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A STRATEGIC MARKET GAME OF A FINITE EXCHANGE ECONOMY WITH A MUTUAL BANK

by

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Abstract:

We introduce a strategic market game for an exchange economy not having enough commodity money. We show the existence of a non-cooperative equilibrium for any finite replication economy with a mutual bank, we then show that efficient trade can be achieved in the limiting economy by expanding the money supply through the use of fractional reserves, where the commodity money is demonetized and used for reserves. The means of exchange becomes bank credit backed in part, by "gold." However, efficiency can not be achieved in general as a non-cooperative equilibrium of a finite player game or a finite exchange economy.

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1. INTRODUCTION. In a one period economy the only need for a money or means of exchange is to finance the mechanism of short term trade. Let the finite economy be given as $E = \{n, \omega^\alpha, \phi^\alpha\}$, where each trader α ($\alpha = 1, \cdots, n$) has an endowment $\omega^\alpha = \{\omega_1^\alpha, \omega_2^\alpha, \cdots, \omega_{m+1}^\alpha\} \in \mathbf{R}_+^{m+1}$ and a utility function $\phi^\alpha (\mathbf{R}_+^{m+1} \to \mathbf{R})$. Assume that the last good be chosen as commodity money (gold), and that traders have to pay in advance using gold for whatever they buy. In a previous paper [Shubik (1990)] examined exchange with such a single commodity money, it was observed that three conditions prevail concerning $\bar{\omega}_{m+1} = \Sigma_{\alpha=1}^n \omega_{m+1}^\alpha$, the amount of gold available as a means of exchange in the society.

They are: (i) Enough money well distributed; (ii) enough money badly distributed and (iii) not enough money. Let (p, x) be a competitive equilibrium (CE) of the economy E such that $p_{m+1} = 1$; the conditions on trade associated with the CE can be given as follows:

(i) E has enough money[1] that is well distributed with respect to the CE (p, x) if:

$$\Sigma_{i=1}^m p_i(x_i^\alpha - \omega_i^\alpha)^+ \leq \omega_{m+1}^\alpha$$

for each trader α. Where $r^+ = \max\{0, r\}$ for any real number r. The left hand side is the amount of gold that trader α needs to pay in advance, ω_{m+1}^α is α's endowment of money.

(ii) E has enough money that is badly distributed with respect to the CE (p, x) if:

$$\Sigma_{\alpha=1}^n \Sigma_{i=1}^m p_i(x_i^\alpha - \omega_i^\alpha)^+ \leq \bar{\omega}_{m+1} = \Sigma_{\alpha=1}^n \omega_{m+1}^\alpha$$

and for at least one α

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[1]. The same concept applies to the strategic market game where traders are assumed to either buy or sell; "selling all" or "buying and selling in the same time" would call for more money.

$$\Sigma_{i=1}^{m} p_i(x_i^{\alpha} - \omega_i^{\alpha})^+ > \omega_{m+1}^{\alpha} .$$

(iii) E has "not enough money" with respect to the CE (p, x) if:

$$\Sigma_{\alpha=1}^{n} \Sigma_{i=1}^{m} p_i(x_i^{\alpha} - \omega_i^{\alpha})^+ > \bar{\omega}_{m+1} .$$

Suppose that the above finite economy does not have enough money for efficient trade. We may consider a modified exchange economy with a mutual bank which can create money in the sense that it issues banknotes (currencies, checking or credit card accounts) which are acceptable as means of exchange and is permitted to create more notes (denominated in gold) than it has gold. We show that a strategic market game can be constructed which enables traders to achieve efficient trade by expanding the money supply through fractional reserves.

In a related paper Shubik and Tsomocos (1990) have considered a model of a mutual bank with fractional reserves with a continuum of economic agents. This model is considerably easier to handle than the present model where there is only a finite set of agents, but both merit examination. In particular it is desirable to show that equilibrium exists with a finite number of agents and that it approaches the continuum solution as the number of agents in a finite economy becomes large.

Figure 1 shows the extensive form of the game defined in the next Section. At the first move all individuals who wish to do so deposit gold in the mutual bank. The bank ownership is in proportion to the amount deposited. The amount of gold deposited is multiplied by a gearing ratio and this determines the total amount of credit offered to all borrowers. At the second move all borrowers bid for bank money. At the third move they exchange goods for bank money.

As we have a finite number of agents the strategy sets for the individuals in this extensive form game can become extremely complicated. We can make a simplification which still enables us to analyze the model without damaging loss of generality. Instead of having the deposit of gold be a strategic move we treat it parametrically. We then collapse the remainder of the game into a single move, by assuming no information availability between the moves.

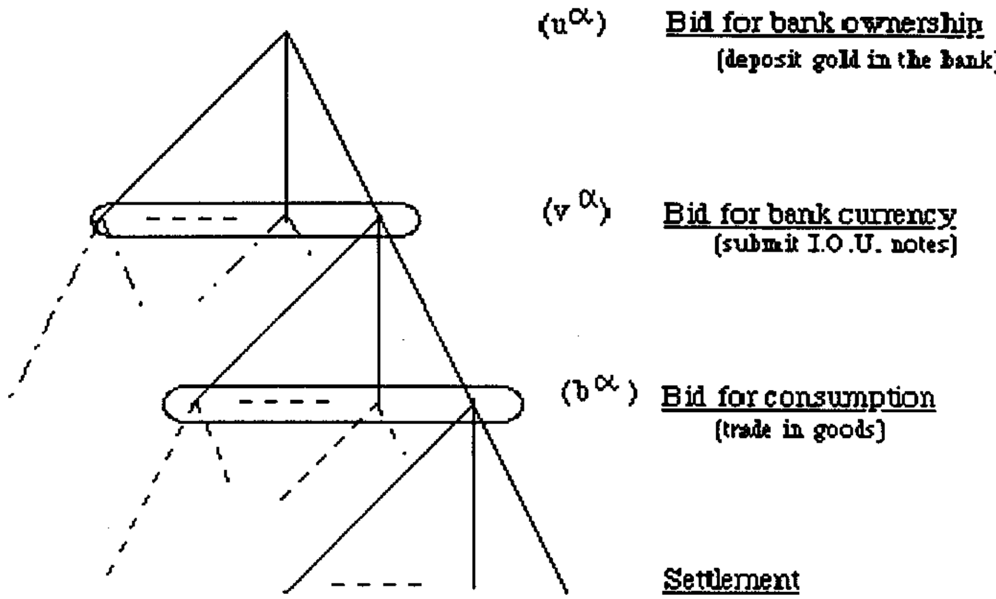

Figure 1.

2. DESCRIPTION OF THE MODEL. Through out the paper traders are indexed by the superscripts α, β, $\cdots$, and goods by the subscripts i, j, $\cdots$. For each vector $x \in \mathbf{R}^n$, $x \geq 0 \Leftrightarrow x_i \geq 0$, all i; $x > 0 \Leftrightarrow x \geq 0$ and $x \neq 0$; and $x >> 0 \Leftrightarrow x_i > 0$, all i, where 0 stands as either a scalar zero or a vector of zeros, its meaning will be clear from the context. For any vector $b = \{b^1, \cdots, b^{\alpha}, \cdots, b^n\} \in \mathbf{R}^n$ (or any collection of vectors with a same dimension) indexed by α, $b^{-\alpha} = \{b^1, \cdots, b^{\alpha-1}, b^{\alpha+1}, \cdots, b^n\} \in \mathbf{R}^{n-1}$ denotes the complement of b^{α}, and b will be written as $b = \{b^{\alpha}, b^{-\alpha}\}$ for convenience.

