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by

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1 Introduction and Summary

We study in this paper *stationary competitive equilibria* in an economy with fiat money, one non-durable commodity, infinitely many discrete-time periods, no credit or futures markets, and a measure space of "non-homogeneous" agents -- who can differ in their preferences and in the distributions of their (random) endowments. These agents are immortal, and hold money in order to affect the random fluctuations in their endowments.[1] In the aggregate, these fluctuations offset each other, and equilibrium prices are constant.

Our two central issues here are *time* and *uncertainty*; we carry out an equilibrium analysis that focuses only on consumption, distribution of wealth, and price formation. From this point of view, the production of the commodity (units of which the agents receive as endowments, from period to period) is assumed to be determined exogenously. Furthermore, in the setup considered here, money can only be hoarded; one can indeed envisage a slightly modified framework in which every agent, in addition to hoarding his money, can also put it in a savings account or invest it in some risky asset(s). Such a framework appears in a series of papers by W. Whitt (1975a-d), again in a setup without active consideration of the institutions associated with these possibilities (banking, stock exchange). In the interest of concreteness, we have preferred to keep the model as simple as possible, in order to concentrate on the above-mentioned features only. We hope that in the future we shall be able to build on this work, and try to capture (and to describe endogenously) additional desirable features such as loan markets, interest rates, insurance, overlapping generations, and the like.

This paper concentrates on the basic mathematical analysis of a class of infinite horizon stochastic strategic market games. Because there is a continuum of agents, each with no influence on price, the analysis is close to that of competitive equilibrium. However, it is noted that we present a well defined process model with an active rôle for fiat money. The equilibrium solution considered is more accurately described as a type-symmetric noncooperative equilibrium which coincides with a competitive equilibrium.

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[1]A nonstochastic infinite horizon version of the strategic market game was first presented in Shubik and Whitt (1973).

As the model defined provides a full description of process, we hope to be able to consider its dynamics as well as its stationarity properties. In this paper, however we limit our main observations to equilibrium.

A natural economic extension to the model presented here is to permit borrowing and lending. When this is done, a money rate of interest may be formed endogenously. If this rate is positive there will be an incentive to lend rather than to hoard money as a means of preserving buying power from period to period. However, in a truly dynamic game of strategy with borrowing, it is possible that the system could attain a state where a borrower is unable to pay the amount he has promised to pay. In such a circumstance in order to well define the game, bankruptcy, settlement and reorganization rules must be specified as part of the game. This is not just an institutional comment, but a logical necessity. There may be many ways to define these extra rules. It is not clear *a priori* that there should be a unique optimal rule.

The existence of a stationary wealth distribution and optimal policy established here involves much of the population inventorying money from period to period. This changes considerably with the introduction of loan markets. In a projected separate expository paper, the economic motivation is presented in more detail together with several examples including models with loan markets and a cyclical supply of the commodity.

In this paper, by limiting our concern to one commodity, we have been able to obtain not only existence but also the uniqueness of equilibrium in some limited instances. As soon as there are two or more goods this no longer holds true. A natural extension of this work is to consider both experimental games and behavioral simulations to see if there is any tendency and any reasons why one attractor is selected over others.

1.1 The Model

In order to describe our model, let us start with an index set $I = [0, 1]$ and a non-atomic probability measure ϕ on it, representing a collection of agents and the "spatial" distribution on this collection, respectively. Each agent α in I has a *utility function* $u^\alpha : [0, \infty) \to [0, \infty)$ with $u^\alpha(0) = 0$ which is increasing, concave and continuously differentiable, and has finite right-hand derivative at the origin.

At the beginning of the nth period of play, the price of the commodity is $p_{n-1}(\omega)$; and each agent α enters with an amount $S^\alpha_{n-1}(\omega)$ in fiat money, and with information $\mathfrak{F}^\alpha_{n-1}$. (The σ-field $\mathfrak{F}^\alpha_{n-1}$ measures past prices $\{p_k, k = 0, 1, ..., n-1\}$, as well as past wealths, endowments and actions $\{S^\alpha_0, S^\alpha_k, Y^\alpha_k, b^\alpha_k, k = 1, ..., n-1\}$ of the agent; it may or may not measure corresponding quantities for other agents.) Based on this information, each agent "bids" a certain amount $b^\alpha_n(\omega) \in [0, S^\alpha_{n-1}(\omega)]$ of fiat money for consumption in the nth period; the total bid is

$$B_n(\omega) = \int_I b^\alpha_n(\omega)\phi(d\alpha) . \tag{1.1}$$

Then the agents' random endowments $\{Y^\alpha_n(\omega)\}_{\alpha \in I}$ are announced for the period $t = n$, denominated in units of the commodity. We shall assume that the *total endowment*

$$Q = \int_I Y^\alpha_n(\omega)\phi(d\alpha) \tag{1.2}$$

is non-random and constant from period to period, whereas for each agent α in I the endowments $Y^\alpha_1, Y^\alpha_2, ...$ are independent, identically distributed (IID) random variables with common law λ^α. A new price is then formed according to the rule

$$p_n(\omega) = \frac{B_n(\omega)}{Q} \tag{1.3}$$

as the ratio of total bid over total endowment. Each agent receives his bid's worth

$$x^\alpha_n(\omega) = \frac{b^\alpha_n(\omega)}{p_n(\omega)}$$

of commodity, denominated in the new price, and consumes it immediately, thereby "earning" utility $u^\alpha(x^\alpha_n(\omega))$, $\alpha \in I$ in that period; he also receives his endowment's worth of fiat money

$$p_n(\omega)Y^\alpha_n(\omega)$$

in this price, and thus goes into the next time period with wealth

$$S^\alpha_n(\omega) = S^\alpha_{n-1}(\omega) - b^\alpha_n(\omega) + p_n(\omega)Y^\alpha_n(\omega) , \quad \alpha \in I . \tag{1.4}$$

This procedure is then repeated ad infinitum, and results in a total reward of

$$(1.5) \qquad v^{\alpha}(\omega) := \sum_{n=0}^{\infty} \beta^n u^{\alpha}(x_{n+1}^{\alpha}(\omega))$$

for agent $\alpha \in I$, where $\beta \in (0, 1)$ is a discount factor. It should be noted that the prescription (1.3) preserves the total amount of wealth (fiat money) in the economy

$$(1.6) \qquad W_n(\omega) = \int_I S_n^{\alpha}(\omega)\phi(d\alpha) ,$$

from period to period, because from Eqs. (1.4) and (1.1)-(1.3) we have

$$(1.6)' \qquad W_n(\omega) = W_{n-1}(\omega) - B_n(\omega) + p_n(\omega)Q = W_{n-1}(\omega) , \text{ for all } n \in \mathbf{N} .$$

All the quantities S_0^{α}, S_n^{α}, b_n^{α}, Y_n^{α} ($\alpha \in I, n \in$ N) are random variables, defined on a probability space $(\Omega, \mathfrak{F}, P)$. For a given $\alpha \in I$, the bids b_n^{α} are $\mathfrak{F}_{n-1}^{\alpha}$-measurable for all $n \in$ N, where $\{\mathfrak{F}_j^{\alpha}\}_{j=0}^{\infty}$ is an increasing sequence of sub-σ-fields of $\mathfrak{F}$, and $\mathfrak{F}_{n-1}^{\alpha}$ represents the information accumulated by agent α up to the beginning of period $t = n$. As we mentioned already, for $n \geq 1$ this information includes past prices $\{p_k, k = 0, 1, ..., n-1\}$, as well as past levels of wealths, endowments, and actions S_0^{α}, S_k^{α}, Y_k^{α}, b_k^{α}, $k = 1, ..., n-1$; it may not include the corresponding quantities for (any) other agents. We set $\mathfrak{F}_0^{\alpha} = \sigma(S_0^{\alpha}, p_0)$.

In this analysis we limit our concern to pure strategies. For consideration of some of the measure theoretic difficulties associated with randomized strategies, see Dubey and Shapley (1980, 1992).

A *strategy* π^{α} for agent $\alpha \in I$ determines the bids b_n^{α} for every $n \in$ N. A collection of strategies $\Pi = \{\pi^{\alpha}\}_{\alpha \in I}$, together with the given distributions, $\{\lambda^{\alpha}\}_{\alpha \in I}$ for the random endowments of different agents and the equations (1.1)-(1.4), determines the joint distribution of all the random variables that we have introduced. In particular, the expected total reward $E[v^{\alpha}(\omega)]$ is determined for every agent $\alpha \in I$, and we have a well-defined stochastic game.

For any collection Π of strategies as above, consider the sequence of random measures

$$(1.7) \qquad \nu_n(A, \omega) := \int_I 1_A(S_n^{\alpha}(\omega))\phi(d\alpha) , \quad A \in \mathscr{B}([0, \infty))$$

which describe the distribution of wealth in the various time periods $n = 0, 1, 2, ...$.

We say that *a collection* $\hat{\Pi} = \{\hat{\pi}^\alpha\}_{\alpha \in I}$ *of strategies results in a stationary competitive equilibrium* (p, μ), where $p \in (0, \infty)$ and μ belongs to the space $\mathcal{M}$ of probability measures on $[0, \infty)$, if

$$(1.8) \qquad \begin{array}{l} \text{(i) with } p_0 = p,\ \nu_0 = \mu \text{ we have } p_n = p,\ \nu_n = \mu,\ \forall n \in \mathrm{N}, \text{ and} \\ \text{(ii) } \hat{\pi}^\alpha \text{ maximizes } E^{\pi^\alpha} \Sigma_{n=0}^{\infty} \beta^n u^\alpha(b_{n+1}(\omega)/p) \text{ over all strategies } \pi^\alpha \text{ for } \alpha \in I . \end{array}$$

1.2 Outline of Results

We shall construct explicitly such a stationary competitive equilibrium in Section 7, by first analyzing in considerable detail the individual agent's optimization (Dynamic Programming) problem with $p_n = p \in (0, \infty)$ fixed from period to period. We set the stage for this analysis in Section 2, where we define the single-agent optimization problem, study some of its elementary properties, and discuss a few examples that can be solved fairly explicitly. The analysis of the *Dynamic Programming Equation*

$$(1.9) \qquad \begin{aligned} V^\alpha(s; p) &= \max_{0 \le c \le s} \left[u^\alpha\left(\frac{c}{p}\right) + \beta \int_0^\infty V^\alpha(s - c + py; p) \lambda^\alpha(dy) \right] \\ &= u^\alpha\left(\frac{c^\alpha(s; p)}{p}\right) + \beta \int_0^\infty V^\alpha(s - c^\alpha(s; p) + py; p) \lambda^\alpha(dy) , \end{aligned}$$

namely, of the *value function* $V^\alpha(s; p)$ and the *optimal* (stationary) *consumption policy* $c^\alpha(s; p)$, is carried out in Sections 3 and 4.[2] Under the conditions

$$(1.10) \qquad 0 < \int_0^\infty y \lambda^\alpha(dy) < h^\alpha , \quad \int_0^\infty y^2 \lambda^\alpha(dy) < \infty$$

on the distribution λ^α, we are able to obtain very precise information about $c^\alpha(\cdot\ ; p)$, which in turn leads to the existence of a unique invariant measure for the associated Markov Chain $\{s_n^\alpha(\omega)\}_{n=0}^\infty$ given by

$$(1.11) \qquad s_{n+1}^\alpha(\omega) = s_n^\alpha(\omega) - c^\alpha(s_n^\alpha(\omega); p) + pY_{n+1}^\alpha(\omega) , \quad n \in \mathrm{N}_0 .$$

In addition, this invariant measure is shown to have finite first moment. Section 5 is devoted

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[2] We have changed notation from b (for bid) to c to emphasize consumption.

to the case when the utility function u^α is strictly increasing on all of $(0, \infty)$. Section 6 treats the case when there is a finite *saturation point* $h^\alpha := \inf\{c > 0;\, u^\alpha(c+s) = u^\alpha(c),\ \forall s \geq 0\}$.

The analysis is quite interesting in itself as an instance of a discounted, infinite-horizon dynamic programming problem, where the ergodic behavior of the resulting optimally controlled Markov Chain as in (1.10) is analyzed in detail.

