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RENEGOTIATION AND SYMMETRY IN REPEATED GAMES

by

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ABSTRACT

It seems reasonable to suppose that in repeated games in which communication is possible, play is determined through a process of negotiation and renegotiation as events unfold. In the absence of a satisfying theory of players' bargaining power, it is unclear how to model this process. Symmetric repeated games are an important class in which the problem is less troublesome. Whatever its source, bargaining power is presumably the same for all players in a symmetric game. We take equal bargaining power to mean that a player can mount a credible objection to a continuation equilibrium in which he receives a particular expected present discounted value, if there are other self enforcing agreements that never give any player such a low continuation value after any history. This is formalized in a solution concept called *consistent bargaining equilibrium.* The definition does not imply strongly symmetric solutions: in some games, after some histories, players will be treated differently from one another. This is commonly the case in games with imperfect monitoring, for example. But there are modest assumptions under which consistent bargaining equilibria of infinitely repeated games with perfect monitoring *are* strongly symmetric. This counters the intuition that with perfect monitoring, there is more reason to expect asymmetric treatment following some histories (since deviations from agreements become common knowledge). Strongly symmetric consistent bargaining equilibria have an unusually elementary characterization in terms of the payoff function of the stage game and the discount factor. Some applications to oligopoly are presented. In the linear Cournot model, for example, closed form expressions for maximally collusive output in consistent bargaining equilibria are available for any discount factor and any number of firms.

1. INTRODUCTION

We are interested in this paper in symmetric infinitely repeated games in which it is possible, after any history, for players to renegotiate their implicit agreement regarding equilibrium play. Our focus is on the joint implications of symmetry and renegotiation in this setting. The symmetry of the supergame makes it reasonable to suppose that players will receive symmetric payoffs in the equilibrium negotiated at the beginning of the game, as long as the Pareto frontier of the set of credible supergame equilibria includes a symmetric element. An asymmetric agreement would seem at odds with the equal bargaining power associated with the symmetric positions of the players. It is tempting to extend this reasoning to say that the symmetry of the subgame in which players find themselves after any history (possibly including deviations from equilibrium play) suggests that the continuation equilibrium in the subgame ought to be symmetric. We argue that there should be no such presumption: *even in the subgame* it may be in the interests of the worst–off player not to insist on symmetry.

The line of reasoning that supports this assertion is an elaboration of the approach to renegotiation theory[1] taken by Pearce (1987). The approach is most easily understood by thinking first of the simple case of a symmetric two–person repeated game all of whose (subgame perfect) equilibria are symmetric. The supergame equilibrium value set can be regarded as a subset of $\mathbb{R}$, say V, with maximum $\bar{v}$ and minimum $\underline{v}$. An equilibrium that achieves $\bar{v}$ may be supported by the "threat" that after certain histories (for example, if someone is observed to deviate, or, with imperfect monitoring, an unfavorable signal arises) the value of the "continuation equilibrium" in the ensuing subgame will be $\underline{v}$. Although the threat is subgame perfect, it is not credible in another sense: it seems plaus-

[1] Quite a distinct theory of renegotiation was initiated by Farrell (1983) and developed by Farrell and Maskin (1987), Bernheim and Ray (1987), van Damme (1987), Benoit and Krishna (1988), and others. We neglect the related problem of coalition formation. The latter is explored by Asheim (1988), whose analysis is based on Greenberg (1988).

ible that the players could convince one another to abandon the worst equilibrium, on the grounds that it constitutes an unnecessarily harsh punishment. While players understand that in order to enjoy mutual cooperation they must accept different continuation payoffs after different histories, they will not accept a "punishment" if there exists another continuation equilibrium that never needs to use such a severe punishment. In other words, because players care about the future rather than the past, they ask themselves not whether a certain punishment was needed to deter deviations earlier in the game, but whether the punishment is inescapable in the sense that in the *future* any equilibrium must inevitably rely on punishments at least as harsh.

Consider now the more general case of a symmetric game in which some supergame equilibrium values are asymmetric. How should players exploit the equal bargaining power associated with the symmetry of their roles? Not, we contend, by insisting on symmetric payoffs in every subgame: in some cases this would result in unnecessarily low payoffs for both players. Rather, a supergame continuation value pair (a,b) with $a < b$ should be acceptable to player 1 as long as there is no other subgame perfect equilibrium in which in all subgames, each player receives at least some value $c > a$. It might happen, for example, that all "strongly symmetric equilibria" (those that treat players symmetrically in every subgame) must use continuation value threats of (2,2) or worse, whereas the asymmetric threats (3,5) and (5,3) sustain a variety of equilibria whose continuation values never drop below 3 for either player. (It is easy to find examples of this sort; a simple one is provided below.) Consequently, if a deviation by player 1 is followed by a path with value (3,5), player 1 accepts the asymmetry because in a symmetric regime, punishments of value 2 would be unavoidable. Formally, we say that a subgame perfect equilibrium σ is a *consistent bargaining equilibrium* (CBE) if the infimum of the values of continuation equilibria (taken over all subgames and players) of σ is at least as great as the corresponding infimum for any other subgame perfect equilibrium. In the context of symmetric games, this specializes the definition of renegotiation–proofness given by Pearce (1987).

While it is intriguing that symmetric bargaining power need not lead to symmetric punishments, it would be tremendously helpful when analyzing a particular game to know that one could restrict attention to strongly symmetric profiles. Abreu (1986) showed in an oligopolistic model that in the traditional perfect monitoring setting without renegotiation, optimization within the class of strongly symmetric profiles yields easily described equilibria with vivid properties. But he further showed that more severe punishments can usually be achieved outside the strongly symmetric class, and that the structure of the optimal punishment tends, unfortunately, to be complex. One of our principal goals is to provide conditions under which renegotiation and equal bargaining power <u>imply</u> strong symmetry, and to explore the properties of strongly symmetric CBE's.

For any finite game G satisfying standard regularity assumptions and having an equilibrium in pure strategies, the associated infinitely repeated discounted game $G^{\infty}(\delta)$ has a consistent bargaining equilibrium (CBE). While much of the paper restricts attention to games with perfect monitoring, the definitions apply equally to imperfect monitoring models. For the latter, we can show that severest CBE punishments are often <u>not</u> strongly symmetric. By contrast, their counterparts in perfect monitoring models usually <u>are</u> strongly symmetric.

It is by now well understood that the crucial step in determining what kind of behavior can be supported in a particular supergame is to compute the worst credible threats that are available to the players. Theorem 2 of Section 2 provides an unexpectedly simple characterization of the worst CBE punishment: it is just the maximized value of a function f defined in an elementary way using the payoff function of the one–shot game G. Specifically, for any symmetric strategy profile x in G, $f(x)$ is the difference between $\frac{1}{1-\delta}$ times payoffs when x is played, and any player's *best response* payoffs against the same profile. If G is the well–known linear Cournot model, for example, this result produces *closed–form expressions* (for any value of the discount factor and any number of firms) for maximally collusive "punishment" values. This degree of tractability is encouraging for the prospects of the theory being useful in a variety of applied areas.

The worst credible punishments in the class of games whose (CBE) solutions we characterize have a "stick and carrot" structure similar to that established by Abreu (1986) for the standard theory without renegotiation. They have two "phases," the first serving to give the players low payoffs for a number of periods, and the second following the equilibrium path of the best strongly symmetric (and *stationary*) CBE.

The assumptions required to generate these results are nontrivial restrictions on the component game. Nonetheless, many games of economic interest satisfy the assumptions. This is documented in the Appendix, which presents classes of games including Cournot oligopolies with convex cost functions and a family of demand functions containing the linear and constant elasticity demands as special cases, as well as price–setting models. A more novel application, found in subsection 2.4, is to the theory of multimarket collusion, an issue recently investigated by Bernheim and Whinston (1987). In contrast to their results, we find that (in symmetric games) given renegotiation with equal bargaining power, multimarket contact has no effect on the extent of collusion: if markets are structurally independent (in terms of firm cost functions and industry demand functions) then, with renegotiation, they will be strategically independent also.

The brief treatment of imperfect monitoring models provided by Section 3 establishes that strong symmetry is *not* a general implication of the definition of a consistent bargaining equilibrium. Fudenberg, Levine and Maskin (1988) identify[2] a large

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[2] See also the references cited in Section 3 below.

class of games with unobservable actions in which first–best outcomes in the supergame can be approached asymptotically in equilibrium as δ approaches 1. By Proposition 7 of Pearce (1987) this is true even if one restricts attention to consistent bargaining equilibria. But we show that imposing strong symmetry leads to inefficiencies that do not vanish asymptotically. Thus, in imperfect monitoring models with patient players, consistent bargaining equilibria will usually violate strong symmetry. Section 4 concludes the paper.

2. PERFECT MONITORING

The class of games considered in this section are symmetric repeated games with *perfect monitoring.*

2.1 PRELIMINARIES

This subsection develops notation, and presents results for *simple strategy profiles* (Abreu (1988)) that will be used below.

The *stage game* is denoted $G = (S_1,...,S_n; \Pi_1,...,\Pi_n)$, where $N = \{1,...,n\}$ is the set of players, S_i is a pure strategy for player i, and $\Pi_i : S_1 \times ... \times S_n \to \mathbb{R}$ is player i's payoff function. The stage game is symmetric in that

(i) $S_i = S_1$ for all i

(ii) for each permutation τ of $\{1,...,n\}$, $\Pi_i(s_{\tau(1)},...,s_{\tau(n)}) = \Pi_{\tau(i)}(s_1,...,s_n)$ for all $s \in S_1 \times ... \times S_n$ and all i.

