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AT YALE UNIVERSITY

Box 2125, Yale University
New Haven, Connecticut 06520

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STRICTLY FAIR ALLOCATIONS IN LARGE EXCHANGE ECONOMIES

by

LIN ZHOU

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Cowles Foundation, Yale University

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ABSTRACT

In this paper we introduce the concept of a strictly fair allocation and investigate the set of strictly fair allocations in large exchange economies. We prove that when agents' utility functions are differentiable, the set of strictly fair allocations coincides with the set of equal-income Walrasian equilibria. This is shown using both the "limit theorem" approach and the "limit economy" approach. We also extend the analysis to economies that have both atoms and an atomless sector. These results substantially improve upon the existing characterizations of equal-income Walrasian equilibria in terms of both economic efficiency and economic equity.

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1. Introduction

In this paper we investigate the economic efficiency and economic equity of equal-income Walrasian equilibrium allocations in large exchange economies. The concept of economic efficiency is well known since Pareto, and equally well known is the fact that Walrasian equilibria are efficient. On the other hand, economists still do not agree on a proper concept of economic equity. We introduce a concept of strictly envy-free allocations, which is stronger than the concept of envy-free allocations but much weaker than the concept of coalitionally envy-free allocations. Once the precise definition is given, it will be obvious that all equal-income Walrasian equilibria are strictly envy-free. The main result we shall prove is that equal-income Walrasian equilibria are the only allocations that are both efficient and strictly envy-free for any differentiable but otherwise standard large exchange economy. This is an important characterization of equal-income Walrasian equilibria, which substantially improves upon existing results of the same nature that have been derived since Foley's [5] original contribution. The remainder of the introduction contains an informal discussion of our results and the existing literature; formal analyses will be developed in later sections.

We consider an exchange economy in which a certain amount of goods is to be divided among many agents. Foley [5] first introduced the following notion of equity. An allocation is envy-free if no agent prefers, or envies, another agent's bundle to his own. An allocation is fair if it is both efficient and envy-free. It is obvious that an equal-income Walrasian equilibrium allocation is fair. But the converse is generally not true, not even in a large economy. Neverthless, since equal-income Walrasian equilibria have a very intuitive appeal, it is desirable to know for what notion of equity they are the only efficient and equitable allocations. This has led many authors to investigate various concepts of fairness that could fully characterize the set of equal-income Walrasian equilibria.

The existing literature features two types of results. In the first type the notion of envy is extended to coalitions of agents. A coalition S envies another coalition T at an allocation x when agents in S could reallocate consumption and achieve a Pareto improvement if S had the average bundle of T at x as its own average. An allocation is coalitionally envy-free if no coalition envies any other coalitions. An allocation is coalitionally fair if it is both efficient and coalitionally envy-free. It has been proved that the set of coalitionally fair allocations in an atomless economy (Varian [14], Vind [16]) or the intersection of the sets of coalitionally fair allocations of all replica economies (Varian [14]) coincides with the set of equal-income Walrasian equilibria. This type of result is not fully satisfactory because "the 'coalition-envy' concept is by no means as compelling as the original envy concept"(Varian [15]). In fact, the need, in the definition of a coalitionally envy-free allocation, to reallocate goods among agents of a coalition reveals that the concept is more a notion of cooperation than a notion of equity. Hence, this type of result is often technically a simple corollary of the core equivalence result.

The study of the second type of result was initiated by Varian and then pursued by other authors (Varian [15], Kleinberg [8], Champsaur and Laroque [3], and Mas-Colell [9]). In these works, various requirements are imposed on an economy to establish the coincidence of the set of fair allocations with the set of equal-income Walrasian equilibria. A common requirement is that there exist a connected continuum of types of agents (instead of just continuum of agents). But this is a drastic deviation from the standard model of exchange economies. It is not met by any economies with countably many types of agents and in particular not by any replica of a finite economy. There is still no parallel result of the second type in replica economies. Hence, if one regards an atomless economy as some sort of limit of large but finite economies, this lack of a parallel result in replica economies is certainly a serious shortcoming.

In this paper we introduce the concept of a strictly fair allocation. As in the case of fair allocations, only individual agents make utility comparisons. Each agent may compare his bundle with any other coalition's average bundle. This kind of comparison is very common in reality and should be taken into account by a proper notion of equity. An allocation is strictly envy-free if no agent prefers any other coalition's average bundle to his bundle. An allocation is strictly fair if it is both efficient and strictly envy-free.

The economies we consider in the paper are standard except for the additional assumption that agents' utility functions are differentiable. Our conclusion is that the set of strictly fair allocations coincides with the set of equal-income Walrasian equilibria. It is derived using both the "limit theorem" and the "limit economy" approaches.

The basic idea behind our results is best illustrated in the case of an atomless economy. An "average" version of the Lyapunov convexity theorem, established in the paper, playes a crucial role here. This result states that the set of all average integrals of an integrable mapping over various sets of positive measures is a convex set and, furthermore, that this set is 'almost' the convex hull of the range of this mapping. Now consider an economy with an average total endowment ω. Assume that x is a strictly fair allocation. Let X be the set of average integrals of x, which obviously includes ω. Since x is efficient, it has a supporting price p. To show that (x, p) is an equal-income Walrasian equilibrium, it suffices to show that the width of X along the direction p is zero since X is almost the convex hull of the range of x. Suppose it is not true. Then a portion of X must be strictly below the budget line passing through ω. Since X is almost the convex hull of the range of x, we assume that there is an agent a with $x(a)$ in X and $p \cdot x(a) < p \cdot \omega$. Given that agent a's utility function is differentiable and that p is the supporting price for x, he must prefer some bundle between $x(a)$ and ω. But X is convex implies that all such bundles are averages of x for some coalitions. So agent a would envy some coalition. This contradicts the assumption that x is strictly fair.