A competitive equilibrium (C.E.) of the economy E is a pair (p, x) (where $p \in \mathbf{R}_+^{m+1}$, $x \in \mathbf{R}_+^{(m+1)\times n}$) consisting of a price vector and a consumption bundle such that

for each $\alpha = 1, \cdots, n$, x^{α} maximizes the utility function ϕ^{α} over the budget set

$$a^{\alpha}(p) = \{y^{\alpha} \in \mathbf{R}_+^{m+1} \mid \Sigma_{i=1}^{m+1} p_i y_i^{\alpha} \leq \Sigma_{i=1}^{m+1} p_i \omega_i^{\alpha}\},$$

and that $\Sigma_{\alpha=1}^{n} x^{\alpha} = \Sigma_{\alpha=1}^{n}\omega^{\alpha}$. For a given C.E. (p, x), the associated "shadow prices of income" can be written as a vector $\lambda_{sp} = \{\lambda^1, \cdots, \lambda^{\alpha}, \cdots, \lambda^n\} \in \mathbf{R}^n$ such that for each α, the same bundle x^{α} also maximizes the function

(1) $\phi^{\alpha}(x) + \lambda^{\alpha}(p\cdot\omega^{\alpha} - p\cdot x)$

over the half space $\mathbf{R}_{+}^{m+1}$.

In the competitive equilibrium the process of exchange for goods is suppressed. If all traders are required by the rules to pay in advance for the goods they bought, the concept of "enough money" or "not enough money" will be crucial to the efficiency of an equilibrium. In order to study the money rate of interest, we should allow traders to borrow or lend money so as to loosen the cash constraint, thus it is necessary to introduce a money market into the strategic market game [Shubik(1972), Shapley and Shubik (1977), Dubey and Shubik (1978)]. A strategic market game representing the above exchange economy is a normal form or extensive form game where market prices and allocations are formed by the strategic behavior of the traders. In Shubik, Tsomocos and Zhao [1990] the money suppliers are the individual traders, while in the present paper the single supplier is a fractional reserve mutual bank. The game is defined as follows:

A trader α's strategy is a vector $s^{\alpha} = (u^{\alpha}, v^{\alpha}; b^{\alpha}) \in \mathbf{R}_{+}^{m+3}$, where $u^{\alpha}, v^{\alpha} \in \mathbf{R}_{+}$ are the amounts of gold deposited in the bank and the amount of I.O.U. notes bid by α respectively, $b^{\alpha} \in \mathbf{R}_{+}^{m+1}$ is the vector of α's bid for consumption measured in bank currency. Let

$$S^{\alpha} = \{s^{\alpha} \in \mathbf{R}_{+}^{m+3} \mid u^{\alpha} \leq \omega_{m+1}^{\alpha},\ v^{\alpha} \leq \bar{M},\ \Sigma_{i=1}^{m+1} b_i^{\alpha} \leq \bar{M}\}$$

denote α's feasible strategy set, where $\bar{M} = K\,\bar{\omega}_{m+1}$ is the upper bound for the credit that can be offered (so it also bounds from above the size of α's repayment and the value of α's consumption), and $K > 0$ is the gearing ratio or multiplicative factor for the reserve ratio.

After all traders have submitted their choices of strategy, the total supply of bank notes, the ex ante rate of interest and market prices are determined according to the following rules:

(2) (Supply of bank notes) $M = M(s) \equiv K\ \bar{u}$,

(3) (Rate of interest, ex ante) $1+\rho = 1+\rho(s) = \dfrac{\bar{v}}{M} = \dfrac{\Sigma_\alpha v^\alpha}{K\ \Sigma_\alpha u^\alpha}$,

(4) (Prices) $p_i = p_i(s) = \dfrac{\bar{b}_i}{\bar{\omega}_i} = \dfrac{\Sigma^n_{\alpha=1} b_i^\alpha}{\Sigma^n_{\alpha=1}\omega_i^\alpha}$ for $i = 1, \cdots, m+1$,

where if $\bar{v} = M = 0$, then $1+\rho = \infty$.

For each trader $\alpha = 1, \cdots, n$, the resulting share of bank ownership, consumption bundle, cash balance, net wealth and payoff function are respectively given as:

(5) (Share of bank ownership) $h^\alpha = u^\alpha / \bar{u}$ ($= 0$ if $\bar{u} = 0$);

(6)(Consumption bundle) For $i = 1, \cdots, m+1$,

$$x_i^\alpha = x_i^\alpha(s) = b_i^\alpha / p_i = b_i^\alpha \bar{\omega}_i / \bar{b}_i \quad \text{if } p_i > 0,$$
$$= \omega_i^\alpha \qquad\qquad \text{if } p_i = 0;$$

(7)(Cash balance)

$$M^\alpha = M^\alpha(s) = v^\alpha/(1+\rho) - \Sigma^{m+1}_{i=1} p_i(x_i^\alpha - \omega_i^\alpha)^+ \quad \text{if } \infty > 1+\rho > 0,$$
$$= -\Sigma^{m+1}_{i=1} p_i(x_i^\alpha - \omega_i^\alpha)^+ \qquad \text{if } 1+\rho = 0 \text{ or } \infty;$$

(8) (Net wealth)

$$d^\alpha = d^\alpha(s) = \Sigma^{m+1}_{i=1} p_i(\omega_i^\alpha - x_i^\alpha) + v^\alpha/(1+\rho) - v^\alpha + \rho^* K u^\alpha \quad \text{if } \infty > 1+\rho > 0,$$
$$= \Sigma^{m+1}_{i=1} p_i(\omega_i^\alpha - x_i^\alpha) \qquad \text{if } 1+\rho = 0 \text{ or } \infty;$$

(9) (Payoff function) $\Phi^\alpha = \Phi^\alpha(s) = \phi^\alpha(x^\alpha) + \lambda^0 (M^\alpha)^- + \lambda^\alpha \cdot (d^\alpha)^-$.

In the above definitions (7-9), $r^+ = \max\{0, r\}$ and $r^- = \min\{0, r\}$ for any real number r; ρ^* in (8) is the ex post rate of interest, its calculation is given in Appendix I (when there is no bankruptcy, we shall always have $\rho^* = \rho$); and $\lambda^0, \lambda^\alpha$ in (9) are the penalty factors for "being overdrawn" and "being bankrupted" respectively.

The assumption underlying the concept of "being overdrawn" is that each trader α establishes a checking or credit card account with the bank. Checks and credit cards are accepted in the trade and the bank will honor all of them. But if a trader α's account were overdrawn, he or she will be punished by the bank through the penalty factor λ^0. If only currency is accepted in the trade, the traders' purchase will be constrained by their available currencies, and each trader α's bid for consumption can be changed to a vector of weights $w^\alpha \in \mathbf{R}_+^{m+1}$ ($\Sigma_{i=1}^{m+1} w_i^\alpha = 1$), which is interpreted as a percentage allocation of funds. In this case, one has to pay the whole value of consumption and the society needs much more currency than before. The actual purchase of each good i is $b_i^\alpha = w_i^\alpha v^\alpha/(1+\rho)$, and $M^\alpha \equiv 0$. All the results in this paper also hold in the new context, but their proofs are a bit more complicated (similar to the proof in Dubey and Shubik (1979)) because the payoff functions are no longer concave.

While the definition of (8) can be interpreted as follows. In the case with an active money market, the first term gives the balance from commodity trading, the next two terms are the size of the loan and the size of the repayment, the final term is the profit or loses accrued to the trader as being a partial owner of the bank. Thus a trader deposits gold for two purposes: to facilitate the trade by creating bank notes, and also to earn the possible bank profit or to take the risk of sharing the loses. Since gold is neither created nor destroyed in the process of trade, the existence of a mutual reserve bank with a positive amount of deposited gold has no effect on the supply of gold as a consumer good. This fact allows us to define one's net debt as that of (8), where each trader α, as a partial owner of the bank, gets back the deposit u^α at the end of all trades and earns as profit $h^\alpha \rho^* M = \rho^* K u^\alpha$ units of bank notes if the bank is profitable (if $\rho^* \geq 0$), and loses an amount if the bank runs into deficits (if $\rho^* < 0$). This is so because at the beginning the bank gives out a value of M units of banknotes and gets back a value of $(1+\rho^*)$ M units of banknotes and thus earns (or loses) a value of ρ^*M units of notes. For example, in the case of $\rho^* \geq 0$, the bank gives out M units of bank notes at the beginning and gets back the same amount of notes but has ρ^*M units to be paid in some other form.

In the above definitions we have assumed that only bank money (in the form of checks or credit cards) be accepted in the trade and either bank notes or any goods be accepted in repaying the bank. Shubik and Tsomocos (1990) will study the case when only bank notes and gold are accepted in the repayments.