In addition we assume that:

(A1) S_i *is compact.*

(A2) Π_i *is continuous.*

The associated *repeated game* is denoted $G^{\infty}(\delta)$, where $\delta \in (0,1)$ is the discount factor. Each player i chooses an action $s_i(t) \in S_i$ in every period $t=1,2,\ldots$. The *perfect monitoring* assumption is that $s_i(t)$ may depend on the entire history of *all* players' previous choices $s(1),\ldots s(t-1)$. We refer to vectors $(z_1,\ldots,z_n)$ by the corresponding unsubscripted symbol z. Also $z_{-i} = (z_1,\ldots,z_{i-1}, z_{i+1},\ldots,z_n)$. A repeated game (pure) strategy for player i is denoted σ_i, and σ denotes a strategy *profile*. Throughout we confine attention to pure strategies. Also,

- $\vec{s} = \left\{s(t)\right\}_{t=1}^{\infty}$ where $s(t) = (s_1(t),\ldots,s_n(t)) \in S$, denotes a path.

- $v_i(\vec{s}) = \sum_{t=1}^{\infty} \delta^t \Pi_i(s(t))$ is the discounted payoff to i from the path $\vec{s}$. Note that first–period payoffs are discounted.

- $\tilde{v}_i(\sigma)$ denotes the payoff to i from the strategy profile σ.

- $v_i(\vec{s};t) = \sum_{\tau=0}^{\infty} \delta^{\tau+1} \Pi_i(s(t+\tau))$ is the payoff to i along the path $\vec{s}$ from period t onwards, discounted to the beginning of period t.

Let $H = \bigcup_{t=0}^{\infty} S^t$ be the set of all histories, where $S^0 = \{\emptyset\}$ and $\emptyset$ denotes the null history. For all $h \in H$, $\sigma|_h$ denotes the strategy profile induced by σ on the subgame following the history h. By convention $\sigma|_{\emptyset} = \sigma$. We will be interested in a subset of the set of *subgame perfect equilibria* (Selten (1965, 1975)).

Let $e = (1,\ldots,1)$. We adopt the convention that $\vec{x} = \left\{x(t)\cdot e\right\}_{t=1}^{\infty}$, $x(t) \in S_1$, denotes a *symmetric* path while $\vec{s} = \left\{s(t)\right\}_{t=1}^{\infty}$, $s(t) \in S$, is, as previously defined, an arbitrary path in S. Let

$$\overline{\Pi}_i(s) = \max\left\{\Pi_i(s'_i,s_{-i}) \mid s'_i \in S_i\right\}$$
$$\pi(x) = \Pi_1(x,\ldots,x)$$
$$\bar{\pi}(x) = \overline{\Pi}_1(x,\ldots,x)\,.$$

DEFINITION: A path $\vec{s}$ is supportable by $w \in \mathbb{R}$ if for each $i = 1,\ldots,n$ and each $t=1,2,\ldots$,

$$\overline{\Pi}_i(s(t)) - \Pi_i(s(t)) \leq v_i(\vec{s};t+1) - w\,.$$

For a symmetric path these conditions reduce to:

$$\bar{\pi}(x(t)) - \pi(x(t)) \leq v_1(\vec{x};t+1) - w \qquad \text{for all } t.$$

We now review some definitions and results from Abreu (1988). As noted there, a strategy profile of $G^{\infty}(\delta)$ may be viewed as a rule specifying an initial path, and punishments for any deviation from the initial path or from a previously prescribed punishment. The next definition identifies a particularly elementary class of strategy profiles.

DEFINITION: Let $\vec{s}^{\,i}$, $i = 0,1,\ldots,n$, be paths in S. The *simple (strategy) profile* $\tilde{\sigma}(\vec{s}^{\,0}, \vec{s}^{\,1},\ldots,\vec{s}^{\,n})$ specifies

(i) play according to $\vec{s}^{\,0}$ until some player deviates singly from $\vec{s}^{\,0}$,

(ii) for any $j \in N$, play $\vec{s}^{\,j}$ if the j^{th} player deviates singly from $\vec{s}^{\,i}$, $i = 0,1,\ldots,n$, where $\vec{s}^{\,i}$ is an ongoing previously specified path. Continue with $\vec{s}^{\,i}$ if no deviations occur or if two or more players deviate simultaneously.

Let $\hat{\sigma}(\vec{s},\vec{x})$ denote the *simple profile* (see Abreu (1988)) with initial path $\vec{s}$ and a single symmetric punishment $\vec{x}$. Any single player deviation from an ongoing path ($\vec{s}$ or $\vec{x}$) is responded to simply by (re)starting $\vec{x}$.

DEFINITION: For each strategy profile σ, $C(\sigma) = \left\{ \tilde{v}(\sigma|_h) \mid h \in H \right\}$ is the set of "continuation values" of σ, including the value of σ itself, and $\ell(\sigma) = \inf\left\{ \min\{w_1,...,w_n\} \mid (w_1,...,w_n) \in C(\sigma) \right\}$.

From Abreu (1988) we have:

(R1) *Let σ be an equilibrium and $\vec{s}$ be its initial path. Then $\vec{s}$ is supportable by $\ell(\sigma)$.*

(R2) *The simple profile $\hat{\sigma}(\vec{s}, \vec{x})$ is a subgame perfect equilibrium if and only if the paths $\vec{s}$ and $\vec{x}$ are supportable by $v_1(\vec{x})$.*

(R3) *The simple profile $\tilde{\sigma}(\vec{s}^0, \vec{s}^1,...,\vec{s}^n)$ is a subgame perfect equilibrium if and only if $\overline{\Pi}_j(s^i(t)) - \Pi_j(s^i(t)) \leq v_j(\vec{s}^i; t+1) - v_j(\vec{s}^j)$ for all $j = 1,...,n$, $i = 0,1,...,n$ and $t = 1, 2,...$.*

Henceforth, we will typically refer to subgame perfect equilibrium simply as equilibrium. We assume that:

(A3)' *$G^\infty(\delta)$ has a (subgame perfect) equilibrium.*

A simple sufficient condition on G which guarantees (A3)' is

(A3) *G has a Nash equilibrium in pure strategies.*

2.2 BARGAINING POWER AND RENEGOTIATION

Our main definition is motivated as follows. Players do not care about symmetry *per se*. Rather, a player exploits his bargaining power by refusing to accept a continuation payoff, say w, unless all (subgame perfect) equilibria rely on punishments at least as harsh as w.

DEFINITION: An equilibrium σ is a *consisent bargaining equilibrium* (*CBE*) if for any equilibrium γ, $\ell(\gamma) \leq \ell(\sigma)$.

If σ is a CBE, it is impossible for any player i to object, following some history h, that his continuation payoff $v_i(\sigma|_h)$ is intolerably low (and to demand renegotiation of the agreement). "Punishments" of at least this severity are an inevitable part of any self–enforcing implicit agreement. Conversely, we interpret equal bargaining power to mean that a player may demand $\ell(\sigma)$ after *any* history h.

THEOREM 1: *(Existence) Under (A1), (A2), and (A3)', a consistent bargaining equilibrium exists.*

PROOF: Let $\underline{r} = sup\ \{\ell(\sigma)\ |\ \sigma$ is an equilibrium$\}$. By (A1) to (A3) $\underline{r}$ is well defined. We complete the proof by exhibiting an equilibrium σ with $\ell(\sigma) = \underline{r}$. Let $\{\sigma^\eta\}_{\eta=1}^\infty$ be a sequence of equilibria such that $(\underline{r} - \ell(\sigma^\eta)) \leq \frac{1}{\eta}$. For each σ^η there exists a history h and a player i such that $(\ell(\sigma^\eta) - \tilde{v}_i(\sigma^\eta|h)) \leq \frac{1}{\eta}$. Since σ^η is an equilibrium, so also is $\sigma^\eta|h$. Furthermore, by symmetry there exists an analogous equilibrium in which player i and player 1 are interchanged. That is, w.l.o.g. we may in addition assume that $(\underline{r} - v_1(\vec{s}^{o\eta})) \leq \frac{1}{\eta}$, where $\vec{s}^{o\eta}$ is the initial path of σ^η.

The rest of the proof mimics Proposition 2 of Abreu (1988). We endow $\Omega = S^{\infty}$ with the product topology. By (A1) and (A2) $v: \Omega \to \mathbb{R}^n$ is continuous, and by Tychonoff's theorem Ω is compact. We may w.l.o.g. take $\{\vec{s}^{o\eta}\}$ to be a convergent sequence. Let $\vec{s}^{o} = \lim \vec{s}^{o\eta}$. By definition, $v_i(\vec{s}^{o\eta};t) \geq \ell(\sigma^{\eta})$ for all i, t. Hence we also have $v_i(\vec{s}^{o};t) \geq \underline{r}$ for all i, t. Let $\vec{s}^{i}$ be obtained from $\vec{s}^{o}$ by interchanging the roles of players 1 and i. That is, writing $\vec{s}^{k} = \left\{(s_1^k(t),\ldots,s_n^k(t))\right\}_{t=1}^{\infty}$ $k = 0,1,\ldots,n$, we have, $s_j^i(t) = s_j^o(t)$ $j \neq 1,i$, $s_i^i(t) = s_1^o(t)$, and $s_1^i(t) = s_i^o(t)$. Clearly $v_i(\vec{s}^{i}) = \underline{r}$, and $v_j(\vec{s}^{i};t) \geq \underline{r}$ for all i,j,t. We now argue that $\tilde{\sigma}(\vec{s}^{o},\vec{s}^{1},\ldots,\vec{s}^{n})$ is an equilibrium. Suppose not. Then by (R3), $\overline{\Pi}_j(s^o(t)) - \Pi_j(s^o(t)) > v_j(\vec{s}^{o};t+1) - \underline{r}$ for some j,t. Since $\vec{s}^{o\eta} \to \vec{s}^{o}$ and $\ell(\sigma^{\eta}) \geq \underline{r}$, by continuity for η large enough $\overline{\Pi}_j(s^{o\eta}(t)) - \Pi_j(s^{o\eta}(t)) > v_j(\vec{s}^{o\eta};t+1) - \ell(\sigma^{\eta})$. But then by (R1), σ^{η} is not an equilibrium, a contradiction. Hence $\tilde{\sigma}(\vec{s}^{o},\vec{s}^{1},\ldots,\vec{s}^{n})$ is an equilibrium. Since its minimum continuation value is $\underline{r}$, it is a CBE.