Since the concept of strictly fair allocations is far more intuitive than that of coalitionally fair allocations, our results represent a considerable improvement over the above-mentioned results of the first type. On the other hand, in sharp contrast to the second type of results, ours rely on no extra assumptions beyond the standard ones for an exchange economy and the differentiability of agents' utility functions. Because, as one will see later, the concept of strictly fair allocations is almost as intuitive as the basic concept of fair allocations, our results with minimal assumptions represent a substantial advance.

Although we mainly focus on strictly fair allocations, parallel analyses apply to strictly fair net trades. In the latter case, instead of considering how to divide a fixed amount of goods among agents, one considers how to conduct trades among agents when individual agents' initial endowments are given. A net trade is strictly fair if it is efficient and no agent prefers the average net trades of any other coalition to his own.[1] The conclusion is thus that a net trade is strictly

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[1] The concept of a strictly fair net trade is different from the concept of a strongly fair net trade of Schmeidler and Vind [12], and Gabszewicz [6]. A net trade is strongly fair if no agent prefers any sum of integer multiples of other agents' net trades to his own. An important consideration behind this notion is that there is some sort of anonymous trading institution at work and an agent can enter the market arbitrarily many times. However, here we are more concerned with distributive justice.

fair if and only if it leads to a Walrasian equilibrium allocation for the given individuals' initial endowments. Hence we have a very nice interpretation of Walrasian equilibria in terms of economic equity, which is more direct and compelling than those given by other authors through the core convergence result, or the Shapley value convergence result (Aumann [2]) .

Here is the plan of our paper. In Section 2 we consider strictly fair allocations in replica economies. We prove that the intersection of the sets of strictly fair allocations of all replica economies of a finite economy coincides with the set of equal-income Walrasian equilibria of the original enonomy. We also show that under slightly stronger conditions the set of strictly fair allocations converges to the set of equal-income Walrasian equilibria at the rate $\frac{1}{n}$, where n is the number of replications. In Section 3 we discuss strictly fair allocations in an atomless economy. First we derive an "average" version of the Lyapunov convexity theorem. This result is then used to prove that the set of strictly fair allocations coincides with the set of equal-income Walrasian equilibria. Section 4 analyzes economies that have both atoms and an atomless sector. Even though an equivalence result no longer holds for such economies, the set of strictly fair allocations still has some nice properties. In the model of oligopolistic mixed economies (Shitovitz [10, 11]), small agents have equal-incomes at any strictly fair allocation when goods are valued at the supporting price. This common income is no more than the income of any large agent despite the fact that small agents do not envy other coalitions' bundles. Finally we conclude the paper with some remarks in Section 5.

2. STRICTLY FAIR ALLOCATIONS IN REPLICA ECONOMIES

We begin with an economy $\mathcal{E}$ of finite agents. The commodity space is R^{ℓ}_{++}. The total endowment of $\mathcal{E}$ is a vector $\omega \in R^{\ell}_{++}$. There are m agents in $\mathcal{E}$, and the set of agents is $A = \{ 1, 2, ..., m \}$. Each agent i has a utility function u_i that is strictly increasing, strictly quasi-concave, and differentiable on R^{ℓ}_{++}.[2]

A feasible allocation x is an m-tuple $(x_1 , x_2 , ... , x_m) \in R^{\ell m}_{++}$ such that

$$\sum_{i=1}^{m} x_i = \omega .$$

A feasible allocation x is efficient if there is no other feasible allocation y such that $u_i(y_i) > u_i(x_i)$ for every agent i.

An agent i envies an agent j at an allocation x if $u_i(x_i) < u_i(x_j)$. An allocation x is envy-free if no agent envies any other agent.

DEFINITION 2.1. An allocation x is fair if it is both efficient and envy-free.

An immediate fact is that any Walrasian equilibrium allocation of $\mathcal{E}$ with equal-incomes for all agents is fair (Foley [5], Schmeidler and Vind [12]). But in general there are fair allocations other than equal-income Walrasian equilibria. Fair allocations do not converge to equal-income Walrasian equilibria even when an economy is replicated because replications of any fair allocation remain fair in replica economies. Hence a stronger notion of fairness has to be employed to get a convergence result. In what follows, each agent now compares his own bundle not just with bundles of other agents, but also with average bundles of various coalitions of other agents. This is very natural since such comparisons

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[2] We can use preference relations instead of utility functions in our discussions throughout this paper. However, the conditions we impose on the preference relations imply the existence of utility function representations. Hence, for ease of exposition we use utility functions .

are constantly observed in reality. Should one consider an agent's envy of other agents undesirable from an equity point of view, one should consider his envy of other coalitions of agents the same way.

Formally, an agent i envies a coalition S ($i \notin S$) at an allocation x if $u_i(x_i) < u_i(\bar{x}_S)$, in which $\bar{x}_S = \frac{1}{|S|} \Sigma_{i \in S} x_i$.

An allocation x is strictly envy-free if no agent envies any other coalition.

DEFINITION 2.2. An allocation x is strictly fair if it is both efficient and strictly envy-free.

It is clear that any equal-income Walrasian equilibrium is strictly fair. More importantly, we now show that strictly fair allocations converge to equal-income Walrasian equilibrium allocations when the economy is replicated.

For an economy $\mathcal{E}$ described at the beginning of this section, we define its n-fold replica $\mathcal{E}_n$ as follows. $\mathcal{E}_n$ has an initial aggregate endowment $n\omega$ and nm agents with exactly n agents having the same utility function u_i for each i from 1 to m. We call those agents who have the same utility function u_i type i agents.