Once this analysis has been carried out, a simple *aggregation* of the various ergodic measures μ^α via the formula

$$\text{(1.12)} \qquad \bar{\mu}(A; p) = \int_I \mu^\alpha(A; p)\phi(d\alpha)\ ,\quad A \in \mathscr{B}([0, \infty))$$

is easily shown (in Section 7) to lead to the stationary competitive equilibria $(p, \bar{\mu}(\cdot\,; p))$, $p \in (0, \infty)$. This is accomplished first in the *homogeneous case* $\lambda^\alpha = \lambda$, $u^\alpha = u$ $(\forall \alpha \in I)$ and then for countably (respectively, uncountably) many *homogeneous classes of identical agents*; cf. Theorems 7.3, 7.6, and Remark 7.7 for the details. Finally, it is seen that such an equilibrium pair $(p, \bar{\mu}(\cdot; p))$ is specified *uniquely* for any given level $W = \int_0^\infty s\bar{\mu}(ds; p)$ of the initial "money supply" (which remains fixed from period to period, as in (1.6)′); see Remark 7.4.

1.3 Relevant Literature

The intellectual progenitor of this work is the unpublished paper of W. Whitt (1975d), which considers the "homogeneous case" $u^\alpha = u$, $\lambda^\alpha = \lambda (\forall \alpha \in I)$ and seeks stationary competitive equilibrium (defined in a sense weaker than ours) as a *fixed point* in $(0, \infty) \times \mathscr{M}$ for the system of equations

$$\text{(1.13)} \qquad p = \frac{\int_0^\infty c(s; p)\mu(ds)}{\int_0^\infty y\lambda(dy)}$$

$$\text{(1.14)} \qquad \mu(A) = \int_0^\infty \lambda\left(\frac{A - s + c(s; p)}{p}\right)\mu(ds)\ ,\quad A \in \mathscr{B}([0, \infty))\ .$$

In this work, we take a different tack, namely, we *construct* a solution to (1.13), (1.14) by first studying in detail the ergodic behavior of the individual Markov Chain as in (1.10); cf. Theorem 7.3. The "non-homogeneous" versions of these equations are then studied by aggregating as in (1.12) the invariant measures across classes of agents; cf. Theorem 7.6.

Related work on stationary competitive Markov equilibria -- always using fixed-point methods -- has been carried out in Lucas (1978), (1980), Shubik (1986), Bewley (1986), and Duffie *et al.* (1988), among others.

One-person dynamic programming problems, similar to that of Sections 2-4, have been treated by several authors including Hakansson (1970), Whitt (1975a-c), Yaari (1976), Schechtman (1976), Schechtman and Escudero (1977), Mendelssohn and Sobel (1980); see also the survey paper of Deaton (1991).

2 Preliminaries for the One-Person Game, and Simple Examples

Consider a single agent, operating in a one-commodity economy, who seeks to maximize his total discounted utility during an infinite stage game by optimally dividing current wealth between immediate consumption and savings for the future. We formulate the problem as a dynamic programming (or Markov decision) problem with the following ingredients:

(2.1) $S = [0, \infty)$ is the *state space* and a state $s \in S$ represents the *wealth* of an agent.

(2.2) The *utility function* $u : S \to S$ is concave, nondecreasing, has a finite derivative from the right at 0, which we write as $\rho := u'(0+)$ or simply $u'(0)$, $0 < u'(0) < \infty$, and $u(0) = 0$. (In later sections, we shall assume additional properties.)

(2.3) An agent with wealth s may select any *action* a from the set $A(s) = [0, s]$ of possible actions. We interpret the agent's choice of an action a as a decision to purchase a/p units of the given commodity, where $p \in (0, \infty)$ is the price of the commodity.

(The commodity is assumed to be nondurable, and is thus consumed immediately, resulting in a reward to the agent of $r(s, a) = u(a/p)$. Here r is the *reward function* of the dynamic programming problem, but we shall use the utility function instead.)

(2.4) The *law of motion* determines the distribution of the next state s_1 for an agent at state s who selects action a, by the rule

(2.5) $$s_1 = s - a + pY .$$

Here Y is a nonnegative random variable with a given distribution λ, which represents the agent's income for the period in units of the commodity. We assume $0 < m := EY = \int_0^\infty y\lambda(dy) < \infty$.

(2.6) The *discount factor* $\beta \in (0, 1)$.

A *plan* π is a sequence $(\pi_0, \pi_1, ...)$ where π_n chooses the action a_n on the nth day based on the history $\mathfrak{F}_n = (s_0, a_0, s_1, a_1, ..., s_{n-1}, a_{n-1}, s_n)$ of previous actions and states. The agent seeks a plan π which will maximize the total expected discounted utility.

For simplicity we set the price p equal to 1. There is no real loss of generality, since this amounts to redefining the utility function and the income variable Y (see also Remark 4.6). With this assumption, the motion formula of the system becomes

(2.7) $$s_{n+1} = s_n - a_n + Y_{n+1} , \quad n = 0, 1, ... ,$$

where $Y = Y_0, Y_1, ...$ are independent and have the same distribution λ.

A plan π, together with this law of motion, determines the distribution of the process $s_0, a_0, s_1, a_1, ...$. We define the *return function* for π to be

(2.8) $$I(\pi)(s) := E^{\pi}_{s_0=s}\left[\sum_{n=0}^{\infty} \beta^n u(a_n)\right] , \quad s \in S .$$

The *optimal return* or *value function* is defined by

(2.9) $$V(s) := \sup_{\pi} I(\pi)(s) , \quad s \in S .$$

A plan π is *optimal*, if $V = I(\pi)$.

If u is bounded, then our problem is a discounted dynamic programming problem as in Blackwell (1965). Even if u is unbounded, we shall see that many of Blackwell's techniques can be successfully adapted.

First, introduce the n-day return from a policy π as

(2.10) $$I_n(\pi)(s) := E^{\pi}_{s_0=s}\left[\sum_{k=0}^{n} \beta^k u(a_k)\right]$$

and the n-day optimal return

$$V_n(s) := \sup_{\pi} I_n(\pi)(s) \tag{2.11}$$

for $s \in S$, $n = 0, 1, \ldots$. Recall the notation $\rho = u'(0)$, $m = E(Y)$.

2.1 Lemma: For $s \in S$ and $n = 0, 1, \ldots$, let $k_n(s) := \rho s \beta^n/(1-\beta) + \rho m \beta^n[n - (n-1)\beta]/(1-\beta)^2$. Then we have

(a) $V(s) \le k_0(s)$,

(b) $V(s) \le V_n(s) + k_{n+1}(s)$, $n = 1, 2, \ldots$.

Thus V is finite, and V_n converges up to V, uniformly on bounded intervals.

Proof: It follows from our assumptions on u that $u(s) \le \rho s$ for all $s \in S$. Also, for $n = 1, 2, \ldots$,

$$\begin{aligned} S_n &= s_0 - a_0 + Y_1 - a_1 + Y_2 - \cdots - a_n + Y_n \\ &\le s_0 + Y_1 + \cdots + Y_n . \end{aligned}$$

Hence,

$$I(\pi)(s_0) \le \sum_{n=\infty}^{\infty} \beta^n E u(s_0 + Y_1 + \cdots + Y_n) \le \rho \sum_{n=0}^{\infty} \beta^n (s_0 + nm) \le k_0(s_0) ,$$

and

$$\begin{aligned} I(\pi)(s_0) &= I_n(\pi)(s_0) + \sum_{k=n+1}^{\infty} \beta^k E u(s_0 + nm) \le k_0(s_0) \\ &\le I_n(\pi)(s_0) + \sum_{k=n+1}^{\infty} \beta^k \rho(s_0 + km) = I_n(\pi)(s_0) + k_{n+1}(s_0) . \end{aligned}$$

Take the supremum over π to get (a) and (b). □

Standard dynamic programming arguments show that V satisfies the optimality equation

$$V(s) = \sup_{0 \le a \le s} [u(a) + \beta E V(s - a + Y)] \tag{2.12}$$

and that the V_n can be calculated by "backward induction" from the formulas

(2.13)
$$V_0(s) = 0\ ,\quad V_1(s) = u(s)$$
$$V_{n+1}(s) = \sup_{0 \le a \le s} [u(a) + \beta E V_n(s - a + Y)]\ .$$

Introduce, as in Blackwell (1965), the operator T defined for Borel functions $\psi : S \to [0, \infty]$ by

$$(T\psi)(s) = \sup_{0 \le a \le s} [u(a) + \beta E \psi(s - a + Y)]\ .$$

Notice that T is monotone (i.e., $\psi_1 \le \psi_2 \to T\psi_1 \le T\psi_2$), and that (2.12) and (2.13) can be rewritten as

(2.14)
$$TV = V\ ,\quad V_{n+1} = TV_n = T^{n+1}0\ .$$

A plan π is *stationary* if there is a function c defined on S such that $0 \le c(s) \le s$, $s \in S$, and π uses action $c(s)$ whenever the current state is s. Sometimes we shall call c the *consumption function* for π. Here is a characterization of optimal *stationary* plans which is well-known for u bounded (Theorem 6 of Blackwell (1965)).

2.2 Theorem: For a stationary plan π with consumption function c, the following conditions are equivalent:

(a) $I(\pi) = V$,

(b) $V(s) = u(c(s)) + \beta E V(s - c(s) + Y)\ ,\ s \in S\ ,$

(c) $T(I(\pi)) = I(\pi)\ .$

Proof: (a) $\to$ (b): For any stationary plan π corresponding to a consumption function c, it follows from the definition of $I(\pi)$ in (2.8) that

$$I(\pi)(s) = u(c(s)) + \beta E I(\pi)(s - c(s) + Y)\ .$$

Now use (a).

(a) $\to$ (c): Immediate from the optimality equation (2.12).

(b) $\to$ (a): Let E denote expectation under π with $s = s_0$. Then (b) can be rewritten as

$$V(s_0) = u(c(s_0)) + \beta E V(s_1)\ .$$

Iterate to get, for $n = 1, 2, \ldots$, in conjunction with Lemma 2.1(a):

$$\begin{aligned}V(s_0) &= E\sum_{k=0}^{n} \beta^k u(c(s_k)) + \beta^{n+1}EV(s_{n+1}\\ &= I_n(\pi)(s_0) + \beta^{n+1}EV(s_{n+1})\\ &\leq I_n(\pi)(s_0) + \beta^{n+1}Ek_0(s_0 + Y_1 + \cdots + Y_{n-1})\\ &= I_n(\pi)(s_0 + \beta^{n+1}k_0(s_0 + (n+1)m) \to I(\pi)(s_0)\end{aligned}$$

as $n \to \infty$. Hence, $V \leq I(\pi)$. The opposite inequality is obvious.

(c) $\Rightarrow$ (a): Iterate (c), to get $I(\pi) = T^n(I(\pi)) \geq T^n 0 = V_n$. By Lemma 2.1, $V_n \uparrow V$. Hence $I(\pi) \geq V$. □

We conclude this introductory section with a few simple examples.

2.3 **Example:** *A linear utility function*. Let

$$u(a) = a\ ,\ \ 0 \leq a < \infty.$$

Intuitively, an agent with this utility function gains nothing by saving for the future and, because of the discount factor, stands to lose by doing so. Indeed, the unique optimal stationary plan π corresponds to the consumption function

$$c(s) = s\ ,\ \ 0 \leq s < \infty.$$

An agent with this plan and initial wealth s will consume s on the first day, and the daily income on each day thereafter. Hence,

$$I(\pi)(s) = u(s) + E\left[\sum_{n=1}^{\infty} \beta^n u(Y_n)\right] = s + \sum_{n=1}^{\infty} \beta^n m = s + \beta\frac{m}{1-\beta}\ .$$

It is easy to check that $I(\pi)$ satisfies the optimality equation (Theorem 2.2(c)). Hence, π is optimal. Uniqueness follows from the fact that, for each s, only $a = c(s) = s$ achieves the supremum in the optimality equation.