Q.E.D.

Recall that a repeated game strategy for player i is a sequence of functions $\sigma_i = (\sigma_i(1),\ \sigma_i(2),\ldots)$ where $\sigma_i(t): S^{t-1} \to S_i$.

DEFINITION: A strategy profile σ is strongly symmetric if for all $i \in N$: (1) $\sigma_i(1) = \sigma_1(1)$; and (2) $\sigma_i(t)(h) = \sigma_1(t)(h)$ for all $t \geq 2$ and $h \in S^{t-1}$.

Does the equal bargaining power assumption imply equal treatment in the sense of identical behavior after all contingencies? Not necessarily, as the following simple example shows.

EXAMPLE: Consider the two–player stage game

	U	D
u	20,20	5,30
d	30,5	7,7

Set $\delta = {}^1/_2$. Given that the one–period gain from cheating at (u,U) is 10, it is clear (see (R1)) that for any strongly symmetric equilibrium σ, $\ell(\sigma) \leq \frac{\delta}{1-\delta} 20 - 10 = 10$. But there exist equilibria which are not strongly symmetric, with a higher infimum value. Let $s^1 = (d,U)$ and $s^2 = (u,D)$, and consider the following strategy profile $\hat{\sigma}$. Start with s^1. If players use s^1 in period t, s^2 is to be played in $(t+1)$, and vice–versa. If row deviates in t, s^2 is to be played in $(t+1)$, and if column deviates in t, s^1 is played in $t+1$. It may be checked directly that $\hat{\sigma}$ is an equilibrium. Furthermore, $\ell(\hat{\sigma}) = \frac{5\delta \,+\, 30\delta^2}{1 \,-\, \delta^2} = \frac{40}{3} > 10 \geq \ell(\sigma)$ for any strongly symmetric equilibrium σ. Thus, in this game it is in the players' self–interest to permit themselves to be treated asymmetrically.

2.3 CHARACTERIZATIONS

While the preceding example demonstrates that strong symmetry cannot be guaranteed *a priori*, it is analytically an extremely attractive restriction, and leads to tractable characterizations. The next assumption plays a central role in establishing that strongly symmetric CBE's exist.

(A4) *For all $s \in S$ there exists $x \in S_1$ such that*

(i) $\pi(x) \geq \frac{1}{n} \sum_i \Pi_i(s)$

(ii) $\quad \bar{\pi}(x)-\pi(x) \leq \frac{1}{n}\sum_i \left[\bar{\Pi}_i(s) - \Pi_i(s) \right].$

This assumption implies that in the stage game asymmetries do not, of themselves, increase aggregate payoffs or reduce aggregate temptations to cheat. While (A4) is a non—trivial assumption, it is satisfied, as we show in the Appendix, in a variety of natural economic settings.

LEMMA 1 *(Smoothing): Given Assumption 4, for any path* $\vec{s}$ *supportable by* $w \in \mathbb{R}$ *there exists a symmetric path* $\vec{x}$ *supportable by* w *such that*

$$v_1(\vec{x};t) \geq \frac{1}{n}\sum_{i=1}^{n} v_i(\vec{s};t) \quad \textit{for} \quad t = 1,2,\ldots .$$

PROOF: Consider $\vec{s} = \left\{s(t)\right\}_{t=1}^{\infty}$ supportable by w. Then for all i, $\bar{\Pi}_i(s(t)) - \Pi_i(s(t)) \leq v_i(\vec{s};t+1) - w$. Hence,

$$\frac{1}{n}\sum_i \left[\bar{\Pi}_i(s(t)) - \Pi_i(s(t)) \right] \leq \frac{1}{n}\sum_i v_i\,(\vec{s};t+1) - w\,.$$

By (A4) there exists a symmetric path $\vec{x} = \left\{ x(t)\cdot e \right\}_{t=1}^{\infty}$ such that

$$\pi(x(t)) \geq \frac{1}{n}\sum_i \Pi_i(s(t)) \text{ and}$$

$$\bar{\pi}(x(t)) - \pi(x(t)) \leq \frac{1}{n}\sum_i \left[\bar{\Pi}_i(s(t)) - \Pi_i(s(t)) \right].$$

Hence $v_1(\vec{x};t+1) \geq \frac{1}{n}\sum_i v_i(\vec{s};t+1)$, and $\bar{\pi}(x(t)) - \pi(x(t)) \leq v_1(\vec{x};t+1) - w$ for all t.

Q.E.D.

The next assumption appears in the proof of Theorem 2. It is used there to resolve an integer problem (time is discrete), and may be dropped if public randomization is allowed. It should be viewed more as a convenience than as an essential component of the basic argument. Note that (A3) and (A4) *imply* that G has a *symmetric* Nash equilibrium. Denote by x^{cn} the symmetric equilibrium which yields the highest payoff. By (A1) and (A2), x^{cn} is well defined.

(A5) *For any* $z \in S_1$ *and* a *such that* $\pi(x^{cn}) \leq a \leq \pi(z)$, *there exists* $y \in S_1$ *such that* $a = \pi(y)$, *and* $\bar{\pi}(y) - \pi(y) \leq \bar{\pi}(z) - \pi(z)$.

DEFINITION: If σ and γ are CBE's, it follows that $\ell(\sigma) = \ell(\gamma)$. Let $\underline{r} = \ell(\sigma)$ for any CBE σ.

The key result of this section is Theorem 2. If one sets aside questions of "openness," the argument is roughly the following. Symmetric CBE paths exist,[3] because by Lemma 1 any asymmetric CBE path can be "averaged" across players while preserving incentive compatibility. It is easy to show that among the best symmetric CBE paths there is at least one that is stationary. For any action x, let $f(x)$ be the greatest value that would support the stationary symmetric path on which action x is always chosen. Such a punishment would be just sufficient to deter a deviation, therefore $\frac{1}{1-\delta}\pi(x) = \bar{\pi}(x) + f(x)$, that is, $f(x) = \frac{1}{1-\delta}\pi(x) - \bar{\pi}(x)$. Let z be the action chosen on some best stationary symmetric CBE path. We know that $\underline{r} \leq f(z)$. Let x^* maximize f. The proof of Theorem 2 exploits (A5) to construct a strongly symmetric SPE with value $f(x^*)$ and continuation values at least $f(x^*)$. Thus, $\underline{r} \leq f(z) \leq f(x^*) \leq \underline{r}$, the last inequality following from the definition of $\underline{r}$. This characterizes $\underline{r}$ as a simple function of δ and the data of the component game.

THEOREM 2: *Let* $f(x) = \frac{1}{1-\delta}\pi(x) - \bar{\pi}(x)$. *Then* $\underline{r} = \max_{x \in S_1} f(x)$.

[3] We refer to a path $\vec{s}$ as a *CBE–path* if there exists a CBE γ with outcome $\vec{s}$.

PROOF: From the definition of a CBE it is clear that any CBE–path is supportable by $\underline{r}$. Hence by Theorem 1 and Lemma 2 there exists a *symmetric* path $\vec{y} = \left\{y(t)\cdot e\right\}_{t=1}^{\infty}$ which is supportable by $\underline{r}$. Let $\tilde{y} \in cl\left\{y(t)\right\}_{t=1}^{\infty}$ (where "cl" is "closure") satisfy $\pi(\tilde{y}) \geq \pi(y(t))$ for all t. Then $\frac{\delta}{1-\delta}\,\pi(\tilde{y}) \geq v_1\,(y;t+1)$, $t=1,2,\ldots$. By the continuity of π and $\bar{\pi}$ it follows that $\bar{\pi}(\tilde{y}) - \pi(\tilde{y}) \leq \frac{\delta}{1-\delta}\,\pi(\tilde{y}) - \underline{r}$. That is, $\underline{r} \leq \frac{1}{1-\delta}\,\pi(\tilde{y}) - \bar{\pi}(\tilde{y}) \leq \max_x f(x)$. Let x^* be any solution to the problem $\max f$, and define

$$v_1^* = \frac{\delta}{1-\delta}\,\pi(x^*).$$

Observe that f may be rewritten as $f(x) = \frac{\delta}{1-\delta}\,\pi(x) - (\bar{\pi}(x) - \pi(x))$, and

$$\frac{\delta}{1-\delta}\,\pi(x^{cn}) = f(x^{cn}) \leq f(x^*) = \frac{\delta}{1-\delta}\,\pi(x^*) - (\bar{\pi}(x^*) - \pi(x^*)) \leq v_1^*\,.$$

Assume that $f(x^*) > f(x^{cn})$. Since x^{cn} is a payoff maximal symmetric Nash equilibrium (NE), this implies that x^* is not an NE. Hence $v_1^* > f(x^*)$. Also, $\pi(x^*) > \pi(x^{cn})$. Let

$$T = \max\left\{\tau \mid (\delta + \delta^2 + \ldots + \delta^\tau)\,\pi(x^{cn}) + \delta^\tau\, v_1^* \geq f(x^*)\right\}.$$

By the continuity of π and (A5) there exists a such that

$$(\delta+\ldots+\delta^T)\,\pi(x^{cn}) + \delta^{T+1}\,\pi(a) + \delta^{T+1}\,v_1^* = f(x^*),$$

where $\bar{\pi}(a) - \pi(a) \leq \bar{\pi}(x^*) - \pi(x^*)$. Let $\vec{\underline{x}} = \left\{\underline{x}(t)\cdot e\right\}_{t=1}^{\infty}$ be the symmetric path where $\underline{x}(t) = x^{cn}$, $t = 1,\ldots,T$; $\underline{x}(T+1) = a$; $\underline{x}(t) = x^*$, $t = T+2,T+3,\ldots$. Then $v_1(\vec{\underline{x}}) = f(x^*)$ and $v_1(\vec{\underline{x}};t+1) > f(x^*)$ all $t = 1,2,\ldots$. Also from above, $\bar{\pi}(x^*) - \pi(x^*) = \frac{\delta}{1-\delta}\,\pi(x^*) - f(x^*)$. Since $\bar{\pi}(x^{cn}) = \pi(x^{cn})$, and $\bar{\pi}(a) - \pi(a) \leq \bar{\pi}(x^*) - \pi(x^*)$, it follows that $\vec{\underline{x}}$ is supportable by $v_1(\vec{\underline{x}}) = f(x^*)$, and hence by (R3), $\hat{\sigma}(\vec{\underline{x}},\vec{\underline{x}})$ is an equilibrium. Since $\ell(\hat{\sigma}(\vec{\underline{x}},\vec{\underline{x}})) = f(x^*)$, $\underline{r} \geq f(x^*)$. Combine this with the earlier inequality $\underline{r} \leq \max f$ to complete the proof for the case $f(x^*) > f(x^{cn})$. The case where x^{cn} maximizes f is trivial. Then $T = \infty$ and $\vec{\underline{x}}$ is the constant symmetric path x^{cn} forever.

