \theta_s)$, where θ_b is the investor's holdings in bonds and θ_s her holdings in stocks. Long positions are denoted by positive values of θ_b or θ_s and short positions by negative values of θ_b or θ_s.

There is a natural one-to-one (linear) operator $R : \Theta \to X$ with range M defined by

$$R(\theta_b, \theta_s) = \theta_b b + \theta_s s.$$

The operator R is known as the *payoff operator.* The payoff operator R induces a natural order $\geq_R$ on Θ by letting $\theta \geq_R \theta'$ if $R(\theta) \geq R(\theta')$. We shall denote the cone of this linear order by Θ_R^+, i.e.,

$$\Theta_R^+ = \{\theta \in \mathbb{R}^2 : R(\theta) \geq 0\} = R^{-1}(M^+),$$

and will be referred to as the cone of *positive* (marketed) *portfolios.*

It should be clear that (Θ, Θ_R^+) is a Riesz space if and only if M is a lattice-subspace of X. An equivalent statement is that M has a Yudin basis if and only if Θ has a Yudin basis.

A Yudin basis for Θ is obtained by the solution to the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & \theta_1 \cdot \theta_2 \\ \text{s.t.} \quad & R(\theta_1) \geq 0 \\ & R(\theta_2) \geq 0 \\ & ||\theta_1|| = ||\theta_2|| = 1, \end{aligned}$$

where $\theta_1 = (\theta_b^1, \theta_s^1)$, $\theta_2 = (\theta_b^2, \theta_s^2) \in \Theta$. Given the obvious continuity and compactness, a solution exists, which we denote by $\{\theta_1^*, \theta_2^*\}$. Then $\{\theta_1^*, \theta_2^*\}$ is a Yudin basis for Θ and $\{R(\theta_1^*), R(\theta_2^*)\}$ is a Yudin basis for M.[1] Note that this argument did not assume a finite state space.

Continuing our discussion of the asset span M generated by a risky security s and a riskless asset b, it follows from the Choquet–Kendall theorem that M is a two-dimensional lattice-subspace of X and the payoff operator $R : \Theta \to M$ is a lattice isomorphism. The ordering induced on Θ via the lattice cone Θ_R^+ will be denoted by $\geq_R$ and the lattice operations by $\vee_R$ and $\wedge_R$. We have, of course, $\theta \geq_R \theta'$ if and only if $\theta - \theta' \in \Theta_R^+$.

[1] For details of the proof see [3], Theorem 9, p. 7.

The space of security prices in the linear vector space $\mathbb{R}^2$. That is, if $\theta \in \Theta$ and $q \in \mathbb{R}^2$, then the value of θ at prices q is simply $q \cdot \theta$. Only arbitrage-free prices are possible in equilibrium. Suppose $q \in \mathbb{R}^2_+$ and $q \cdot \theta < 0$ for some portfolio $\theta \in \Theta^+_R$. Then q admits an arbitrage opportunity $\theta' \in \Theta^+_R$, i.e., the payoff $R(\theta')$ of θ' is positive in all states and $q \cdot \theta' < 0$. Hence, the arbitrage-free price vectors are the vectors $q \in \mathbb{R}^2_+$ such that $q \cdot \theta \geq 0$ for all positive portfolios $\theta \in \Theta^+_R$. In other words, the dual cone $(\Theta^+_R)'$ of Θ^+_R, defined by

$$(\Theta^+_R)' = \{q \in \mathbb{R}^2_+ : q \cdot \theta \geq 0 \text{ for each } \theta \in \Theta^+_R\},$$

is the cone of arbitrage-free price vectors.

