COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 976

AN "AVERAGE" LYAPUNOV CONVEXITY THEOREM AND SOME CORE EQUIVALENCE RESULTS

Lin Zhou

May 1991

AN "AVERAGE" LYAPUNOV CONVEXITY THEOREM AND SOME CORE EQUIVALENCE RESULTS *

Lin Zhou
Cowles Foundation, Yale University
P. O. Box 2125, New Haven, CT 06520

May 1991

Abstract. I prove an "average" version of the Lyapunov convexity theorem and apply it to establish some core equivalence results for an atomless economy.

* I thank Robert Anderson, Alvin Klevorick, and Nicholas Yannelis for their helpful comments.

1. Introduction

The purpose of this paper is twofold. First, I prove an "average" version of the Lyapunov convexity theorem. Inspired by the original Lyapunov convexity theorem, many authors have derived other useful results in the literature. In a recent paper (Zhou (1991)) I proved the following result: The set of average integrals of a point-valued mapping f is convex; moreover, it is the essential convex hull of the range of f. The latter feature of this result provides a characterization that previous convexity results do not. In Section 2, I will extend it to average integrals of a set-valued correspondence.

Second, I use this new convexity result to investigate the core of an atomless economy. It is well-known that the core coincides with the set of Walrasian equilibria for an atomless economy in which agents' preferences are continuous and locally nonsatiated (Aumann (1964), Hildenbrand (1982)). Many economists might consider this conclusion very general since the assumptions of continuity and local nonsatiation seem quite weak. But these two assumptions still impose substantial restrictions on an economy. For instance, they rule out indivisibility of consumption goods, an issue many economists believe should be seriously addressed in economic theory.[1] Hence, an even more general core equivalence theorem is desirable. In Section 3, I apply the new convexity result to derive a simple yet powerful core equivalence result for an atomless economy without making any explicit assumptions on individual agents' preferences. It yields the original Aumann core equivalence theorem as a special case but covers a much wider range of economies, and in particular, economies with indivisible goods.

---

[1] In fact, the literature on core equivalence as a whole has few results to offer for general economies with indivisible goods other than some special cases (Shapley and Scarf (1974)).

I also strengthen the Aumann theorem in another direction by establishing an equivalence result for some concepts of approximate cores and approximate equilibria.

Finally, in Section 4, I try to relate the analysis in previous sections to the core convergence of general large but finite economies. Recently, Manelli (1991) constructed some sequences of finite economies without monotonic preferences for which the cores do not converge even in the conventionally weak sense (as in Anderson (1978)). But the cores in some of these sequences do converge according to an even weaker notion of convergence suggested by our new core equivalence result. It remains to be seen when this type of core convergence holds for more general large but finite economies.

2. A Convexity Result on Average Integrals of a Correspondence

Consider a complete measure space ($\Omega, \mathcal{A}, \mu$) of a finite atomless measure μ. The Lyapunov convexity theorem asserts that for an integrable point-valued mapping f from Ω to R^l, the following set A is convex and compact,

$$A = \{ x \in R^l \mid x = \int_E f d\mu \text{ for some } E \in \mathcal{A} \}.$$

In Zhou (1991) I considered the set of all average integrals of an integrable point-valued mapping f. I proved:

(i) the following set B is convex,

$$B = \{ x \in R^l \mid x = \frac{\int_E f d\mu}{\mu(E)} \text{ for some } E \in \mathcal{A} \text{ with } \mu(E) > 0 \}; \text{ and}$$

(ii) $f(\omega) \in \overline{B}$ for almost all $\omega \in \Omega$, in which $\overline{B}$ is the closure of B.

Conclusion (i) is quite standard. Conclusion (ii), however, is a novel feature that previous Lyapunov-type results do not possess; it relates the set of average integrals of an integrable mapping to its essential convex hull. This second feature makes this result very useful in many applications. I now extend it to a set-valued correspondence.