Let us denote by SF_n the set of strictly fair allocations of $\mathcal{E}_n$. There are two simple facts about SF_n. First, given the assumption of strict quasi-concavity of utility functions, SF_n has the property of "equal treatment," i.e., any allocation in SF_n assigns agents of the same type exactly the same consumption bundle (Varian [14]). Hence we can represent every strictly fair allocation in SF_n , which is a vector in $R^{n\ell m}_{++}$, by an allocation in the original economy $\mathcal{E}$, which is just a vector in $R^{\ell m}_{++}$. Without introducing a new notation, we will use SF_n to denote its representation in the original economy $\mathcal{E}$. The second fact is that the sequence of sets $\{SF_n\}$ is non-increasing. This can be easily derived from the definition with the help of the first fact. Finally, let us denote by W_{ei} the set of equal-income Walrasian equilibrium allocations of the original economy $\mathcal{E}$.

PROPOSITION 2.3. $\bigcap_{n=1}^{\infty} SF_n = W_{ei}$.

Proof. It is obvious that $W_{ei} \subseteq \bigcap_{n=1}^{\infty} SF_n$. To complete the proof we have to show $\bigcap_{n=1}^{\infty} SF_n \subseteq W_{ei}$. Let us take any $x \in \bigcap_{n=1}^{\infty} SF_n$. Since x is efficient, we can find a price vector p that supports x given the assumptions we have imposed on all u_i . If we can show that $p \cdot x_i = p \cdot \frac{\omega}{m}$ for every type i , then (p, x) is an equal-income Walrasian equilibrium. Now suppose $p \cdot x_i \neq p \cdot \frac{\omega}{m}$ for some i , we shall derive a contradiction.

Assume that for some type i , $p \cdot \frac{\omega}{m} > p \cdot x_i$, or $p\,(\frac{\omega}{m} - x_i) > 0$. Let $l = \frac{\omega}{m} - x_i$. Since *grad* $u_i(x_i)$ is a positive multiple of p , $\frac{du_i}{dl}(x_i) > 0$. Thus there exists an $\alpha > 0$ such that

$$u_i(x_i + \varepsilon l) > u_i(x_i), \quad \text{for any } 0 < \varepsilon < \alpha .$$

Take any n-fold replica economy $\mathcal{E}_n$ with $n > \frac{m}{\alpha} + 2 - m$. Consider an agent of type i and a coalition S_n of $(n - 2)$ agents of type i plus a single copy of the original economy $\mathcal{E}$. The aggregate bundle of S_n is $\omega + (n - 2)x_i$, hence its average is

$$\bar{x}_{Sn} = \frac{\omega + (n-2)\,x_i}{m + (n - 2)} = x_i + (\frac{m}{n + m - 2})l .$$

Since $n > \frac{m}{\alpha} + 2 - m$, we have $\frac{m}{n + m - 2} < \alpha$. Hence $u_i(\bar{x}_{Sn}) > u_i(x_i)$. But this means that i envies S_n at x in $\mathcal{E}_n$, which is a contradiction to the assumption $x \in \bigcap_{n=1}^{\infty} SF_n$. Q. E. D.

The differentiability of agents' utility functions is necessary for Proposition 2.3. One can check this by looking at an Edgeworth economy $\mathcal{E}$ depicted in Figure 1.

The allocation x is efficient in $\mathcal{E}$ and thus in any n-fold replica $\mathcal{E}_n$. It is also strictly envy-free in any $\mathcal{E}_n$ since the average bundle of any coalition S must lie on the segment xy, which neither type 1 nor type 2 agent prefers to x. Hence, x is strictly fair in any $\mathcal{E}_n$. But, it is not an equal-income Walrasian equilibrium allocation. Of course, it is apparent from the proof that this can happen only at an allocation where at least one agent's utility function has a kink.

Proposition 2.3 represents a considerable improvement of Varian's result [14]. First, the concept of strictly fair allocations is superior to that of coalitionally fair allocations as a pure notion of equity. Second, the set of strictly fair allocations is much larger that the set of coalitionally fair allocations. Hence, Proposition 2.3 implies Varian's result. (To be precise, the concept of coalitional fairness used here is slightly different from that in Varian. Our claim, nevertheless, is still valid. See the appendix for a discussion of this point.)

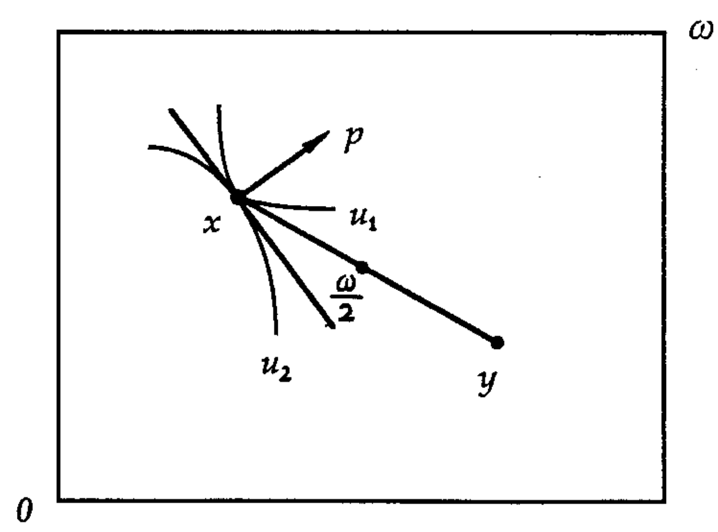

FIGURE 1

A direct corollary of Proposition 2.3 is that the set of strictly fair allocations converges to the set of equal-income Walrasian equilibrium allocations as the original economy is replicated. It is interesting to know how fast it converges. This issue has been thoroughly investigated in the literature on core convergence since Debreu's seminal work [4]. Under some assumptions slightly stronger than what we have made so far, Debreu proved that the core converges to the set of Walrasian equilibria at a rate equal to the inverse of the number of replications. Here we show that under similar assumptions the same result holds for convergence of the set of strictly fair allocations. We first derive a basic inequality that any allocation in SF_n must satisfy.