As mentioned in the introduction, in the rest of this paper we shall concentrate on the so called "Fixed Share Mutual Bank Model". This assumes that each trader α deposits a fixed portion θ $(0< \theta <1)$ of the endowment ω^{α}_{m+1}. Under such assumption, α's deposit $u^{\alpha} \equiv \theta\, \omega^{\alpha}_{m+1}$ and share of the bank ownership $h^{\alpha} \equiv u^{\alpha} / \bar{u} \equiv \omega^{\alpha}_{m+1} / \bar{\omega}_{m+1}$ are both constants. We shall call θ the deposit ratio. The case when deposit appears as strategic variable is examined in Shubik and Tsomocos (1990).

The above game will be denoted by $\Gamma_{\lambda}(E)$, where $\lambda = \{K, \theta, \lambda^0, \lambda_{sp}\} = \{K, \theta, \lambda^0, \lambda^1, \cdots, \lambda^n\}$, to remind us that this game depends on the information of E and the parameters in λ. The non-cooperative solution of $\Gamma_{\lambda}(E)$, often called Nash Equilibrium (N.E.), is a joint strategy $\bar{s} = \{\bar{s}^1, \bar{s}^2, \cdots, \bar{s}^n\}$ such that each $\bar{s}^{\alpha}$ is α's best response to all others' strategies $\bar{s}_{-\alpha}$ (the complementary strategies) in the sense that $\bar{s}^{\alpha}$ maximizes the payoff function $\Phi^{\alpha}(s^{\alpha}, \bar{s}_{-\alpha})$ over all the feasible strategies of α, that is, $\bar{s}$ is an N.E. of $\Gamma_{\lambda}(E)$ if for all $\alpha = 1, \cdots, n$, $\Phi^{\alpha}(\bar{s}) = \Phi^{\alpha}(\bar{s}^{\alpha}, \bar{s}_{-\alpha}) = \mathrm{Max}\{\Phi^{\alpha}(s^{\alpha}, \bar{s}_{-\alpha}) \mid s^{\alpha} \in S^{\alpha}\}$.

3. THE EXISTENCE THEOREMS.

In Part 1 of this section we shall study the equilibria of the market game derived from the finite economy E, in Part 2 we shall study that from the r-fold replication economy E^r, and in Part 3 that from the limiting economy E^∞. The following Assumption I is the sufficient condition for all of our results except Lemma 1.

Assumption I: (i) All ϕ^α are continuous, non-decreasing, non-satiated and quasi-concave, (ii) all goods are "essential"[2], (iii) $\bar{\omega} = \Sigma_\alpha \omega^\alpha >> 0$, and (iv) there are at least two traders who have positive endowments of the last good, which is called gold.

Conditions (i) - (iii) are straight forward, condition (iv) says that there are at least two depositors in the Fixed Share Mutual Bank, this will guarantee a rate of interest strictly larger than minus one.

3.1. EQUILIBRIA OF THE GAME $\Gamma_\lambda(E)$. By a very harsh punishment we shall mean that the punishment factors $\{\lambda^0, \lambda^1, \cdots, \lambda^n\}$[3] are very large and the payoff function of a trader α is

(10)[4] $\quad \Phi^\alpha = \Phi^\alpha(s) = \phi^\alpha(x^\alpha) + \lambda^0 \,\mathrm{Sign}\{(M^\alpha)^-\} + \lambda^\alpha \cdot \mathrm{Sign}\{(d^\alpha)^-\},$

where $\mathrm{Sign}\{r\} = +1$ if $r > 0$, $= 0$ if $r = 0$, and $= -1$ if $r < 0$; in such case no trader would like to be overdrawn or to be bankrupted.

Theorem 1. If the economy E satisfies Assumption I and if the penalty factors are very harsh, then for any deposit ratio θ ($0 < \theta < 1$), any reserve ratio K ($0 < K$) and any harsh punishment factor $\{\lambda^0, \lambda^1, \cdots, \lambda^n\}$,

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2. The intuition of an essential good is that there is at least one trader who has to consume at least a small positive quantity of that good. To be more specific, by " the good k is essential" it is meant that there is a small $\varepsilon > 0$ ($0 < \varepsilon < \min\{\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_{m+1}\}$) and there is at least one trader who has to consume at least ε quantity of the good k.

3. For the purpose of a harsh punishment, it is sufficient to have only two punishment factors λ^0 and λ^* ($\lambda^* = \lambda^1 = \lambda^2 = \cdots = \lambda^n$). But traders should be punished differently when an N.E. and a C.E. are equivalent (Theorem 3).

4. Note that the particular payoffs in (10) are homogeneous of degree 0 when the deposits are not fixed. In this case we can always set the price of the last good, the price of gold equal to one by scaling up or down the joint strategies. However, this property does not hold in the fixed share case or in the general case when payoffs are given by (9).

(i) the corresponding game $\Gamma_\lambda(E)$ has at least one N.E., and

(ii) at any N.E., $1+\rho > 0$.

Proof of Theorem 1. The existence part follows claims 1-3 below. To prove part (ii), let $s = \{s^1, s^2, \cdots, s^n\}$ be an N.E. of $\Gamma_\lambda(E)$. Clearly $1+\rho \geq 0$. Assume by way of contradiction that $1+\rho = 0$. Then $v^\alpha = 0$ for all α. It follows from the harsh penalty that $M^\alpha = M^\alpha(s) = -\Sigma_{i=1}^{m+1} (b_i^\alpha - p_i\, \omega_i^\alpha)^+ = 0$, $d^\alpha = d^\alpha(s) = \Sigma_{i=1}^{m+1} (p_i\omega_i^\alpha - b_i^\alpha) = 0$. By part (iv) in the assumption, $\bar{\omega}_{m+1} > \omega_{m+1}^\alpha$ for all α. If α chooses $\tilde{s}^\alpha = \{ u^\alpha, \tilde{v}^\alpha; b^\alpha \}$, where $0 < \tilde{v}^\alpha < \theta K \bar{\omega}_{m+1}$, then

$$M^\alpha(\tilde{s}^\alpha, s_{-\alpha}) = \theta K \bar{\omega}_{m+1} > 0 = M^\alpha(s),$$

$$d^\alpha(\tilde{s}^\alpha, s_{-\alpha}) = \theta K \,(\bar{\omega}_{m+1} - \omega_{m+1}^\alpha)\,(1 - \frac{\tilde{v}^\alpha}{\theta K \bar{\omega}_{m+1}}) > 0 = d^\alpha(s).$$

Thus by the nonsatiation assumption, trader α can strictly increase his payoff and keep M^α and d^α both non-negative by bidding more (say $\varepsilon > 0$) on each b_i^α. This contradicts the definition of an N.E., and we must have $1+\rho > 0$. **Q.E.D.**

Claim 1. Let $\Gamma_\lambda^\varepsilon(E)$ $(\varepsilon > 0)$ be the same game as $\Gamma_\lambda(E)$ in Theorem 1 except that the rate of interest (3) is changed to

$$\text{(3-1)} \qquad 1+\rho = \frac{\bar{v} + \varepsilon}{M} = \frac{\Sigma_\alpha v^\alpha + \varepsilon}{\theta K \bar{\omega}_{m+1}} > 0.$$

Then under the conditions of Theorem 1, $\Gamma_\lambda^\varepsilon(E)$ has at least one N.E. for any $\varepsilon > 0$.