Assume that F is a set-valued correspondence F from Ω to R^l. We say that f is a measurable selection of F if f is a measurable point-valued mapping from Ω to R^l such that $f(\omega) \in F(\omega)$ for almost all $\omega \in \Omega$. Let $\mathcal{F}$ be the set of all measurable selections of F. The integral of F is then defined by

$$\int_\Omega F d\mu = \{ x \in R^l \mid x = \int_\Omega f d\mu \text{ for some } f \in \mathcal{F} \}.$$

For a systematic treatment of integrals of set-valued correspondences, readers are referred to the book by Klein and Thompson (1984).

Vind (1964) (also see Hildenbrand (1974)) generalized the Lyapunov theorem to integrals of a set-valued correspondence. He proved: For any set-valued correspondence F from Ω to R^l, the following set C is convex:

$$C = \{ x \in R^l \mid x = \int_E F\, d\mu \text{ for some } E \in \mathcal{A} \text{ with } \mu(E) > 0 \}.$$

Vind's result is valid for any correspondence F even if the set C might be empty. To guarantee that integrals such as $\int_\Omega F d\mu$ are nonempty, however, one has to introduce other conditions on F. A natural and important condition is the measurability of F.

A correspondence F from Ω to R^l is measurable if its graph $\{ (\omega, x) \in \Omega \times R^l \mid \omega \in \Omega, \text{ and } x \in F(\omega) \}$ is a measurable set in $\mathcal{A} \otimes \mathcal{B}$, in which $\mathcal{B}$ is the σ-algebra generated by open sets in R^l and $\mathcal{A} \otimes \mathcal{B}$ is the product σ-algebra of $\mathcal{A}$ and $\mathcal{B}$.

I now state and prove the main result in this section.

Theorem 1. Assume that F is a measurable set-valued correspondence from Ω to R^l, which is also bounded from below by an integrable mapping. Then:

(i) *the following set D is nonempty and convex:*

$$D = \{ x \in R^l \mid x = \frac{\int_E f\, d\mu}{\mu(E)} \text{ for some } f \in \mathcal{F}, \text{ and } E \in \mathcal{A} \text{ with } \mu(E) > 0 \} ;$$

(ii) *$F(\omega) \subseteq \bar{D}$ for almost all $\omega \in \Omega$.*

Proof. (i) Let us construct a correspondence G from Ω to R^{l+1} as follows:

$$G(\omega) = (F(\omega), 1) \text{ for every } \omega \in \Omega .$$

Now consider the set C in R^{l+1} defined by:

$$C = \{ (x, z) \in R^{l+1} \mid x = \int_E f\, d\mu, \text{ and } z = \mu(E) \text{ for some } E \text{ in } \mathcal{A} \text{ with } \mu(E) > 0 \} .$$

Since F is measurable and bounded from below by an integrable set-valued mapping, so is G. By a measurable selection theorem of Aumann (1969), there is a measurable selection g of G. Being a selection of G, g is obviously bounded from below. Hence g is integrable. This shows that C is nonempty. In addition, C is convex by Vind's result. We now form *cone*(C) -- the cone generated by C. Since C is convex, *cone*(C) is convex. So is the intersection of *cone*(C) with the hyperplane $L = \{ (x, z) \in R^{l+1} \mid z = 1 \}$. It is clear that

$$cone(C) \cap L = (D, 1) .$$

Therefore, D is convex.

(ii) For each $x \notin \bar{D}$, there is an open ball O_x around x such that $O_x \cap D = \emptyset$. Consider H_x, the subset of Ω, such that $\omega \in H_x$ if and only if $O_x \cap F(\omega) \neq \emptyset$. Since F is measurable and O_x is open, H_x is a measurable subset of Ω by the projection theorem (see Hildenbrand (1974)). Suppose $\mu(H_x) > 0$. We consider $F_x = O_x \cap F$ as a correspondence from (H_x, $\mathcal{A}|_{H_x}$) to R^{l+1}. Then again by the Aumann measurable selection theorem, there is a measurable selection f_x of F_x that is also integrable because it is bounded. Since $f_x(\omega) \in O_x$ for all $\omega \in H_x$, it is easy to verify that $x = \frac{\int_H f_x d\mu}{\mu(H_x)} \in O_x$. This contradicts the assumption $O_x \cap D = \emptyset$. Hence, we have shown

$$\mu(H_x) = \mu(\omega \in \Omega \mid O_x \cap F(\omega) \neq \emptyset) = 0.$$