Q.E.D.

Before we comment on this rather striking formula it is useful to have some further results. Let $\vec{\underline{x}}$ be as defined in the proof above. First note that the strongly symmetric simple profile $\hat{\sigma}(\vec{\underline{x}},\vec{\underline{x}})$ is a CBE, and yields the payoff $\underline{r}$. Furthermore,

LEMMA 2: *A path* $\vec{s}$ *is a CBE path if and only if*

(i) $\vec{s}$ *is supportable by* $\underline{r}$, and

(ii) $v_i(\vec{s};t) \geq \underline{r}$ *for* $i = 1,\ldots,n$ *and* $t = 1,2,\ldots$.

Under these conditions the simple profile $\hat{\sigma}(\vec{s},\vec{\underline{x}})$ *is a CBE.*

PROOF: *(Necessity)* Let σ be a CBE with initial path $\vec{s}$. By the definition of a CBE, $\ell(\sigma) \geq \underline{r}$. This establishes (ii). Finally, by (R1) $\vec{s}$ is supportable by $\ell(\sigma)$, and therefore by $\underline{r}$.

(Sufficiency) By (R2) $\hat{\sigma}(\vec{s},\vec{\underline{x}})$ is an equilibrium. Its continuation values are $\left\{ v(\vec{a};t) \mid \vec{a} = \vec{s}, \vec{\underline{x}} \text{ and } t = 1,2,\ldots \right\}$. Hence, $\ell(\sigma(\vec{s},\vec{\underline{x}})) \geq \underline{r}$, and $\hat{\sigma}(\vec{s},\vec{\underline{x}})$ is a CBE.

Q.E.D.

An immediate consequence of Lemmas 1 and 2 is:

THEOREM 3: *For any CBE* γ *there exists a strongly symmetric CBE* σ *such that* $\tilde{v}_1(\sigma) \geq \frac{1}{n}\sum_i \tilde{v}_i(\gamma)$.

Thus, under our assumption players' self–interest does not force them to accept asymmetries (recall the example and the earlier discussion). The minimum payoff $\ell(\sigma)$ is not improved by permitting asymmetric treatment. Our principle of equal bargaining power therefore implies that players may, without loss, insist on symmetric payoffs both prior to and following a deviation. We therefore restrict attention to $R = \{\tilde{v}_1(\sigma) \mid \sigma$ is a strongly symmetric CBE$\}$. It is straightforward to show that R is a compact set.

LEMMA 3: *R is compact.*

PROOF: From (A1) and (A2), R is bounded. Consider a sequence of symmetric CBE–paths $\{\vec{x}^{\eta}\}$ such that $\lim v_1(\vec{x}^{\eta}) = a$. We need to show that $a \in R$. Endow $\Omega = S^{\infty}$ with the product topology. By Tychonoff's theorem, Ω is compact. Assume w.l.o.g. that $\vec{x}^{\eta} \to \vec{x}^*$. By continuity, $v_i(\vec{x}^*;t) = \lim v_i(\vec{x}^{\eta};t) \geq \underline{r}$ for all i and $t \geq 1$. By (R1) and the definition of a CBE $\vec{x}^{\eta}$ is supportable by $\underline{r}$. Hence by Lemma 2 $\vec{x}^*$ is a CBE–path with $v_1(\vec{x}^*) = a$.

Q.E.D.

Two numbers of special interest are $\underline{r} = \min R$ $(= \ell(\sigma)$ for any CBE $\sigma)$, and $\bar{r} = \max R$. The former is the worst credible punishment payoff, and the latter the best, or most "collusive," payoff. It is remarkable that both these numbers which emerge from a potentially complex intertemporal incentive compatibility problem, may be expressed in terms of explicit, trivially computable formulae.

Let x^* satisfy $x^* \in \arg\max f$ and $\pi(x^*) \geq \pi(y)$ for all $y \in \arg\max f$. This notation is useful in characterizing $\bar{r}$.

THEOREM 4: $\bar{r} = \frac{\delta}{1-\delta}\pi(x^*)$.

PROOF: Let $\vec{y}$ be a symmetric CBE path with payoff $\bar{r}$. By Lemma 2, $\vec{y}$ is supportable by $\underline{r}$. Let $\tilde{y} \in \mathrm{cl}\{y(t)\}$ satisfy $\pi(\tilde{y}) \geq \pi(y(t))$ for all t. By continuity of π and $\bar{\pi}$ (this step is analogous to the proof of Theorem 2), $\bar{\pi}(\tilde{y}) - \pi(\tilde{y}) \leq \frac{\delta}{1-\delta}\pi(\tilde{y}) - \underline{r}$. Hence, $\underline{r} \leq \frac{1}{1-\delta}\pi(\tilde{y}) - \bar{\pi}(\tilde{y})$. It follows from Theorem 2 that $\tilde{y}$ maximizes f. Thus, $\pi(x^*) \geq \pi(\tilde{y})$, and $v_1(\vec{x}^*) \geq \frac{\delta}{1-\delta}\pi(\tilde{y}) \geq v_1(\vec{y})$. Since $\vec{x}^*$ is supportable by $\underline{r}$, it follows from Lemma 2 that $\vec{x}^*$ is a CBE path. Therefore, $\bar{r} = v_1(\vec{x}^*) = \frac{\delta}{1-\delta}\pi(x^*)$ as required.

Q.E.D.

To summarize, $\underline{r} = \max f$, $\bar{r} = \frac{\delta}{1-\delta}\pi(x^*)$, and $\vec{x}^*$, the constant path x^* forever, is the most collusive symmetric CBE–path.

Let x^m denote any maximizer of π. The proofs of the next two results assume that S_1 is an interval, and use the following regularity assumptions:

(A6) *The functions* π *and* $\bar{\pi}$ *are continuously differentiable.*

(A7) *The points* x^{cn}, x^m *lie in the interior of* S_1, *and* $\bar{\pi}'(x^{cn})$, $\bar{\pi}'(x^m) \neq 0$. *Furthermore, for all* $x \notin \mathrm{int}\, S_1$, $\pi(x^{cn}) > \pi(x)$.

THEOREM 5: $\underline{r} > \frac{\delta}{1-\delta}\pi(x^{cn})$, *and* $\bar{r} < \frac{\delta}{1-\delta}\pi(x^m)$ *for all* $\delta \in (0,1)$.

PROOF: By Theorem 2, $\underline{r} = \max f = \max \{\frac{\delta}{1-\delta}\pi(x) - [\,\bar{\pi}(x) - \pi(x)\,]\}$. Hence, by (A6) and (A7), arg max $f \subset \mathrm{int}\, S_1$ and $f'(z) = 0$ for all $z \in \arg\max f$. Obviously, $\underline{r} \geq \frac{\delta}{1-\delta}\pi(x^{cn})$. Now observe that, if

$$\underline{r} = \frac{\delta}{1-\delta}\pi(x^{cn}) = \frac{\delta}{1-\delta}\pi(x^{cn}) - [\,\bar{\pi}(x^{cn}) - \pi(x^{cn})] = \frac{\pi(x^{cn})}{1-\delta} - \bar{\pi}(x^{cn}),$$

then $x^{cn} \in \arg\max f$ and $f'(x^{cn}) = 0$. Noting that $\bar{\pi}'(x^{cn}) = \pi'(x^{cn})$ we have $\bar{\pi}'(x^{cn}) = 0$, contradicting (A7). Finally, let x^* be defined as in Theorem 4, and suppose that $\bar{r} = \frac{\delta}{1-\delta}\pi(x^*) = \frac{\delta}{1-\delta}\pi(x^m)$. Then, x^* maximizes π and $\pi'(x^*) = 0$. By assumption $\bar{\pi}'(x^*) \neq 0$ which contradicts the requirement $f'(x^*) = 0$.

Q.E.D.

This should be contrasted with the usual theory without renegotiation, where the first inequality of Theorem 5 (with $\underline{r}$ replaced by $\underline{v}$, the minimum of the equilibrium value set) is reversed, and the second inequality (with $\bar{r}$ replaced by $\bar{v}$, the symmetric maximum of the equilibrium value set) holds with equality for sufficiently high δ.

THEOREM 6: *As functions of* δ, $\frac{1-\delta}{\delta}\,\underline{r}(\delta)$ *and* $\frac{1-\delta}{\delta}\,\bar{r}(\delta)$ *are strictly increasing. Furthermore,* $\lim_{\delta\to 1} \frac{1-\delta}{\delta}\,\underline{r}(\delta) = \lim_{\delta\to 1} \frac{1-\delta}{\delta}\,\bar{r}(\delta) = \pi(x^m)$.