2.4 Example: *A generalization of Example 2.3.* Assume there exist positive numbers L and M such that $\beta M \le L$ and that $u(\cdot)$ is continuously differentiable, with $L \le u'(s) \le M$ for all $s \in S$. Then again the optimal stationary plan π corresponds to $c(s) = s$, for all s. To see this, set

$$w(s) := I(\pi(s)) = u(s) + E\left[\sum_{n=1}^{\infty} \beta^n u(Y_n)\right] = u(s) + \beta\frac{Eu(Y)}{1-\beta} .$$

To check that w satisfies that optimality equation, consider the function

$$\begin{aligned}\varphi(a) &:= u(a) + \beta Ew(s - a + Y) \\ &= u(a) + \beta Eu(s - a + Y) + \beta^2\frac{Eu(Y)}{1-\beta} , \quad 0 \le a \le s .\end{aligned}$$

It suffices to show that φ has its maximum at $a = s$. But

$$\varphi'(a) = u'(a) - \beta Eu'(s - a + Y) \ge L - \beta M \ge 0 , \quad 0 \le a \le s .$$

2.5 Example: *A piecewise-linear utility function with saturation.* Assume

$$u(a) = \begin{cases} a , & 0 \le a \le 1 , \\ 1 , & a > 1 . \end{cases}$$

For an agent with wealth $s \in [0, 1]$, the same intuition as for Example 2.4 suggests there is no point in saving for the future. However, if $s > 1$, it seems equally clear that there is no point in consuming more than 1 unit of wealth because consuming more results in no additional utility. So define π to be the stationary plan corresponding to

$$c(s) = \begin{cases} s , & 0 \le s \le 1 , \\ 1 , & s > 1 . \end{cases} \tag{2.15}$$

It can be shown that π is optimal in complete generality (cf. Appendix). However, we shall only consider here the special case when the income variable Y has the *Bernoulli distribution*

$$P[Y = 0] = 1-\gamma , \quad P[Y = 2] = \gamma \quad \text{with} \quad 0 < \gamma < \frac{1}{2}$$

(so that $E(Y) < 1$, where $h = 1$ is the point at which $u(\cdot)$ "saturates"). It can be seen that, in this simple case, *the return function* $I(\pi)$ for the plan π *is given by*

(2.16) $$I(\pi)(s) = \frac{1}{1-\beta} - \frac{\theta^s}{1-\theta}, \quad s \in \mathrm{N}_0$$

on the integers, and by linear interpolation between them:

(2.17) $$I(\pi)(s) = \frac{1}{1-\beta} - \left(\frac{1}{1-\theta} - (s - [s])\right)\theta^{[s]}, \quad s \in [0, \infty) .$$

Furthermore, the resulting Markov chain

$$s_{n+1} = s_n - c(s_n) + Y_{n+1}, \quad n = 0, 1, 2, \ldots$$

has a unique invariant probability measure $\mu = \{\mu_j\}_{j=1}^{\infty}$, *concentrated on the nonnegative integers as follows:*

(2.18) $$\mu_0 = c(1-\gamma), \quad \mu_1 = c\gamma, \quad \mu_j = c\left(\frac{\gamma}{1-\gamma}\right)^{j-1} \text{ for } j = 2, 3, \ldots .$$

In the above, we have set

(2.19) $$\theta := \frac{1 - \sqrt{1 - 4\beta^2\gamma(1-\gamma)}}{2\beta\gamma}, \quad c := \frac{1-2\gamma}{1-\gamma} .$$

To see that π is optimal, it suffices, by Theorem 2.2(c) to show that $TI(\pi) = I(\pi)$. The details of this verification are somewhat laborious, but straightforward.

To verify (2.18) suppose first that $s_0 \in \mathrm{N}$. The state space of the Markov Chain $\{s_n\}_{n=0}^{\infty}$ is N_0; the chain is positive recurrent, irreducible and aperiodic, with transition probability matrix

$$\begin{pmatrix} 1-\gamma & 0 & \gamma & 0 & 0 & 0 & \cdots \\ 1-\gamma & 0 & \gamma & 0 & 0 & 0 & \cdots \\ 0 & 1-\gamma & 0 & \gamma & 0 & 0 & \cdots \\ 0 & 0 & 1-\gamma & 0 & \gamma & 0 & \cdots \\ 0 & 0 & 0 & 1-\gamma & 0 & \gamma & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and a unique invariant probability measure μ, which satisfies $\mu_0 = (1-\gamma)(\mu_0 + \mu_1)$,

$\mu_1 = (1-\gamma)\mu_2$, $\mu_2 = \gamma(\mu_0 + \mu_1) + (1-\gamma)\mu_3$, and $\mu_j = \gamma\mu_{j-1} + (1-\gamma)\mu_{j+1}\}$ for $j \geq 3$. The solution of this system is given by (2.18).

Finally, suppose that the initial state s_0 is an arbitrary number in $(0, \infty)\mathbf{N}$ and let $x := s_0 - [s_0]$. Then the state space of the Markov chain becomes $\mathbf{N}_x \cup \mathbf{N}_0$, where all the states in $\mathbf{N}_x := \{x, x+1, x+2, ...\}$ are transient, and $\mathbf{N}_0$ is a (communicating) class of positive recurrent states. More precisely, the chain starts out in s_0, and stays in the lattice $\mathbf{N}_x$ until the first time it visits the origin (which is almost surely finite); from that time onward, the chain remains in $\mathbf{N}_0$. It is thus clear that the probability measure μ of (2.18) is the invariant measure for this case as well.

2.6 Example: *A piecewise-linear utility function without saturation.* Assume that

$$u(a) = \begin{Bmatrix} a & ; & 0 \leq a \leq 1 \\ 1 + \alpha(a-1) & ; & 1 < a < \infty \end{Bmatrix}$$

for some $0 < \alpha < 1$, and

$$P[Y = 0] = \frac{1}{2} = P[Y = 2] .$$

We shall see that, under appropriate conditions on the positive constants α and β, namely

$$\beta^2 \leq \frac{2\alpha}{1+\alpha} \leq \beta < 1 < \frac{2}{1+\alpha} , \tag{2.20}$$

an optimal consumption function is given by

$$c(s) = \begin{Bmatrix} s & ; & 0 \leq s \leq 1 \\ 1 & ; & 1 < s \leq 2 \\ s-1 & ; & 2 < s < \infty \end{Bmatrix} . \tag{2.21}$$

For instance, (2.20) is satisfied by $\alpha = 1/2$, $\beta = 3/4$.

Let $\underset{\sim}{Q}(\cdot) = I(\pi)(\cdot)$ be the return function of the stationary plan π corresponding to $c(\cdot)$ of (2.21); then $\underset{\sim}{Q}(\cdot)$ satisfies

$$Q(s) = \begin{Bmatrix} s + (\beta/2)[Q(0) + Q(2)] & ; & 0 \le s \le 1 \\ 1 + (\beta/2)[Q(s-1) + Q(s+1)] & ; & 1 \le s \le 2 \\ 1 + \alpha(s-2) + (\beta/2)[Q(1) + Q(3)] & ; & 2 \le s < \infty \end{Bmatrix} . \tag{2.22}$$

Since $Q(0) = (\beta/2)[Q(0) + Q(2)]$, $Q(2) = 1 + (\beta/2)[Q(1) + Q(3)]$, $Q(1) = 1 + Q(0)$, and observing that for $1 \le s \le 2$ we have

$$Q(s-1) = s - 1 + (\beta/2)[Q(0) + Q(2)]$$

and

$$Q(s+1) = 1 + \alpha(s-1) + (\beta/2)[Q(1) + Q(3)] ,$$

we may re-cast (2.2) in the more explicit form

$$Q(s) = \begin{Bmatrix} Q(0) + s & ; & 0 \le s \le 1 \\ Q(1) + (\beta/2)(1+\alpha)(s-1) & ; & 1 \le s \le 2 \\ Q(2) + \alpha(s-2) & ; & 2 \le s < \infty \end{Bmatrix} . \tag{2.22$'$}$$

Because $1 \ge \frac{\beta}{2}(1+\alpha) \ge \alpha$ from (2.20), the piecewise-linear function $Q(\cdot)$ is *concave*. It remains to check that $Q(\cdot)$ satisfies the optimality equation

$$Q(s) = \max_{0 \le a \le s}\left[u(a) + \frac{\beta}{2}\{Q(s-a) + Q(s+a)\}\right] , \quad s \in [0, \infty) \tag{2.12$'$}$$

(recall Theorem 2.2), or equivalently that the function

$$q_s(a) := u(a) + \frac{\beta}{2}[Q(s-a) + Q(s+a)] , \quad 0 \le a \le s \tag{2.23}$$

is increasing on $[0, c(s)]$ and decreasing on $(c(s), s]$, for any given $s \in (0, \infty)$.

This is straightforward by considering each interval $(0, 1]$, $(1, 2)$, $[2, \infty)$ separately, under the conditions of (2.20). We omit the details.

3 The Basic Recursion Formula

In order to establish properties of the value function V, we will first show that many properties of u are inherited by the functions V_n given by the recursion (2.13). To do this, we introduce a general recursion formula

$$v(s) = v_w(s) := \sup_{0 \le a \le s} [u(a) + \beta E w(s - a + Y)] \ , \quad s \in S \ . \tag{3.1}$$

The following assumptions are made on u and w in this section

A.1: In addition to the properties (2.2) already assumed for it, the utility function u has a continuous strictly positive and strictly decreasing derivative on $S = [0, \infty)$.

A.2: The function $w : S \to S$ has properties (a) and (b) also.

A.3: $u'(0) = w'(0)$.

Notice that we write $u'(0)$ and $w'(0)$ for the derivatives from the right at 0. Likewise, we say that a function is differentiable on $[0, \infty)$ if it is differentiable on $(0, \infty)$ with a right derivative at 0.

For $0 < s < \infty$ and $0 \le a \le s$, define

$$\psi_s(a) = \psi_{s,w}(a) := u(a) + \beta E w(s - a + Y) \ . \tag{3.2}$$

Then

$$\psi_s'(a) = u'(a) - \beta E w(s - a + Y)$$

is continuous and strictly *decreasing* for $0 \le a \le s$. (As above, we are writing $\psi_s'(0)$ and $\psi_s'(s)$ for the right and left derivatives at 0 and s, respectively. Also, there is no difficulty with taking the derivative inside the expectation, because $0 < w'(s) \le w'(0)$ and $w(s) \le w(0) + w'(0)s$, $s \in S$.) Furthermore,

$$\begin{aligned} \psi_s'(0) &= u'(0) - \beta E w'(s+Y) \\ &\ge u'(0) - \beta w'(0) = (1-\beta)u'(0) > 0 \ . \end{aligned} \tag{3.3}$$

Thus, for $s > 0$, ψ_s has its maximum on $[0, s]$ at a unique point $c(s) = c_w(s) \in (0, s]$. Set $c(0) = 0$. Then, for all s, (3.4) $$v(s) = u(c(s)) + \beta Ew(s - c(s) + Y) .$$

Our primary goal in this section is to establish the following result.

3.1 Proposition: The function v inherits all the properties assumed for w. This is, v satisfies A.1 and $v'(0) = u'(0)$. Furthermore,

(3.5) $$v'(s) = u'(c(s)) , \quad \text{on} \quad S . \qquad \Box$$

A secondary goal is to establish a number of properties for the consumption function c. Several of these properties will be used in our proof of the proposition.

Extend w to be C^1 and strictly concave on $(-\infty, \infty)$, in such a way that $w'(s) \to +\infty$ as $s \to -\infty$. Then ψ_s is defined on $[0, \infty)$, and ψ_s' strictly decreases to $-\infty$ as $a \to \infty$. Thus, by (3.3) the equation $\psi_s'(a) = 0$ has a unique solution $a = \check{c}(s)$, and the function $\check{c}$ is continuous. Hence,

(3.6) $$c(s) = \check{c}(s) \wedge s$$

is continuous also.

3.2 Lemma: Let $0 \le a \le s_1 < s_2$. Then

(a) $\psi_{s_1}'(a + s_2 - s_1) < \psi_{s_1}'(a) < \psi_{s_2}'(a)$,

(b) $c(s_1) < c(s_2)$,

(c) $s_1 - c(s_1) \le s_2 - c(s_2)$.