Because $R^l \setminus \bar{D}$ is open and separable, we can find countably many O_x like this such that their union is $R^l \setminus \bar{D}$. Since $\{ \omega \in \Omega \mid F(\omega) \subseteq \bar{D} \} = \Omega \setminus \bigcup O_x$ and $\mu(\Omega)$ is finite, we have

$$\mu(\omega \in \Omega \mid F(\omega) \subseteq \bar{D}) = \mu(\Omega).$$ **Q. E. D.**

Theorem 1 can also be viewed from a slightly different angle. It presents an explicit expression for the essential convex hull of the range of an integrable correspondence in terms of average integrals of this correspondence. The convex hull of the range of a correspondence is not intrinsic in the sense that two correspondences that are identical almost everywhere can have different convex hulls for their ranges. But one can define *ess co* (F), the essential convex hull of the range of a correspondence F, which is analogous to the essential bound of an integrable real-valued function, as follows:

$$ess\ co\ (F) \equiv \bigcap_{\mu(E) = \mu(\Omega)} co\ \Big(\bigcup_{\omega \in E} F(\omega) \Big).$$

Corollary. Given the assumptions and the notation of Theorem 1,

$$ess\ co\ (F) = \bar{D}\ .$$

Proof. Obviously, (ii) of Theorem 1 implies *ess co* $(F) \subseteq \bar{D}$. We now show that the converse is also true.

Take any $x \in D$. By definition, $x = \dfrac{\int_E f\,d\mu}{\mu(H)}$ for some $f \in \mathcal{F}$, and $H \in \mathcal{A}$ with $\mu(H) > 0$. Suppose that $x \notin co\,(\bigcup_{\omega \in E} F(\omega))$ for some E with $\mu(E) = \mu(\Omega)$. By the Minkowski separation theorem, there is a vector $p \in R^l$ such that $p \cdot x > p \cdot y$ for any $y \in co\,(\bigcup_{\omega \in E} F(\omega))$. In particular, $p \cdot x > p \cdot f(\omega)$ for any $\omega \in H \cap E$. But this contradicts the condition that $x = \dfrac{\int_E f\,d\mu}{\mu(H)}$. Hence $x \in co\,(\bigcup_{\omega \in E} F(\omega))$ for all E with $\mu(E) = \mu(\Omega)$. This shows that $D \subseteq$ *ess co* (F). Therefore, $\bar{D} \subseteq$ *ess co* (F) since *ess co* (F) is closed. **Q. E. D.**

3. Some Core Equivalence Theorems for An Atomless Economy

In this section we apply Theorem 1 to investigate the core of an atomless exchange economy.

Aumann, in his pioneering paper (1964), formalized the notion of perfect competition by introducing the model of an atomless economy. He proved that in such an economy the core coincides with the set of Walrasian equilibria if agents' preferences are continuous and monotonically increasing. A different proof was given in Hildenbrand (1974) who ascribed it to Schmeidler. The original proof of Aumann was later modified by Hildenbrand (1982) to derive an improved core equivalence result in which the condition of monotonicity used in Aumann's result was replaced by local nonsatiation.

Here I offer yet another approach, one which is extremely intuitive and simple given Theorem 1. Furthermore, it leads naturally to some important strengthenings of the Aumann core equivalence theorem. First, I show that a single measurability condition is virtually all one needs for a core equivalence result. No conditions on individual agents' preferences are necessary. This enlarges considerably the domain of economies to which the core equivalence relation applies. Second, I show that the core equivalence relation is quite robust. It also holds for some properly defined concepts of approximate cores and approximate equilibria. To make both points conceptually clear, I discuss them separately although they can be formally combined into one result.

3.1. A core equivalence result with no explicit conditions on agents' preferences

The set of agents is represented by a complete measure space ($\Omega, \mathcal{A}, \mu$) with a finite atomless measure. For simplicity, we assume that the commodity space is R^l_+ -- the nonnegative orthant of some Euclidean space.