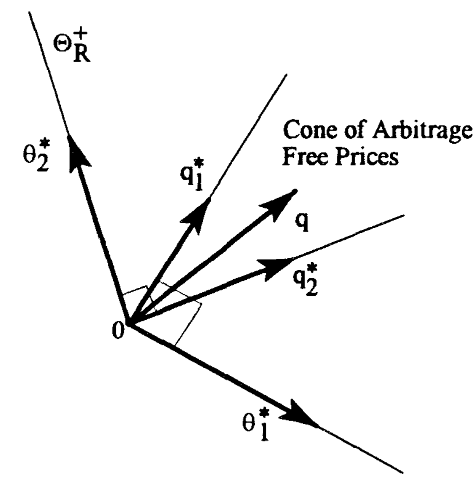


Figure 1

If the Yudin basis for Θ is the pair of portfolios $\{\theta^*_1, \theta^*_2\}$ and the non-zero vectors $q^*_1, q^*_2 \in (\Theta^+_R)'$ satisfy $q^*_1 \cdot \theta^*_1 = q^*_2 \cdot \theta^*_2 = 0$, then $\{q^*_1, q^*_2\}$ is a Yudin basis for the price space $\mathbb{R}^2$. The arbitrage-cone $(\Theta^+_R)'$ as well as the Yudin basis for Θ are shown in Figure 1. From [[2], Theorem 2.2], we know that

$$\theta \geq_R \theta' \Leftrightarrow q \cdot \theta \geq q \cdot \theta' \text{ for every arbitrage-free price } q.$$

This equivalence is central to our analysis of portfolio insurance in incomplete derivative markets.

The following example illustrates the preceding discussion.

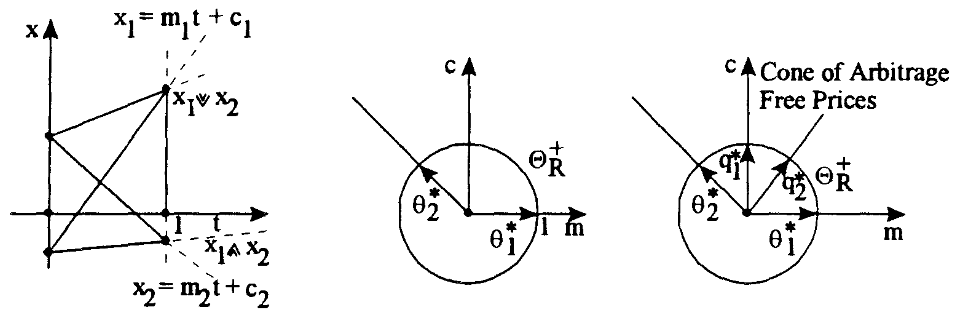
Example 3.1. Assume that the payoff space X is $C[0,1]$, i.e., $X = C[0,1]$. The bond b is the constant function one, i.e., $b = \mathbf{1}$, and the stock s is the function $t \mapsto t$, i.e., $s(t) = t$ for each $t \in [0,1]$. Notice that the asset span M consists of all linear functions, i.e., $x \in M$ if and only if $x(t) = mt + c$ for all t and some constant m and c. The payoff operator $R : \Theta \longrightarrow M$ is given by $R(\theta)(t) = mt + c$ for each $\theta = (m, c) \in \Theta$.

A direct verification shows that $R(\theta)(t) = mt + c \geq 0$ for all $t \in [0,1]$ if and only if $m + c \geq 0$ and $c \geq 0$. That is,

$$\Theta_R^+ = \{(m,c) \in \Theta : m + c \geq 0 \text{ and } c \geq 0\}.$$

The drawing of this cone Θ_R^+ can be seen in Figure 2b. The normalized vectors of the Yudin basis $\{\theta_1^*, \theta_2^*\}$ for Θ satisfy $\theta_1^* = (1,0)$ and $\theta_2^* = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The cone of arbitrage-free prices is shown in Figure 2c. Notice that the vectors $q_1^* = (0,1)$ and $q_2^* = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ form a (normalized) Yudin basis for the price space $\mathbb{R}^2$.

The lattice operators of M are shown in Figure 2a. The vectors $\{R(\theta_1^*), R(\theta_2^*)\}$ form a Yudin basis for M^+. Observe that the vectors $R(\theta_1^*)(t) = t$ and $R(\theta_2^*)(t) = -\frac{1}{\sqrt{2}}t + \frac{1}{\sqrt{2}}$ for each $t \in [0,1]$ form a Yudin basis for M.



