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# A NOTE ON THE DUAL APPROACH TO THE EXISTENCE AND CHARACTERIZATION OF OPTIMAL CONSUMPTION DECISIONS UNDER UNCERTAINTY AND LIQUIDITY CONSTRAINTS

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by

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## Abstract

This paper considers a dual approach to the problem of maximizing lifetime utility subject to liquidity constraints in a discrete time setting. These constraints prohibit the decision maker from borrowing against future endowment income. The dual approach allows us to exploit directly the supermartingale property of the marginal utility of expenditure and to establish existence and uniqueness of the optimal solution. The optimal solution is interpreted as deriving from a version of the problem that is subject to a single lifetime budget constraint, where expenditures and incomes are discounted to the beginning of the horizon by means of individualized Arrow-Debreu prices.

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# 1 INTRODUCTION

The present paper demonstrates the usefulness of duality theory for solving a fairly standard problem of consumption decisions under uncertainty in the presence of liquidity constraints. We modify suitably for our setting new research in finance that has exploited the consequences of no arbitrage opportunities in the presence of alternative sets of institutional restrictions, such as asset incompleteness, prohibition of short sales, etc. Here we explore the consequences of a restriction that plays a key role in enhancing the attractiveness of the model of lifecycle consumption and savings decisions. We assume that individuals receive labor income but are prohibited from borrowing against that income. That is, they may not hold negative wealth at any point during their lifetimes.

Models of optimal consumption, portfolio, and investment decisions under uncertainty have served as the backbone of modern macroeconomics and finance theory. Maximizing the expectation of a time-separable lifetime utility index (that ranks lifetime consumption bundles) subject to well-defined investment opportunities lends itself neatly to a dynamic programming formulation. The elegant contributions by Samuelson (1969) and Merton (1969; 1971) have been instrumental in popularizing this tool in studying consumption, savings and investment decisions. Use of this basic workhorse has been hindered, however, because closed-form solutions have been available only for the cases of linear-quadratic utility, which lead to the certainty equivalence case, and fully diversifiable wealth, which is tractable, for all members of the HARA class of utility functions. Naturally, imposing constraints such as liquidity constraints further diminishes the likelihood that closed-form solutions may be obtained in general [Zeldes (1989a); Hajivassiliou and Ioannides (1990)].

The problem of the optimal consumption decisions under uncertainty has received much attention, especially because of its importance for monetary theory. See, for example the classic formulation by Bewley (1977). Yet, it was only relatively recently that the problem was solved explicitly for a general utility function by Karatzas, Lehoczky, Sethi and Shreve (1986). Zeldes (1989b) has aptly demonstrated that the general case of maximizing lifetime utility with income uncertainty yields results substantially different from certainty equivalence. Zeldes' method is numerical solution by means of the dynamic programming algorithm. Deaton (1991) demonstrates the feasibility of the computational approach in the presence of liquidity constraints when consumption as the single decision and several alternative assumptions are made about the stochastic process describing (exogenous) labor income.

The powerful consequences of the methods employed by the new finance literature may be explained quite simply as follows. The maximization of expected lifetime utility as a function of the consumption path subject to liquidity constraints is vastly simplified if it is formulated by means of duality theory. This simplification is possible because the conditions that no arbitrage opportunities are present in a continuous-time model are equivalent to the condition that wealth must be non-negative at every point in time. This result, due to Dybvig and Huang (1988), depends critically on the assumptions of continuous time. The lifetime budget constraint along with the non-negativity of wealth constraint imply a sequence of budget constraints involving expectations. A particularly significant feature of the dual approach is that it enables us to interpret the optimal solution of the liquidity-constrained problem as an unconstrained problem under an (individual-specific) implicit set of Arrow-Debreu state prices. In general, this dual problem is much easier to characterize and to solve, in the case of continuous time, and readily leads to the optimal solution to the original (primal) problem.

This advantage of duality theory has not been recognized before even in the context of deterministic models. Pissarides (1978), Artle and Varayia (1978), and Jackman and Sutton (1982) were the first to show in deterministic settings that the optimal consumption policy takes the form of a sequence of subproblems. These subproblems alternate between consuming all income as it is received, and consuming at a rate that is optimized over a suitably defined subhorizon. As He and Pagés (1990) show, the interpretation of the optimal consumption and investment model within the dual approach is that the decision maker behaves as if he could sell his future endowment of income at the implicit Arrow-Debreu prices and set his consumption in an unconstrained fashion. The implicit Arrow-Debreu prices are obtained by transforming the Lagrange multipliers that adjoin the original sequence of constraints. The dual statement of the constrained problem is rather standard and particularly simple to obtain, as it rests on the saddle point property of the Lagrangean. Nevertheless, this particular interpretation had not been pointed out before He and Pagés.

Under uncertainty, the advantage of duality theory is even greater. The interpretation of the solution as an unconstrained problem supported by a suitable set of Arrow-Debreu prices is retained. But, in addition, the partial differential Hamilton-Jacoby equation obtained by dynamic programming for the dual problem in continuous time is easier to solve than the well-known one for the primal problem. Finally, and quite importantly, when stated in terms of the solution to the dual problem, this framework is actually identical to that of the theory of marginal utility of wealth-constant demand functions or *Frisch* demands. This connection has not been made previously and is particularly interesting in the context of empirical applications.

The new breed of the finance literature uses continuous time models and owes its origin to Cox and Huang (1989), Dybvig and Huang (1988), and Karatzas, Lehoczky and Shreve (1987).[1] Dybvig and Huang prove that a nonnegativity-of-wealth constraint precludes arbitrage opportunities when quite general trading strategies are considered in continuous time models. This is a very significant result because the existence of arbitrage possibilities in a continuous time model would render asset pricing theory rather vacuous. Our ability to rule out getting something for nothing by imposing a constraint that may be institutionally motivated is particularly welcome.