Each agent ω has a preference relation $>_\omega$ on R^l_+ and an initial endowment $e(\omega) \in R^l_+$. It is assumed that $\int_\Omega e \, d\mu << \infty \mathbf{1}$, where $\mathbf{1}$ is the vector in which all components are one.

An allocation f is an integrable point-valued mapping from Ω to R^l_+ such that $\int_\Omega f \, d\mu = \int_\Omega e \, d\mu$.

A coalition E of agents is a measurable set in $\mathcal{A}$. A coalition E can improve upon an allocation f if there is another allocation g such that $g(\omega) >_\omega f(\omega)$ for almost every $\omega \in E$, and $\int_E g \, d\mu = \int_E e \, d\mu$. An allocation f is a core allocation if no coalition E of positive measure can improve upon it.

A quasi-equilibrium is a pair (p, f) in which p is a nonzero vector in R^l, and f an allocation such that $inf_{x >_\omega f(\omega)} p \cdot x = p \cdot f(\omega) = p \cdot e(\omega)$ for almost all $\omega \in \Omega$.

The Aumann core equivalence theorem (Hildenbrand (1982)) states that any core allocation is a quasi-equilibrium allocation when agents' preferences are continuous and locally nonsatiated. For many people this result is considered quite general.

In my view, however, continuity and local nonsatiation are still two rather restrictive assumptions. For example, when there is consumption indivisibility, an agent may have a preference relation $>$ on the set of all integer points only. Assuming free-disposal of any good in fractional amount, $>$ can be extended to a preference relation $>'$ on R^l_+ by letting $x >' y$ if $\lfloor x \rfloor > \lfloor y \rfloor$.[2] Obviously, $>'$ is neither continuous nor locally nonsatiated. Another example is as follows. Let $>$ be a preference relation that is represented by a continuous and increasing utility function u on R^l_+ . Again the preference relation $\lfloor > \rfloor$ represented by the utility function $\lfloor u \rfloor$ is neither continuous nor locally nonsatiated, where $\lfloor u \rfloor$ is defined by $\lfloor u \rfloor(x) = \lfloor u(x) \rfloor$ for all $x \in R^l_+$. Notice that reference relations $>'$ and $\lfloor > \rfloor$ in these two examples are quite natural. In fact, a continuous and increasing preference relation is usually thought as an idealization of preference relations of these types.

Of course when discontinuous and locally satiated preference relations are allowed, a core allocation may not be a quasi-equilibrium. Hence we have to find another weaker yet natural notion of equilibrium to which core allocations are related. The next example is suggestive to this point.

Let the set of agents be the unit interval [0, 1] with the Lebesgue measure. The commodity space is R^2_+. All gents in interval [0, 0.5] have an identical

[2] $\lfloor x \rfloor$ is the vector in which $\lfloor x \rfloor_i$ is the largest integer that is less than or equal to x_i for every i.

preference relation represented by $u(x_1,x_2) = x_1 \cdot x_2$, and agents in [0.5, 1] have a preference relation represented by $v(x_1,x_2) = \lfloor x_1 \cdot x_2 \rfloor$. All agents have the same initial endowment (1.5, 1.5). It is easy to verify that the allocation that assigns (2, 2) to agents in interval [0, 0.5], and (1, 1) to agents in [0.5, 1] is a core allocation. This is not a quasi-equilibrium since the only efficient price for this allocation is $p = (1, 1)$ under which $inf_{x >_\omega f(\omega)} p \cdot x = 4$, $p \cdot f(\omega) = 2$, and $p \cdot e(\omega) = 3$ for agents in interval [0, 0.5], and $inf_{x >_\omega f(\omega)} p \cdot x = 4$, $p \cdot f(\omega) = 4$, and $p \cdot e(\omega) = 3$ for agents in [0.5, 1]. But under $p = (1, 1)$ this allocation is still at equilibrium in a weak sense that no agent can afford a better consumption bundle given his initial income since $inf_{x >_\omega f(\omega)} p \cdot x = 4 > 3 = p \cdot e(\omega)$ for all agents. This leads to our definition of a pseuo-equilibrium.