Proof: (a) Because w' is strictly decreasing,

$$\begin{aligned} \psi_{s_2}' &= u'(a) - \beta Ew'(s_2 - a + Y) \\ &> u'(a + s_2 - s_1) - \beta Ew'(s_2 + s_2 - s_1) + Y) \\ &= \psi_{s_1}'(a) ; \end{aligned}$$

and because u' is strictly decreasing,

$$\psi_{s_1}'(a) = u'(a) - \beta E w'(s_1 - a + Y)$$
$$> u'(a + s_2 - s_1) - \beta E w'(s_2 - (a + s_2 - s_1) + Y)$$
$$= \psi_{s_2}'(a + s_2 - s_1) \ .$$

(b) By (a) and the definition of $c(s_1)$, $\psi_{s_2}'(c(s_1)) > \psi_{s_1}'(c(s_1)) \geq 0$. Hence, $c(s_2) > c(s_1)$.

(c) If $c(s_1) = s_1$, the inequality is trivial. Suppose $c(s_1) < s_1$. Then $\psi_{s_1}'(c(s_1)) = 0$ and, by (a), $\psi_{s_2}'(c(s_1) + s_2 - s_1) < 0$. Hence,

$$c(s_2) < c(s_1) + s_2 - s_1 \ . \qquad \square$$

For every w, there is a nonempty interval $[0, s^*]$, in which it is optimal to consume everything available.

3.3 Lemma: Let s^* be the supremum of those $s \in S$ such that $\psi_s'(s) \geq 0$. Then $0 < s^* \leq \infty$, and one of the following is true;

I. $0 < s^* < \infty$ and s^* is the unique element of S such that $u'(s^*) = \beta E w'(Y)$. Furthermore, $c(s) = s$ or $c(s) < s$ according as $0 \leq s \leq s^*$ or $s > s^*$.

II. $s^* = \infty$ and, for every $s \in S$, $u'(s) > \beta E w'(Y)$ and $c(s) = s$.

In both cases I and II,

$$v(s) = u(s) + \beta E w(Y) \ , \quad 0 \leq s \leq s^* \ , \quad s < +\infty \ . \tag{3.7}$$

Proof: Observe that $c(s) = s$ if and only if $\psi_s'(s) = u'(s) - \beta E w'(Y) \geq 0$. Also,

$$u'(0) - \beta E w'(Y) \geq u'(0) - \beta w'(0) > 0.$$

Everything follows easily. $\square$

If $0 < s^* < \infty$ as in case I of the lemma, then we can write

$$s^* = I(\beta E w'(Y))$$

where I is the inverse of the continuous, strictly decreasing function u'.

We shall need to compare various v_w's and c_w's. Here are some useful facts.

3.4 Lemma: (a) $w \le w_1 \Rightarrow v \le v_1$.

(b) $w' \le w_1' \Rightarrow c \ge c_1$ and $s^* \ge s_1^*$.

(c) If w_n' increases to w' and $w_n(0)$ increases to $w(0)$ as $n \to \infty$, then s_n^* decreases to s^*, $v_n(s)$ increases to $v(s)$ and $c_n(s)$ decreases to $c(s)$, $s \in S$, as $n \to \infty$.

(In the above, v_n has been written for v_{w_n}, c_n for c_{w_n}, and s_n^* for $s_{w_n}^*$.)

Proof: (a) is obvious from the definitions of v and v_1 in (3.1). To check (b), notice that, if $w' \le w_1'$, then $\psi_{s,w}' \ge \psi_{s,w_1}'$ and it follows easily that $c \ge c_1$. It's also easy to use the previous lemma, together with the fact that u' is decreasing, to see that $s^* \ge s_1^*$. Now consider (c). If w_n' increases to w', then

$$w_n(s) = w_n(0) + \int_0^s w_n'(t)dt$$

increases to

$$w(s) = w(0) + \int_0^s w'(t)dt \ .$$

So by (a) and (b), v_n increases to a limit $\tilde{v}$, c_n decreases to a limit $\tilde{c}$, and s_n^* decreases to some $\tilde{s}$. We need to show that $\tilde{c} = c$, $\tilde{v} = v$, and $\tilde{s} = s^*$.

If $\tilde{c}(s) = s$, then $c_n(s) = s$ for all n and so $u'(s) \ge \beta E w_n'(Y)$ for all n. By the monotone convergence theorem, $Ew_n'(Y) \to Ew'(Y)$ and therefore $u'(s) \ge \beta Ew'(Y)$, which implies that $c(s) = s = \tilde{c}(s)$.

If $\tilde{c}(s) < s$ then, for sufficiently large n, $c_n(s) < s$ and $c_n(s)$ satisfies

$$u'(c_n(s)) = \beta E w_n'(s - c_n(s) + Y).$$

Let $n \to \infty$ to get

$$u'(\tilde{c}(s)) = \beta E w'(s - \tilde{c}(s) + Y)$$

and, hence, $\tilde{c}(s) = c(s)$.

To see that $\tilde{v} = v$, calculate as follows:

$$\begin{aligned} v(s) &= u(c(s)) + \beta E w(s - c(s) + Y) \\ &= \lim_n [u(c_n(s)) + \beta E w_n(s - c_n(s) + Y)] \\ &= \lim_n v_n(s) . \end{aligned}$$

The first and last equalities are by the definitions of c and c_n, respectively. The second equality uses monotone convergence, together with the fact that c_n decreases to c.

To see that $\bar{s} = s^*$, first assume $\bar{s} < \infty$. Then $s_n^* < \infty$ for n large and $w' \geq w_n'$ for all n. Thus, by (b), $s^* < \infty$ and, by Lemma 3.3, part I,

$$s^* = I(\beta E w'(Y)) = \lim_n I(E w_n'(Y)) = \lim_n s_n^* .$$

Now assume $\bar{s} = \infty$; i.e. $s_n^* = \infty$ for all n. Then by Lemma 3.3, part II,

$$u'(s) > \beta E w_n'(Y) , \quad s \in S , \quad \text{all } n .$$

Hence,

$$u'(s) \geq \beta E w'(Y) , \quad s \in S .$$

However, u' is strictly decreasing on S and it follows that the last inequality must be strict as well. So $s^* = \infty$. □

Proof of Proposition 3.1. If $s^* = \infty$, the proposition is immediate from (3.7). So assume $0 < s^* < \infty$ as in Lemma 3.3, part I. Again, it is immediate that v has the desired properties on $[0, s^*]$. On $[s^*, \infty)$, $c = \check{c}$ by (3.6), and $c(\cdot)$ is given implicitly by

$$u'(c(s)) = \beta E w'(s - c(s) + Y) . \tag{3.8}$$

If w is bounded and twice continuously differentiable, then c is continuously differentiable on $[s^*, \infty)$ as follows from the implicit function theorem. Now differentiate (3.4) and use (3.8) to get

$$\begin{aligned} v'(s) &= c'(s) u'(c(s)) + \beta(1 - c'(s)) E w'(s - c(s) + Y) \\ &= u'(c(s)) . \end{aligned}$$

Furthermore, the left and right derivatives at s^* of v' agree, because $c(s^*) = s^*$.

For the general case, when w may be only once continuously differentiable, we can approximate w by a sequence $\{w_n\}$ of bounded C^2 functions which satisfy our Assumptions A.2 and A.3, such that $w_n(0)$ increases up to $w(0)$ and also w_n' increases up to w' as $n \to \infty$. Then, by Lemma 3.4,

$$v(s) = \lim_n v_n(s) = \lim_n [v_n(0) + \int_0^s u'(c_n(t))dt]$$

$$= v(0) + \int_0^s u'(c(t))dt \ .$$

It is now easy to check that v has the desired properties. □

4 The Value Function and the Optimal Stationary Plan

In this section, we continue to assume that the utility function u satisfies Assumption A.1, and we shall use the techniques of Section 3 to study the dynamic programming problem. Here is our main result.

4.1 Theorem: Assume that u satisfies

(a) The value function V is concave, strictly increasing, and continuously differentiable on S.

(b) There is a unique optimal stationary plan π, corresponding to a continuous function $c : S \to S$ such that $0 \le c(s) \le s$, and $s \to c(s)$, $s \to s - c(s)$ are nondecreasing

(c) $V'(s) = u'(c(s))$, $s \in S$.

(d) There exists s^* $(0, \infty]$ such that $c(s) = s$ for $s \le s^*$ and $c(s) < s$ for $s > s^*$.

Proof: As in Section 2, let $V_0 = u$, $V_{n+1} = TV_n$ and set $c_{n+1} = c_{V_n}$, $s^*_{n+1} = s^*_{V_n}$ for $n = 0, 1, \ldots$. By Proposition 3.1 and induction, we see that the functions V_n satisfy A.1, and

$$(4.1) \qquad \begin{aligned} V_{n+1}(s) &= u(c_{n+1}(s)) + \beta E V_n(s - c_{n+1}(s) + Y) \ , \\ V_{n+1}'(s) &= u'(c_{n+1}(s)) \ , \quad s \in S \ . \end{aligned}$$

Notice that, for all s,

$$V_1'(s) = u'(c_1(s)) \ge u'(s) = V_0'(s)$$

so that, by Lemma 3.4, $c_2(s) \le c_1(s)$. Thus

$$V_2'(s) = u'(c_2(s)) \ge u'(c_1(s)) = V_1'(s) .$$

Using induction, one easily shows that V_n' are increasing in n, and the c_n are decreasing in n. By Lemma 3.4, the s_n^* are decreasing also. Define

$$c(s) := \lim_n c_n(s) , \quad s \in S .$$

By Proposition 3.1, and the monotone convergence theorem,

$$V(s) - V(0) = \lim_n [V_n(s) - V_n(0)]$$

$$= \lim_n \int_0^s u'(c_n(t))dt = \int_0^s u'(c(t))dt .$$

This establishes (a) and (c).

Now $c(s)$ and $s - c(s)$ are nondecreasing because, by Lemma 3.2, $c_n(s)$ and $s - c_n(s)$ are nondecreasing for every n. It follows that c is also continuous. Furthermore, a passage to the limit in (4.1) gives

$$V(s) = u(c(s)) + \beta EV(s - c(s) + Y) \tag{4.2}$$

and, by Theorem 2.2, c corresponds to an optimal stationary plan. Thus, $c(s)$ maximizes

$$\varphi_s(a) = u(a) + \beta EV(s - a + Y) , \quad 0 \le a \le s .$$

Notice

$$\varphi_s'(a) = u'(a) - \beta EV'(s - a + Y)$$

is strictly decreasing in a because u' is strictly decreasing and, by (c), V' is nonincreasing. It follows that $c(s)$ is the unique maximizing value for every s and, therefore, c is the unique optimal stationary plan. This completes the proof of (b).

For (d), let $s^* = \infty$ if $u'(s) > \beta EV'(Y)$ for all s, and let $s^* = I(\beta EV'(Y))$ otherwise, where I is the inverse function for u'. In the second case, $s^* > 0$ because $u'(0) = V'(0) > \beta EV'(Y)$. □

4.1.1 Comment: Matthew Sobel observes that Theorem 4.1.(d) is valid under the weaker assumption that u satisfies only (2.2).

4.2 Corollary: If $0 < s^* < \infty$, then $s^* = I(\beta EV'(Y))$, where I is the inverse function of u'.

Proof: $c(s) = s$ if and only if $u'(s) \geq \beta EV'(Y)$. □

Here is another property of the optimal stationary plan.

4.3 Theorem: If u satisfies A.1 and c corresponds to the optimal plan as in Theorem 4.1, then $\lim_{s\to\infty} c(s) = \infty$.

Proof: Suppose first that u is unbounded. Then, obviously, V is unbounded also. Now if c is bounded by b, $b < \infty$ and π is the corresponding stationary plan, then

$$V(s) = I(\pi)(s) \leq u(b) + \beta u(b) + \beta^2 u(b) + \cdots = \frac{u(b)}{1-\beta} .$$

We conclude that c is unbounded. The result follows, since c is increasing.

Next assume that u is bounded and set $u(\infty) := \lim_{s\to\infty} u(s)$, $V(\infty) := \lim_{s\to\infty} V(s)$.

4.4 Lemma: $V(\infty) = u(\infty)/(1-\beta)$.

Proof: For $s \in S$ and $n = 1, 2, ...$, $V(ns) \geq \Sigma_{j=0}^{n-1} \beta^j u(s) = (1-\beta^n)u(s)/(1-\beta)$. Let $n \to \infty$ and then let $s \to \infty$ to get $V(\infty) \geq u(\infty)/(1-\beta)$. The opposite inequality is even easier.