The first specific consideration of liquidity constraints in the context of finance literature appears in Chapter 3, "Consumption of an Endowment," of Pagés (1989). Pagés considers the case of consumption when labor income is uncertain. With a finite horizon and complete markets, this would be a standard case of the optimal allocation of lifetime resources, were it not for an additional restriction he imposes on financial (i.e., non-human) wealth, which is required to be non-negative. Even though the endowment of leisure is modelled so as not to constitute, in itself, a new source of uncertainty in the economy, the non-negativity restriction on non-human wealth embodies a certain kind of market incompleteness. The agent is prevented from trading his endowment in ways that may cause a negative market value of his portfolio at any point in time.[2]

Somewhat more general is the approach of He and Pearson (1989a,b) who assume incomplete markets and include short-sale constraints. It is known that when markets are complete, there exists a *unique* measure (that is, state price process) that is used to form the equivalent budget constraint. When, on the other hand, markets are incomplete infinitely many equivalent martingale measures are consistent with the absence of arbitrage. Therefore, the equivalent static problem of maximizing lifetime utility is subject to infinitely many budget constraints. He and Pearson show, however, that the feasible set of consumption policies in the finite-dimensional case is generated by finitely many budget constraints, which correspond to the extreme points of the closure of the set of state prices consistent with no arbitrage.[3] It is then much easier to characterize the optimum. Most recently, Karatzas,

[1] On a bit of history, Cox and Huang was available as a working paper since 1986. Karatzas *et al.* were unaware of this work and worked independently. They differ from Cox and Huang by their avoidance of $L_2$ theory.

[2] Before one prohibits entirely trading of the endowment process, one may consider trading the expected value of the endowment process, that is by allowing borrowing against that expected value. An analogue is pursued by Clarida (1987) who examines the possibility that one may borrow against the lower support of the distribution of labor income.

[3] It would be interesting to know whether or not continuous probability distributions, expressing income or

Lehoczky, Shreve and Xu (1991) go further than He and Pearson by using duality theory and local martingale methods developed by Xu (1990) and Xu and Shreve (1990).[4]

This paper poses the problem of optimal consumption under uncertainty subject to liquidity constraints in a discrete time setting. We do so because we are primarily interested in existence and characterization of the optimal solution in estimable models [Hajivassiliou and Ioannides (1990)]. An alternative and much harder approach would be to consider the problem in a continuous time setting and pose the estimation problem in the context of sampling at discrete points in time. An interesting advantage of the approach taken in the present paper is to set the optimal solution to the problem directly in terms of the dual specification of preferences. This, in turns, leads to Frisch demand theory, a familiar tool for applied theorists. We establish existence and uniqueness of the optimal policy. We characterize it by means of a threshold value of assets, below which expenditure exhausts all beginning-of-period assets.

# 2 THE MODEL

We consider the decisions of individuals who live finite lifetimes of length $T$ in an economy that runs for a countable number of discrete periods $t \in \{1, \ldots, \bar{T}\}$, where $\bar{T}$ may be infinite. Uncertainty in this economy is characterized by means of a probability space $(\boldsymbol{\Omega}, \boldsymbol{N}, \boldsymbol{P})$, where the element $\omega \in \boldsymbol{\Omega}$ stands for a particular realization of all random variables in this economy from 1 to $\bar{T}$. Information in this economy is represented by a sequence of partitions of $\boldsymbol{\Omega}$, $\{N_t \mid t = 0, 1, \ldots, \bar{T}\}$. The interpretation of this information structure is that at time $t$ the agent knows which cell of $N_t$ contains the true state. Information increases through time; $N_{t+1}$ is at least as fine as $N_t$. Without loss of generality we assume that $N_0$ is trivial (i.e., $N_0 = \boldsymbol{\Omega}$) and $N_{\bar{T}}$ is the discrete partition, i.e., $N_{\bar{T}} = \{\omega \mid \omega \in \boldsymbol{\Omega})$. The $\sigma$-field of events generated by $N_t$ is denoted by $\mathcal{N}_t$, and N = $\{N_t$; t$\in \{0,1,\ldots,\bar{T}\}\}$ is the filtration generated by the sequence of partitions $N_t$.

The notion of filtration is a standard concept of the revelation of information over time. That is, the increasing sequence of $\sigma-$fields of events $\mathcal{N}_0 \subseteq \mathcal{N}_1 \ldots \subseteq \mathcal{N}_t$ characterizes how information accumulates over time in this economy by becoming increasingly finer. This

demographic shocks, may be represented by approximate discrete distributions, as in Deaton (1991). Deaton's approach is explicitly cast in terms of a dynamic programming formulation, which is quite different from the more abstract martingale approach of He and Pearson.

[4]See Karatzas, *et al.* (1991) for full details on the relationship of their work with He and Pearson's work.

information structure is rather standard and may be easily and intuitively represented by an event tree. We use $E_t$ to denote the expectations operator associated with $N_t$.

The typical agent faces a wage rate $W_h(t)$ and a vector of prices of all other goods, $W_g(t)$, and is characterized by an exogenous maximum amount of leisure in period $t$, $\bar{L}(t)$. The vector $W(t) = (W_h(t), W_g(t))$ is a stochastic process that is adapted to the filtration $\mathcal{N}$. Let $\bar{L}(t)$, the endowment of leisure in period $t$, be a stochastic process that is also adapted to $\mathcal{N}$. The randomness of prices introduces no additional source of uncertainty.

Let $u_t(\ell(t), G(t) \mid \mathcal{N}_t)$ denote a utility function as a function of leisure $\ell(t)$ and the vector of other goods, $G(t)$, which is conditional on all new information available to the individual as of time $t$, $\mathcal{N}_t$. Utility per period, $u_t(\cdot \mid \cdot)$ is assumed to be concave and increasing with respect to all of its arguments $(\ell(t), G(t))$. To $u_t(\cdot \mid \cdot)$ there corresponds an indirect utility function with the standard properties: $v_t(b; W) = max_{\{\ell,G\}} : u_t(\ell, G)$, subject to the constraint: $b = W_h\ell + W_g'G$.

The assumption that follows modifies suitably for our setting the assumptions usually made about utility as a function of a scalar decision variable [ *c.f.* Arkin and Evstigneev, *op. cit.* ; He and Pagés, *op. cit.* ]. We will rely on this Assumption to prove existence and uniqueness of the optimal solution, and it will be retained for the remainder of the paper.