LEMMA 2.4. *Let $x \in SF_n$ and p be the efficient price that supports x with $\|p\| = 1$. Assume that there exists a positive number ρ such that for each type i,*

$$u_i(y) > u_i(x_i), \quad \textit{for every } y \textit{ satisfying } d(y, x_i + \rho p) < \rho. \qquad (*)$$

Then for each type i,

$$\left| p \cdot \left(\frac{\omega}{m} - x_i\right) \right| \le \frac{m(m-1)\|\omega\|^2}{2\rho(n+m-2)} .$$

Proof. First take any type i that satisfies $p \cdot (\frac{\omega}{m} - x_i) > 0$. Let $l = \frac{\omega}{m} - x_i$. We look at Figure 2. Let b denote the intersection of the ball $B(O, \rho)$ and segment $x_i\omega'$, in which $O = x_i + \rho p$ and $\omega' = \frac{\omega}{m}$. We can write $b = x_i + \delta l$. To determine the value of δ, we consider two triangles $\Delta O a x_i$ and $\Delta x_i c \omega'$. Since they are similar and $|Ox_i| = \rho$, $|ax_i| = \frac{\delta\|l\|}{2}$, $|x_i\omega'| = \|l\|$, and $|c\omega'| = p \cdot l$, we have

$$\frac{\delta\|l\|}{2\rho} = \frac{p \cdot l}{\|l\|} .$$

Hence $\delta = \frac{2\rho\, p \cdot l}{\|l\|^2}$. Now consider an agent of type i and the coalition S_n defined in the proof of Proposition 2.3 with $\bar{x}_{S_n} = x_i + (\frac{m}{n+m-2})l$. Since $x \in SF_n$, $u_i(x_i) \geq u_i(\bar{x}_{S_n})$. Therefore, $\frac{m}{n+m-2} \geq \delta = \frac{2\rho\, p \cdot l}{\|l\|^2}$. This leads to

$$p \cdot (\frac{\omega}{m} - x_i) = p \cdot l \leq \frac{m\,\|l\|^2}{2\rho\,(n+m-2)} \leq \frac{m\,\|\omega\|^2}{2\rho\,(n+m-2)}.$$

Now we consider agents with $p \cdot (\frac{\omega}{m} - x_i) < 0$. The above inequality and the fact that $\sum_{i=1}^{m} x_i = \omega$ imply

$$p \cdot (\frac{\omega}{m} - x_i) \geq -\frac{m\,(m-1)\,\|\omega\|^2}{2\rho\,(n+m-2)},$$

for any type i that satisfies $p \cdot (\frac{\omega}{m} - x_i) < 0$. Q. E. D.

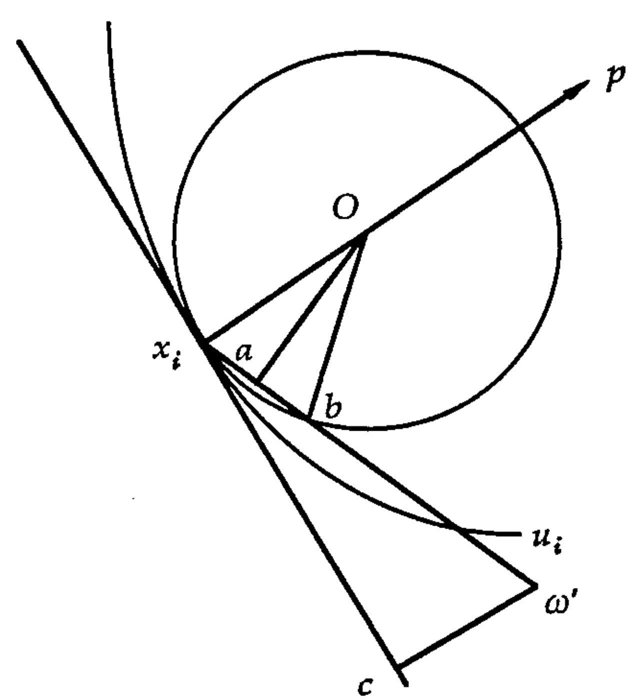

FIGURE 2

Lemma 2.4 says that when condition (*) is met, the value of any x_i at an allocation $x \in SF_n$ evaluated by its supporting price is close to the value of one m-th of the initial endowments evaluated by the same price. Condition (*) however does not follow from the assumption that utility functions are differentiable. Even if it is met, the conclusion of Lemma 2.4 is still short of the statement that SF_n converges to the set of equal-income Walrasian equilibria at the rate of $\frac{1}{n}$. To close these gaps one has to introduce additional assumptions. Then one can prove the following result.

PROPOSITION 2.5. *If the original economy $\mathcal{E}$ is smooth and regular, then the set of strictly fair allocations of the n-fold replica economy $\mathcal{E}_n$ converges to the set of equal-income Walrasian equilibrium allocations of $\mathcal{E}$ at the rate $\frac{1}{n}$.*

For a proof of Proposition 2.5 it suffices to point out that Debreu's [4] proof for the rate of core convergence also applies here. A key step there is equation (3) in [4] for core allocations. The same holds for strictly fair allocations because of Lemma 2.4. Readers are referred to Debreu's paper for the precise definitions of the terms in Proposition 2.5 and a detailed proof.

3. STRICTLY FAIR ALLOCATIONS IN AN ATOMLESS ECONOMY WITH A CONTINUUM OF AGENTS

In this section we consider strictly fair allocations an atomless economy with a continuum of agents. The set of agents is an atomless measure space $(A, \mathcal{F}, \mu)$ with $\mu(A) = 1$. A coalition S is a measurable set in $\mathcal{F}$. The commodity space is still R^{ℓ}_{++}. Every agent a has a utility function $u(.\,; a)$ that is strictly increasing and differentiable on R^{ℓ}_{++}. Notice that the of convexity of utility functions is now dispensed with. The following measurability condition (Aumann [1]) is also assumed: the set $\{ a \in A \mid u(x(a); a) > u(y(a); a) \}$ is measurable for any measurable functions x and y from A to R^{ℓ}_{++}.