Return to the proof of Theorem 4.1, and use the lemma to get

$$\begin{aligned} V(s) &= u(c(s)) + \beta EV(s - c(s) + Y) \\ &\leq u(c(s)) + \beta u(\infty)/(1-\beta) , \end{aligned}$$

and in the limit as $s \to \infty$

$$\frac{u(\infty)}{1-\beta} \leq u(c(\infty)) + \beta \frac{u(\infty)}{1-\beta} ,$$

where $c(\infty) := \lim c(s)$. Since u is strictly increasing, we conclude $c(\infty) = \infty$. □

4.5 Remark: The function $V'(s)$ is equal to $u'(s)$ on $[0, s^*]$, and equal to $u'(c(s)) = \beta EV'(s - c(s) + Y) > u'(s)$ on (s^*, ∞). Therefore, we have

$$V'(s) = \max[u'(s), \beta EV'(s - c(s) + Y)] , \quad \forall s \in S .$$

We deduce that for the Markov chain $\{s_n\}_{n=0}^{\infty}$ given by $s_{n+1} = s_n - c(s_n) + Y_{n+1}$, and with $\xi_n := c(s_n)$, $\theta := \inf\{n \leq 0; s_n \leq s^*\}$, $M_n := \beta^n V'(s_n) = \beta^n u'(\xi_n)$, $n \in \mathbf{N}_0$ is a supermartingale and $M_{\theta \wedge n}$, $n \in \mathbf{N}_0$ is a martingale.

4.6 Remark: Suppose that, in (2.5) the price $p \in (0, \infty)$ of the commodity is not equal to one, so that the motion formula (2.7) takes the form: $s_{n+1} = s_n - a_n + pY_{n+1}$, $n = 0, 1, \ldots$. Then the return function of (2.8) becomes $I(\pi)(s; p) = E^{\pi}_{s_0=s}[\Sigma_{n=0}^{\infty} \beta^n u(a_n/p)]$, and the corresponding value function

$$(4.3) \qquad \begin{aligned} V(s; p) &= \sup_{0 \leq a \leq s} \left[u\left(\frac{a}{p}\right) + \beta EV(s - a + pY; p) \right] \\ &= u\left(\frac{c(s; p)}{p}\right) + \beta EV(s - c(s; p) + pY; p) \end{aligned}$$

by analogy with (2.11) and Theorem 2.2(b). It is quite obvious that the value function $V(s; p)$ and the (stationary) optimal strategy $c(s; p)$ have the *scaling properties*

$$(4.4) \qquad V(sp; p) = V(s; 1) = V(s) , \quad \frac{1}{p} c(sp; p) = c(s; 1) = c(s)$$

for all $s \in (0, \infty)$, $p \in (0, \infty)$.

Furthermore, assume that the Markov chain

$$(4.5) \qquad s_n = s_{n-1} - c(s_{n-1}; p) + pY_n , \quad n = 1, 2, \ldots$$

has a unique stationary distribution $\mu(A) := \mu(A; 1)$, if $p = 1$ (cf. Section 5 for a study of this question); then it also has a unique stationary distribution $\mu(A; p)$ for any other value of p in $(0, \infty)$, and this is given by

$$(4.6) \qquad \mu(A; p) = \mu\left(\frac{1}{p} A\right) ; \quad A \in \mathscr{B}([0, \infty)) .$$

5 The Stationary Distribution

Assume that the utility function u satisfies Assumption A.1 of Section 3, and let π be the optimal stationary plan with consumption function c given by Theorem 4.1. In this section we assume that the agent uses π, and study the resulting Markov chain of successive states of wealth $s_0, s_1, \ldots$ given by $s_{n+1} = s_n - c(s_n) + Y_n$, $n = 0, 1, \ldots$. Our main result is that the chain is *positive recurrent*, with a stationary distribution μ *which can be regarded as an equilibrium distribution of wealth for many independent agents facing the same problem* (cf. Remark 7.4).

5.1 Theorem: Under the optimal plan π, the Markov chain $\{s_n\}_{n=0}^{\infty}$ has a unique stationary distribution μ. □

The proof will be based on the renewal theorem or, more precisely, one of its corollaries (Theorem 3.5, p. 153 in Asmussen (1987)).

Let $s^* \in (0, \infty]$ be as in Theorem 4.1. If $s^* = \infty$, then $s_n = Y_n$ for $n = 1, 2, \ldots$ and the stationary distribution is just the common distribution λ of the Y_n's. So assume, for the rest of this section, that $0 < s^* < \infty$ and define $R = [0, s^*]$. The set R is a *regeneration set* for the chain, because whenever $s_n \in R$ we have $c(s_n) = s_n$ and $s_{n+1} = Y_{n+1}$. Thus, *whenever the chain visits R, it starts over with initial distribution* λ. Our theorem will follow from that of Asmussen (1987), if we show that regenerations occur in finite expected time. Define

$$\tau^* := \inf\{n \geq 1; s_n \in R\}, \tag{5.1}$$

so that τ^* is the time of the first regeneration and, if $s_0 \in R$, τ^* is the length of a typical cycle. Theorem 5.1 will follow from the proposition below.

5.2 Proposition: If s_0 is constant, or if s_0 is random and $Es_0 < \infty$, then $E\tau^* < \infty$. □

The proof will be given in several lemmas. First, observe that since, by Theorem 4.3, $c(s) \to \infty$ as $s \to \infty$, there exists $\bar{s}$ such that $s^* < \bar{s} < \infty$ and $s \geq \bar{s} \Rightarrow c(s) \geq EY + 1$. We shall first show that the expected time to reach $[0, \bar{s}]$ is finite. Define

$$\bar{\tau} := \inf\{n \geq 0; s_n \leq \bar{s}\}. \tag{5.2}$$

5.3 Lemma: For s_0 constant, $E\tau \le (s_0 - \bar{s})^+ + c(\bar{s})$.

Proof: The idea is to compare $\{s_n\}_{n=0}^{\infty}$ with the random walk $\{X_n\}_{n=0}^{\infty}$ defined by

$$X_0 = s_0\,,\quad X_{n+1} = X_n - c(\bar{s}) + Y_{n+1}\,,\quad n \ge 0\,. \tag{5.3}$$

Notice $X_n \ge s_n$ for all $n \le \tau$, because $c(s) \ge c(\bar{s})$ for $s \ge \bar{s}$. So, if

$$\tau_X := \inf\{n \ge 0;\, X_n \le \bar{s}\}\,, \tag{5.4}$$

then $\tau_X \ge \tau$. Now the random walk has drift $-c(\bar{s}) + EY \le -1$, and it follows from a version of the optional sampling theorem (or Wald's equation) that, for $s_0 > \bar{s}$,

$$\begin{aligned} E\tau_X &= \frac{s_0 - EX_{\tau_X}}{c(\bar{s}) - EY} \\ &\le s_0 - EX_{\tau_X} \le s_0 - \bar{s} + c(\bar{s})\,. \end{aligned} \tag{5.5}$$

We are using here the inequality $E(X_{\tau_X}) = E[X_{\tau_X - 1} - c(\bar{s}) + Y_{\tau_X}] \ge \bar{s} - c(\bar{s})$. □

5.4 Lemma: For s_0 random, we have $E\tau \le E(s_0 - \bar{s})^+ + c(\bar{s})$.

Proof: Condition on s_0, and use Lemma 5.3. □

The next idea is that, once the chain reaches $[0, \bar{s}]$, it will go on to reach R in a fixed, finite number of periods, with a probability which is bounded away from 0. We shall first show that the income variable Y is less than s^* with positive probability. Define

$$\varepsilon_0 := \inf\{\varepsilon > 0;\, P[Y \ge \varepsilon] = 1\}\,.$$

5.5 Lemma: $s^* > \varepsilon_0$.

Proof: Recall that I is the inverse of u' and is strictly decreasing. Now use Theorem 4.1 and Corollary 4.2 to see that

$$I(\beta EV'(Y)) = s^* = I(u'(s^*)) = I(u'(c(s^*))) = I(V'(s^*))\,.$$

Hence,

$$\beta V'(s^*) < V'(s^*) = \beta EV'(Y) \le \beta V'(\varepsilon_0)\,,$$

where the final inequality uses the definition of ε_0 together with the fact that V' is nonincreasing. The desired inequality now follows from the same fact. □

5.6 Lemma: There is a positive integer N and a number $\eta \in (0, 1)$ such that

$$P[\tau* \leq N | s_0 = s] \geq \eta \ , \quad \forall s \in (s^*, \mathfrak{T}] \ . \tag{5.6}$$

Proof: Choose $\delta \in (\varepsilon_0, s^*)$ by the previous lemma. By definition of ε_0, $\alpha = P[Y < \delta] > 0$. Then, for $t \geq s^*$, $c(t) \geq c(s^*) = s^*$ and

$$P[-c(t) + Y < \delta - s^*] \geq \alpha \ .$$

So the process s_n -- whenever it is to the right of s^* -- moves at least a distance $\delta - s^*$ (to the left) with probability at least α, namely $P[s_{n+1} \leq s_n - (s^* - \delta) | s_n > s^*] \geq \alpha$. Now choose N so that $(-\delta + s^*)N > \mathfrak{T} - s^*$, and set $\eta := \alpha^N$. □

We are ready at last for the

Proof of Proposition 5.2: First write

$$\tau^* = \bar{\tau} + \tau \ , \tag{5.7}$$

where τ is the additional time to reach $[0, s^*]$ after reaching $[0, \mathfrak{T}]$. By Lemma 5.4, $E\bar{\tau} < \infty$, and we need only show that $E\tau < \infty$.

To see this, think of an entry of the process into $[0, \mathfrak{T}]$ as the start of a "trial"; call the trial a "success," if the process goes on to reach $[0, s^*]$ within the next N days and call it a "failure" otherwise. Notice that, by Lemma 5.6 the probability of success is at least η and, in the event of a failure, the process will be at a random state which is stochastically smaller than

$$\mathfrak{T} + Y_1 + \cdots + Y_N \ ,$$

a random variable with expectation $\mathfrak{T} + NE(Y) < \infty$. Thus, by Lemma 5.4, again the expected time until the next entry into $[0, \mathfrak{T}]$ is a random variable σ with finite expectation. Therefore, τ is stochastically dominated by a sum

(5.8) $$Z = \sum_{k=1}^{T} W_k \ ,$$

where the W_k are independent random variables; each of them is equal to N with probability at least η, equal to a variable $\sigma^{(k)}$ distributed like σ with the remaining probability. Finally, T is a geometric stopping variable distributed like the number of Bernoulli trials until the first success. By Wald's equation, $E\tau \le EZ = E(T)\cdot E(W_1) < \infty$.

This completes the proof. □

It is well-known (cf. Asmussen (1987), p. 152) that the stationary measure μ of Theorem 5.1 can be represented as

(5.9) $$\mu(A) = \frac{1}{E_\lambda(\tau^*)} E_\lambda \sum_{k=0}^{\tau^*-1} 1_A(s_k) \ , \quad A \in \mathscr{B}([0, \infty))$$

or equivalently as

(5.10) $$\int_0^\infty f(s)\mu(ds) = \frac{1}{E_\lambda(\tau^*)} E_\lambda \sum_{k=0}^{\tau^*-1} f(s_k) \ ,$$

for every measurable $f : [0, \infty) \to [0, \infty)$. Here τ^* is the stopping time of (5.1), and the subscript λ in E_λ means that the initial state s_0 of the Markov chain has the distribution λ (of the random variables $Y, Y_1, Y_2, \ldots$). From Lemma 5.4 and the assumption $\int_0^\infty y\lambda(dy) < \infty$, we have $E_\lambda(\tau^*) < \infty$. On the other hand, (5.10) yields

(5.11) $$\int_0^\infty s\mu(ds) = \frac{1}{E_\lambda(\tau^*)} E_\lambda \sum_{k=0}^{\tau^*-1} s_k \ .$$

We shall need conditions, under which this expression is finite.