Assumption 1: *Preferences are such that there exist functions of the price vector* $W$, $\Delta_1$, $\Delta_2$, $\Delta_1'$, $\Delta_2'$, $\delta$, *and* $\delta'$, *such that:*

$$\Delta_1' - \Delta_2' b^{\delta'} \le v(b; W) \le \Delta_1 + \Delta_2 b^{\delta}$$

*where* $\Delta_2$ *and* $\Delta_2'$ *are positive, and* $0 < \delta < 1$ *and* $\delta' < 0$ .

The standard statement of the problem faced by the typical individual is to choose the $\mathcal{N}_t$-measurable vector $(\ell(t), G(t))$ for each $t \in \{1, \ldots, T\}$ to solve the problem

$$\max_{\{\ell(t),G(t)\}} : E_0 \left\{ \sum_{t=1}^{T} \frac{1}{(1+\rho)^{t-1}} \, u_t(\ell(t), G(t) \mid \mathcal{N}_t) \right\}, \tag{1}$$

subject to a lifetime budget constraint and to a terminal financial wealth constraint, and given financial wealth at the beginning of the process, to be clarified below. The expectations operator $E_0$ is associated with $\mathcal{N}_0$.

We introduce additional notation and rewrite (1) in terms of the indirect utility function

$v_t(b; W)$. Let the $\mathcal{N}_t$-measurable variables be defined as follows: $F(t)$ is the market value of *financial* and $A(t)$ the market value of *total* wealth at the beginning of period $t$ (before expenditure decisions for that period are made); $F^*(t)$ is the end-of-period financial wealth (after expenditure decisions in period $t$ have been incurred); $b(t)$ is net asset decumulation in period $t$. We then have:

$$b(t) = W_g'(t)G(t) + W_h(t)\ell(t). \tag{2}$$

The household's budget constraint in period $t$ is:

$$A(t) \equiv F(t) + W_h(t)\bar{L}(t) = b(t) + F^*(t); \quad t = 1, \ldots T. \tag{3}$$

To complete the description of problem (1), let us also assume that there exist $k+1$ securities, $k \in \{0, 1, 2, \ldots, K\}$, that households may trade costlessly each period for current consumption and labor. To simplify matters, we assume that $K$ is countable [*c.f.* Altug and Miller (1990); He and Pearson (1988)]. Let $s_k(t)$ denote the quantity of security $k$ held by the individual from period $t-1$ to period $t$, $q_k(t)$ its period $t$ price, and assume for simplicity (and without loss of generality) that securities pay no dividends. Therefore, beginning-of-period wealth comes from liquidating all asset holdings:

$$F(t) = \sum_{k=0}^{K} s_k(t)q_k(t); \quad t = 1, \ldots, T. \tag{4}$$

End-of-period wealth is invested in financial assets:

$$F^*(t) = \sum_{k=0}^{K} s_k(t+1)q_k(t); \quad t = 1, \ldots, T. \tag{5}$$

Conditions (4) and (5) must be satisfied for all measurable sets of $\mathcal{N}_t$ and $t \in \{1, \ldots, T\}$. The securities trading strategy, that is portfolio strategy, is a predictable $(K+1)-$ dimensional process $s$, with $s = \{(s_0(t), \ldots, s_K(t)), t = 1, \ldots, T\}$, where predictable means that $(s_0(t), \ldots, s_K(t))$ is measurable with respect to $\mathcal{N}_{t-1}$.

Problem (1) may now be restated as follows. Find a net asset decumulation path $b(t)$ and a portfolio strategy $(s_0(t), s_1(t), \ldots, s_K(t))$, $1 \leq t \leq T$ to maximize:

$$v_t(b(t); W(t) \mid \mathcal{N}_t) + E_t \left\{ \sum_{j=t+1}^{T} \frac{1}{(1+\rho)^{j-t}} v_j(b(j); W(j) \mid \mathcal{N}_t) \right\}, \tag{6}$$

subject to constraints (3), (4), and (5), and a given value of A(1). The consumption of leisure and of other elements of the consumption bundle $(\ell(t), G(t))$ follows from $v_t(\cdot \mid \cdot)$ and Roy's identity once $b(t)$ is known.

As Altug and Miller (1990) emphasize, in a competitive economy with complete markets, problem (1) is vastly simplified. All trades occur at time period 1 by choosing an $\mathcal{N}_t$-measurable vector $(\ell(t), G(t))$ and an $\mathcal{N}_t$-measurable vector $(q_0(t), \ldots, q_k(t))$ for each $t \in \{1, \ldots, T\}$ subject to a single lifetime budget constraint. This constraint is obtained by requiring that the expectation of the present value of lifetime expenditure minus the expectation of the present value of lifetime receipts not exceed initial wealth $A(1)$. Following Altug and Miller, *op. cit.* , we use $P_t$, a measure defined on $\mathcal{N}_t$ for each $t \in \{1, \ldots, T\}$, to denote prices of contingent claims defined in terms of the numeraire good in period $t$. For example, for any set $C_t \in \mathcal{N}_t$, the nonnegative real number $p_t(C_t)$ denotes the period 1 price of a unit of the numeraire good to be delivered on date $t$, contingent on $C_t$ occurring. The usual assumption is that $p_t$ is absolutely continuous with respect to $P$. This assumption implies a "density" (strictly speaking, a Radon-Nikodym derivative of $p_t$), denoted by $\frac{1}{(1+\rho)^{t-1}}\lambda_t$, of contingent claims prices. Hence we may write:

$$p_t(C_t) = \frac{1}{(1+\rho)^{t-1}} \int_{C_t} \lambda_t(\omega) P(d\omega). \tag{7}$$

We now write the lifetime budget constraint corresponding to (3-5) under complete markets as follows:

$$E_0 \left\{ \sum_{t=1}^{T} \frac{1}{(1+\rho)^{t-1}} \lambda_t \left[ b(t) - W_h(t)\bar{L}(t) \right] \right\} \leq A(1). \tag{8}$$

The necessary conditions for the maximization of (6), $t = 1, \ldots, T$, subject to (8) are:

$$\frac{\partial}{\partial b(t)} v_t(b(t); W(t) \mid \mathcal{N}_t) = \mu \lambda_t; \tag{9}$$

$$\lambda_t = \frac{1}{1+\rho} E_t \left\{ \lambda_{t+1} \frac{q_k(t+1)}{q_k(t)} \right\}. \tag{10}$$

Here $\mu$ is a Lagrange multiplier associated with the individual's lifetime budget constraint (8) and reflects individual characteristics. Equation (9) holds with probability one. Once the sequence $(b(1), \ldots, b(T))$ has been computed the consumption bundle may be obtained from Roy's identity.