PROOF: Let $g(x;\delta) = \pi(x) - \frac{1-\delta}{\delta}\,[\bar{\pi}(x) - \pi(x)]$. By Theorem 2, $\frac{1-\delta}{\delta}\,\underline{r}(\delta) = \max g(x;\delta)$. Consider $\delta_1 < \delta_2$ and let x_i maximize $g(x;\delta_i)$. By Theorem 5, x_1 does not define a symmetric NE and $\bar{\pi}(x_1) > \pi(x_1)$. It follows that

$$\frac{1-\delta_2}{\delta_2}\,\underline{r}\,(\delta_2) = \max g(x;\delta_2) \geq g(x_1;\delta_2) > g(x_1;\delta_1) = \frac{1-\delta_1}{\delta_1}\,\underline{r}\,(\delta_1).$$

Hence, $\frac{1-\delta}{\delta}\,\underline{r}(\delta)$ is strictly increasing in δ. From the first order conditions to the problem $\max g(x;\delta_i)$, it is clear that $x_1 \neq x_2$ and

$$g(x_1;\delta_1) > g(x_2;\delta_1)$$
$$g(x_2;\delta_2) > g(x_1;\delta_2)\,.$$

Noting that $\frac{1-\delta_1}{\delta_1} > \frac{1-\delta_2}{\delta_2}$, these inequalities may be seen to imply $\pi(x_2) > \pi(x_1)$. Hence, $\frac{1-\delta}{\delta}\,\bar{r}(\delta)$ is strictly increasing in δ. Finally, observe that $\pi(x^m) \geq \pi(x) - \frac{1-\delta}{\delta}\,(\bar{\pi}(x) - \pi(x))$ for all x and δ, and $\lim_{\delta\to 1} [\pi(x^m) - \frac{1-\delta}{\delta}\,(\bar{\pi}(x^m) - \pi(x^m))] = \pi(x^m)$.

Q.E.D.

The limit results of Theorem 6 are in the spirit of Proposition 3 of Pearce (1987). Our additional structure yields the new result that $\frac{1-\delta}{\delta}\,\bar{r}(\delta)$ is strictly monotonic.

Recall the punishment path $\vec{\underline{x}}$ constructed in the proof of Theorem 2, and the interpretation in Theorem 4 of x^* forever, as the most collusive CBE path. Then $\vec{\underline{x}}$ has two phases: an initial phase of low payoffs (the "stick") followed by a phase of the highest (renegotiation proof) payoffs (the "carrot") available. This is analogous to the symmetric "stick–and–carrot" punishments of Abreu (1986).

Under an additional assumption this structure may be expressed more crisply : phase 1 consists of exactly one period.

(A8) *If* $\delta(\pi(x^{cn}) + \bar{r}) > \underline{r}$, *there exists* $y \in S_1$ *such that* $\delta(\pi(y) + \bar{r}) = \underline{r}$ *and* $\bar{\pi}(y) \leq \bar{\pi}(x^{cn})$.

DEFINITION: For all $x_1, x_2 \in S_1$, let $\xi(x_1,x_2)$ denote the symmetric path in which x_1 is played in the first period, and x_2 in all subsequent periods.

Let x^* be as defined in Theorem 4.

THEOREM 7: *There exists* $\underline{x}_1 \in S_1$ *such that* $\hat{\sigma}(\xi(\underline{x}_1,x^*), \xi(\underline{x}_1,x^*))$ *is a CBE and* $v_1(\xi(\underline{x}_1,x^*)) = \underline{r}$.

PROOF: If the hypothesis of (A8) is false, then in the proof of Theorem 2, $T = 0$ and we may set $\underline{x}_1 = a$, as defined there. If not, let y be as in (A8) and set $\underline{x}_1 = y$. Then $\delta(\bar{\pi}(\underline{x}_1) + \underline{r}) - \delta(\pi(\underline{x}_1) + \bar{r}) \leq \delta(\pi(x^{cn}) + \underline{r}) - \underline{r} < 0$ since $\delta(\pi(\underline{x}_1) + \bar{r}) = \underline{r}$ and $\underline{r} > \frac{\delta}{1-\delta}\pi(x^{cn})$. Hence $\bar{\pi}(\underline{x}_1) - \pi(\underline{x}_1) \leq \bar{r} - \underline{r}$. Therefore, $\vec{z} \equiv \xi(\underline{x}_1,x^*)$ is supportable by $\underline{r}$, and by (R2), $\hat{\sigma}(\vec{z},\vec{z})$ is an equilibrium. Also, $v_1(\vec{z}) = \underline{r}$ and $v_1(\vec{z};t) = \bar{r}$, $t = 2,3,\ldots$. Thus, $\hat{\sigma}(\vec{z},\vec{z})$ is a CBE, and the proof is complete.

Q.E.D.

Theorems 2 and 4 emphasize how easily the best and worst renegotiation proof *payoffs* may be computed. The simple stick–and–carrot structure of the associated *strategies* is given a sharp expression in Theorem 7.

2.4 AN APPLICATION: MULTIMARKET CONTACT

A determinant of the degree of collusion in a particular market is the extent to which the participants interact in other markets. If the markets are *inherently* linked this inter–dependence is, of course, unavoidable. However, even if they are not, the markets might nevertheless be coupled by *strategic* interactions: a deviation in one market could be responded to in several markets. This issue has been formally addressed in recent papers by Bernheim and Whinston (1987) and Harrington (1986) in the context of infinitely and finitely repeated game models, respectively. The treatment below follows Bernheim and Whinston. They deal with a number of different types of multimarket contact. One they analyze fairly extensively is the case where each of the different markets is symmetric. Given the setting of the present paper this is the only case we address. A key insight of their work is that it may be beneficial to *pool* incentive constraints. This is helpful both for optimal collusion and optimal (i.e., worst) *punishment.* With renegotiation as conceived here, the objective is to *maximize* the *lowest* payoff. If pooling constraints help, it is conceivable that multimarket contact serves to improve the lowest payoff by so much that the most collusive payoff actually *declines.* In fact, it turns out that with renegotiation, multi–market contact (when markets are symmetric) has absolutely no impact. We proceed to details.

Consider n firms, $i = 1, \ldots, n$, each of which operates in two markets, a and b. The issue we wish to consider is whether a less (or more) collusive outcome prevails if the two markets are completely segmented in the sense that the set of firms in market a is disjoint from that in market b.

Let $G^k = (S_1^k, \ldots, S_n^k; \Pi_1^k, \ldots, \Pi_n^k)$, where G^k, $k = a,b$ satisfies (A1) to (A7). In the multi–market situation where the firms in the two industries have the same identity, we have a single game: $G = (S_1, \ldots, S_n; \Pi_1, \ldots, \Pi_n)$ with

$$S_i = S_i^a \times S_i^b, \; s_i = (s_i^a, s_i^b),$$

$$\Pi_i(s_1,\ldots,s_n) = \Pi_i^a(s_1^a,\ldots,s_n^a) + \Pi_i^b(s_1^b,\ldots,s_n^b)$$

$$\overline{\Pi}_i(s_1,\dots,s_n) = \overline{\Pi}_i^a(s_1^a,\dots,s_n^a) + \overline{\Pi}_i^b(s_1^b,\dots,s_n^b)$$
$$\overline{\pi}(x_a,x_b) \equiv \overline{\pi}^a(x_a) + \overline{\pi}^b(x_b)\,.$$

Since S_1 is now a subset of $\mathbb{R}^2$, we need to reformulate (A7). In particular, "$\overline{\pi}'(x^m)$, $\overline{\pi}'(x^{cn}) \neq 0$" needs to be replaced by "$\frac{\partial\overline{\pi}(x^{cn})}{\partial x_k}$, $\frac{\partial\overline{\pi}(x^m)}{\partial x_k} \neq 0$ for $k = a,b$." With these changes, if G^a, G^b satisfy (A1) to (A7) then so does G, and all our results apply. In particular, Theorem 2 implies

$$\underline{r} = \max_x f(x) = \max_{x_a} f^a(x_a) + \max_{x_b} f^b(x_b) = \underline{r}_a + \underline{r}_b\,,$$

and by Theorem 4

$$\overline{r} = \frac{\delta}{1-\delta}\,(\pi^a(x_a^*) + \pi^b(x_b^*)) = \overline{r}_a + \overline{r}_b\,.$$

Thus, in symmetric markets the optimal collusion problem is completely separable when players are able to renegotiate with equal bargaining power.

If the markets are structurally independent (in terms of firm cost functions and industry demand functions) then they are strategically independent also.

3. IMPERFECT MONITORING

In repeated games with perfect monitoring, some histories of play discriminate sharply amongst the participants: perhaps one player has deviated from cooperative behavior, while all others have conformed to some agreement. Evidence distinguishing one player from another tends to be less conclusive in models in which publicly observed signals are only stochastically related to players' private decisions. This suggests that there is if anything less reason to treat players asymmetrically after certain histories in imperfect monitoring models than in supergames with perfect monitoring. We show that on the contrary, consistent bargaining equilibria under imperfect monitoring will commonly violate strong symmetry, unlike their counterparts in perfect monitoring environments. In other words, sometimes players find it in their interest to submit gracefully to asymmetric treatment.

The results presented below are chosen with the intention of conveying as succinctly as possible the idea that asymmetric continuation payoffs arise quite naturally despite the presumption of equal bargaining power. No attempt is made to describe the specific structure of optimally collusive equilibria.

The model is a repeated partnership. We set out the model and notation below, emphasizing only those aspects which do not overlap with Section 2. It is now assumed that player i selects an action in period t from a *finite* set S_i. His choice $s_i(t)$ is unobservable to $j \neq i$ but the realization of a random variable $\theta(t)$ is publicly observed at the end of period t. The signal θ can take one of m values $\theta_1,\ldots,\theta_m$, and $p_i(s)$ denotes the probability of signal θ_i given the action profile $s \in S$. We assume that θ has *constant support*: $p_i(s) > 0$ for all $i = 1,\ldots,m$ and $s \in S$. Player i's payoff $\Pi_i(s(t))$ in the component game in period t is the expectation of his realized payoff $\pi_i(s_i(t), \theta(t))$. Thus a player cares about the unobserved actions of others only insofar as these determine the distribution of the payoff relevant signal θ. The component game is symmetric in that $S_i = S_1$ for all i, and $\Pi_{\tau(i)}(s_1,s_2,\ldots,s_n) = \Pi_i(s_{\tau(1)},s_{\tau(2)},\ldots,s_{\tau(n)})$ for any permutation τ of $\{1,\ldots,n\}$.