Proof. In the fixed share model, M, M^α and d^α ((2), (7) and (8)) are changed respectively to

$$\text{(2-1)} \qquad M = K\,\bar{u} = \theta K\, \bar{\omega}_{m+1} > 0$$

$$\text{(7-1)} \qquad M^\alpha = \frac{v^\alpha}{\bar{v} + \varepsilon}\, \theta K \bar{\omega}_{m+1} - \Sigma_{i=1}^{m+1} (b_i^\alpha - p_i\, \omega_i^\alpha)^+$$

$$\text{(8-1)} \quad d^\alpha = \Sigma_{i=1}^{m+1} (p_i\omega_i^\alpha - b_i^\alpha) + \frac{v^\alpha}{\bar{v} + \varepsilon}\, \theta K \bar{\omega}_{m+1} - v^\alpha + (\frac{\bar{v} + \varepsilon}{\theta K\, \bar{\omega}_{m+1}} - 1)\, \theta K \omega_{m+1}^\alpha$$

because $1+\rho>0$. For each α, let

$$\delta^{\alpha}=\delta^{\alpha}(s_{-\alpha}) = \text{Arg-Max}\{\Phi^{\alpha}(s^{\alpha}, s_{-\alpha}) \mid s^{\alpha}\in S^{\alpha}\}$$

denote the set of α's best responses to any complementary bids $s_{-\alpha}$, where Φ^{α} is given by (10). Let $S=S^1\times S^2\times\cdots\times S^n$ be written as $S=(S^{\alpha}, S_{-\alpha})$ for convenience. Since the penalty factors are very harsh, the map $\delta^{\alpha}:\ S_{-\alpha}\to 2^{S^{\alpha}}$ can be rewritten as

$$\delta^{\alpha}(s_{-\alpha}) = \underset{s^{\alpha}\in S^{\alpha}\ \text{S.T.}\ M^{\alpha}\geq 0,\ d^{\alpha}\geq 0.}{\text{Arg- Max}}\ \Phi^{\alpha}(s)=\phi^{\alpha}(x^{\alpha})=\phi^{\alpha}(x^{\alpha}(s^{\alpha}, s_{-\alpha})) \tag{11}$$

for each $s_{-\alpha}\in S_{-\alpha}$. Where x^{α}, M^{α} and d^{α} are given by (6), (7-1) and (8-1) respectively. Under the assumption that all goods are essential, it can be verified [Zhao (1990)] that the prices for all goods are bounded from below by a small positive number. Then under Assumption I, it is easy to show that

$$\phi^{\alpha}(x^{\alpha}(s^{\alpha},s_{-\alpha})) = \phi^{\alpha}(b_1^{\alpha}/p_1,\ b_2^{\alpha}/p_2,\ \cdots,\ b_{m+1}^{\alpha}/p_{m+1})$$

$$=\phi^{\alpha}\left(\bar{\omega}_1\left(1-\frac{\Sigma_{\beta\neq\alpha}\, b_1^{\beta}}{\Sigma_{\beta}\, b_1^{\beta}}\right),\ \bar{\omega}_2\left(1-\frac{\Sigma_{\beta\neq\alpha}\, b_2^{\beta}}{\Sigma_{\beta}\, b_2^{\beta}}\right),\ \cdots,\ \bar{\omega}_{m+1}\left(1-\frac{\Sigma_{\beta\neq\alpha}\, b_{m+1}^{\beta}}{\Sigma_{\beta}\, b_{m+1}^{\beta}}\right)\right)$$

is quasiconcave in s^{α} and continuous in $s=(s^{\alpha}, s_{-\alpha})$. Because $1+\rho>0$, the constraint functions $M^{\alpha}=M^{\alpha}(s)$ and $d^{\alpha}=d^{\alpha}(s)$ are clearly both concave in s^{α} and continuous in s $=(s^{\alpha}, s_{-\alpha})$. Thus by the standard results regarding the stability in mathematical programming {Berge(1963), Fiacoo(1983)], the map $\delta: S\to 2^S$ defined by

$$\delta(s) = \delta^1(s_{-1})\times\delta^2(s_{-2})\times\cdots\times\delta^n(s_{-n}) \tag{12}$$

(for each $s\in S$) has a compact and convex value and a closed graph . Thus by Kakutani fixed point theorem there exists at least one fixed point $\bar{s}=\{\bar{s}^1,\ \bar{s}^2,\ \cdots,\ \bar{s}^n\}\in\delta(\bar{s})$, which is a Nash Equilibrium of the game $\Gamma_{\lambda}^{\varepsilon}(E)$. **Q.E.D.**

Claim 2. Let $s(\varepsilon)=\{s^1(\varepsilon), s^2(\varepsilon), \cdots, s^n(\varepsilon)\}$ be an N.E. of $\Gamma_{\lambda}^{\varepsilon}(E)$, and $\rho(\varepsilon)$ be the rate of interest resulting from $s(\varepsilon)$. For any sequence $\{\varepsilon_t\}$ such that $\varepsilon_t\to 0$ as $t\to\infty$, if $\rho(\varepsilon_t)\to\bar{\rho}$ as $t\to\infty$, then $1+\bar{\rho}>0$.

Proof. It follows from $1+\rho(s(\varepsilon)) = 1+\rho(\varepsilon) = \frac{\bar{v}(\varepsilon)+\varepsilon}{M} > 0$ that $1+\bar{\rho} \geq 0$. Assume by way of contradiction that $1+\bar{\rho} = 0$, then $\bar{v}(\varepsilon_t) = \Sigma_\alpha v^\alpha(\varepsilon_t) \to 0$ as $t \to \infty$. Without loss of generality, we can find trader α such that for all t, $v^\alpha(\varepsilon_t) = \min\{ v^\beta(\varepsilon_t) \mid 1 \leq \beta \leq n\}$ (Otherwise consider any subsequence of $\{\varepsilon_t\}$ satisfying this, such a subsequence must exist because of the finiteness of traders). Note that

$$\frac{\partial}{\partial v^\alpha} d^\alpha(s(\varepsilon_t)) = \theta K \bar{\omega}_{m+1} \frac{\Sigma_{\beta\neq\alpha} v^\beta(\varepsilon_t) + \varepsilon_t}{(\bar{v}(\varepsilon_t)+\varepsilon_t)^2} + \frac{\omega^\alpha_{m+1}}{\bar{\omega}_{m+1}} - 1.$$

It follows from $v^\alpha \leq \frac{1}{n-1} \Sigma_{\beta\neq\alpha} v^\beta \leq \frac{1}{n-1} (\Sigma_{\beta\neq\alpha} v^\beta + \varepsilon)$ that

$$\frac{\Sigma_{\beta\neq\alpha} v^\beta + \varepsilon}{\bar{v} + \varepsilon} \geq \frac{1}{\frac{1}{n-1} + 1} = \frac{n-1}{n}.$$

By the previous two expressions we have

$$\frac{\partial}{\partial v^\alpha} d^\alpha(s(\varepsilon_t)) \geq \frac{\theta K \bar{\omega}_{m+1}(n-1)}{n} \frac{1}{\bar{v}(\varepsilon_t)+\varepsilon_t} + \frac{\omega^\alpha_{m+1}}{\bar{\omega}_{m+1}} - 1.$$

Then $\frac{\partial}{\partial v^\alpha} d^\alpha(s(\varepsilon_t)) \to +\infty$ as $t \to \infty$. Note that $\frac{\partial}{\partial v^\alpha} M^\alpha(s(\varepsilon_t)) > 0$ for all t. Thus there is a t such that $\frac{\partial}{\partial v^\alpha} d^\alpha(s(\varepsilon_t)) > 0$ and $\frac{\partial}{\partial v^\alpha} M^\alpha(s(\varepsilon_t)) > 0$. Then by increasing v^α, trader α can increase all b_i^α and keep M^α and d^α both non-negative. By the non-satiation assumption, $s^\alpha(\varepsilon_t)$ is not a best response. This proves Claim 2. **Q.E.D.**

Claim 3. Let $s(\varepsilon) = \{s^1(\varepsilon), s^2(\varepsilon), \cdots, s^n(\varepsilon)\}$ be the N.E. in Claim 2. Then there is a sequence $\{\varepsilon_t\}$ ($\varepsilon_t \to 0$ as $t \to \infty$) such that: (i) $s(\varepsilon_t) \to \bar{s} \in S$ as $t \to \infty$; and (ii) $\bar{s}$ is an N.E. of the original game $\Gamma_\lambda(E)$.

Proof. For any sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0$ as $m \to \infty$, the corresponding sequence $\{s(\varepsilon_m)\}$ is a non-negative and bounded sequence, thus there is a subsequence $\{\varepsilon_t\}$ such that $s(\varepsilon_t) \to \bar{s} = \{\bar{s}^1, \bar{s}^2, \cdots, \bar{s}^n\} \in S$ and $\varepsilon_t \to 0$ as $t \to \infty$. It follows from Claim 2 that $1+\rho(\varepsilon_t) = \frac{\bar{v}(\varepsilon_t)+\varepsilon_t}{M} \to 1+\bar{\rho} > 0$ as $t \to \infty$. By the assumption that all goods are essential, all prices are bounded from below by a small positive number [See Zhao(1990)].