In the absence of complete markets, Equations (3)–(5) collapse to:

$$\sum_k q_k(t)[s_k(t) - s_k(t+1)] + W_h(t)\bar{L}(t) \geq b(t). \tag{11}$$

The corresponding first-order conditions become:

$$\frac{\partial}{\partial b(t)} v_t(b(t); W(t) \mid \mathcal{N}_t) = \lambda(t); \tag{12}$$

$$\lambda(t) = \frac{1}{1+\rho} E_t \left\{ \lambda(t+1) \frac{q_k(t+1)}{q_k(t)} \right\}; \tag{13}$$

where $\frac{1}{(1+\rho)^{t-1}} \lambda(t)$ is the Radon-Nikodym derivative of the period $t$ Lagrange multiplier associated with the individual's period $t$ constraint (11).

It is important to point out the difference between the definitions and properties of $\lambda_t$ and $\lambda(t)$. The Lagrange multiplier $\lambda(t)$ is a time-dependent variable, a stochastic process, that depends on individual characteristics with a domain in the dual space corresponding to the space where the consumption bundle, securities and prices are defined. Clearly if markets are complete, (13) implies (10). The Lagrange multiplier $\lambda(t)$ satisfies $\lambda(t) = \mu\lambda_t$, and is, thus, multiplicatively separable with respect to individual characteristics and market information by being expressed as the product of an individual-specific variable, $\mu$, and the (market-dependent) density $\lambda_t$ defined in the complete markets case above. This multiplicative separability of the Lagrange multiplier in the complete markets case is emphasized by Altug and Miller (1990) and lies at the heart of their innovative estimation procedure. In the remainder of the present paper we demonstrate that the solution to the liquidity constrained problem may be interpreted in a way that utilizes the important intuition obtained from the complete markets case.

## A. Liquidity Constraints and Incomplete Markets

Matters are substantially complicated if liquidity constraints are introduced. Here liquidity constraints express the (reasonable) restriction that the individual may not borrow against his future labor income and consequently the individual's end-of-period assets may not become negative in any period.

As can be seen from conditions (11)–(13), if markets are incomplete, the necessary conditions for the maximization of (6) over an individual's lifetime with respect to $\{b(t), A(t); s_0(t), s_1(t), \ldots, s_K(t)\}$ and subject to (3)–(5) are substantially more complicated than (9). Now trades take place as uncertainty evolves sequentially, and the sequence of budget constraints must be satisfied for all measurable subsets of $\mathcal{N}_t$ and $t \in \{1, \ldots, T\}$.

For notational simplicity, recall we have defined $A(t)$ to be the market value of total wealth at the beginning of period $t$, which includes financial wealth $F(t)$ as of the beginning of period $t$, plus the value of the endowment of leisure, $W(t)\bar{L}(t)$. Net expenditure in period $t$, $b(t)$, is thus defined as:

$$b(t) = W_g'(t)G(t) + W_h(t)\ell(t). \tag{14.1}$$

The household's budget constraint for period $t$ is:

$$A(t) = b(t) + F^*(t). \tag{14.2}$$

The individual may not hold negative financial wealth at the end of period $t$ and must thus satisfy a sequence of (liquidity) constraints:

$$A(t) - b(t) \geq 0, \quad t = 1, \ldots, T. \tag{14.3}$$

That is, with probability one the individual can never run into debt. We will not restrict trading in assets in any other way. It is possible, in particular, that the individual can short one asset and long another asset.[5]

[5]The results derived by Dybvig and Huang (1988), He and Pagés (1990), and Huang and Pagés (1990) imply that the alternative conditions that there exist no arbitrage opportunities (i.e., it is not possible to create wealth by starting out with zero wealth) and that wealth may not become negative (i.e., (14.3) holds) are equivalent. He and Pagés (1990), in particular, work in a continuous time model with assets whose prices are assumed to follow standard Brownian motions. They show that the non-negativity of wealth condition may be expressed equivalently in terms of expectations as follows:

$$\sup_{\tau \in \mathcal{T}} E\left\{ \int_0^\tau \frac{\xi(t)}{B(t)} \left[b(t) - W_h(t)L(t)\right] dt \right\} \leq A(0),$$

where $\mathcal{T}$ denotes the set of stopping times,[6] $\xi(t)$ is the total normalized return to the risky asset, which may

In the remainder of this paper we work from first principles and modify the He-Pagés approach to analyze the problem of maximizing:

$$E_0 \left\{ \sum_{j=0}^{T} \frac{1}{(1+\rho)^j} v_j(b(j); W(j) \mid \mathcal{N}_j) \right\}, \tag{15}$$

subject to a sequence of budget constraints that ensure that the individual may not borrow against his future income, that is, that he must hold nonnegative wealth. For notational simplicity, we assume that there exists in every period only one asset, which yields a random return that is represented by a stochastic process $r(t)$, a random variable adapted to $\mathcal{N}_t$. We assume, in addition, that $1 + r(t) \geq 0,\ \boldsymbol{P} - a.s.$

The dynamic evolution of $A(t)$ is described as:

$$A(t+1) = (1 + r(t+1))[A(t) - b(t)] + W_h(t+1)\bar{L}(t+1),\ t = 0, 1, \ldots, T, \tag{16}$$

where $b(t)$, the period $t$ expenditure, is a decision variable adapted to $\mathcal{N}_t$. By using (16) and (14.1-2), we may rewrite the sequence of liquidity constraints (14.3) equivalently as:

$$F_0 \,+\, \sum_{k=0}^{t} \pi_k W_h(k)\bar{L}(k) \,\geq\, \sum_{k=0}^{t} \pi_k b(k), \quad t = 0, 1, \ldots, T; \tag{17}$$

where $\pi_k$ is stochastic process adapted to $\mathcal{N}_t$, defined as $\pi_k \triangleq (1 + r(1))^{-1} \ldots (1 + r(k))^{-1}$, $k = 1, \ldots, t$; $\pi_0 = 1$ ; $F_0$ denotes financial assets as of the beginning of period 0; and $A(0) \equiv F_0 + W_h(0)\bar{L}(0)$. Note that since assets left at the end of the lifetime horizon are not valued by the individual, (17) implies $A(T) = b(T)$. The interpretation of constraints (17) is straightforward. The present value of the endowment of leisure plus initial financial assets must not exceed the present value of net expenditure for any realization of the relevant random variables $\pi$ and $W$.