A strategy σ_i for player i in the repeated game is a sequence of measurable functions $\sigma_i(1), \sigma_i(2),\ldots$, where $\sigma_i(1) \in S_1$ and for $t \geq 2$, $\sigma_i(t) : (S_i \times \Theta)^{t-1} \to S_i$, where $\Theta = \{\theta_1,\ldots,\theta_m\}$. Let $\psi(\sigma)$ denote the vector of (expected) present discounted payoffs when the strategy profile σ is used. It is convenient to work with *average* discounted payoffs $v(\sigma) = \frac{1-\delta}{\delta}\psi(\sigma)$. Let $p(s) = (p_1(s),\ldots,p_m(s))$ and $P^i(s) = \left\{ p(s_i', s_{-i}) \mid s_i' \neq s_i, s_i' \in S_i \right\}$. Denote by $G^\infty(\delta)$ the repeated game with discount factor δ. Fudenberg, Levine and Maskin (1988) have introduced the following condition.

DEFINITION: The stage game satisfies the *pairwise full rank condition* if the vectors $\left\{ p(s) \right\} \cup P^1(s) \cup P^2(s)$ are linearly independent for all $s \in S$.

We will assume that:

(M1) *The stage game has a symmetric Nash equilibrium in pure strategies.*

(M2) *For all* $s^* \in argmax \sum_i \Pi_i(s)$, *there exists* j *such that* $\overline{\Pi}_j(s^*) > \Pi_j(s^*)$.

(M3) *The stage game satisfies the pairwise full rank condition.*[4]

We first record a simple corollary of a folk theorem of Fudenberg, Levine and Maskin (1988) which implies that a symmetric first–best payoff can be approached in equilibrium as δ tends to 1. Related results are given by Matsushima (1988) and, in a static setting, Radner and Williams (1987).

The pairwise full rank condition is required for the result below. Let $e = (1,\ldots,1) \in \mathbb{R}^n$, $\pi^* = \frac{1}{n} \max_s \sum_i \Pi_i(s)$, and $V(\delta) = \{v(\sigma) \mid \sigma$ is a (perfect Bayesian) equilibrium of $G^\infty(\delta)\}$.

PROPOSITION 1 (*Fudenberg, Levine and Maskin, 1988*): *For all* $\varepsilon > 0$ *there exists* $\underline{\delta}$ *such that for all* $\delta \geq \underline{\delta}$ *there exists* $u \in V(\delta)$ *with* $\mid \pi^* \cdot e - u \mid \leq \varepsilon$.

[4] It might seem at first sight that the pairwise full rank condition is inconsistent with symmetry. This is emphatically not the case. To construct a simple example, let $\theta = (\xi_1,\ldots,\xi_n)$ be a multi–dimensional random variable where the ξ_i's are independent, and the probability distribution of each ξ_i depends only on s_i. Of course, in a non–trivial example π_i will depend on all the ξ_j's.

The preceding theorem is easily translated to a statement about consistent bargaining equilibria by appealing to a limiting characterization of renegotiation–proof equilibria.

PROPOSITION 2 *(Pearce, 1987): For all* δ', $u \in V(\delta')$ *and* $\varepsilon > 0$ *there exists* $\underline{\delta}$ *such that for all* $\delta \geq \underline{\delta}$ *there exists an S.E.* σ *such that for all* i, $\inf_h v_i(\sigma | h) \geq u_i - \varepsilon$.

The next result on consistent bargaining equilibria is an immediate corollary of the preceding propositions.

COROLLARY: *For all* $\varepsilon > 0$ *there exists* $\underline{\delta}$ *such that for all* $\delta \geq \underline{\delta}$ *there exists a consistent bargaining equilibrium* $\tilde{\sigma}$ *with* $\ell(\tilde{\sigma}) \geq \pi^* - \varepsilon$.

We now show that restricting attention to *strongly symmetric* equilibria bounds payoffs away from efficiency uniformly in δ. Our main result then follows immediately. Theorem 3 is closely related to the work of Radner, Myerson and Maskin (1986) and various subsequent papers (see, for example, Abreu, Milgrom and Pearce (1988) and Fudenberg, Levine and Maskin (1988)).

THEOREM 8: *There exists* $\Delta > 0$ *such that for all* δ *and for all strongly symmetric sequential equilibria* σ *of* $G^\infty(\delta)$, $v_1(\sigma) \leq \pi^* - \Delta$.

PROOF: Fix δ. Let σ be a payoff maximal strongly symmetric (sequential) equilibrium. By Corollary 2 of Abreu, Pearce, and Stacchetti (1986) (hereafter APS), such a σ exists. Let $\bar{v} \equiv v_1(\sigma)$. As argued in APS we may w.l.o.g. take players' behavior in σ to be a function only of publicly observed outcomes and not on the history of their own past choices. This implies in particular that $\sigma|_\theta$, the behavior induced by σ after a first–period outcome θ, is an equilibrium. Hence, $\underline{v} \leq w_\theta \leq \bar{v}$, where $w_\theta = v_1(\sigma|_\theta)$ and $\underline{v}$ is the worst strongly symmetric equilibrium payoff. Let $\alpha_\theta \in (0,1)$ be defined by $w_\theta = \bar{v} - \alpha_\theta(\bar{v}-\underline{v})$.

Let $\sigma_1(1) = x$ be the action played in the first period. Then $\bar{v} = (1-\delta)\pi(x) + \delta \sum_\theta p_\theta w_\theta$, where p_θ is the probability of the outcome θ when all players use action x. Suppose $x \cdot e \notin \mathrm{A} \equiv \operatorname{argmax} \sum_i \Pi_i(s)$. Then $\pi(x) < \pi^*$, and $\bar{v} \leq (1-\delta)\pi(x) + \delta\bar{v}$, or $\bar{v} \leq \pi(x) < \pi^*$. Hence $\bar{v} \leq \pi^* - \Delta_1$, where $\Delta_1 = \min \left\{ \pi^* - \pi(x) \mid x \cdot e \in S \setminus \mathrm{A} \right\} > 0$. (Recall that S_1 is a finite set.) Now suppose $x \cdot e \in \mathrm{A}$. Then by (M2) there exists $s_1 \neq x$ such that $\Pi_1(s_1,x,\ldots,x) > \Pi_1(x,\ldots,x)$. Let $g > 0$ be the difference between these payoffs. Let q_θ be the probability of outcome θ when the action profile is $(s_1,x,\ldots,x)$. Since σ is an equilibrium, it follows that

$$g \leq \frac{\delta}{1-\delta} \sum_\theta (p_\theta - q_\theta)(\bar{v} - \alpha_\theta(\bar{v}-\underline{v})) = \frac{\delta}{1-\delta}(Q-P)(\bar{v}-\underline{v})$$

where $P = \sum \alpha_\theta p_\theta$, $Q = \sum \alpha_\theta q_\theta$. Let $m = \frac{Q-P}{P}$. Then $(1-\delta)\frac{g}{m} \leq \delta(\bar{v}-\underline{v})P$. By the constant support assumption, $P > 0$. Together with the finite outcome, finite action assumption it follows that m ($(m+1)$ is a likelihood ratio) is bounded above by some finite $\overline{m}$, independent of α_θ, $x \cdot e \in \mathrm{A}$, and the profitable deviation s_1. But $\bar{v} = (1-\delta)\pi^* + \delta \sum p_\theta [\bar{v} - \alpha_\theta(\bar{v}-\underline{v})]$. Therefore, $(1-\delta)\bar{v} = (1-\delta)\pi^* - \delta P(\bar{v}-\underline{v}) \leq (1-\delta)\pi^* - (1-\delta)\frac{g}{\overline{m}}$. Hence $\bar{v} \leq \pi^* - \frac{g}{\overline{m}}$. Let $\Delta_2 = g / \overline{m} > 0$. Set $\Delta = \min\{\Delta_1, \Delta_2\}$ to complete the proof.

Q.E.D.

It is now easy to argue that equal bargaining power need not imply strong symmetry; in many supergames *none* of the consistent bargaining equilibria is strongly symmetric. In the notation of Theorem 8 $\ell(\sigma) \leq \pi^* - \Delta$ for every strongly symmetric equilibrium σ of $G^\infty(\delta)$ (regardless of the values of δ). Setting $\varepsilon = \frac{\Delta}{2}$ in the Corollary to Propositions 1 and 2, we find a $\underline{\delta} < 1$ such that for all $\delta \geq \underline{\delta}$ there exists an equilibrium $\tilde{\sigma}$ with $\ell(\tilde{\sigma}) \geq \pi^* - \frac{\Delta}{2}$. That is, $\ell(\tilde{\sigma}) > \ell(\sigma)$ for any strongly symmetric equilibrum σ, therefore no CBE can be strongly symmetric. This result for patient players is recorded in Theorem 9.

THEOREM 9: *Under (M1), (M2) and (M3) there exists* $\underline{\delta} < 1$ *such that for all* $\delta \geq \underline{\delta}$ *there exists no strongly symmetric consistent bargaining equilibrium.*

The benefit of treating players asymmetrically after certain histories is easily explained. Players' incentives to cooperate depend upon their payoffs varying with the realizations of the random signal. In a strongly symmetric regime this means that surplus is systematically thrown away; with a finite signal space (or whenever the relevant likelihood ratios are bounded above) there is consequently an inescapable efficiency loss. Relaxing symmetry introduces the possibility of passing surplus from player to player instead of destroying it. Thus, rather than punish both players in a two–person game whenever certain signals arise, one may reward one player at the expense of the player whose "record" is less favorable. Of course, this is not possible if there is no information in the signal that distinguishes the two players, in which case consistent bargaining equilibria are again likely to exhibit strong symmetry.