A pseudo-equilibrium is a pair (p, f) in which p is a nonzero vector in R^l, and f an allocation such that $inf_{x >_\omega f(\omega)} p \cdot x \geq p \cdot e(\omega)$ for almost all $\omega \in \Omega$.

As we have seen, it is possible at a pseudo-equilibrium that $inf_{x >_\omega f(\omega)} p \cdot x > p \cdot e(\omega)$ or $p \cdot f(\omega) < p \cdot e(\omega)$ for some agents ω. But when agents' preferences are continuous and locally nonsatiated, $inf_{x >_\omega f(\omega)} p \cdot x = p \cdot f(\omega) = p \cdot e(\omega)$ must be true for almost all $\omega \in \Omega$ at any pseudo-equilibrium; so a pseudo-equilibrium is now a quasi-equilibrium. Finally, when agents' preferences are continuous and increasing, a pseudo-equilibrium becomes a Walrasian equilibrium.

Our main result here is that any core allocation is a pseudo-equilibrium. In order to prove it, we still need some conditions on the economy or on the allocation under consideration. So far we have not assumed any conditions on *individual* agents' preferences, and we won't. The only condition we need is an *aggregate* measurability condition (M) on an allocation f:

(M) The set-valued correspondence F from Ω to R^l is measurable:

$$F(\omega) = \{ x \in R^l \mid x + e(\omega) >_\omega f(\omega) \}.$$

Condition (M) enables us to use Theorem 1 to establish the following result.

Theorem 2. *Assume that f is a core allocation. If f satisfies the measurability condition (M), then there is a nonzero vector $p \in R^l$ such that (p, f) is a pseudo-equilibrium.*

Proof. For such an f we consider the correspondence F:

$$F(\omega) = \{ x \in R^l \mid x + e(\omega) >_\omega f(\omega) \}.$$

F is measurable since f satisfies condition (M). F is bounded from below by $-e$ from definition. Thus, according to (i) of Theorem 1, the set D is convex:

$$D = \{ x \in R^l \mid x = \frac{\int_E g \, d\mu}{\mu(E)} \text{ for some } g \in \mathcal{F}, \text{ and } E \in \mathcal{A} \text{ with } \mu(E) > 0 \}.$$

f is a core allocation implies $0 \notin D$. By the Minkowski's separation theorem, there is a nonzero vector $p \in R^l$ that separates 0 from D. According to (ii) of Theorem 1, $F(\omega) \subseteq \bar{D}$ for almost all $\omega \in \Omega$. So we have

$$p \cdot F(\omega) \geq 0, \quad \text{or equivalently,}$$

$$inf_{x >_\omega f(\omega)} \; p \cdot x \geq p \cdot e(\omega) \quad \text{for almost all } \omega \in \Omega.$$

Hence (p, f) is a pseudo-equilibrium allocation. **Q.E.D.**

Since Theorem 2 needs only the measurability condition (M), it highlights the significance of the use of various measurability conditions in the core equivalence literature; they are far from being as mild and merely technical as one might have thought. Our condition (M) is similar to the Aumann-

Hildenbrand measurability condition (as in Hildenbrand (1982)) in that they both are imposed on the core allocation itself. Strictly speaking, our condition (M) is slightly stronger because it requires that the set $\{ (\omega, x) \in \Omega \times R^l \mid \omega \in \Omega$ and $x \in F(\omega) \}$ be measurable in $\mathcal{A} \otimes \mathcal{B}$ while the A-H condition only requires that almost all x-sections of this set be measurable in $\mathcal{A}$. But the gain is considerable since we no longer assume the continuity of agents' preferences that Aumann's proof needed to reach the conclusion of Theorem 2. If one considers the space of continuous and irreflexive preferences endowed with the topology of closed convergence, then condition (M) can be derived from the condition that the preference mapping is measurable, which was assumed in the proof of the Aumann theorem by Hildenbrand-Schmeidler (Hildenbrand (1974)). But condition (M) or the A-H condition does not have a direct interpretation in terms of its economic content, which the H-S measurability condition possess (see Anderson (1991)).