Under some mild restrictions existence and uniqueness of the optimal solution may be established by means of Theorem 6, Arkin and Evstigneev (1987), p.108. Their work constitutes the only contribution in the literature that integrates inequality constraints into the standard stochastic dynamic programming. Still, they do not explore the supermartingale properties of the corresponding Lagrange multipliers. Moreover, their theorem says nothing

**be interpreted as an implicit Arrow-Debreu contingent-claim price at $t = 1$ for a unit of purchasing power in period $t$; and $B(t)$ the total compounded return to the (locally) riskless asset. [*ibid.* p.18].**

about the interesting economic aspects of the problem and it is for this reason, too, that we turn to a dual formulation.

B. Dual Formulation

We now introduce the dual formulation of the problem of maximizing (15) subject to constraints (17). Let $Y(t)$ be Lagrange multipliers corresponding to (17), $t = 0, 1, \ldots, T$. These Lagrange multipliers are positively valued random variables that are elements of the dual of the period $t$ constraint space, which is $\mathcal{N}_t$-adapted. That is, as far as information is concerned, $Y(t)$ depends only on information available as of time $t$. By adjoining constraints (17) and by simplifying notation, we may write problem (15) as:

$$\min_{\{Y(t)\geq 0\}} : \max_{\{b(t)\}} : E_0\left\{\sum_{j=0}^{T} \frac{1}{(1+\rho)^j} v_j\big(b(j); W(j) \mid \mathcal{N}_j\big) + Y(0)\big[F_0 + W_h(0)\bar{L}(0) - b(0)\big]\right.$$
$$\left. + \sum_{t=1}^{T} Y(t)\big[F_0 + W_h(0)\bar{L}(0) - b(0) + \sum_{k=1}^{t} \pi_k\big(W_h(k)\bar{L}(k) - b(k)\big)\big]\right\}. \tag{18}$$

The term that adjoins the sequence of all constraints in (18) may be rewritten as:

$$\begin{aligned}
E_0\Big\{F_0 \quad &+ \quad \big[W_h(0)\bar{L}(0) - b(0)\big]\big[Y(0) + Y(1) + \cdots + Y(T)\big] \\
&+ \quad \pi_1\big[W_h(1)\bar{L}(1) - b(1)\big]\big[Y(1) + \cdots + Y(T)\big] \\
&+ \quad \pi_{T-1}\big[W_h(T-1)\bar{L}(T-1) - b(T-1)\big]\big[Y(T-1) + Y(T)\big] \\
\cdots \quad &+ \quad \pi_T\big[W_h(T)\bar{L}(T) - b(T)\big]Y(T)\Big\}.
\end{aligned}$$

This allows us to rewrite (18) as follows:

$$\min_{\{Y(t)\geq 0\}} : \max_{\{b(t)\}} : E_0\left\{\sum_{j=0}^{T} \frac{1}{(1+\rho)^j} v_j\big(b(j); W(j) \mid \mathcal{N}_j\big) + F_0\big[Y(0) + E_0\{Y(1) + \cdots + Y(T)\}\big]\right.$$
$$\left. + \sum_{t=0}^{T} \pi_t\big(W_h(t)\bar{L}(t) - b(t)\big)\big[Y(t) + E_t\{Y(t+1) + \cdots + Y(T)\}\big]\right\}. \tag{19}$$

Let us define the auxiliary variable $X(t)$ in terms of the Lagrange multipliers introduced in (18):

$$X(t) = Y(t) + E_t\{Y(t+1) + \ldots + Y(T)\}, \quad t = 1, \ldots, T-1; \quad X(T) = Y(T).$$

$\{X(t) \mid \mathcal{N}_t\}$ is a non-negatively valued stochastic process adapted to $\mathcal{N}_t$. This definition of $X(t)$ is a very critical step: As the Lagrange multipliers $\{Y(t) \mid \mathcal{N}_t\}$ are non-negative, this definition implies that $\{X(t) \mid \mathcal{N}_t\}$ is actually a supermartingale.

# 3 RESULTS

Our results are summarized in four propositions, which are presented in this section. We start with a formal statement of the problem, as it has been developed in the previous section:

Proposition 1.

*The problem of maximizing (15) subject to constraints (16) and (17) is equivalently stated as:*

$$\max_{\{b(t)\}} : E_0 \left\{ \sum_{j=0}^{T} \left( \frac{1}{(1+\rho)^j} v_j(b(j); W(j) \mid \mathcal{N}_j) - X(j)\pi_j b(j) \right) + X(0)F_0 + \sum_{t=0}^{T} X(t)\pi_t W_h(t)\bar{L}(t) \right\}. \tag{20}$$

We now explore the saddle point property of the Lagrangean to rewrite Problem (20) as the problem of finding $X = \{X(t); t = 0, 1, \ldots, T\}$, a non-negative, predictable supermartingale process, $X \in \mathcal{D}$, such that:

$$\min_{X \in \mathcal{D}} : E_0 \left\{ \sum_{j=0}^{T} \tilde{v}(X(j); W(j) \mid \mathcal{N}_j) + \sum_{t=0}^{T} X(t)\pi_t W_h(t)\bar{L}(t) + X(0)F_0 \right\}, \tag{21}$$

where the profit function $\tilde{v}_j(\cdot;\cdot \mid \mathcal{N}_j)$ is defined as:

$$\tilde{v}_j(X(j); W(j) \mid \mathcal{N}_j) \triangleq \max_{b(j)} : \frac{1}{(1+\rho)^j} v_j(b(j); W(j) \mid \mathcal{N}_j) - X(j)\pi_j b(j); \tag{22}$$

and $\mathcal{D}$ denotes the set of non-negative, predictable supermartingales. Under the assumption that the indirect utility function $v_j(b; w)$ is concave with respect to $b$, then the profit function $\tilde{v}_j(X; w)$ is convex and decreasing in $X$.