Thus for all i, $p_i(\varepsilon_t) \rightarrow \bar{p}_i > 0$ as $t \rightarrow \infty$. This will guarantee the continuity of ϕ^α, M^α and d^α at $\bar{s}$ for all α. We shall show that for each trader α, each $\bar{s}^\alpha$ is α's best response to $\bar{s}_{-\alpha}$. It follows from the continuity of M^α and d^α at $\bar{s}$ that

$$M^\alpha = M^\alpha(\bar{s}^\alpha, \bar{s}_{-\alpha}) = \frac{\bar{v}^\alpha}{1+\bar{\rho}} - \Sigma_{i=1}^{m+1} (\bar{b}_i^\alpha - \bar{p}_i \omega_i^\alpha)^+ \geq 0,$$

$$d^\alpha = d^\alpha(\bar{s}^\alpha, \bar{s}_{-\alpha}) = \Sigma_{i=1}^{m+1} (\bar{p}_i \omega_i^\alpha - \bar{b}_i^\alpha) + \frac{\bar{v}^\alpha}{1+\bar{\rho}} - \bar{v}^\alpha + \bar{\rho}\, \theta K \omega_{m+1}^\alpha \geq 0.$$

Then by reducing some b_i^α and increasing v^α for some small quantities, we can find $\tilde{s}^\alpha \in S^\alpha$ such that

(13) $\qquad M^\alpha(\tilde{s}^\alpha, \bar{s}_{-\alpha}) > 0, \ d^\alpha(\tilde{s}^\alpha, \bar{s}_{-\alpha}) > 0.$

Let $\hat{s}^\alpha \in \delta^\alpha = \delta^\alpha(\bar{s}_{-\alpha})$ be one of α's best responses to $\bar{s}_{-\alpha}$, then we have $M^\alpha(\hat{s}^\alpha, \bar{s}_{-\alpha}) \geq 0$, $d^\alpha(\hat{s}^\alpha, \bar{s}_{-\alpha}) \geq 0$. By (13), by the concavity of M^α and d^α on s^α, and by the continuity of M^α and d^α, we can find $s_t^\alpha \in S^\alpha$ such that

(14) $\qquad M^\alpha(s_t^\alpha, s(\varepsilon_t)_{-\alpha}) > 0, \ d^\alpha(s_t^\alpha, s(\varepsilon_t)_{-\alpha}) > 0.$ and $|s_t^\alpha - \hat{s}^\alpha| < \varepsilon_t$.

This implies

$$\phi^\alpha(s_t^\alpha, s(\varepsilon_t)_{-\alpha}) \leq \phi^\alpha(s(\varepsilon_t)) = \phi^\alpha(s^\alpha(\varepsilon_t), s(\varepsilon_t)_{-\alpha}),$$

because $s(\varepsilon_t)$ is an N.E.. From the continuity of ϕ^α at $\bar{s}$, (14) and the previous expression, by letting $t \rightarrow \infty$, we get

$$\phi^\alpha(\hat{s}^\alpha, \bar{s}_{-\alpha}) \leq \phi^\alpha(\bar{s}) = \phi^\alpha(\bar{s}^\alpha, \bar{s}_{-\alpha}),$$

Thus $\bar{s}^\alpha$ is also α's best response to $\bar{s}_{-\alpha}$. This proves Claim 3 and completes the proof of Theorem 1. **Q.E.D.**

3.2. EQUILIBRIA OF THE GAME $\Gamma_\lambda(E^r)$. For any positive integer $r \geq 2$, the r-fold replication economy $E^r = \{nr, \omega^\alpha, \phi^\alpha\}$ is an economy with n types of traders and r traders of each type, where for each type $\alpha = 1, \cdots, n$, all the r traders of the α-th type have the same endowment ω^α and the same utility function ϕ^α. That is, for each type α, $\omega^{\alpha\beta} = \omega^\alpha$, $\phi^{\alpha\beta} = \phi^\alpha$, for all $\beta = 1, \cdots, r$. The game $\Gamma_\lambda(E^r)$ for exchange economy E^r is defined similarly, where all traders face the same penalty λ^0 for being short of cash, and for each type $\alpha = 1, \cdots, n$, all the r traders of the α-th type face the same bankruptcy penalty λ^α and have the same strategy set:

(15) $$S^{\alpha\beta} = S^\alpha = \{s^\alpha \in \mathbf{R}^{m+3}_+ \mid u^\alpha \leq \omega^\alpha_{m+1},\ v^\alpha \leq r\bar{M},\ \Sigma^{m+1}_{i=1} b^\alpha_i \leq r\bar{M}\ \}.$$

A Type-Symmetric Non-cooperative Equilibrium (T.S.N.E.) [Shapley and Shubik (1977), Dubey and Shubik (1978)] of the game $\Gamma_\lambda(E^r)$ is a non-cooperative equilibrium such that traders of the same type choose the same strategies. A T.S.N.E. of $\Gamma_\lambda(E^r)$ will also be represented as a vector $s = \{s^1, \cdots, s^n\} \in \mathbf{R}^{(m+3)\times n}_+$, where each $s^\alpha \in \mathbf{R}^{m+3}_+$ is the strategy that all traders in the α-th type choose.

We can prove the following Theorem 2 by using similar arguments as in the proof of Theorem 1 and by using the Type-Symmetric technique developed in Shapley and Shubik (1977) or Dubey and Shubik (1978).

Theorem 2. If the economy E satisfies Assumption I and if the penalty factors are very harsh, then for any deposit ratio θ ($0< \theta <1$), any reserve ratio K ($0<K$) and any harsh punishment factor $\{\lambda^0, \lambda^1, \cdots, \lambda^n\}$, the game $\Gamma_\lambda(E^r)$ has at least one T.S.N.E. with $1+\rho > 0$.

The next question that arises naturally is what happens if the replication number r goes to infinity. This is addressed by Lemmas 1-2 below and by Lemma 3 in the next sub-section. By Theorem 2, for each $r = 1, 2, \cdots$, $\Gamma_\lambda(E^r)$ has at least one T.S.E.P. $s_r = \{s^1_r, \cdots, s^n_r\}$. Thus a sequence of T.S.E.P. $\{s_r\} = \{s^1_r, \cdots, s^n_r\}$ always exists. Lemma 1 provides a sufficient condition for the existence of a limiting point of the sequence $\{s_r\}$; Lemma 2 establishes the relation between the limiting point of $\{s_r\}$ and the C.E. of the original economy E; and Lemma 3 establishes the relation between the limiting point of $\{s_r\}$ and the N.E. of the game $\Gamma_\lambda(E^\infty)$. Lemma 1 assumes Assumption II below:

Assumption II: (i) All ϕ^α are continuous, non-decreasing, non-satiated and quasi-concave, (ii) all goods are "essential", (iii) $\bar{\omega} = \Sigma_\alpha \omega^\alpha >> 0$, (iv) for each good i, there are at least two traders who are i-furnished, and there is at least one trader α who strongly desires good i, and (v) there is a large number $\bar{c} > 0$ such that $v^{\alpha} \leq \bar{c}$ for all α.

By "α is i-furnished" we mean that α has a positive endowment of the good i; By "trader α strongly desires good i"[5] we mean that for any $x^{\alpha} \in R_{+}^{m+1}$, if $x_i^{\alpha} < c+\omega_i^{\alpha}$ (where $c > 0$ is any constant such that $c+\omega_i^{\alpha} < \bar{\omega}_i$), then for any $j \neq i$ such that $0 < p_i / p_j < \infty$, there is a small $\delta > 0$ such that $\phi^{\alpha}(z^{\alpha}) > \phi^{\alpha}(x^{\alpha})$ for $z^{\alpha} = x^{\alpha} + \{\delta p_j / p_i\} e_i - \delta e_j$, where for any k, e_k is the vector of $\mathbf{R}^{m+1}$ whose i-th component is one if i = k and zero otherwise.