3. 2. *Equivalence of approximate cores and approximate equilibria*

Another advantage of our approach is that it immediately leads to an equivalence result for approximate cores and approximate equilibria. The concept of an approximate core recognizes that there exist various tangible or intangible costs that prevent free coalition formation. Similarly, by using the concept of an approximate equilibrium we recognize that many forces might keep individual agents from full utility maximization and the market from complete clearing. If the core equivalence relation is robust, it should extend in some form to approximate cores and approximate equilibria.

For ease of exposition, we assume that each agent ω has a preference relation $>_\omega$ on R^l_+ that satisfies continuity -- $\{ (x, y) \in R^l_+ \times R^l_+ \mid x >_\omega y \}$ is open in $R^l_+ \times R^l_+$ -- and monotonicity -- $x \geq y$ and $x \neq y \Rightarrow x >_\omega y$. We also assume that the preference mapping is measurable with respect to the topology of closed convergence on preference space.

A coalition E can ε-improve upon an allocation f if there is an allocation g such that $g(\omega) >_\omega f(\omega)$ for almost all $\omega \in E$, and $\int_E g \, d\mu \leq \int_E e \, d\mu \, \Theta \varepsilon\mu(E)\mathbf{1}$. (The operator Θ is defined by $(a \Theta b)_i = \max \{ a_i - b_i , 0 \}$.) Here $\varepsilon\mathbf{1}$ can be interpreted as the cost vector per capita of coalition formation. An allocation f is an ε-core allocation if no coalition E of positive measure can ε-improve upon it. Let $C(\varepsilon)$ denote the set of all ε-core allocations.

An ε-equilibrium is a pair (p, f) in which p is a price vector ($p_i \geq 0$, $\Sigma p_i = 1$) and f an allocation such that $\inf_{x >_\omega f(\omega)} p \cdot x \geq p \cdot e(\omega) - \varepsilon$ for almost all $\omega \in \Omega$. An allocation f is an ε-equilibrium allocation if there is a p such that (p, f) constitutes an ε-equilibrium. Let $W(\varepsilon)$ denote the set of all ε-equilibrium allocations.

The definition of the ε-core used here is standard (Kannai (1970)) with the 0-core being the conventional core.

Similarly, given the assumptions on agents' preferences, a 0-equilibrium is a Walrasian equilibrium. The definition of a general ε-equilibrium here is close to, but weaker than that used by Khan (1974). Although both require that compensated expenditures be no less than initial incomes by more than ε for almost all agents, we do not require, as Khan did, that budget deviations -- $| p \cdot f(\omega) - p \cdot e(\omega) |$ -- be less than ε for almost all agents. But parts of Khan's conclusions with his notion of ε-equilibria are incorrect.[3] I soon will give an example that illustrates this point.

The equivalence of the ε-core and the set of ε-equilibrium allocations can be easily established.

Theorem 3. (i) $C(\varepsilon) \subseteq W(\varepsilon)$ for all ε;

(ii) $W(\varepsilon) \subseteq C(\varepsilon)$ *when* $e(\omega) \geq \varepsilon \mathbf{1}$ *for almost all* $\omega \in \Omega$.

Proof. (i) The proof is the same as that of Theorem 2.

(ii) When the initial endowment e is uniformly bounded from below by ε, $\int_E e \, d\mu \ominus \varepsilon\mu(E)\mathbf{1} = \int_E e \, d\mu - \varepsilon\mu(E)\mathbf{1}$. The proof is straightforward. **Q.E.D.**

Notice that (i) in Theorem 3 is as strong as possible. For any ε-core allocation one can only find a price p such that compensated expenditures are no less than initial incomes by more than ε for almost all agents. But one cannot conclude that budget deviations are less than ε for almost all agents. The intuition is clear. By definition, an ε-core allocation only prevents almost every agent from being worse off by ε. It does not, however, prevent a sizable set of agent from being better off by more than ε. Here we give a simple example. Let the set of agents be the unit interval [0, 1] with the Lebesgue measure. The commodity space is R^2_+. All agents have an identical preference relation represented by the utility function $u(x_1,x_2) = x_1 \cdot x_2$ and an identical initial endowment (1, 1). For any $\varepsilon < 1$, consider an allocation f_ε that assigns $(1-\varepsilon, 1-\varepsilon)$ to a set of agents with a measure of $\frac{2}{3}$ and $(1+2\varepsilon, 1+2\varepsilon)$ to all other agents. This is obviously an ε-core allocation. But the set of agents whose budget deviations are more than ε has a measure of $\frac{1}{3}$, regardless of how small ε is. In fact, for any large number M, we

[3] Consequently, Khan's comments on his conclusions and the result by Arrow and Hahn (later generalized by Anderson (1978)) were also incorrect.

can find an ε-core allocation in which there is sizable set of agents who are better off by more than $M\varepsilon$.