The total endowment of the economy is a vector $\omega \in R^{\ell}_{++}$. A feasible allocation x is an integrable function from A to R^{ℓ}_{++} such that

$$\int_A x(a) d\mu = \omega .$$

A feasible allocation x is efficient if there is no other feasible allocation y such that

$$u(y(a), a) > u(x(a), a), \qquad \text{a.e. in } A .$$

Let $\mathcal{G}$ denote the set of all coalitions of positive measures:

$$\mathcal{G} = \{ S \in \mathcal{F} \mid \mu(S) > 0 \}.$$

An agent a is envious at an allocation x if $u(x(a); a) < u(\bar{x}(S); a)$ for some S in $\mathcal{G}$ ($a \notin S$), where $\bar{x}(S)$ is the average bundle of S: $\bar{x}(S) = \dfrac{\int_S x(a)\, d\mu}{\mu(S)}$.[3] An allocation x is strictly envy-free if the set of agents who are envious at x has measure zero.

DEFINITION 3.1. An allocation x is strictly fair if it is both efficient and strictly envy-free.

As a concept of economic equity, strict fairness is stronger than fairness, but not by much. First, in both concepts single agent's utility comparisons only are

3 There is a possibly stronger notion of envy-free. One can choose an arbitrarily small positive number ε and say that agent a is ε-envious if he prefers the average bundle of a coalition S of measure no more than ε. The interpretation is that an agent's ability to aggregate is limited for reasons unspecified here. An allocation is ε-envy-free if no agent is ε-envious. But this actually makes no difference in an atomless economy because of the Lyapunov convexity theorem.

made. Second, both concepts have the same informational requirement. Each agent needs to know only consumption bundles all agents receive, but not the utility functions others have. Third, even the stronger requirement in strictly fair allocations that agents know how to aggregate can be weakened considerably as footnote 2 has indicated.

DEFINITION 3.2. An allocation x is an equal income Walrasian equilibrium allocation if there is a price p such that $x(a)$ maximizes $u(.\,; a)$ on the set $\{ x \in R^{\ell}_{++} \mid p{\cdot}x = p{\cdot}\omega \}$ almost everywhere in A.

As in replica economies, any equal income Walrasian equilibrium allocation here is obviously strictly fair. We now show that the converse is also true. To do this we need an 'average' version of the Lyapunov convexity theorem. The original Lyapunov theorem states that for an integrable function on an atomless measure space the set of all integrals on various measurable sets is convex. However, in the definition of a strictly envy-free allocation, each agent compares his own bundle to all average bundles of coalitions with positive measures. Hence, we want a result that is similar to the Lyapunov theorem but deals with the set of average integrals of an integrable function on an atomless measure space.

Assume that h is an integrable function from an atomless measure space (Ψ, $\mathcal{B}, \lambda$) to R^{ℓ}, in which $0 < \lambda(\Psi) < \infty$. Denote by H the set of all average integrals of h on measurable sets of positive measures:

$$H = \{ x \in R^{\ell} \mid x = \bar{h}(T) \text{ for some } T \text{ in } \mathcal{B} \text{ with } \lambda(T) > 0 \}.$$

LEMMA 3.3. (i) *H is convex; and*

(ii) $\lambda(h^{-1}(cl(H))) = \lambda(\Psi)$, *where $cl(H)$ is the closure of H ; hence,* *there is some* $b \in \Psi$ *such that* $h(b) \in cl(H)$.

Proof. (i) Let us construct a mapping d from Ψ to $R^{\ell+1}$ as follows:

$$d(c) = (h(c), 1), \quad \text{for every } c \in \Psi.$$

By the Lyapunov theorem, the following set D is convex in $R^{\ell+1}$,

$$D = \{ z \in R^{\ell+1} \mid z = (\int_T h_1 d\lambda, \ldots, \int_T h_\ell d\lambda, \lambda(T)) \text{ for some } T \text{ in } \mathcal{B} \}.$$

We then form $Cone(D)$, the cone generated by D. $Cone(D)$ is convex, and so is its intersection with the hyperplane $L = \{ z \in R^{\ell+1} \mid z_{\ell+1} = 1 \}$. It is obvious that

$$Cone(D) \cap L = (H, 1).$$

Therefore, H is convex.

(ii) For each $x \notin cl(H)$, there is a ball O_x around x such that $O_x \cap H = \emptyset$. Hence, the set of c's for which $h(c)$'s lie in O_x has zero measure, i.e.,

$$\lambda(c \in \Psi \mid h(c) \in O_x) = 0.$$

Since $R^\ell \backslash cl(H)$ is separable, we can find countably many O_x like this such that their union covers $R^\ell \backslash cl(H)$. Hence, $\lambda(c \in \Psi \mid h(c) \in R^\ell \backslash cl(H)) = 0$. Therefore,

$$\lambda(h^{-1}(cl(H))) = \lambda(c \in \Psi \mid h(c) \in cl(H)) = \lambda(\Psi).$$

The existence of some b with $h(b) \in cl(H)$ follows from the fact $\lambda(\Psi) > 0$. Q. E. D.

Notice that Lemma 3.3 is tight: First, H is not necessarily closed; second, there may not exist b such that $h(b) \in H$. For example, let h be a function from $[0, 1]$ to R^2: $h(b) = (\sin 2\pi b, \cos 2\pi b)$. It is clear that $H = \{ \|x\| < 1 \}$. So H is open and $h([0, 1]) \cap H$ is empty.