5.7 Theorem: Under the additional assumption $E(Y^2) = \int_0^\infty y^2\lambda(dy) < \infty$, the stationary distribution μ of Theorem 5.1 has finite first moment: $\int_0^\infty s\mu(ds) < \infty$. □

The rest of the section will be devoted to a proof of this result. Clearly from (5.11) it suffices to show

$$E_\lambda\left(\sum_{k=0}^{\tau-1} s_k\right) < \infty, \tag{5.12}$$

$$E_\lambda\left(\sum_{k=\tau}^{\tau^*-1} s_k\right) < \infty, \tag{5.13}$$

where τ is the stopping time of (5.2). Now in the notation of (5.3), (5.4) we have

$$E_\lambda\left(\sum_{k=0}^{\tau-1} s_k\right) \le E_\lambda\left(\sum_{k=0}^{\tau-1} X_k\right) \tag{5.14}$$

and thus, in order to prove (5.12), it suffices to establish the analogous property for the random walk of (5.3).

5.8 Lemma: Let $\xi = \xi_0, \xi_1, \ldots$ be a sequence of IID random variable with $P(\xi \ge -b) = 1$ for some $b \in (0, \infty)$, $E(\xi) < 0$ and $E(\xi^2) < \infty$. For any given $x \in (0, \infty)$, consider the random walk $X_n = x + \Sigma_{k=0}^{n-1}\xi_k$, $n \in \mathbf{N}_0$ and the stopping time $\sigma_x := \inf\{n \in \mathbf{N}_0; X_n \le 0\}$. We have

$$f(x) := E\left(\sum_{n=0}^{\sigma_x-1} X_n\right) < \infty, \quad \forall x \in (0, \infty) \tag{5.15}$$

and

$$\int_0^\infty f(x)\lambda(dx) < \infty \tag{5.16}$$

for any distribution λ on $[0, \infty)$ with finite second moment.

Proof: By analogy with Lemma 5.3, we have $E(\sigma_x) < \infty$. On the other hand, $X_{\sigma_x-1} \le X_{\sigma_x-1} - X_{\sigma_x} = -\xi_{\sigma_x-1} \le b$ and inductively: $X_{\sigma_x-2} \le 2b$, $X_{\sigma_x-3} \le 3b$, ..., $X_0 \le b\sigma_x$, almost surely. Therefore, $0 \le \Sigma_{k=0}^{\sigma_x-1}X_k \le b\sigma_x(1 + \sigma_x)/2$ a.s. and thus

$$0 \le f(x) \le \frac{b}{2}[E(\sigma_x^2) + E(\sigma_x)] < \infty \tag{5.17}$$

since $E(\sigma_x^2) < \infty \leftarrow E(\xi^2) < \infty$ (cf. Gut (1988), p. 78).

This proves (5.15); for (5.16), let us start by observing that σ_x is stochastically larger than σ_y, if $x > y$. Thus, it suffices to show

(5.18) $$\sum_{n=1}^{\infty} f(n)\lambda((n-1, n]) < \infty .$$

To this end, fix an initial position $x = n \in \mathbf{N}$, and introduce the stopping time

$$R_1 := \inf\{k \geq 1; X_k \leq n-1\} = \inf\left\{k \geq 1; \sum_{j=0}^{k-1} \xi_1 \leq -1\right\}$$

and its independent copies

$$R_2 := \inf\{k \geq 1; X_{k+T_1} \leq X_{T_1} - 1\}, \ldots,$$

$$R_{j+1} := \inf\{k \geq 1; X_{k+T_j} \leq X_{T_j} - 1\} .$$

Obviously $\sigma_n \leq \Sigma_{j=1}^{n} R_j$, and thus $E(\sigma_n) \leq n \cdot E(R_1)$, $E(\sigma_n^2) \leq Cn^2$, for all $n \in \mathbf{N}$, where $C \in (0, \infty)$ is a constant depending only on $E(R_1)$, $E(R_1^2)$. Now from (5.17) we obtain $f(n) \leq bCn^2$, and (5.18) is obviously satisfied. □

Proof of Theorem 5.7: Take $\xi_j = Y_j - b$ in Lemma 5.8, where $b = c(\bar{s})$; it is easily seen that the conditions of the lemma are satisfied. We conclude that, for the random walk $\{X_n\}_{n=0}^{\infty}$ of (5.3), the right-hand side of (5.14) is finite. This proves (5.12).

To prove (5.13), it suffices to show that there exists a constant $C \in (0, \infty)$, such that

$$g(s) := E_{s_0=s}\left(\sum_{k=0}^{\tau^*-1} s_k\right) \leq C , \quad \forall s \in [0, \bar{s}] .$$

But from the fact that $s \to x - c(s)$ is nondecreasing, one can check easily that the mapping $s \to g(s)$ is nondecreasing as well, so it suffices to prove $g(\bar{s}) < \infty$. Now let us recall the setup and the notation of the proof of Proposition 5.2. Repeating the type of reasoning that we employed there, we observe that, by analogy with (5.5), the random variable $\Sigma_{k=0}^{\tau^*-1} s_k$ (with $s_0 = \bar{s}$) is stochastically dominated by a sum of the form

$$(5.5) \qquad \tilde{Z} = \sum_{k=1}^{T} \tilde{W}_k .$$

Here T is as in (5.8), and the $\tilde{W}_k$'s are IID random variables; each of them is equal to A_k with probability η, and equal to $A_k + B_k$ with probability $1-\eta$, $k = 1, 2, \ldots$. The sequences $\{A_k\}_{k=1}^{\infty}$, $\{B_k\}_{k=1}^{\infty}$ are independent, and consist of IID random variables; A_1 is distributed like $\Sigma_{j=0}^{N-1} s_j$ (with $s_0 = \tilde{s}$), whereas B_1 is distributed like $\Sigma_{j=0}^{N-1} s_j$ (with s_0 having the distribution of $\tilde{s} + Y_0 + \cdots Y_{N-1}$). Obviously $E(A_1) < \infty$, and $E(B_1) < \infty$ from (5.14) and Lemma 5.8. Therefore, $E(\tilde{W}_1) < \infty$ and

$$g(\tilde{s}) = E_{s_0=\tilde{s}}\left(\sum_{k=0}^{\tau^*-1} s_k\right) \leq E(T)\cdot E(\tilde{W}_1) < \infty .$$

6 The Saturated Case

In this section we consider a utility function which saturates at a finite value. More precisely, we impose the following condition in place of Assumption A.1.

B In addition to the properties (1.2), we assume the following:

(i) There exists $b \in (0, \infty)$ such that $u(x) = u(h)$ for $h \leq x < \infty$.

(ii) u has a continuous derivative on $S = [0, \infty)$.

(iii) u' is strictly positive and strictly decreasing on $(0, h)$.

If the income variable Y is greater than or equal to h with probability one, then the problem is trivial. After the first period, the agent can always attain the maximum possible utility and, consequently,

$$V(s) = u(s) + \sum_{n=1}^{\infty} \beta^n u(b)$$

$$= u(s) + \frac{\beta u(h)}{(1-\beta)} .$$

Furthermore, the optimal plan is not unique because, for $s > h$, any action in the interval $[h, s]$ is optimal.

The situation is more interesting if Y is less than h with positive probability. Indeed, the methods of Sections 3 and 4 can be adapted to prove all the conclusions of Theorem 4.1 if Assumption A.1 is replaced by B together with the condition $P[Y < h] > 0$. (Part (d) of Theorem 4.1 can be strengthened to say $0 < s^* < \infty$.) However, it is no longer true as in Theorem 4.3 that the optimal consumption function $c(s)$ approaches infinity as $s \to \infty$. It is intuitively clear that $c(s) \leq h$, and it can be shown that $c(s) \nearrow h$, as $s \to \infty$.

The method of Section 5 can also be adapted to prove the existence of a unique stationary distribution as in Theorem 5.1 under assumption B and the additional assumption $EY < h$; furthermore, the additional condition $E(Y^2) = \int_0^\infty y^2 \lambda(dy) < \infty$ guarantees that this stationary distribution has a finite first moment). If, instead, $EY \geq h$, the Markov chain $\{s_n\}_{n=0}^\infty$ is not positive recurrent, and has no stationary distribution.

7 Stationary Competitive Equilibrium

Consider now an index set $I = [0, 1]$ and a non-atomic probability measure ϕ on $\mathcal{B}(I)$; the set I represents an uncountable collection of agents, whereas ϕ represents the "spatial" distribution of these agents on I. Each agent $\alpha \in I$ has a utility function $u^\alpha(\cdot)$ and receives a sequence of random endowments $Y_1^\alpha, Y_2^\alpha, \ldots$; these are independent copies of the nonnegative random variable Y^α, whose distribution we shall denote by λ^α. For each $\alpha \in I$, the utility function $u^\alpha(\cdot)$ and the distribution λ^α satisfy the assumptions imposed on them in Theorems 5.1, 5.7.

In particular, if the price $p \in (0, \infty)$ of the non-durable commodity is announced in advance and fixed from period to period, each agent $\alpha \in I$ faces an infinite-horizon discounted dynamic programming problem of the type (2.7), (2.8). For this problem, the value function $V^\alpha(s; p)$ and the optimal consumption level $c^\alpha(s; p)$ satisfy, by analogy with (4.3), the Bellman equation

$$(7.1)\qquad \begin{aligned} V^\alpha(s;p) &= \max_{0\le c\le s}\left[u^\alpha\left(\frac{c}{p}\right) + \beta\cdot EV^\alpha(s - c + pY^\alpha; p)\right] \\ &= u^\alpha\left(\frac{c^\alpha(s;p)}{p}\right) + \beta\cdot EV^\alpha(s - c^\alpha(s;p) + pY^\alpha; p)\ . \end{aligned}$$

According to Theorem 5.7, the Markov chain

$$(7.2)\qquad s_n^\alpha = s_{n-1}^\alpha - c^\alpha(s_{n-1}; p) + pY_n^\alpha\ ;\ \ n = 1, 2, \ldots$$

has then a unique invariant measure $\mu^\alpha(ds; p)$ with finite first moment, and

$$(7.3)\qquad \int_0^\infty c^\alpha(s;p)\mu^\alpha(ds;p) = p\cdot E(Y^\alpha)$$

$$(7.4)\qquad \mu^\alpha(A;p) = \int_0^\infty P[(s - c^\alpha(s;p) + pY^\alpha) \in A]\mu^\alpha(ds;p)\ ;\ \ A \in \mathscr{B}([0,\infty))\ .$$

(This last identity merely states the invariance of the measure $\mu^\alpha(\cdot\,; p)$, whereas (7.3) follows from (7.2) by taking expectations, and recalling that $E(s_n^\alpha) = \int_0^\infty s\mu^\alpha(ds;p) = \int_0^\infty s\mu^\alpha(ds)$ is independent of n and finite; cf. Theorem 5.7.)

Consider now a situation, in which all these agents interact with each other in the following way. At the beginning of period $t = n$ ($n \in$ N), with price $p_{n-1}(\omega)$ and random wealths $S_{n-1}^\alpha(\omega)$, $\alpha \in I$ across agents, each agent α bids the amount $b_n^\alpha(\omega) = c^\alpha(S_{n-1}^\alpha(\omega); p_{n-1}(\omega))$. Thus, the total amount bid is given, by analogy with (1.1), as

$$(7.5)\qquad B_n(\omega) = \int_I c^\alpha(S_{n-1}^\alpha(\omega); p_{n-1}(\omega))\phi(d\alpha)\ .$$

Then the endowments $\{Y_n^\alpha(\omega)\}_{\alpha\in I}$ for the period $t = n$ are revealed for all the different agents; *it is assumed that the total endowment in each period $t = n$ namely*

$$(7.6)\qquad Q_n = \int_I Y_n^\alpha(\omega)\phi(d\alpha)\ ,$$

is *it non-random*; in particular,

$$(7.7) \qquad Q_n = Q := \int_I E(Y^\alpha)\phi(d\alpha)\ , \quad \forall n \in \mathbb{N}_0\ .$$

A new price is then formed, as the ratio of total bid over total endowment:

$$(7.8) \qquad p_n(\omega) := \frac{B_n(\omega)}{Q_n} = \frac{1}{Q}\int_I c^\alpha(S^\alpha_{n-1}(\omega);\, p_{n-1}(\omega))\phi(d\alpha)\ .$$

Finally, each agent consumes his bid's worth $x^\alpha_n(\omega) = c^\alpha(S^\alpha_{n-1}(\omega);\, p_{n-1}(\omega))/p_n(\omega)$ and receives his endowment's worth (denominated in the new price), thereby receiving a reward $u^\alpha(x^\alpha_n(\omega))$ for the nth period and starting the next period $t = n+1$ with price $p_n(\omega)$ and wealth

$$(7.9) \qquad S^\alpha_n(\omega) = S^\alpha_{n-1}(\omega) - c^\alpha(S^\alpha_{n-1}(\omega);\, p_{n-1}(\omega)) + p_n(\omega)Y^\alpha_n(\omega)\ , \quad \alpha \in I\ .$$

This procedure is then repeated ad infinitum.