We shall see now how duality simplifies the characterization of the solution of (20). From duality we have that:

Proposition 2.

*If $X^*$ is a solution to Problem (21), then*

$$b^*(t) = f(X^*(t)(1+\rho)^t \pi_t \mid \mathcal{N}_t), \tag{23}$$

*is a solution to Problem (20), where $f(\cdot \mid \cdot)$[7] is defined as:*

$$f(x \mid \mathcal{N}_t) = \inf_b : \left\{ b \geq 0;\ \frac{\partial v}{\partial b}(b; W) \leq x \right\}. \tag{24}$$

Proof: Let $\mathcal{L}(X^*)$ denote the minimand in (21) as a function of $X^*$. We first show that the sequence $b^* = \{b^*(t);\ t = 0, 1, \ldots, T\}$, defined according to (23), is feasible. For any $0 \leq \tau \leq T$, we define the process $X^\epsilon$ as $X^\epsilon \equiv X^* + \epsilon\, \mathbf{1}[0, \tau] \in D$, where $\epsilon > 0$ and $\mathbf{1}[0, \tau]$ is a function defined as equal to 1 if $0 \leq t \leq \tau$, and equal to 0 otherwise. Since, by the definition of $X^*$, $\mathcal{L}(X^*) \leq \mathcal{L}(X^\epsilon)$, we have that:

$$\lim_{\epsilon \downarrow 0} \sup \frac{\mathcal{L}(X^\epsilon) - \mathcal{L}(X^*)}{\epsilon} \geq 0.$$

This, in turn, implies that:

$$\lim_{\epsilon \downarrow 0} \sup E_0 \left\{ \sum_{j=0}^{T} \frac{\tilde{v}(X^\epsilon(j); W(j) \mid \mathcal{N}_j) - \tilde{v}(X^*(j); W(j) \mid \mathcal{N}_j)}{\epsilon} + A(0) + \sum_{t=0}^{t} \pi_t W_h(t) \bar{L}(t) \right\}.$$

Since, by definition, $\tilde{v}(X_j; W_j \mid \mathcal{N}_j)$ is decreasing in $X$, it follows that:

$$\tilde{v}(X^\epsilon(j); W(j) \mid \mathcal{N}_j) \leq \tilde{v}(X^*(j); W(j) \mid \mathcal{N}_j).$$

Moreover, $\tilde{v}(.)$ satisfies:

[7] $f$ is the inverse function of $\frac{\partial v(b;W)}{\partial b}$ with respect to $b$.

$$\frac{\partial}{\partial X(j)}\,\tilde{v}(X(j);W(j)\mid \mathcal{N}_j) = -\pi_j b(j).$$

Hence we obtain:

$$\sup_{\tau}\; E_0\left\{F_0 + \sum_{k=0}^{\tau}\pi_k W_h(k)\bar{L}(k) - \sum_{k=0}^{\tau}\pi_k b^*(k)\right\} \geq 0, \tag{25}$$

for $0 \leq \tau \leq T$. The next step in showing that $\mathbf{b}^*$ is feasible is to prove a counterpart of Lemma 1, *ibid.* p. 20. That is, we show that if $\{b(k)^*\}$ is feasible, then we may find a non-negative wealth process, $\{A(t)\mid \mathcal{N}_t\}$, satisfying (16) for all $t$.

In order to prove the counterpart of that Lemma we work as follows. Let us define:

$$\Gamma(t) \equiv \sum_{k=0}^{t}\pi_k[b(k) - W_h(k)\bar{L}(k)].$$

From (25) it follows that:

$$E_0\left\{\sum_{k=0}^{T}\pi_k b(k)\right\} < \infty.$$

Hence, the stochastic process $\Gamma$ is of class $D\{0,T\}$. That is, the sets $\Gamma(\tau)$, $\tau \in \mathcal{T}$ are uniformly integrable, where $\mathcal{T}$ is the set of stopping times.

Define as $V(t)$ the Snell envelope of $\Gamma(t)$, the smallest supermartingale that majorizes $\Gamma(t)$. See Dellacherie and Meyer (1982), Appendix I, pp.22-23, and Snell (1952) for precise definitions. By the Doob-Meyer Decomposition Theorem [Doob (1953), p. 296], $V(t)$ may be written as:

$$V(t) = V(0) + M(t) - N(t),$$

where $M$ is a uniformly integrable martingale under $\boldsymbol{P}$ with $M(0) = 0$, and $N(t)$ is an increasing process with $N(0) = 0$.

From the definition of $V$ as the Snell envelope of $\Gamma$ we have that

$$V(0) = \sup_{\tau\epsilon\mathcal{T}} E_0\{\Gamma(\tau)\}; \quad V(T) = \Gamma(T).$$

Condition (25) implies that $F_0 \geq V(0)$. Since by their definition, $V(t) \geq \Gamma(t)$ and $N(t) \geq 0$, by adding up, we get:

$$F_0 - V(0) + V(t) - \Gamma(t) + N(t) \geq 0.$$

By using the Doob-Meyer result that $V(t) + N(t) - V(0) = M(t)$, the above can written as:

$$F_0 - \Gamma(t) + M(t) \geq 0.$$

Now define $A(t+1)$ as follows:

$$A(t+1) = W_h(t+1)\bar{L}(t+1) + \pi_{t+1}^{-1}\left[F_0 - \Gamma(t) + M(t)\right],$$

where by the martingale representation theorem, $M(t) = \sum_{k=0}^{t} \psi_k$, with $\psi_0 = 0$. Clearly, by construction, $A(t+1)$ is non-negative, $\boldsymbol{P} - a.s.$, and so are both of its individual components. By substituting in from the definition of $\Gamma(t)$ and rearranging, we have:

$$\pi_{t+1}A(t+1) + \sum_{k=0}^{t} \pi_k b^*(k) = F_0 + \sum_{k=0}^{t} \pi_k W_h(k)\bar{L}(k) + \pi_{t+1}W_h(t+1)\bar{L}(t+1) + \sum_{k=0}^{t} \psi_k. \quad (26)$$

This is the counterpart of (17) in the form of equality constraints in realization. This completes the proof of the claim that $\{b^*(k)\}$, given by (23), is feasible.