The first conclusion in Lemma 3.3 on the convexity of H is standard as all versions of the Lyapunov theorem prove the convexity of various sets. What is special about Lemma 3.3, however, is its second conclusion that establishes a close connection between H and the range of h. This feature is really crucial to our analysis.

We are now ready to state and prove the main result of this section. Let SF and W_{ei} denote the set of strictly fair allocations and that of equal income Walrasian equilibrium allocations respectively.

PROPOSITION 3.4. $SF = W_{ei}$.

Proof. Assume that x is strictly fair. Since x is efficient, it is well-known that under the assumptions we have made, there is a price vector p that supports x, i.e., $x(a)$ maximizes $u(.\,; a)$ on the set $\{ x \in R^{\ell}_{++} \mid p{\cdot}x = p{\cdot}x(a) \}$ almost everywhere in A. Hence, if we show that $p{\cdot}x(a) = p{\cdot}\omega$ almost everywhere in A, then x is an equal income Walrasian equilibrium allocation.

Let A_1 be the set of agent a's for whom $x(a)$'s do not maximize $u(.; a)$ on the sets $\{ x \in R^{\ell}_{++} \mid p{\cdot}x = p{\cdot}x(a) \}$ and A_2 the set of agents who are envious. Both sets have zero measures. Consider $A' = A \setminus (A_1 \cup A_2)$. Thus if we can show that $p{\cdot}x(a) = p{\cdot}\omega$ a.e. in A', then $p{\cdot}x(a) = p{\cdot}\omega$ a.e. in A. Suppose that it is not the case, we shall derive a contradiction.

Since $\int_{A'} x(a)\, d\mu = \omega$, $\int_{A'} p{\cdot}x(a)\, d\mu = p{\cdot}\omega$. If $p{\cdot}x(a) = p{\cdot}\omega$ is not true a.e. in A', then $\mu(a \in A' \mid p{\cdot}x(a) < p{\cdot}\omega) > 0$. Because $\{ a \in A' \mid p{\cdot}x(a) < p{\cdot}\omega \} = \bigcup_{n=1}^{\infty} A'_n$, where $A'_n = \{ a \in A \mid p{\cdot}x(a) \leq p{\cdot}\omega - \frac{1}{n} \}$, there is an n such that $\mu(A'_n) > 0$.

Applying (ii) of Lemma 3.3 to x on A'_n, we can find an agent $a_0 \in A'_n$ with $x(a_0) \in cl(Y)$, where

$$Y = \{ x \in R^{\ell}_{++} \mid x = \bar{x}(S) \text{ for some } S \text{ in } \mathcal{F} \cap A'_n \text{ with } \mu(S) > 0 \}.$$

Since $x(a_0)$ maximizes a differentiable function $u(.\,; a_0)$ on the set $\{ x \in R^{\ell}_{++} \mid p{\cdot}x =$

$p{\cdot}x(a_0)$ } and $p{\cdot}x(a_0) \leq p{\cdot}\omega - \frac{1}{n}$, there exists a number α with $0 < \alpha < 1$ such that

$$u(x(a_0); a_0) < u(\alpha x(a_0) + (1-\alpha)\omega ; a_0) .$$

We now consider the set

$$X = \{ x \in R^{\ell}_{++} \mid x = \bar{x}(T) \text{ for some } T \text{ in } \mathcal{F} \text{ with } \mu(T) > 0 \} .$$

Obviously, $X \supseteq Y$. Hence, $x(a_0) \in cl(Y)$ implies that there is a sequence $\{x_k\}$ in X that converges to $x(a_0)$. Since $\omega \in X$, and X is convex by (i) of Lemma 3.3, $\alpha x_k + (1-\alpha)\omega \in X$ for every k. That agent a_0 is not envious at x implies that $u(x(a_0); a_0) \geq u(\alpha x_k + (1-\alpha)\omega ; a_0)$. Let k go to infinity, we have

$$u(x(a_0), a_0) \geq u(\alpha x(a_0) + (1-\alpha)\omega ; a_0) .$$

This is a contradiction. Q. E. D.

Our earlier comment on Proposition 2.3 and Varian's result on coalitionally fair allocations in replica economies also applies to Proposition 3.4 and Vind's [16] result on coalitionally fair allocations in atomless economies. It is very intuitive though not proved yet that a coalitionally fair allocation must be strictly fair. Hence our result is stronger. A rigorous proof is included in the appendix.

We also want to comment on Proposition 3.4 and other results by Varian [15], Kleinberg [8], Champsaur and Laroque [3], Mas-Colell [9] etc. They have shown under various sorts of assumptions that the set of fair allocations coincides with the set of equal-income Walrasian equilibria. These assumptions, however, are either strong or complicated or both. In sharp contrast, assumptions needed for Proposition 3.4 are minimal. Thus, considering that strictly fairness as a concept of economic equity is not stronger than fairness by much, Proposition 3.4 could be viewed as an indirect improvement of these results.

4. Economies that Have Both Atoms and an Atomless Sector

In this section we extend the analysis to economies that have both atoms and an atomless sector. As in Section 3 the set of agents is a measure space (Q , $\mathcal{H}$, ν) with $\nu(Q) = 1$. But Q now has both no-null atoms and a no-null atomless sector. Such a model is most useful in describing a market with coexistence of large traders and an ocean of small traders. Let Q_0 be the non-null atomless sector and $Q_1 = Q \setminus Q_0$ be the set of atoms. The commodity space is still R^{ℓ}_{++}. Every agent has a utility function that is strictly increasing, differentiable, and furthemore strictly quasi-concave if the agent is in Q_1. Agents in the same atom have the same utility function. The measurability condition holds for agents in Q_0: the set $\{ a \in Q_0 \mid u(x(a); a) > u(y(a); a) \}$ is measurable for any measurable functions x and y from Q to R^{ℓ}_{++}. The economy has a total endowment $\omega \in R^{\ell}_{++}$.