This is a major correction that Theorem 3 makes to results in Khan (1974). There are also other differences. First, Theorem 3 does not assume a uniformly boundedness condition that Khan needed to assume for his nonstandard analysis approach. Second, Theorem 3 can be proved under condition (M) only, but Khan's proofs required the monotonicity of agents' preferences.

4. A Remark

The notion of an approximate equilibrium (or a pseudo-equilibrium) used here is different from those used by other authors (Khan (1974), Arrow-Hahn (1971) and Anderson (1978)). Instead of considering both budget deviations and compensated expenditure gaps, an approximate equilibrium here considers only compensated expenditure gaps and, furthermore, only gaps from below. One advantage of this definition, as shown in Theorem 3, is that this definition fully characterizes the standard approximate cores in an atomless economy. I now want to speculate on a possible implication of this notion for the core convergence of large but finite economies.

Anderson (1978) proved a core convergence result for finite economies in which agents' preferences are monotonic and weakly transitive. The notion of convergence used there was the mean convergence of both budget deviations and compensated expenditure gaps. Recently Manelli (1991) constructed several examples of sequences of finite economies without monotonic preferences for which the cores do not converge in that sense.

A natural question suggested by Manelli's examples is: In what sense, if not in Anderson's, do cores of general large economies converge? Based on the analysis in Section 3, especially Theorem 2, we introduce the following notion of core convergence, which is even weaker than that used in Anderson (1978):

A sequence of large finite economies weakly converges if for any $\delta > 0$, there exists an integer n such that any core allocation in economies after the n-th has a normalized vector p for which the proportion of the agents whose compensated incomes fall by more than δ below their initial incomes is less than δ.

This type of convergence does hold in some of Manelli's examples, although it fails in some other examples of his. But given Theorem 2, it is tempting to conjecture that this type of convergence should hold in a large class of well-behaved sequences of large finite economies. At the present time, however, we are still unable to provide a precise answer, one which might require first a meaningful finite version of Theorem 1.

References

Anderson, R., 1978, An elementary core equivalence theorem, Econometrica 46, 1483-1487.

Anderson, R., 1991, The core in perfectly competitive economies, in: Handbook of Game Theory with Economic Applications (North-Holland, Amsterdam).

Arrow, K. and F. Hahn, 1971, General competitive analysis (Holden Day, San Francisco).

Aumann, R., 1964, Markets with a continuum of traders, Econometrica 32, 39-50.

Aumann, R., 1969, Integrals of set-valued functions, Journal of Mathematical Analysis and Applications 12, 1-12.

Hildenbrand, W., 1974, Core and equilibria of a large economy (Princeton University Press, Princeton).

Hildenbrand, W., 1982, Core of an economy, in: Handbook of Mathematical Economics (North-Holland, Amsterdam).

Kannai, Y., 1970, Continuity properties of the core of a market, Econometrica 38, 791-815.

Khan, A., 1974, Some equivalence theorems, Review of Economic Studies 41, 549-565.

Klein, E., and A. Thompson, 1984, Theory of correspondences (Wiley, New York).

Manelli, A., 1991, Monotonic preferences and core convergence, Econometrica 59, 123-138.

Shapley. L., and H. Scarf, 1974, On cores and indivisibility, Journal of Mathematical Economics 1, 23-38.

Vind, K., 1964, Edgeworth-allocations in an exchange economy with many traders, International Economic Review 5, 165-177.

Zhou, L., 1991, Strictly fair allocations in large exchange economies, Cowles Foundation Discussion Paper #972, Yale University.