Conditions (i) - (iii) are straight forward, conditions (iv) and (v) together guarantee an upper bound for the rate of interest and all prices. Condition (iv) will lead to a positive transaction in each of the m+1 markets at any equilibrium, and condition (v) provides an upper bound for the size of the repayment.[6]

Lemma 1. Suppose that the economy E satisfies Assumption II and the penalty factors are very harsh. Then for any deposit ratio θ ($0 < \theta < 1$), any reserve ratio K ($0 < K$) and any harsh punishment factor $\{\lambda^0, \lambda^1, \cdots, \lambda^n\}$, there exists a sequence of T.S.E.P. $\{s_r\} = \{s_r^1, \cdots, s_r^n\}$ of $\Gamma_{\lambda}(E^r)$ which has at least one limiting point.

Lemma 2. Suppose that the economy E satisfies Assumption I and the penalty factors are very harsh. Then for any deposit ratio θ ($0 < \theta < 1$), any reserve ratio K ($0 < K$) and any harsh punishment factor $\{\lambda^0, \lambda^1, \cdots, \lambda^n\}$, there exists apparently a sequence of T.S.E.P. $\{s_r\} = \{s_r^1, \cdots, s_r^n\}$ of $\Gamma_{\lambda}(E^r)$. If $\{s_r\}$ has a subsequence $\{s_t\}$ such that $s_t \to \bar{s}$ and $1+\rho(t) \to 1$ as $t \to \infty$, then the price and allocation pair ($\bar{p}$, $\bar{x}$) resulting from $\bar{s}$ is a C.E. of the original economy E.

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[5]. Compare the similar assumption in Dubey and Shubik (1978).

[6]. Note that condition (v) can possibly be weakened. For example, as the proof in Appendix II shows, Lemma 1 still hold when (v) is changed as: (v)* there exists a $c^* > 0$ such that $v^{\alpha} \leq c^* \Sigma_{i=1}^{m+1} b_i^{\alpha}$ for all α.

Lemmas 1-2 are proved respectively in Appendix II and Appendix III.

3.3. THE EQUILIBRIA OF THE LIMITING GAME $\Gamma_\lambda(E^\infty)$. The limiting economy E^∞ is the limit of the r-fold replication economy E^r as r goes to ∞. A C.E. of the economy E^∞ is a pair consisting of a price vector p ($p \in R_+^{m+1}$) and a sequence of consumption bundle $\{x^{\alpha\beta}\}$ in R_+^{m+1} ($\alpha=1,\cdots,n$, $\beta= 1, 2,\cdots$) such that

(i) For each trader $\alpha\beta$, $x^{\alpha\beta}$ maximizes the utility function $\phi^{\alpha\beta} = \phi^\alpha$ over the budget set $a^{\alpha\beta}(p) = a^\alpha(p) = \{y^\alpha \in R_+^{m+1} \mid \Sigma_{i=1}^{m+1} p_i y_i^\alpha \leq \Sigma_{i=1}^{m+1} p_i \omega_i^\alpha\}$; and that

(16) (ii) $$\lim_{r\to\infty} \frac{1}{r} \Sigma_{\beta=1}^{r} \Sigma_{\alpha=1}^{n} x^{\alpha\beta} = \Sigma_{\alpha=1}^{n} \omega^\alpha.$$

Note that an important class of the competitive equilibria of E^∞ (or E^r) is the Type-Symmetric Competitive Equilibrium (T.S.C.E.). A T.S.C.E. is a C.E. such that all traders in the same type choose the same consumption bundle. We shall denote a T.S.C.E. of E^∞ (or E^r) as a pair (p, x) (where $p \in R_+^{m+1}$, $x \in R_+^{(m+1)\times n}$), where for each type α, $x^\alpha \in R_+^{m+1}$ is the consumption bundle that all traders of the α-th type choose.

Clearly, any such T.S.C.E. (p, x) is also a C.E. of the original economy E.

The game $\Gamma_\lambda(E^\infty)$ is defined similarly as in the game $\Gamma_\lambda(E^r)$: all players face the same penalty λ^0 for being short of cash, and all the traders in each type α face the same bankruptcy penalty λ^α and have the same strategy set:

(17) $$S^{\alpha\beta} = S^\alpha = \{s^\alpha \in R_+^{m+3} \mid u^\alpha \leq \omega_{m+1}^\alpha,\ v^\alpha < \infty,\ \Sigma_{i=1}^{m+1} b_i^\alpha < \infty\}.$$

The interest rate (3) and prices (4) are now given as:

(3-2) $$1+\rho = 1+\rho(s) = \frac{1}{\theta K \bar{\omega}_{m+1}} \inf.\lim_{r\to\infty} \frac{1}{r} \Sigma_{\beta=1}^{r} \Sigma_{\alpha=1}^{n} v^{\alpha\beta}.$$

(4-2) $$p_i = p_i(s) = \frac{1}{\bar{\omega}_i} \inf.\lim_{r\to\infty} \frac{1}{r} \Sigma_{\beta=1}^{r} \Sigma_{\alpha=1}^{n} b^{\alpha\beta} \qquad \text{for} \quad i = 1, \cdots, m+1.$$

Since all the strategy variables are non-negative, the above two formula are both well defined. It is clear that by the above two expressions, any finite variations of the individual strategy $s^{\alpha\beta}$ has no influence on either interest rate or prices. Thus the game

$\Gamma_\lambda(E^\infty)$ not only saves us from the burden of the "measurability assumption", which is essential in the literature with a continuum of traders, but also satisfy the requirement of "perfect competition", which is the basic motivation of Aumann's paper (1964). Further more, judging from the fact that any real economy has only finite population and finite number of other players(firms, parties, clubs, etc.), the game $\Gamma_\lambda(E^\infty)$ (even though very unrealistic and too abstract) is closer to the real world than the model with a continuum of traders.

Under a harsh punishment, an individual trader $\alpha\beta$'s optimization problem is simply to take prices and interest rate as given and to maximize $\phi^{\alpha\beta}(x^\alpha) = \phi^\alpha(x^\alpha(s^\alpha, p, \rho))$ over S^α (given in (17)) subject to two constraints:

(18) $$M^{\alpha\beta} = M^\alpha(s^\alpha, p, \rho) \geq 0,$$
$$d^{\alpha\beta} = d^\alpha(s^\alpha, p, \rho) \geq 0.$$

A T.S.N.E. of the game $\Gamma_\lambda(E^\infty)$ is a joint strategy when traders of the same type choose the same strategies. Such a T.S.N.E. of $\Gamma_\lambda(E^\infty)$ will be represented as a vector $s=\{s^1,\cdots, s^n\} \in \mathbf{R}_+^{(m+3)\times n}$, where each $s^\alpha \in \mathbf{R}_+^{m+3}$ is the strategy that all traders in the α-th type choose. Similarly as before, the price and allocation resulting from a T.S.N.E. of $\Gamma_\lambda(E^\infty)$ will also be represented by (p, x), meaning that all traders in the same type have the same consumption bundle.

With the above definitions we are now ready to present our result regarding the relationship between the T.S.N.E. of the game $\Gamma_\lambda(E^\infty)$ and the limit of a sequence $\{s_r\}$ of the T.S.E.P. of $\Gamma_\lambda(E^r)$ (Lemma 3), and the results regarding the equivalence between a C.E. of E and the T.S.N.E. of the game $\Gamma_\lambda(E^\infty)$ (Lemma 4 and Theorem 3).

Lemma 3. Suppose that the economy E satisfies Assumption I and the penalty factors are very harsh. Then for any given parameters λ, any limiting point $\bar{s}$ of a sequence of T.S.E.P. $\{s_r\} = \{s_r^1,\cdots, s_r^n\}$ of $\Gamma_\lambda(E^r)$ is a T.S.N.E. of the game $\Gamma_\lambda(E^\infty)$.

By "A given N.E. coincides with some C.E." we shall mean that the resulting price and allocation of (4) and (6) is a C.E. of the original economy E (or a T.S.C.E. of E^{∞}).

Lemma 4. Let the penalty factor $\lambda=\{\lambda^0, \lambda_{sp}\}=\{\lambda^0, \lambda^1,\cdots, \lambda^n\}$ be large enough that no trader wants to be either short of cash or bankrupted. Then under the condition of Assumption I, any T.S.N.E. of $\Gamma_\lambda(E^{\infty})$ with a zero rate of interest coincides with some C.E. of the original finite economy E.