Definitions such as allocations, strictly envy-free, efficiency, strict fairness, and Walrasian equilibria are exactly the same as in Section 3.

In such an economy the set of strictly fair allocations is generally larger than the set of equal income Walrasian equilibria. The economy E in Figure 3 is an

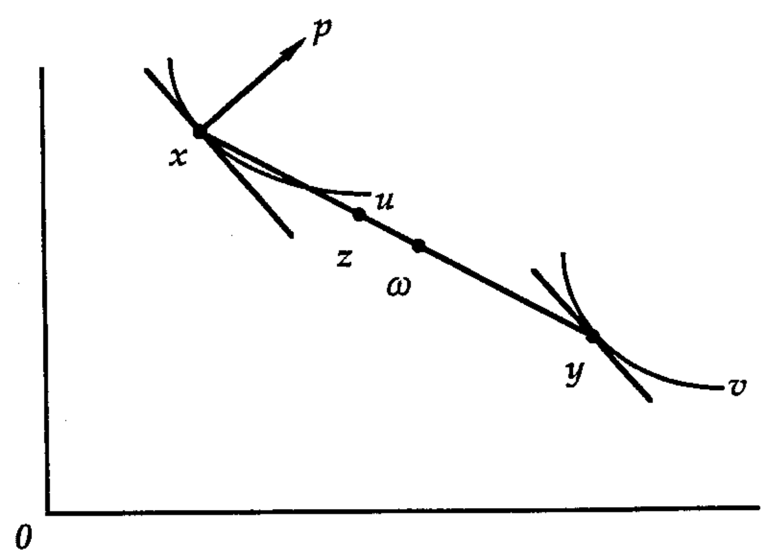

FIGURE 3

example. The atomless sector Q_0 has a measure of $\frac{1}{2}$. All agents in Q_0 have the same utility function u. There are two atoms: $Q_1 = \{ A_1, A_2 \}$. Each atom has a measure of $\frac{1}{4}$. All agents in Q_1 have the same utility function v. We claim that the allocation $x(a) = x$ for all $a \in Q_0$ and $x(b) = y$ for all $b \in Q_1$ is strictly fair. It is clearly efficient. For each $a \in Q_0$ the bundles with which he compares his own lie on the segment zy, where $z = \frac{2}{3}x + \frac{1}{3}y$, and for each $b \in Q_1$ such bundles lie on the segment xy. Hence no agents are envious. So x is a strictly fair allocation. But x is not an equal income Walrasian equilibrium allocation.

Despite the lack of general coincidence of the set of strictly fair allocations and that of equal income Walrasian equilibria, there are some interesting properties of the set of strictly fair allocations.

First for any efficient allocation x, there is a price p that supports x, i.e., $x(a)$ maximizes $u(., a)$ on the set $\{ x \in R^{\ell}_{++} \mid p{\cdot}x = p{\cdot}x(a) \}$ almost everywhere in Q. It is obvious that our result in Section 3 still applies to the atomless sector in terms of the supporting price.

PROPOSITION 4.1. *If x is a strictly fair allocation and p is its supporting price, then* $p{\cdot}x(a) = p{\cdot}\bar{x}(Q_0)$ *almost everywhere in* Q_0.

The values of the bundles that agents in the atoms receive are different. Some may be greater than $p{\cdot}\bar{x}(Q_0)$ and others may be less. But for an important model of oligopolistic mixed economies (Shitovitz [10, 11], Gabszewicz [6]), there is a more definite answer. In such a model, we assume that for each atom A_i in Q_1 there is always another atom A_j such that agents in both have the same utility functions.

PROPOSITION 4.2. *Assume that x is a strictly fair allocation in an oligopolistic mixed economy and p is its supporting price. Then* $p{\cdot}x(a) \geq p{\cdot}\bar{x}(Q_0)$ *for all* $a \in Q_1$.

Proof. Take any agent $a \in Q_1$. Let his bundle be $x(a)$. Let A_i be the atom he is in and A_j be the atom such that agents in both A_i and A_j have the same utility function u . As in Section 2 equal treatment property holds for atoms of the same type. Hence all agents in A_i and A_j have the same bundle $x(a)$. Now consider the segment connecting $x(a)$ and $\bar{x}(Q_0)$. Because Q_0 is atomless, by the Lyapunov theorem, for any arbitrarily small ε there exists a coalition Q_ε in Q_0 such that $\mu(Q_\varepsilon) = \varepsilon$ and $\bar{x}(Q_\varepsilon) = \bar{x}(Q_0)$. Hence, coalition $S_\varepsilon = A_j \cup Q_\varepsilon$ has an average $\bar{x}(S_\varepsilon)$ on segment $x(a)\bar{x}(Q_0)$ that can be arbitrarily close to $x(a)$. So if $p \cdot x(a) < p \cdot \bar{x}(Q_0)$, then agent a and all agents in A_i would envy S_ε for very small ε. This contradicts the assumption that x is a strictly fair allocation. Thus $p \cdot x(b) \geq p \cdot \bar{x}(Q_0)$. Q. E. D.

The inequality in Proposition 4.2 can be strict as the above example shows. This is a very intuitive result. It is generally believed that large traders always have many advantages over small ones. It is apparent from the proof that in our case such an advantage is gained by sheer size. Large traders can absorb small traders' consumption bundles piece by piece while small traders can only take it or leave it when a consumption bundle is offered to them by some large trader. As a result small traders always end with less income than any large trader even though the allocation is strictly fair bundlewise. In case some readers find this trivial, they are reminded that the core equivalence result paradoxically holds in such an economy (Shitovitz [10], Greenberg and Shitovitz [7]).