It should be noted that, with $\mathcal{F}_0 := \sigma(p_0,\, S^\alpha_0;\ \alpha \in I)$ and $\mathcal{F}_n := \sigma(p_0, S^\alpha_0,\, Y^\alpha_k;\ \alpha \in I,\ k = 1, \ldots, n)$ for $n \in \mathbb{N}$, the random variables $p_n(\omega)$, $S^\alpha_n(\omega)$ $(\alpha \in I)$ are measurable with respect to $\mathcal{F}_{n-1}$ and $\mathcal{F}_n$, respectively, for $n \in \mathbb{N}$. Let us also introduce the random measure

$$(7.10) \qquad \nu_n(A,\, \omega) := \int_I 1_A(S^\alpha_n(\omega))\phi(d\alpha)\ ;\ \ A \in \mathcal{B}([0,\, \infty))\ , \quad n \in \mathbb{N}_0$$

which describes the spatial distribution of wealth across agents, at time $t = n$.

7.1 Remark: If the agents are *homogeneous* (i.e., they all have the same utility function $u = u^\alpha$ and all the random variables Y^α have the same distribution λ as the random variable Y), then the formula (7.8) takes the simpler form

$$(7.11) \qquad p_n(\omega) = \frac{1}{Q}\int_0^\infty c(s;\, p_{n-1}(\omega))\nu_{n-1}(ds;\, \omega)$$

where $Q = \int_0^\infty y\lambda(dy)$, and (7.3), (7.4) become

$$(7.12) \qquad \int_0^\infty c(s;\, p)\mu(ds;\, p) = pQ\ ,$$

(7.13) $$\mu(A; p) = \int_0^\infty \lambda\left(\frac{A - s + c(s; p)}{p}\right)\mu(ds; p) \ , \quad A \in \mathscr{B}([0, \infty)) \ .$$

We are now in a position to introduce the notion of a "stationary competitive equilibrium" for the system of interacting agents (*strategic market game*) described in this section, and to construct such an equilibrium explicitly in terms of the invariant measures μ^α, $\alpha \in I$ of the individual Markov Chains in (7.2). We deal first with the homogeneous case of Remark 7.1, and then extend that result to the case of countably (Theorem 7.6) and uncountably (Remark 7.7) many homogeneous classes of agents.

7.2 Definition: A *stationary competitive equilibrium* is a pair (p, μ), where $p \in (0, \infty)$ and μ is a probability measure on $\mathscr{B}([0, \infty))$, such that with $p_0 = 0$ and $\nu_0 = \mu$ we have $p_n = p$ and $\nu_n = \mu$, for all $n = 1, 2, \ldots$ in (7.8), (7.10).

It is relatively straightforward to construct a stationary competitive equilibrium in the *homogeneous case* of Remark 7.1. To accomplish this, let $Y_1^\alpha(\omega)$, $Y_2^\alpha(\omega)$, ... be independent copies of the measurable mapping $Y : I \times \Omega \to [0, \infty)$ with marginals

(7.14) $$P[\omega \in \Omega; Y^\alpha(\omega) \in A] = \lambda(A) \ , \quad \forall \alpha \in I$$

(7.15) $$\phi[\alpha \in I; Y^\alpha(\omega) \in A] = \lambda(A) \ , \quad \forall \omega \in \Omega$$

for any $A \in \mathscr{B}([0, \infty))$ (e.g., Feldman & Gilles (1985), Proposition 2).

7.3 Theorem: In the homogeneous case $u^\alpha = u$, $\lambda^\alpha = \lambda$ $(\forall \alpha \in I)$, for any given $p \in (0, \infty)$ and with $\mu(\cdot) = \mu(\,\cdot\,; p)$ the ergodic probability measure of (7.13), the pair (p, μ) is a stationary competitive equilibrium and satisfies the equations (1.12), (1.13).

Proof: It suffices to show $p_1(\omega) = p$, $\nu_1(\cdot\,, \omega) = \mu(\cdot\,; p)$. Indeed, from (7.11) and (7.12) we have

(7.16) $$p_1(\omega) = \frac{1}{Q}\int_0^\infty c(s; p)\nu_0(ds; \omega) = \frac{1}{Q}\int_0^\infty c(s; p)\mu(ds; p) = p \ ,$$

and from (7.10), (7.9), (7.13):

$$\nu_1(A, \omega) = \phi\{\alpha \in I;\ S_1^\alpha(\omega) \in A\}$$

$$= \phi\{\alpha \in I;\ (S_0^\alpha(\omega) - c(S_0^\alpha(\omega); p) + pY_1^\alpha(\omega)) \in A\}$$

(7.17)
$$= \int_0^\infty \phi\{\alpha \in I;\ (s - c(s; p) + pY_0^\alpha(\omega)) \in A\}\cdot\phi\{\alpha \in I;\ S_0^\alpha(\omega) \in ds\}$$

$$= \int_0^\infty \lambda\left(\frac{A - s + c(s; p)}{p}\right)\mu(ds; p) = \mu(A; p)\ ,\quad \forall A \in \mathscr{B}([0, \infty))\ . \quad \square$$

7.4 Remark: In the setting of Theorem 7.3, the constant size of "money supply" W_n in (1.6) is given by $W = \int_0^\infty s\mu(ds; p) = p\int_0^\infty s\mu(ds; 1)$; with this fixed size specified in advance, there is only *one* equilibrium pair $(p, \mu) \in (0, \infty) \times \mathcal{M}$ in Theorem 7.3, namely

(7.18)
$$p = W/\int_0^\infty s\mu(ds; 1)\ ,\quad \mu(ds; p) = \mu\left(\frac{ds}{p}; 1\right).$$

On the other hand, the quantity of (7.5) becomes: $B = \int_0^\infty c(s; p)\mu(ds; p)$, and the ratio

$$\theta := \frac{B}{W} = \frac{pQ}{\int_0^\infty s\mu(ds; p)} = \frac{Q}{\int_0^\infty s\mu(ds; 1)}$$

acquires the significance of a "societal measure of relative risk-aversion" for the economy as a whole.

7.5 Open Question: Suppose that, in the homogeneous case of Proposition 7.3, we set $p_0(\omega) = p \in (0, p)$ and $\nu_0(\cdot\ , \omega) = \mu(\cdot\ , q)$ for some $q \in (0, \infty)$, $q \neq p$. What can be said about the limiting behavior of the (non-random) sequences $\{p_n\}_{n\in\mathbf{N}_0} \subseteq (0, \infty)$, $\{\nu_n\}_{n\in\mathbf{N}_0} \subseteq \mathcal{M}$? $\square$

In order to deal with the *non-homogeneous case*, let us imagine that the space I of agents has been partitioned into countably many disjoint "clans" I_k, $k \in \mathbf{N}$ so that $I = \bigcup_{k=1}^\infty I_k$, $w_k := \phi(I_k) > 0$, $\Sigma_{k=1}^\infty w_k = 1$. Agents in each clan I_k receive endowments with the same distribution λ_k, and have the same utility function u^k (both λ^k, u^k satisfy the conditions of Theorems 5.1 and 5.7). More precisely, consider for each $k \in \mathbf{N}$ a measurable function ${}^kY : I_k \times \Omega \to [0, \infty)$ with marginals

$$(7.14)' \qquad P[\omega \in A;\ {}^kY^\alpha(\omega) \in A] = \lambda^k(A)\ , \quad \forall \alpha \in I_k$$

$$(7.15)' \qquad \phi[\alpha \in I_k;\ {}^kY^\alpha(\omega) \in A] = \lambda^k(A)\ , \quad \forall \omega \in \Omega$$

for any $A \in \mathscr{B}([0, \infty))$; let ${}^kY_1^\alpha(\omega)$, ${}^kY_2^\alpha(\omega)$, ... be independent copies of this function (for fixed $\alpha \in I_k$, these represent the independent daily endowments for agent α), and set

$$(7.19) \qquad Q_k := \int_0^\infty y\lambda^k(dy)\ , \quad Q := \sum_{k=1}^\infty w_k Q_k\ , \quad \lambda := \sum_{k=1}^\infty w_k \lambda^k\ .$$

Agents in the "clan" I_k have the same utility function u^k, and face the same dynamic programming problem

$$(7.1)' \qquad \begin{aligned} V^k(s; p) &= \max_{0 \le a \le s}\left[u^k\left(\frac{s}{p}\right) + \beta \int_0^\infty V^k(s - a + py; p)\lambda^k(dy)\right] \\ &= u^k\left(\frac{c^k(s; p)}{p}\right) + \beta \int_0^\infty V^k(s - c^k(s; p) + py; p)\lambda^k(dy) \end{aligned}$$

corresponding to a price $p \in (0, \infty)$ that is held fixed from period to period, with optimal stationary consumption plan $c^k(s; p)$ and invariant measure $\mu^k(ds; p)$ that satisfy

$$(7.3)' \qquad \int_0^\infty c^k(s; p)\mu^k(ds; p) = pQ_k$$

$$(7.4)' \qquad \mu^k(A; p) = \int_0^\infty \lambda^k\left(\frac{A - s + c^k(s; p)}{p}\right)\mu^k(ds; p)\ , \quad \forall A \in \mathscr{B}([0, \infty))\ .$$

We introduce also the aggregate of these invariant measures:

$$(1.11)' \qquad \bar{\mu}(A; p) := \sum_{k=1}^\infty w_k \mu^k(A; p) = \int_I \mu^\alpha(A; p)\phi(d\alpha)\ , \quad A \in \mathscr{B}([0, \infty))\ .$$

7.6 Theorem: *The countably non-homogeneous case.* With the above assumptions and notation, the pair (p, μ) with $\mu(\cdot) = \bar{\mu}(\cdot\ ; p)$ as in (1.11)′ is a stationary competitive equilibrium, for any $p \in (0, \infty)$. And as in Remark 7.4, there is only *one* such pair for any given, fixed level of the "money supply" $W = \int_0^\infty s\bar{\mu}(ds; p) = p\int_I\int_0^\infty s\mu^\alpha(ds; 1)\phi(d\alpha)$.