We now prove that $\{b^*(k)\}$ is optimal and thus a solution to Problem (20). Since $\boldsymbol{X}^*$ is, by definition, a non-negative supermartingale, then we have that

$$X^*(t) - E_t\{X^*(t+1)\} \geq 0, \quad (27)$$

$\boldsymbol{P} - a.s.$ From (26) and the fact that by construction $A(t) \geq 0$, $\boldsymbol{P} - a.s.$, we have that

$$\sum_{k=0}^{t} \pi_k b^*(k) \leq F_0 + \pi_{t+1}W_h(t+1)\bar{L}(t+1) + \sum_{k=0}^{t} \pi_k W_h(k)\bar{L}(k) + \sum_{k=0}^{t} \psi_k.$$

By multiplying both sides of the above by $X^*(t) - E_t\{X^*(t+1)\}$, for $t$, by summing up over all $t$'s, by using the fact that $M(t)$ is a uniformly integrable martingale with $M(t)$, and by taking expectations as of time 0, we finally obtain an equivalent single lifetime budget constraint:

$$X^*(0)F_0 + E_0\left\{\sum_{t=1}^{T} X^*(t)\pi_t W_h(t)\bar{L}(t)\right\} \geq E_0\left\{\sum_{t=1}^{T} X^*(t)\pi_t b^*(t)\right\}. \tag{28}$$

Since it follows from our assumptions that the decision-maker will use up unused slack, expression (28) actually holds as an equality. The optimality of $\boldsymbol{b}^*$ follows from the saddle point property of the Lagrangean. □

The interpretation of this solution is particularly revealing. Equation (23) is the solution to the problem of maximizing (15), subject to a single constraint that expresses the transformation of the sequence of constraints (17) into a single one resembling (8). $X^*(t)\pi_t$ is an *implicit* system of Arrow-Debreu prices (or shadow prices) for this particular individual. $X^*(t)$ is an individual-specific supermartingale which has been defined in terms of the Lagrange multipliers adjoining the liquidity constraints in realization (17). $\pi_t$, on the other hand, is a price. If $X^*(t)\pi_t$ are interpreted as Arrow-Debreu state prices and the individual were allowed to sell his labor at these prices at time $t = 0$, then the individual's optimal consumption decisions would be identical to those of the original problem with liquidity constraints [*c.f.* He and Pagés (1990), p. 22].

The relationship of the formulation of the problem according to Proposition 1 is, rather naturally, closely related to Frisch demand theory. This is established by:

Proposition 3.
*The optimal expenditure function (23) yields a vector of demand functions for leisure and other goods that coincide with the Frisch demands.*

Proof: For a proof, it suffices to recognize that according to its definition in (22), the profit function $\tilde{v}(x; W)$ coincides to the profit function defined by McFadden (1978) for production settings, and adapted for the underlying system of preferences by Browning, Deaton and Irish (1985). □

To proceed further and fully characterize existence and uniqueness of the solution to Problem (21), we use stochastic dynamic programming methods [Bertsekas (1987)]. The dual problem, defined in (21), may be expressed in terms of the value function in a standard stochastic dynamic programming approach as follows:

$$
\begin{aligned}
J_T(X(T-1)) &= \\
\inf_{X(T)\geq 0,\, E_{T-1}\{X(T)\}\leq X(T-1)} &: \quad \left\{\tilde{v}_T(X(T); W(T) \mid \mathcal{N}_T) + X(T)\pi_T W_h(T)\bar{L}(T)\right\} \qquad (29.1)
\end{aligned}
$$

$$
\begin{aligned}
J_t(X(t-1)) &= \\
\inf_{X(t)\geq 0,\, E_{t-1}\{X(t)\}\leq X(t-1)} &: \quad \left\{\tilde{v}_t(X(t); W(t) \mid \mathcal{N}_t) + X(t)\pi_t W_h(t)\bar{L}(t) + E_t\left\{J(X(t))\right\}\right\} \qquad (29.2)
\end{aligned}
$$

$$
\begin{aligned}
J_0(F_0) &= \\
\inf_{X(0)\geq 0} &: \quad \left\{\tilde{v}_0(X(0); W \mid \mathcal{N}_0)) + X(0)F_0 + X(0)\pi_t W_h(0)\bar{L}(0) + E_0\left\{J_1(X(0))\right\}\right\} \\
&= \inf_{X(0)\geq 0} : \{X(0)F_0 + J_0(X(0))\} \qquad (29.3)
\end{aligned}
$$

where the infimum in (29.2) is taken with respect to non-negative supermartingale process $\{X(j)\}_{j=t-1}^{T}$, and $X(t-1)$ is given. This would be a standard stochastic dynamic programming problem, were it not for constraining the unknown function $X(t)$ to be a supermartingale, $E_{t-1}\{X(t)\} \leq X(t-1)$.

We now put our results formally in terms of Propositions 4 and 5 below. Proposition 4 may be proven by means of a trivial modification of Theorem 4, He and Pagés, *op. cit.* Its proof is omitted here.