The above two Lemmas are proved respectively in Appendices IV and V.

Theorem 3. Given any C.E. (x, p) of the original economy E, let the associated shadow price of income be $\lambda_{sp}=\{\lambda^1,\cdots, \lambda^n\}$. Let $\lambda=\{\lambda^0, \lambda_{sp}\}$, where λ^0 is large enough to guarantee the rule of "cash in advance". Suppose that the economy E satisfies Assumption I. Then for any deposit ratio θ ($0<\theta<1$), there exists a reserve ratio K such that the corresponding game $\Gamma_\lambda(E^{\infty})$ has a T.S.E.P. which coincides with (x,p) .

Proof of Theorem 3. Given the C.E. (p, x) and any deposit ratio q ($0<\theta<1$), let the gearing ratio $K > 0$ be such that

(19) $$\theta K \bar{\omega}_{m+1} = \Sigma_{\alpha=1}^{n}\Sigma_{i=1}^{m+1} p_i(x_i^{\alpha} - \omega_i^{\alpha})^+.$$

For each trader α, let a type-symmetric strategy $\hat{s}^{\alpha}= \{\theta\omega_{m+1}^{\alpha}, \hat{v}^{\alpha}; \hat{b}^{\alpha}\}$be given by:

(20) $$\hat{u}^{\alpha} = \theta\omega_{m+1}^{\alpha}, \quad \hat{v}^{\alpha} = \Sigma_{i=1}^{m} p_i(x_i^{\alpha} - \omega_i^{\alpha})^+, \quad \hat{b}_i^{\alpha} = p_i x_i^{\alpha} \quad \text{for } i = 1, \cdots, m+1.$$

Then it follows from (4-2) and the market clearing condition $\Sigma_{\alpha=1}^{n} x^{\alpha} = \Sigma_{\alpha=1}^{n}\omega^{\alpha}$ that the pair $(\hat{p}, \hat{x})$ resulting from (18) coincides with (p, x). Thus we need only to show that the strategy of (20) is α's best response.

It follows from (3-2) and (19) that $1+\rho = 1+\rho(\hat{s}) = 1$. Since a single trader alone in the game $\Gamma_\lambda(E^{\infty})$ can influence neither price nor interest rate, and because both the rate of interest and the individual share of bank ownership are zero, the individual optimization problem becomes the following parametric mathematical programming problem:

(21) $$\text{Max} \quad \Phi^{\alpha} =\Phi^{\alpha}(s^{\alpha}, \hat{s}_{-\alpha}) = \phi^{\alpha}(x^{\alpha}(s^{\alpha}, \hat{s}_{-\alpha})) + \lambda^{\alpha} d^{\alpha}$$

S.T. $0 \leq v^{\alpha} < \infty$, $0 \leq b_i^{\alpha} < \infty$ for all i, and

$$M^{\alpha}(s^{\alpha}, \hat{s}_{-\alpha}) = v^{\alpha} - \Sigma_{i=1}^{m+1} (b_i^{\alpha} - \hat{p}_i \omega_i^{\alpha})^{+} \geq 0, \tag{22}$$

which is equivalent to

$$\text{Max}\{ \phi^{\alpha}(x^{\alpha}) + \lambda^{\alpha}(\hat{p} \cdot \omega^{\alpha} - \hat{p} \cdot x^{\alpha}) \mid x^{\alpha} \in \mathbf{R}_{+}^{m+1} \}. \tag{23}$$

By the definition of shadow price (1), for all α, the above strategy $\hat{s}^{\alpha}$ is indeed a solution of (22). Thus the joint strategy $\hat{s}$ of (24) is a T.S.N.E. of $\Gamma_{\lambda}(E^{\infty})$. **Q.E.D.**

4. CONCLUDING REMARKS. We have presented a mechanism (a strategic market game) through which an exchange economy not having enough commodity money (gold) achieves an equilibrium. In the framework of "Fixed Share Mutual Bank Model", we show the existence of a non-cooperative equilibrium for any finite replication economy with a mutual bank (Theorems 1-2). Efficient trade can be achieved in the limiting economy by expanding money supply through fractional reserves (Theorem 3), however, it can not be achieved in general as a non-cooperative equilibrium of a finite player game.

It is of interest to observe the following three remarks. (i) As the gearing ratio K becomes large the commodity cover or store of value aspect of the money or means of payment approaches zero.

(ii) This model retains the merits of both the "sell all" model and the "either sell or bid" model. Under the assumption that each trader needs to pay in advance the whole value of each of the goods consumed, the "sell all" model provides a simple price formation rule (4). On the other hand, the "either sell or bid" model allows a trader to pay only the value of goods purchased but at the cost of introducing more variables and losing the concavity of the payoff function. While with the price formation rule (4) and the cash requirement (7) in the present model, a trader needs only to pay the value of "net purchase" of each good and the concavity of payoff function is also retained.

(iii) The model discussed also reveals the underlying price-value relationship. The price in the present model (also in the "sell all" model) reflects the "Total use value", while the price in the "bid or sell" model reflects only the "Transaction value" (or the "value of exchange"). These two values, the transaction value and the use value, are always equal in the present model[7], while it is not so clear in other context.

Appendix I. The Calculation of the Ex-post Rate of Interest.

Let $\rho_1 = \rho$, and $I_0 = \varnothing$.

Start. $t =: 1$. For each trader α, let

(A-1) $$d_t^\alpha = \Sigma_{i=1}^{m+1} p_i(\omega_i^\alpha - x_i^\alpha) + \frac{v^\alpha}{1+\rho} - v^\alpha + \rho_t \theta K \omega_{m+1}^\alpha,$$

$$I_t = \{ \alpha \mid d_t^\alpha < 0 \}, \quad J_t = \{ \alpha \mid d_t^\alpha \geq 0 \}.$$

If $I_t = I_{t-1}$, then Stop and let $\rho^* = \rho_t$;

Otherwise if $I_t \neq I_{t-1}$, define

(A-2) $$\rho_{t+1} = \frac{\Sigma_{\alpha \in I_t}(d_t^\alpha + v^\alpha) + \Sigma_{\alpha \in J_t} v^\alpha}{M} - 1,$$

and go back to Start with $t = t+1$. It follows from

$$d_t^\alpha + v^\alpha < v^\alpha \text{ for all } \alpha \in I_t$$

that $\rho_{t+1} < \rho_t$, and thus $I_t \subseteq I_{t+1}$. Since there are only finite number of trades, the above procedure will terminate at most at $t = n+1$.

Appendix II. Proof of Lemma 1.

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7. For each good $i = 1, \cdots, m+1$, its transaction value in the present model can be given as

$$p^*_i = \frac{\Sigma_{\alpha=1}^n p_i (x_i^\alpha - \omega_i^\alpha)^+}{\Sigma_{\alpha=1}^n (\omega_i^\alpha - x_i^\alpha)^+} \quad \text{if } \Sigma_{\alpha=1}^n (\omega_i^\alpha - x_i^\alpha)^+ > 0, \quad = p_i \quad \text{if } \Sigma_{\alpha=1}^n (\omega_i^\alpha - x_i^\alpha)^+ = 0.$$

By the market clearing condition $\Sigma_{\alpha=1}^n (x_i^\alpha - \omega_i^\alpha)^+ = \Sigma_{\alpha=1}^n (\omega_i^\alpha - x_i^\alpha)^+$ it is clear that $p_i = p^*_i$ for all i.