The results in this section can also be compared with some recent results by Shitovitz [11] on coalitionally fair allocations in mixed economies. Although our differentiability condition is slightly stronger than his, our results are much sharper.

5. Concluding Remarks

We conclude the paper with several remarks on some possible generalization of the results derived here and on the relevance of these results to other topics in economic theory.

In the definition of strictly fair allocations, every agent is required to know the bundles of all other agents and to compare various averages of these bundles with his own consumption bundle. This is a strong requirement, especially in the context of large economies. A more reasonable assumption is that an agent is aware of and sensitive to consumption bundles of only those agents to whom he can relate himself. We are convinced that under proper conditions the main results in this paper, or some variants of them, generalize in such a model. An accurate and detailed treatment, however, remains to be given.

Our results can also be viewed from another angle. Some authors recently considered the issue of consistency for allocation rules in exchange economies. For economies with finitely many agents and using a set of assumptions that includes two forms of consistency properties, Thomson [13] proved a result that is in fact equivalent to our Proposition 2.3. In a paper still in progress, we apply the result in Section 3 to address the same issue in an atomless economy with a continuum of agents.

Finally, although Lemma 3.3 in the paper was used only as a tool to prove Proposition 3.4, it is of interest in its own right. Its most distinct feature as compared with other versions of the Lyapunov convexity result is that it determines an intrinsic convex hull of the range of any integrable function on an atomless measure space. In another paper (Zhou [17]), we extend this result to the case of integrable correspondences and then use it to derive a very simple and intuitive proof of a stronger version of the Aumann core equivalence theorem.

We believe that there are more interesting problems to which this result can be successfully applied.

APPENDIX

In the appendix we consider the relationship between the set of strictly fair allocations and the set of coalitionally fair allocations.

We first consider replica economies. In Varian [14] the concept of coalitionally fair allocations is defined as follows. Each coalition can compare its bundle to bundles of other coalitions of the same size only. Suppose that S and T are two disjoint coalitions of the same size. It is said that S envies T at an allocation x if there is a vector y such that $\Sigma_{i\in S}\, y_i = \Sigma_{j\in T}\, x_j$, and $u_i(y_i) > u_i(x_i)$ for every $i \in S$. An allocation x is coalitionally envy-free if no coalition S envies other coalitions of the same size.

DEFINITION A.1. An allocation x is coalitionally fair if it is both efficient and coalitionally envy-free.

In general, the set of coalitionally fair allocations defined this way is larger than that we used in the text. And it has no obvious relation to the set of strictly fair allocations. However, there does exist an inclusion relation in the context of replica economies. Let CF_k be the set of coalitionally fair allocations of the k-fold replica economy $\mathcal{E}_k$.

LEMMA A.2. $CF_{n(m+1)-2} \subseteq SF_n$.

Proof. Assume $x \notin SF_n$. If x is inefficient, then clearly $x \notin CF_{n(m+1)-2}$. Otherwise, there is an agent of type i who envies a coalition T at x. Since T is a coalition in $\mathcal{E}_n$ and $i \notin T$, $\#T \leq nm - 1$. So one can find in $\mathcal{E}_{n(m+1)-2}$ a coalition S with $\#T$ agents of type i that envies T at x. Hence, $x \notin CF_{n(m+1)-2}$. Q. E. D.

Next we consider an atomless economy $(A, \mathcal{F}, \mu)$. Each coalition compares its average bundle with those of all other coalitions. The restriction on coalitions' sizes no longer matters because of the Lyapunov theorem. We say that S envies T at an allocation x if there is some y such that $\bar{y}(S) = \dfrac{\int_S y}{\mu(S)} = \dfrac{\int_T x}{\mu(T)} = \bar{x}(T)$, and

$$u(y(a), a) > u(x(a), a), \quad \text{a.e. in } S.$$

An allocation x is coalitionally envy-free if no coalition S envies other coalitions, and x is coalitionally fair if it is both efficient and coalitionally envy-free. Let CF be the set of coalitionally fair allocations of $(A, \mathcal{F}, \mu)$.

LEMMA A.3. $CF \subseteq SF$.

Proof. Assume $x \notin SF$. If x is inefficient, then $x \notin CF$. Otherwise, let

$$X = \{ x \in R^\ell \mid x = \bar{x}(T) \text{ for some } T \text{ in } \mathcal{F} \text{ with } \mu(T) > 0 \},$$

and $\tilde{A}$ be the set of agents who are envious at x,

$$\tilde{A} = \{ a \in A \mid u(x(a), a) < u(x, a) \text{ for some } x \in X \}.$$

We should have $\mu(\tilde{A}) > 0$. Since X is separable, we can find a countable dense subset Y of X. Because each agent's utility function is differentiable, we have

$$\tilde{A} = \{ a \in A \mid u(x(a), a) < u(y, a) \text{ for some } y \in Y \}.$$

Because $\mu(\tilde{A}) > 0$ and Y is countable, there is a $z \in Y \subseteq X$ such that $\mu(S) > 0$, where $S = \{ a \in A \mid u(x(a), a) < u(z, a) \}$, the set of agents who prefer y to their bundles. By the definition of X, there exists some T in $\mathcal{F}$ with $\mu(T) > 0$ and $z = \bar{x}(T)$.

We still do not know S envies T yet because they might have a nonempty intersection. But by the Lyapunov theorem, for every integer n there is a subset T_n of T such that $\mu(T_n) = \frac{1}{n}\mu(T)$, and $\bar{x}(T_n) = \bar{x}(T) = z$. Let us consider $S_n = S \cap (T_n)^c$. Since $\mu(T_n) \to 0$, $\mu(S_n) \to \mu(S) > 0$. Hence, there is some n with $\mu(S_n) > 0$. Now we know that S_n envies T_n at x. Therefore, $x \notin CF$. Q. E. D.

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