4. CONCLUSION

This paper suggests a particular approach to the problem of renegotiation in symmetric repeated games. The theory developed is based on the idea that players will tolerate asymmetries in continuation payoffs precisely to the extent that even the worst–off player in any subgame finds this to be in his interests. In games with imperfect monitoring, our solution concept often leads to asymmetric continuation payoffs, despite

the equal bargaining power of the players. But we give conditions under which solutions of games with perfect monitoring are strongly symmetric. Under these conditions we provide simple formulae for the computation of the most collusive credible equilibrium and the value of the severest credible punishment. In contrast to the traditional theory without renegotiation, severest punishment paths here take an almost naively intuitive form: following a number of periods of "Cournot–Nash reversion," play returns to (constrained) maximal collusion. A variety of oligopolistic models are shown to satisfy the required conditions. Cournot oligopolies with modest restrictions are in this class; the linear Cournot supergame is solved fully in closed form, for all discount factors and any number of firms. Finally, the theory has strong, simple implications for collusion amongst firms having multi–market contact.

APPENDIX

The purpose of this appendix is to demonstrate briefly that the assumptions of Section 2 will be satisfied in a range of natural repeated economic models. There is no attempt at generality. We consider, in turn, oligopolistic supergames with quantities and prices, respectively, as strategic variables.

A. Quantities

We consider here a class of quantity setting oligopolistic supergames of the sort studied in Abreu (1986). Identical firms produce a homogeneous product at constant marginal cost $c > 0$. The industry inverse demand function is denoted p. Then $\Pi_i(s_1,\dots,s_n) = (p(\Sigma s_j)-c)s_i$, where s_i is the output of firm i.

Under reasonable assumptions this model fits into the framework above. These assumptions are:

(Q1) $p : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous and strictly decreasing. Also,* $\lim_{z\to 0} p(z) > c$, *and* $\lim_{z\to\infty} p(z) = 0$.

(Q1) implies that there exists $\overline{M}(\delta)$ such that: $-\Pi_1(\overline{M}(\delta),0,\dots,0) > \frac{\delta}{1-\delta} \sup_z \Pi_1(z,0,\dots,0)$.

The loss to a firm from producing an output of $\overline{M}(\delta)$ or more cannot be recouped by any possible future gain. Thus w.l.o.g. we may restrict firms to output choices in the interval $[0,\overline{M}(\delta)]$.

(Q2) $S_i = [0, \overline{M}(\delta)] \qquad i = 1,\dots, n.$

(Q3) *$G = (S_1, \ldots, S_n; \Pi_1, \ldots, \Pi_n)$ has a symmetric pure strategy Nash equilibrium.*

(Q1) to (Q3) imply (A1) to (A3) of the previous section. They also imply (A4), since $\pi(\frac{1}{n}\Sigma_i s_i) = \frac{1}{n}\Sigma_i \Pi_i(s)$, and by Lemma 21 of Abreu (1986) $\overline{\Pi}_i$ is a convex function. If in addition we assume that π is concave (A5) holds.

(Q4) *π is concave*

Sufficient conditions (on primitives) for (Q4) are that the demand function is linear, or has constant elasticity greater than unity. Corresponding to (A6) we now have:

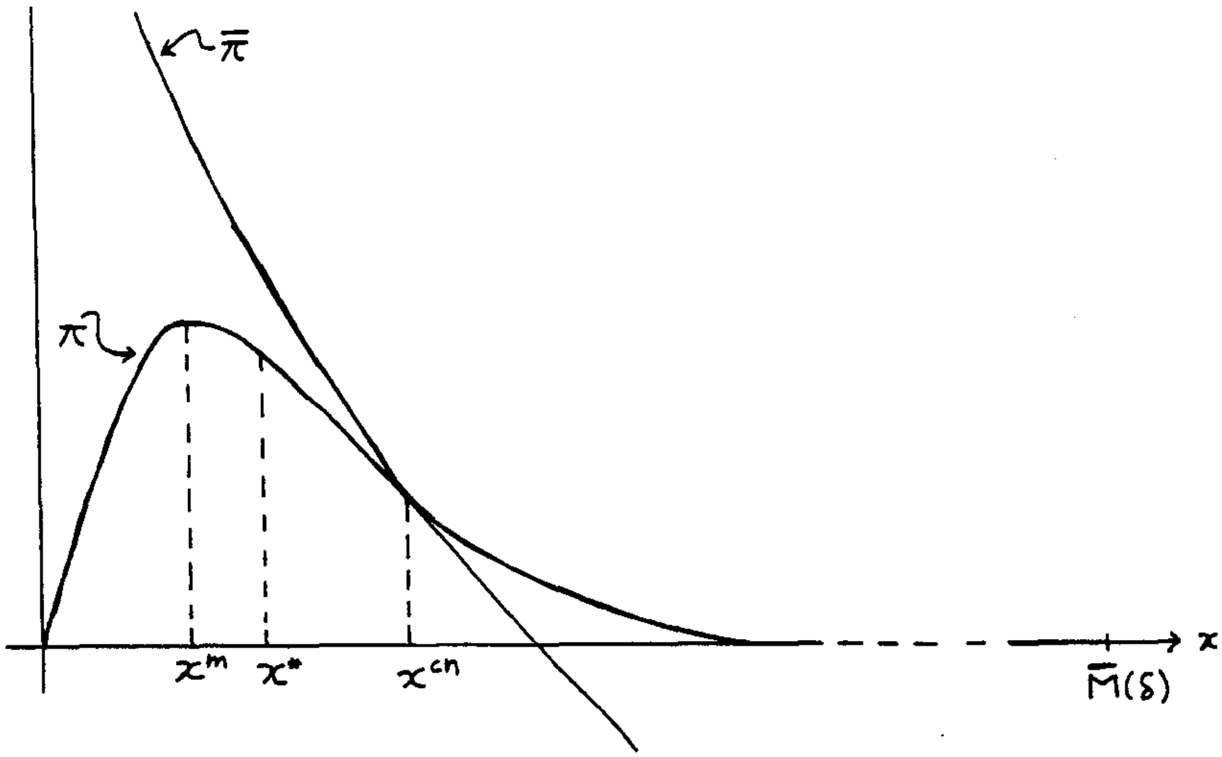
(Q5) *π and $\overline{\pi}$ are continuously differentiable.*

If the inverse demand function is differentiable then, of course, so is π; since $\overline{\pi}$ is convex it follows that it is differentiable almost everywhere.

(Q1) to (Q3) imply $\pi(x^{cn}) > 0$. Also $\pi(0) = 0$ and $\pi(\overline{M}(\delta)) < 0$. Together with (Q5), Corollary 3 (reproduced below) and Lemma 21 of Abreu (1986), (A7) is implied.

COROLLARY 3 (Abreu 1986): Let $x_2 > x_1 \geq 0$. Then $\overline{\pi}(x_1) = \overline{\pi}(x_2) = 0$, or $\overline{\pi}(x_1) > \overline{\pi}(x_2)$.

Finally (A8) follows from Corollary 3 (above) and the fact that if $\delta[\pi(x^{cn}) + v_2] > v_1 \geq 0$, we may always choose $y \geq x^{cn}$ such that $\delta[\pi(y) + v_2] = v_1$. Thus, the symmetric oligopolistic quantity–setting supergame with assumptions (Q1) to (Q5) satisfies all the assumptions of the previous section. The relevant picture is:



For illustrative purposes we compute a linear example. The inverse demand function is:

$$p(z) = \begin{cases} \alpha - \beta z & z \leq \alpha/\beta \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha, \beta > 0$ and $\alpha > c$. Then

$$\pi(x) = (\alpha - c - n\beta x)x \text{ when } x \leq \frac{\alpha - c}{n\beta} \text{ and zero otherwise, and}$$

$$\bar{\pi}(x) = \frac{1}{4\beta}(\alpha - c - (n-1)\beta x)^2 \text{ when } x \leq \frac{\alpha - c}{(n-1)\beta} \text{ and zero otherwise.}$$

By definition, $f(x) = \frac{1}{1-\delta}\pi(x) - \bar{\pi}(x)$, and by Theorem 2, $\underline{r} = \max f(x)$. This problem has a unique maximum at $x^* = \frac{\alpha - c}{\beta}\left[\frac{(n+1) - \delta(n-1)}{4n + (1-\delta)(n-1)^2}\right]$. Hence

$\underline{r} = \frac{\delta}{1-\delta}\frac{(\alpha - c)^2}{\beta}\frac{1}{4n + (1-\delta)(n-1)^2}$. By Theorem 4, $\bar{r} = \frac{\delta}{1-\delta}\pi(x^*)$. That is,

$\bar{r} = \frac{\delta}{1-\delta}\frac{(\alpha - c)^2}{\beta}\frac{(n+1)^2 - \delta^2(n-1)^2}{(4n + (1-\delta)(n-1)^2)^2}$. It follows directly that in agreement with Theorem 6,

$$\lim_{\delta\to 1}\frac{1-\delta}{\delta}\,\underline{r}(\delta) = \lim_{\delta\to 1}\frac{1-\delta}{\delta}\,\overline{r}(\delta) = \frac{(\alpha-c)^2}{4n\beta} = \pi(x^m)\,.$$

B. Prices

We now consider price–setting supergames with differentiated products. We provide two examples which satisfy our assumptions and permit us to use the characterizations provided above. In particular, it is trivial to compute explicit solutions for $\underline{r}$, $\overline{r}$, and the collusive price level p^*. The examples below consider the polar cases of substitutes and complements, respectively.

A Spatial Model

There are two firms located at the poles of a circle of unit length. They produce at constant marginal cost $c > 0$. Consumers are uniformly distributed, demand at most one unit, and have a reservation value v for either firm's product. Consumers pay the transportation cost t per unit of distance.