Proof: For $k \in \mathbf{N}$, introduce the random measures

$$ {}^k\nu_n(A,\omega) := \frac{1}{w_k}\phi(\alpha \in I_k;\ S_n^\alpha(\omega) \in A)\ ,\quad n \in \mathbf{N}_0 \tag{7.20}$$

so that $\nu_n(A,\omega) = \Sigma_{k=1}^\infty w_k\,{}^k\nu_n(A,\omega)$ in (7.10). Suppose that $p_0(\omega) = p$ and ${}^k\nu_0(\cdot\,,\omega) = \mu^k(\cdot\,;p)$, for all $k \in \mathbf{N}$ (so, in particular, $\nu_0(\cdot\,,\omega) = \bar{\mu}(\cdot\,;p)$). In order to establish equilibrium, it suffices to show $p_1(\omega) = p$, ${}^k\nu_1(\cdot\,,\omega) = \varphi^k(\cdot\,;p)$ (and thus, in particular, $\nu_1(\cdot\,,\omega) = \bar{\mu}(\cdot\,;p)$). Indeed, from (7.8), (7.3)', and (7.19) we have

$$\begin{aligned} p_1(\omega) &= \frac{1}{Q}\sum_{k=1}^\infty \int_{I_k} c^k(S_0^\alpha(\omega);p)\phi(d\alpha) \\ &= \frac{1}{Q}\sum_{k=1}^\infty w_k \int_0^\infty c^k(s;p)\mu^k(ds;p) \\ &= \frac{1}{Q}p\sum_{k=1}^\infty w_k Q_k = p\ , \end{aligned} \tag{7.8'}$$

and from (7.20), (7.9), (7.15)', and (7.4)'

$$\begin{aligned} {}^k\nu_1(A,\omega) &= \frac{1}{w_k}\phi\left\{\alpha \in I_k;\ S_0^\alpha(\omega) - c^k(S_0^\alpha(\omega);p) + p\,{}^kY_1^\alpha(\omega)) \in A\right\} \\ &= \frac{1}{w_k}\int_0^\infty \phi\left\{\alpha \in I_k;\ {}^kY_1^\alpha(\omega) \in \frac{A - s + c^k(s;p)}{p}\right\}\phi(\alpha \in I_k;\ S_0^\alpha(\omega) \in ds) \\ &= \int_0^\infty \lambda^k\left(\frac{A - s + c^k(s;p)}{p}\right)\mu^k(ds;p) = \mu^k(A;p)\ ,\quad \forall k \in \mathbf{N}\ . \end{aligned}$$

□

7.7 Remark: Finally, we consider the *uncountably nonhomogeneous case* as follows: let I, K be copies of $[0, 1]$ and set $\mathfrak{I} = I \times K$. One thinks of an element (α, k) of $\mathfrak{I}$ as an agent $\alpha \in I$, of type $k \in K$. We endow I, K, and $\mathfrak{I}$ with their Borel σ-fields, and consider a non-atomic probability measure ψ on $\mathscr{B}(\mathfrak{I})$ with marginal w on K and with regular conditional probability ϕ^k on I, given $k \in K$:

$$\psi(B) = \int_K \phi^k(B_k) w(dk) \ , \quad \forall B \in \mathscr{B}(\mathfrak{S}) \ .$$

Agents of the same type are supposed to have the same utility function u^k, and the same distribution λ^k of daily endowment; $u^{(\alpha,k)} = u^k$, $\lambda^{(\alpha,k)} = \lambda^k$ for every $k \in K$, $p \in (0, \infty)$, so we can define

$$\bar{\mu}(A; p) := \int_K \mu^k(A; p) w(dk) = \int_{\mathfrak{S}} \mu^{\alpha,k}(A; p) \phi(d\alpha, dk) \ , \quad A \in \mathscr{B}([0, \infty)) \ .$$

It can be shown then, by analogy with the proof of Theorem 7.6, that $(p, \bar{\mu}(\cdot\, ; p))$ is a *stationary competitive equilibrium, for every* $p \in (0, \infty)$. We omit the (straightforward) details.

8 Appendix

We shall establish in this section the optimality of the stationary plan π of Example 2.5, for the problem treated there, and for an arbitrary sequence of nonnegative random variables $Y_1, Y_2, \ldots$. In order to set the stage for these investigations, let us consider a random sequence $(s_0, e_0), (s_1, e_1), \ldots$ with

$$\text{(A.1)} \qquad \begin{aligned} s_{n+1} &= s_n - c(s_n) + y_{n+1} \ , \quad s_0 \geq 0 \\ e_{n+1} &= e_n + \beta^{n+1} u(c(s_n)) \ , \quad e_0 \geq 0 \ . \end{aligned}$$

Let us also look at another sequence $(t_0, d_0), (t_1, d_1), \ldots$ generated by the same mechanism, but with possibly different initial conditions:

$$\text{(A.2)} \qquad \begin{aligned} t_{n+1} &= t_n - c(t_n) + y_{n+1} \ , \quad t_0 \geq 0 \\ d_{n+1} &= d_n + \beta^{n+1} u(c(t_n)) \ , \quad d_0 \geq 0 \ . \end{aligned}$$

A.1 Lemma: If $s_0 \leq t_0$, $e_0 \leq d_0$, then $s_1 \leq t_1$, $e_1 \leq d_1$ and by induction $s_n \leq t_n$, $e_n \leq d_n (n \in \mathbf{N})$. If furthermore, $0 \leq t_0 - s_0 \leq d_0 - e_0$, then we have $0 \leq t_1 - s_1 \leq d_1 - e_1$, and by induction $0 \leq t_n - s_n \leq d_n - e_n (n \in \mathbf{N}\})$. □

In the proof of this lemma, and of other related results, we shall find it convenient to consider separately the following three cases:

(A)
$$s_0 \geq 1 \,. \quad \text{Then} \quad s_1 = s_0 - 1 + y_1 \,, \quad e_1 = e_0 + \beta$$
$$t_1 = t_0 - 1 + y_1 \,, \quad d_1 = d_0 + \beta \,,$$

(B)
$$0 \leq s_0 \leq t_0 \leq 1 \,. \quad \text{Then} \quad s_1 = y_1 \,, \quad e_0 = e_0 + \beta s_0$$
$$t_1 = y_1 \,, \quad d_1 = d_0 + \beta t_0 \,,$$

(C)
$$0 \leq s_0 \leq 1 \leq t_0 \,. \quad \text{Then} \quad s_1 = y_1 \,, \quad e_1 = e_0 + \beta s_0$$
$$t_1 = t_0 - 1 + y_1 \,, \quad d_1 = d_0 + \beta \,.$$

Proof of Lemma A.1. Since $t_0 - s_0 \geq 0$, $d_0 - e_0 \geq 0$, we have from cases (A)-(C) above:

(A.3)
$$t_1 - s_1 = \begin{Bmatrix} t_0 - s_0 \,; & (A) \\ 0 \,; & (B) \\ t_0 - 1 \,; & (C) \end{Bmatrix} \geq 0$$

and

(A.4)
$$d_1 - e_1 = \begin{Bmatrix} d_0 - e_0 \,; & (A) \\ (d_0 - e_0) + \beta(t_0 - s_0) \,; & (B) \\ (d_0 - e_0) + \beta(1 - s_0) \,; & (C) \end{Bmatrix} \geq 0$$

establishing the first claim. For the second, just notice that (A.3), (A.4) imply

(A.5)
$$(d_1 - e_1) - (t_1 - s_1) = \begin{Bmatrix} (d_0 - e_0) - (t_0 - s_0) \,; & (A) \\ (d_0 - e_0) + \beta(t_0 - s_0) \,; & (B) \\ (d_0 - e_0) + \beta(1 - s_0) + (1 - t_0) \,; & (C) \end{Bmatrix} \geq 0 \,.$$

Under the inductive assumption $d_0 - e_0 \geq t_0 - s_0 \geq 0$, the first two expressions are nonnegative and the third one dominates $(t_0 - s_0) + \beta(1 - s_0) + (1 - t_0) = (1+\beta)(1 - s_0) \geq 0$, as we are in case (C); thus, the expression of (A.5) is nonnegative.

A.2 Lemma: If $0 \leq t_0 - s_0 \leq e_0 - d_0$, then $0 \leq t_1 - s_1 \leq e_1 - d_1$ and by induction $0 \leq t_n - s_n \leq e_n - d_n$ $(n \in \mathbf{N})$.

Proof: From (A.3), (A.4) we obtain

$$(\text{A.6}) \qquad (d_1 - e_1) - (t_1 - s_1) = \begin{cases} (e_0 - d_0) - \beta(t_0 - s_0) \; ; & (A) \\ (e_0 - d_0) + \beta(t_0 - s_0) \; ; & (B) \\ (e_0 - d_0) + \beta(1 - s_0) \; + (1 - t_0) \; ; & (C) \end{cases} \geq 0$$

Thanks to the assumption $e_0 - d_0 \geq t_0 - s_0 \geq 0$, the first term is nonnegative, the second dominates $(1-\beta)(t_0 - s_0) \geq 0$, whereas the third one dominates $(t_0 - s_0) - \beta(1 - s_0) + (1 - t_0) = (1-\beta)(1 - s_0) \geq 0$. Consequently, the expression of (A.6) is nonnegative. □

Suppose now that we start at $s_0 \geq 0$, and employ the strategy π; schematically.

$$(\text{A.7}) \qquad \pi = \begin{bmatrix} s_0, \, s_1 = s_0 - c(s_0) + Y_1, \, s_2, \, \ldots \\ c(s_0), \, c(s_1), \, c(s_2), \, \ldots \end{bmatrix} .$$

For an arbitrary but fixed $c_0 \in [0, s_0]$, consider now the strategy $\tilde{\pi}$ that first uses c_0, and then switches to the strategy π for $n \geq 1$; schematically,

$$(\text{A.8}) \qquad \tilde{\pi} = \begin{bmatrix} s_0, \, t_1 = s_0 - c_0 + Y_1, \, t_2, \, \ldots \\ c_0, \, c(t_1), \, c(t_2), \, \ldots \end{bmatrix} .$$

Denote by $I(\pi)$, $I(\tilde{\pi})$ the rewards that correspond to these two strategies, starting at s_0. If we manage to show that

$$(\text{A.9}) \qquad I(\tilde{\pi}) \leq I(\pi)$$

holds, for any given $s_0 \in [0, \infty)$ and $c_0 \in [0, s_0)$, then Blackwell's verification theorem guarantees that π is *optimal*. Notice also that both strategies are stationary, so the expectations that appear below refer exclusively to the random sequence $\{Y_n\}_{n=0}^{\infty}$. We have

$$(\text{A.10}) \qquad \begin{aligned} I(\pi) &= E\sum_{n=0}^{\infty} \beta^n u(c(s_n)) = u(c(s_0)) + E_{s_0}\left[E_{s_1}\sum_{n=0}^{\infty} \beta^{n+1} u(c(s_n))\right] \\ &= E(\lim_n \tilde{z}_n) , \end{aligned}$$

where $\{\tilde{y}_n, \tilde{z}_n\}_{n=0}^{\infty}$ is generated as in (A.1), with initial conditions

$$(\text{A.11}) \qquad \tilde{y}_0 = s_1 \, , \;\; \tilde{z}_0 = u(c(s_0)) = u(s_0) \, .$$

Similarly,

$$I(\pi) = E\sum_{n=0}^{\infty} \beta^n u(c_n) = u(c_0) + E_{s_0}\left[E_{s_1}\sum_{n=0}^{\infty} \beta^{n+1} u(c(\mathcal{T}_n))\right] \tag{A.12}$$

$$= E(\lim_n \mathcal{D}_n) ,$$

where $\{\mathcal{T}_n, \mathcal{D}_n\}_{n=0}^{\infty}$ is generated as in (A.2), with initial conditions

$$\mathcal{T}_0 = t_1 , \ \mathcal{D}_0 = u(c_0) . \tag{A.13}$$

A.3 Theorem: For any $s_0 \in [0, \infty)$, $c_0 \in [0, s_0]$ we have the comparison (A.9). In particular, π is an optimal (stationary) strategy.

Proof: In view of (A.10), (A,12) it certainly suffices to show the "pathwise comparison"

$$\mathcal{Z}_n \geq \mathcal{D}_n , \quad \forall n \in N_0 \tag{A.14}$$

almost surely.

Obviously from (A.11), (A.13), (A.7), (A.8): $\mathcal{T}_0 - \mathcal{S}_0 = t_1 - s_1 = u(s_0) - c_0$ and $\mathcal{Z}_0 - \mathcal{D}_0 = u(s_0) - u(c_0) \geq 0$. We distinguish the following cases:

(i) $u(s_0) \geq c_0$. Then $0 \leq \mathcal{T}_0 - \mathcal{S}_0 \leq \mathcal{Z}_0 - \mathcal{D}_0$ (the first inequality by assumption, the second amounts to $u(s_0) - c_0 \leq u(s_0) - u(c_0)$ which is obviously true), and the pathwise comparison (A.14) follows from Lemma A.2.

(ii) $u(s_0) \leq c_0 \leq 1$. Then we have $0 \leq \mathcal{S}_0 - \mathcal{T}_0 \leq \mathcal{Z}_0 - \mathcal{D}_0$. The first inequality by assumption; the second amounts to $2u(s_0) \geq u(c_0) + c_0$, which holds because

$$2u(s_0) - u(c_0) - c_0 = \begin{cases} 2(s_0 - c_0) \geq 0; \ s_0 \leq 1 \\ 2(1 - c_0) \geq 0; \ s_0 \geq 1 \end{cases} ,$$

and the pathwise comparison (A.14) follows from Lemma A.1.

(iii) $c_0 \geq 1$. In this case we have $\mathcal{S}_0 - \mathcal{T}_0 = c_0 - u(s_0) = c_0 - 1 \geq 0$, $\mathcal{Z}_0 - \mathcal{D}_0 = u(s_0) - u(c_0) \geq 0$, and (A.14) follows again from Lemma A.1.

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