Proposition 4.
*Under Assumption 1 there exists a unique solution to Problem (21).*

Proposition 5.
*The optimal solution to Problem (21) has the following form:*

(a) *There exists a unique solution $X_c(t)$ to (31) below, which is a random variable adapted to $\mathcal{N}_t$, such that:*

i. *If $E_t\{X_c(t+1)\} \leq X(t)$, then $X(t+1) = X_c(t+1)$;*

ii. *If $E_t\{X_c(t+1)\} > X(t)$, then $X(t+1) = X(t)$*

*We say that (i) is the constrained case, with (14.3) holding as an equality and $A(t) = b(t)$, and (ii) is the unconstrained case, with (14.3) holding as an equality, and $A(t) > b(t)$.*

(b) *The threshold values $\{X_c(t) \mid \mathcal{N}_t\}$ are defined recursively from (31), for $t = T, T-1, \ldots$ and form a predictable and $\mathcal{N}_t$-measurable random variable and characterize fully the optimal policy.*

(c) *To the threshold values $\{X_c(t) \mid \mathcal{N}_t\}$ there correspond threshold values for $A(t)$, $A_c(t)$, in terms of the state variable of the primal problem, such that $b^*(t) = A(t)$, if $A(t) \leq A_c(t)$, and $b(t) < A(t)$, otherwise.*

Proof: We note first that the minimization problem in the RHS of (29.1) involves a convex function of $X(T)$ and is subject to a convex constraint. The solution takes the form:

$$X^*(T) = X_c(T), \quad \text{if} \quad E_{T-1}\{X_c(T)\} \leq X^*(T-1);$$

$$X^*(T) = X(T-1), \quad \text{if} \quad E_{T-1}\{X_c(T)\} > X^*(T-1);$$

where $X_c(T)$ is the solution to:

$$\frac{\partial}{\partial X(T)} \tilde{v}_T(\cdot \mid \mathcal{N}_T) + \pi_T W_h(T) \bar{L}(T) = 0.$$

Clearly, if a solution to the above equation exists, it is unique. Working recursively we may establish that since $\tilde{v}_t$ is convex and decreasing in $X(t)$, $J_{t+1}(X(t))$ is also convex and decreasing in $X(t)$. Since $\tilde{v}_t$ is also convex and decreasing in $X(t)$, the optimal value of $X(t)$ may be expressed as:

$$X^*(t) = X_c(t), \quad \text{if} \quad E_{t-1}\{X_c(t)\} \leq X^*(t-1); \tag{30.1}$$

$$X^*(t) = X^*(t-1), \quad \text{if} \quad E_{t-1}\{X_c(t)\} > X^*(t-1); \tag{30.2}$$

where $X_c(t)$ is the unique solution to:

$$\frac{\partial \tilde{v}}{\partial X}(X_c(t); W(t) \mid \mathcal{N}_t) + \pi_t W_h(t)\bar{L}(t) + E_t\{\frac{\partial}{\partial X} J_{t+1}(X_c(t))\} = 0, \tag{31}$$

and $J_{t+1}(X(t))$ has been defined as the dual value function corresponding to (21) and is defined in (29.2).

Clearly, the threshold value $X_c(t)$ is conditional on all information available as of time $t$. (30.2) follows from observing that since $X^*(t)$ may not be equal to $X_c(t)$ and the minimand is a convex decreasing function of $X(t)$, then $X^*(t)$ should be set at as low a value as possible, which is $X^*(t-1)$. Finally, to show uniqueness of the optimal solution, it suffices to show that (29.3) implies a unique $X(0)$. Since $\tilde{v}$ and $J_1(X(0))$ are both convex decreasing functions of $X(0)$ then generically a unique non-negative $X(0)$ exists for which the infimum is attained.

We now recall the properties of the Lagrange multipliers introduced in (18), where $Y(t) \equiv X(t) - E_t\{X(t+1)\}$. Note that $Y(t) = 0$ implies that the liquidity constraint is not binding. Equivalently, this may be stated as $X(t) = E_t\{X(t+1)\}$. Therefore, (30.1) implies that $A(t) > b(t)$, that is unconstrained behavior, and (30.2) implies that $A(t) = b(t)$, that is constrained behavior.

Thus, Parts a and b of Proposition 5 have been proven. To establish Part c, note that from Proposition 2 and (30.1-2) that the existence of $X_c(t)$ implies the existence of $b_c(t)$, $b_c(t) = f(X_c(t)(1+\rho)^t\pi_t \mid \mathcal{N}_t)$ such that if $X(t) = X_c(t)$, then $A_c(t) \equiv b_c(t) = b^*(t)$. It then follows from the monotonicity and concavity of $v$ with respect to b that:

$$b^*(t) = A(t), \quad \text{if} \quad A(t) \leq A_c(t); \tag{32.1}$$

$$b^*(t) < A(t), \quad \text{if} \quad A(t) > A_c(t). \tag{32.2}$$

The proof of Proposition 5 is thus complete. □

The threshold value $X_c(t)$ defines a critical boundary in the nonnegative halfspace. This boundary splits the dual halfspace into two regions, the lower one corresponding to unconstrained behavior and the higher one to constrained behavior. Returning to the primal space, this implies a separation in the space of the market value of wealth in the beginning of each period, which includes financial assets plus the value of the endowment of leisure. The threshold value of assets need not be 0; in fact, in general, it would not. This is an important result, as much of the previous literature identified, quite arbitrarily and restrictively, constrained behavior with holding no financial assets as of the beginning of the period.[8]

## 4 CONCLUSIONS

The simplicity of the optimal solution of the problem of consumption decisions under uncertainty and liquidity constraints in its dual formulation has eluded researchers for a long time. In this paper we have suitably modified tools developed by finance theorists to prove existence and uniqueness and to provide a full characterization of the optimal solution for a general stochastic, discrete-time version of a proposition known to hold in a deterministic setting. At the same time, we have been able to interpret the solution in an economically insightful way. That interpretation alone is sufficiently powerful to warrant further research into the decision problem of an individual as well as its aggregate implications.

The interpretation of the solution to the liquidity constrained problem in terms of an equivalent unconstrained problem is seemingly akin to the permanent income hypothesis. The present value of the lifetime endowment of leisure looks like a linear function of the endowment of leisure in each period, but it is not. The respective discount factors, that is the coefficients that multiply each of the terms in the sequence, are themselves functions of initial assets and of individual characteristics that enter as preference parameters. Furthermore, it is very important that those same coefficients, which we have interpreted as individualized implicit Arrow-Debreu prices, are also used in discounting expenditures.

[8] See Deaton (1991) for an exception.

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