2a: The Lattice Operations in M 2b: The Cone Θ_R^+ 2c: The Dual Cone

Figure 2

4 Lattice-Subspaces and Positive Projections

Recall that an operator $P : X \to X$ on a partially ordered vector space is said to be a *positive projection* if P is positive (i.e., $P(X^+) \subseteq X^+$) and idempotent (i.e., $P^2 = P$).

H. H. Schaefer ([16], Proposition 11.5, p. 214) proved that the range of any positive projection on a vector lattice is always a lattice-subspace. As before, if M is a lattice-subspace of a Riesz space X and $x, y \in M$, then we shall denote the supremum and infimum of the set $\{x, y\}$ in M by $x \curlyvee y$ and $x \curlywedge y$, respectively.

Theorem 4.1 [Schaefer]. *If $P : X \to X$ is a positive projection on a Riesz space, then its range $M = P(X)$ is a lattice-subspace of X. Its lattice operations are given by*

$$x \curlyvee y = P(x \vee y) \quad \text{and} \quad x \curlywedge y = P(x \wedge y).$$

In particular, we have $x^{++} = x \curlyvee 0 = P(x^+)$.

The converse of Theorem 4.1 is not true, i.e., there exist lattice-subspaces of vector lattices which are not the range of a positive projection; see ([1], Example 4.3,

p. 250). Nevertheless, Miyajima ([13], Theorem 1, p. 84) proved an important partial converse to Schaefer's theorem.

Theorem 4.2 [Miyajima]. *A vector subspace M of a Riesz space X is a lattice-subspace if and only if there exists a unique positive projection $P : \mathcal{L}_V(M) \to \mathcal{L}_V(M)$ whose range is M, where $\mathcal{L}_V(M)$ is the Riesz space generated by M. The unique positive projection P on $\mathcal{L}_V(M)$ is given by*

$$P(\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} x_{ij}) = \veebar_{i=1}^{n} \barwedge_{j=1}^{m} x_{ij},$$

where $\mathcal{L}_V(M) = \{\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} x_{ij} : x_{ij} \in M\}$.

In the intended applications of our theory, we assume X is some $C(\Omega)$ space — R^S is the spacial case, where S is finite — of state-contingent claims (or price contingent contracts) over a compact topological space of states Ω; M is a finite dimensional vector subspace of marketed securities (or marketed portfolios); and b, the bond or riskless asset, is marketed, i.e., $b \in M$ and serves as an order unit for X. Under these circumstances, it follows from the previously cited theorems of Schaefer and Miyajima and ([1], Lemma 3.4) that M is a lattice subspace of X if and only if M is the range of a positive projection P. The restriction of this projection to $\mathcal{L}_V(M)$ is unique. Moreover, it is unique on all of X if the elements of M are resolving, a consequence of the Stone–Weierstrass theorem.

The space of price-contingent contracts is the Riesz space of continuous functions on the normalized cone of arbitrage-free price vectors, denoted Δ. Y is the Banach lattice $C(\Delta)$ with order unit the riskless asset b. Given the duality between the cone of positive portfolios and the cone of arbitrage-free vectors, every portfolio θ is a price-contingent contract in $C(\Delta)$, where $\theta(q) \equiv q \cdot \theta$. Thus Θ is the range of a positive projection on Y. Similarly, assuming a finite state space, we see that the asset span M is the range of a positive projection on the space of state-contingent claims X.

For finite dimensional spaces, we have the following basic result.

Theorem 4.3 [Abramovich–Aliprantis–Polyrakis]. *For a subspace M of some finite dimensional space $\mathbb{R}^s$, the following are equivalent.*

1. *M is a lattice subspace of $\mathbb{R}^s$.*

2. *$\exists N$ non-negative vectors $\{e_1, ..., e_N\}$ in M and a subset of N states $F = \{s_1, ..., s_N\}$ such that $e_i(s_j) = \delta_{ij}$ for all $i, j = 1, ..., N$.*

3. *$\exists$ a basis of non-negative vectors $\{e_1, ..., e_N\}$ in M such that a vector $x = \sum_{i=1}^{N} \lambda_i e_i \in M$ satisfies $x \geq 0$ iff $\lambda_i \geq 0$ $\forall i = 1, ..., N$.*

4. *M is the range of a positive projection on $\mathbb{R}^s$.*

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