By Theorem 2 for any integer r, the game $\Gamma_\lambda(E^r)$ has at least one T.S.E.P. $s_r = \{s_r^1, \cdots, s_r^n\}$, thus a sequence $\{s_r\}$ of T.S.E.P. of $\Gamma_\lambda(E^r)$ apparently exists. By the market clearing property of the consumption (6), the corresponding allocation sequence $\{x_r\}$ ($x_r \in \mathbf{R}_+^{(m+1)\times n}$) is a bounded and non-negative sequence, thus without loss of generality, we can assume that $x_r \to \bar{x}$ as $r \to \infty$. By condition (iv) in assumption II,

(A-3) $$\Sigma_{\alpha=1}^{n} (\bar{x}_i^\alpha - \omega_i^\alpha)^+ > 0 \text{ for all i.}$$

Otherwise for sufficiently large r, the trader type who desires good i would have improved and thus contradict to the definition of T.S.N.E.. By summing up all $M^\alpha \geq 0$, we have

(A-4) $$\theta K \bar{\omega}_{m+1} \geq \Sigma_{\alpha=1}^{n} \Sigma_{i=1}^{m+1} (b_i^\alpha - p_i \omega_i^\alpha)^+ = \Sigma_{i=1}^{m+1} p_i \Sigma_{\alpha=1}^{n} (x_i^\alpha - \omega_i^\alpha)^+ .$$

It follows from (A-3) and (A-4) that the corresponding sequence of prices $\{p_r\}$ is bounded, and then by the price formation rule (4) the sequence $\{b_r\}$ ($b_r \in \mathbf{R}_+^{(m+1)\times n}$) is also bounded. Then by condition (v) of Assumption II, $\{v^\alpha(r)\}$ is also a bounded sequence for all α. Thus $\{s_r\}$ is a bounded and non-negative sequence and thus has at least one limiting point. **Q.E.D.**

Appendix III. Proof of Lemma 2.

Let the sequence $\{s_t\}$ of T.S.E.P. of $\Gamma_\lambda(E^t)$ be such that $s_t \to \bar{s}$ and $1+\rho_t \to 1$ as $t \to \infty$, then $1+\rho(\bar{s}) = 1$. Let $(\bar{p}, \bar{x})$ be the price and allocation pair resulting form $\bar{s}$, then by the assumption that all goods are essential, we have $\bar{p}_i > 0$ for all i. Assume by way of contradiction that $(\bar{x}, \bar{p})$ is not a C.E., then there is a trader α and consumption bundle x^α in α's budget set corresponding to the price $\bar{p}$ such that $\phi^\alpha(x^\alpha) > \phi^\alpha(\bar{x}^\alpha) = \phi^\alpha(x^\alpha(\bar{s}))$. Thus there is $\hat{x}^\alpha \in \mathbf{R}_+^{m+1}$ such that

(A-5) $$\phi^\alpha(\hat{x}^\alpha) > \phi^\alpha(\bar{x}^\alpha),$$

(A-6) $$\Sigma_{i=1}^{m+1} \bar{p}_i \hat{x}_i^\alpha < \Sigma_{i=1}^{m+1} \bar{p}_i \omega_i^\alpha .$$

Now consider a trader β of the type α, which is denoted as $\alpha\beta$. Let $\hat{s}^{\alpha\beta} = \{\theta\omega_{m+1}^\alpha, \hat{v}^{\alpha\beta}; \hat{b}^\alpha\}$ be defined as:

$$\hat{v}^{\alpha\beta} = 1 + \Sigma_{i=1}^{m+1} \bar{p}_i (\hat{x}_i^{\alpha} - \omega_i^{\alpha})^+, \quad \hat{b}_i^{\alpha} = \bar{p}_i \hat{x}_i^{\alpha}, \text{ for all i.}$$

Let $\hat{s}_t = (\hat{s}^{\alpha\beta}, s(t)_{-\alpha\beta})$ be a joint strategy of the game $\Gamma_\lambda(E^t)$ and $x^{\alpha\beta}(\hat{s}_t)$ be $\alpha\beta$'s consumption from $\hat{s}_t$. It follows from (A-6), the definitions (7) and (8) and the following two expressions

$$1 + \rho = 1 + \rho(\hat{s}_t) = \frac{1}{\theta K \bar{\omega}_{m+1}} \{\frac{1}{t} \hat{v}^{\alpha\beta} + \frac{t-1}{t} v^{\alpha}(t) + \Sigma_{\beta \neq \alpha} v^{\beta}(t) \},$$

$$p_i = p_i(\hat{s}_t) = \frac{1}{\bar{\omega}_i} \{\frac{1}{t} \hat{b}_i^{\alpha\beta} + \frac{t-1}{t} b_i^{\alpha}(t) + \Sigma_{\beta \neq \alpha} b_i^{\beta}(t) \} \text{ for all i}$$

that for any $\delta > 0$, there is a $T_0 > 0$ such that

(A-7) $\quad | x^{\alpha\beta}(\hat{s}_t) - \hat{x}^{\alpha} | < \delta, \quad M^{\alpha\beta}(\hat{s}_t) > 0$, and $d^{\alpha\beta}(\hat{s}_t) > 0$

for all $t \geq T_0$. For any $\varepsilon > 0$, by (A-5,7) and $x^{\alpha}(s(t)) \to \bar{x}^{\alpha}$, there is $T_1 > T_0$ such that

$$\phi^{\alpha}(x^{\alpha\beta}(\hat{s}_t)) \geq \phi^{\alpha}(\hat{x}^{\alpha}) - \frac{\varepsilon}{2} \geq \phi^{\alpha}(x^{\alpha}(s_t)) + \frac{\varepsilon}{2} > \phi^{\alpha}(x^{\alpha}(s_t)) = \phi^{\alpha\beta}(x^{\alpha}(s_t))$$

for all $t \geq T_1$, which contradicts to the definition that s_t is a T.S.N.E.. Thus $(\bar{p}, \bar{x})$ must be a C.E. of the original economy E. **Q.E.D.**

Appendix IV. Proof of Lemma 3.

Suppose that for each r, $\{s_r\} = \{s_r^1, \cdots, s_r^n\}$ is a T.S.N.E. of $\Gamma_\lambda(E^r)$ and that $s_r \to \bar{s} = \{\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_n\}$ as $r \to \infty$. It is easy to verify that $\bar{p}_i = \bar{p}_i(\bar{s}) > 0$ for all i, $1 + \bar{\rho} = 1 + \bar{\rho}$ $)(\bar{s}) > 0$, and that for each trader type α, there is $\tilde{s}^{\alpha} \in S^{\alpha}$ such that $M^{\alpha\beta}(\tilde{s}^{\alpha}, \bar{p}, \bar{\rho}) > 0$ and $d^{\alpha\beta}(\tilde{s}^{\alpha}, \bar{p}, \bar{\rho}) > 0$ (see (18)) for all β. Then repeating the arguments following (13) in the proof of Claim 3 will lead to the conclusion. **Q.E.D.**

Appendix V. Proof of Lemma 4.

Suppose $\bar{s} = \{\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_n\}$ is a T.S.N.E. of $\Gamma_\lambda(E^{\infty})$ with $1 + \bar{\rho} = 1$. By Assumption I $\bar{p}_i = \bar{p}_i(\bar{s}) > 0$ for all i. The optimization problem facing each trader $\alpha\beta$ is

$$\text{Max } \phi^{\alpha\beta}(x^{\alpha}(s^{\alpha\beta}, \bar{p}, \bar{\rho}))$$

$$\text{S.T. } M^{\alpha\beta}(s^{\alpha\beta}, \bar{p}, \bar{\rho}) \geq 0, \text{ and } d^{\alpha\beta}(s^{\alpha\beta}, \bar{p}, \bar{\rho}) \geq 0.$$

Where the constraint functions are $M^{\alpha\beta}(s^{\alpha\beta}, \bar{p}, \bar{\rho}) = v^{\alpha\beta} - \Sigma_{i=1}^{m+1} (b_i^{\alpha\beta} - \bar{p}_i \omega_i^{\alpha})^+$, and $d^{\alpha\beta}(s^{\alpha\beta}, \bar{p}, \bar{\rho}) = \Sigma_{i=1}^{m+1} (\bar{p}_i \omega_i^{\alpha} - b_i^{\alpha\beta})$. Since any finite variation of the individual strategy $s^{\alpha\beta}$ has no influence on either interest rate or prices, the above problem is equivalent to

$$\text{Max} \quad \phi^{\alpha}(x_1^{\alpha\beta}, x_2^{\alpha\beta}, \cdots, x_{m+1}^{\alpha\beta}) \quad \text{S.T.} \quad \Sigma_{i=1}^{m+1} \bar{p}_i (\omega_i^{\alpha} - x_i^{\alpha\beta}) \geq 0.$$

Thus $(\bar{p}, \bar{x})$ resulting from $\bar{s}$ is a C.E. of the original economy E. **Q.E.D.**

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