The analysis of this model is straightforward but can become tedious. To simplify we assume:

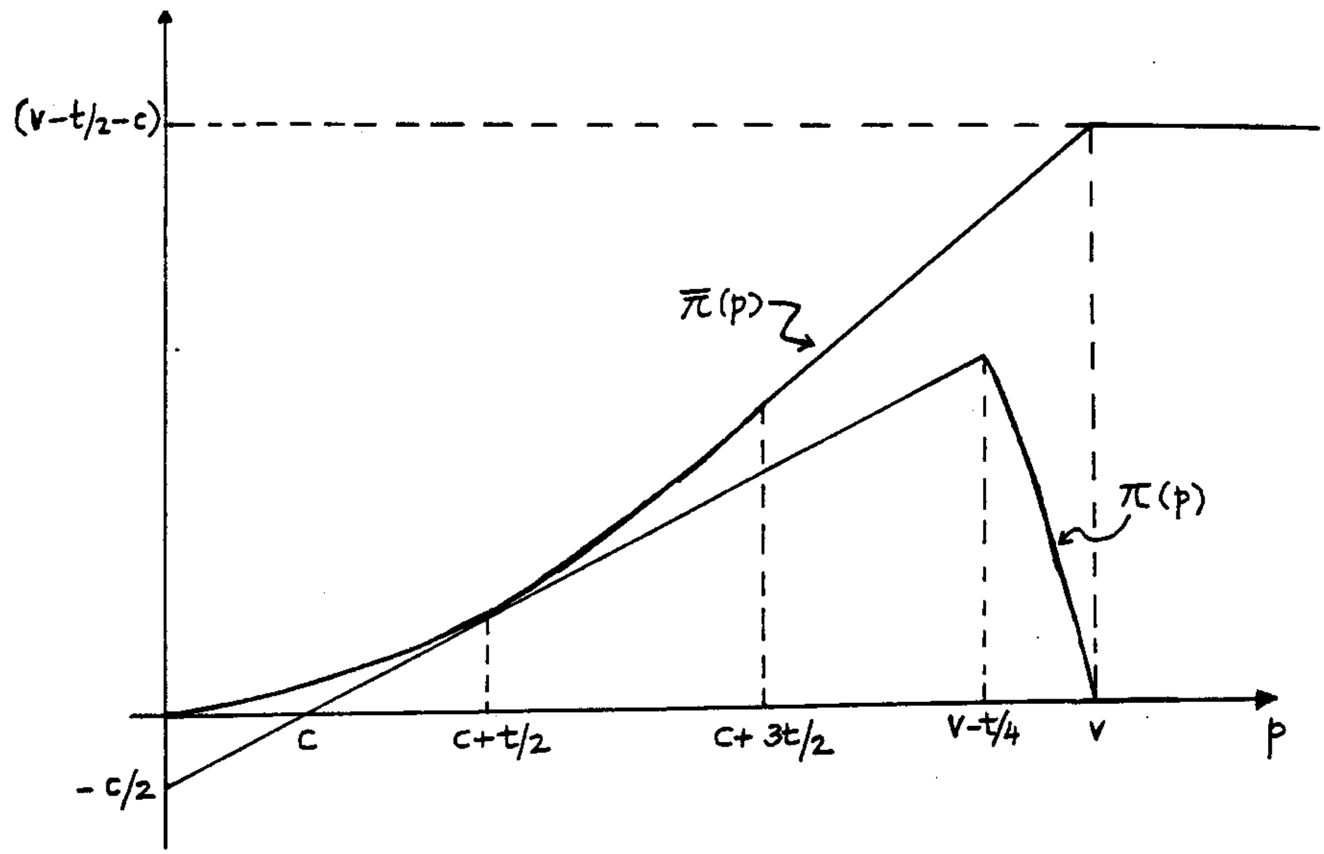
$$\frac{t}{2} \geq c \text{ and } v \geq \frac{3t}{2} + c\,.$$

Firms are constrained to choose prices in the range $[0,v]$.

$$\pi(p) = \begin{cases} (p-c)/2 & 0 \leq p < v - \frac{t}{4} \\ \frac{2}{t}(v-p)(p-c) & v - \frac{t}{4} \leq p \leq v \end{cases}$$

$$\overline{\pi}(p) = \begin{cases} \frac{1}{4t}(p + \frac{t}{2} - c)^2 & 0 \leq p \leq c + \frac{3t}{2} \\ (p - \frac{t}{2} - c) & c + \frac{3t}{2} \leq p \leq v \end{cases}$$

It is helpful to plot these functions:



The symmetric NE of this stage game is of course: $p^{cn} = c + \frac{t}{2}$. Assumptions (A1) to (A3) are obviously satisfied. It is diagramatically clear that (A5) is satisfied also. Somewhat involved but straightforward calculations show that if $p = \frac{p_1+p_2}{2}$ then $\pi(p) \geq \frac{1}{2}\Pi_1(p_1,p_2) + \frac{1}{2}\Pi_2(p_1,p_2)$. Since $\bar{\pi}$ is convex, $\bar{\pi}(p) \leq \frac{1}{2}\bar{\pi}(p_1) + \frac{1}{2}\bar{\pi}(p_2) = \frac{1}{2}(\bar{\Pi}_2(p_1,p_2) + \bar{\Pi}_1(p_1,p_2))$. These two inequalities imply (A4). All the results up to and including Theorem 4 are therefore available. Thus, $\underline{r} = max\ f(p)$. For $\delta \leq \frac{1}{2}$, the optimum is unique and lies in the range $[c+\frac{t}{2}, c+\frac{3t}{2}]$. Precisely, $p^* = c + \frac{t(1+\delta)}{2(1-\delta)}$ and $\bar{r} = \frac{\delta}{1-\delta}\frac{t(1+\delta)}{4(1-\delta)}$. When $\delta \geq \frac{1}{2}$, $p^* = p^m = v - \frac{t}{4}$. That is, $\bar{r} = \frac{\delta}{1-\delta}\frac{1}{2}(v - \frac{t}{4} - c)$.

A Linear Model with Complements

Consider two firms which produce complementary products:

$$q^1 = \max\{0,\ \alpha-\beta p_1 - \gamma p_2\}$$

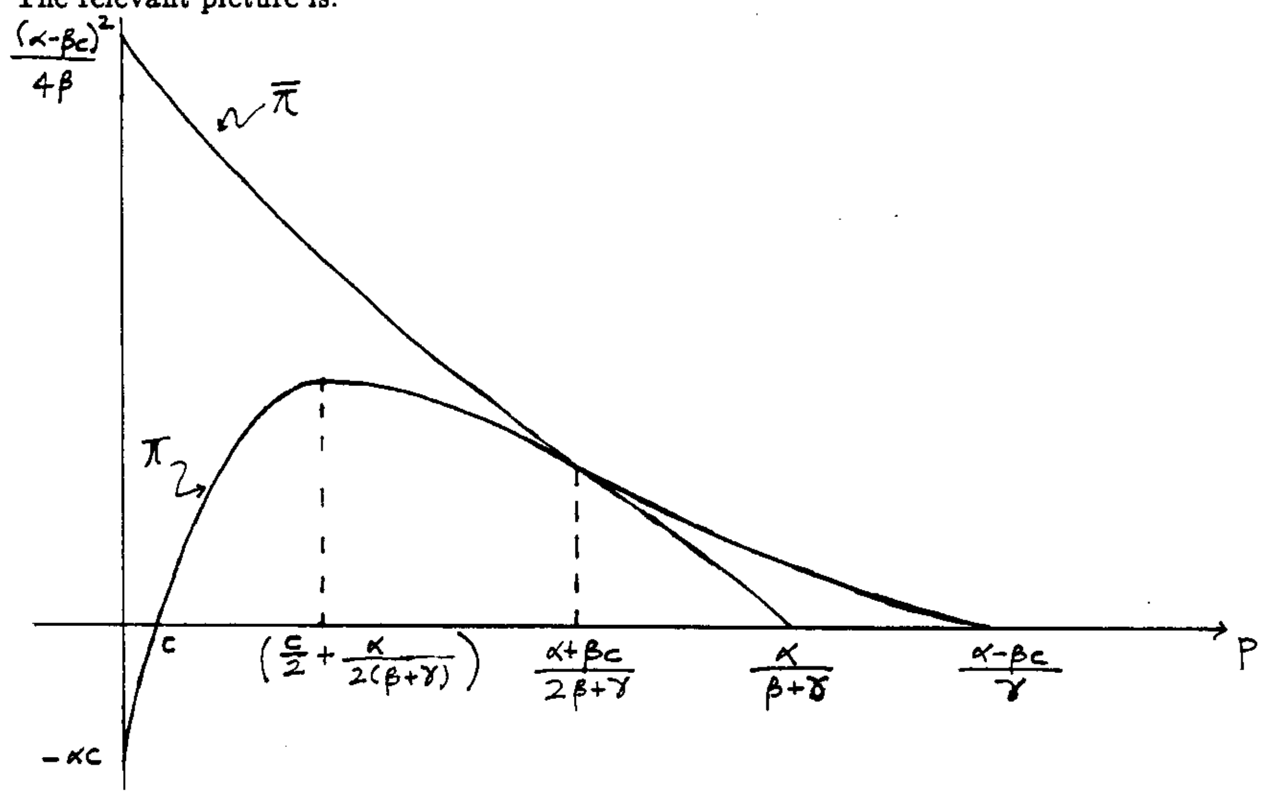
$$q^2 = \max\{0,\ \alpha-\beta p_2 - \gamma p_1\}$$

where $\alpha, \beta, \gamma > 0$, $\beta \geq \gamma$ and $\alpha > (\beta+\gamma)c$. Set $S_i = [0,\ \alpha/\beta]$. Then

$$\pi(p) = (\alpha - (\beta+\gamma)p)(p-c) \text{ if } p \leq \frac{\alpha}{\beta+\gamma} \quad \text{and zero otherwise.}$$

$$\bar{\pi}(p) = \frac{1}{4\beta}(\alpha-\beta c-\gamma p)^2 \text{ if } p \leq \frac{\alpha-\beta c}{\gamma} \quad \text{and zero otherwise.}$$

The relevant picture is:



It is clear that (A1) – (A3) and (A5) are satisfied. So also is (A4) (though this takes some argument). The by now familiar maximization of f yields

$$p^* = \frac{2\beta[\alpha+c(\beta+\gamma)] + (1-\delta)\gamma(\alpha-\beta c)}{4\beta^2 + 4\beta\gamma + \gamma^2(1-\delta)}.$$

As $\delta \to 1$, $p^* \to p^m = \frac{c}{2} + \frac{\alpha}{2(\beta+\gamma)}$, in conformity with Theorem 6.

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