COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

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Peter C. B. Phillips

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P. C. B. Phillips

Cowles Foundation for Research in Economics
Yale University

0. ABSTRACT

We offer a new and straightforward proof of F. B. Knight's [3] theorem that the Cauchy type is characterized by the fact that it has no atom and is invariant under the involution $i : x \to -1/x$. Our approach uses the representation $X = \tan \theta$ where θ is uniform on $(-\pi/2, \pi/2)$ when X is standard Cauchy. A matrix generalization of this characterization theorem is also given.

Key Words: Cauchy distribution; Involution; Matrix variate; Uniform distribution.

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1. INTRODUCTION

E. J. Williams [4] showed that X is standard Cauchy with density $f(x) = [\pi(1 + x^2)]^{-1}$ (we shall write $X \equiv C(0,1)$) iff

$$X \equiv \frac{1 + bX}{b - X} \tag{1}$$

where b is some constant which is not the tangent of a rational multiple of 2π. (In (1) and hereafter we use the symbol " $\equiv$ " to signify equivalence in distribution.) F. B. Knight [3] sharpened this result considerably by proving that X is of the Cauchy type (i.e. belongs to the equivalence class of distributions of $a + pX$, $p \neq 0$, where X is standard Cauchy) iff

$$\frac{aX + b}{cX + d} \equiv \alpha X + \beta$$

for some constants $\alpha \neq 0$ and β whenever $ad - bc \neq 0$. C. Hassenforder [2] pointed out that this is equivalent to the statement that elements of the type have no atoms and the type is invariant under the involution $x \to -1/x$. This leads to the central result:

1.1. **THEOREM**

$X \equiv C(0,1)$ i.e. X *is standard Cauchy*

iff

$X \equiv -1/X$

and the distribution has no atoms.

2. A NEW PROOF OF THEOREM 1.1

We offer a new and straightforward proof of Theorem 1.1. Our approach is to use the representation of $C(0,1)$ in terms of $X = \tan\theta$, where θ is uniform on $(-\pi/2, \pi/2)$. We have:

2.1. PROPOSITION. *The following statements are equivalent*:

(i) $X \equiv -1/X$ *and has no atoms;*

(ii) θ *is uniform on* $(-\pi/2, \pi/2)$;

(iii) $X \equiv C(0,1)$.

PROOF. Direct calculation shows that (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii). It remains to prove that (i) $\Rightarrow$ (ii). Observe that (i) implies that X is symmetric so that (i) also implies that $X \equiv 1/X$. The transformation $X = \tan\theta$ induces a unique (and, under (i), continuous) probability distribution for θ on $(-\pi/2, \pi/2)$. By replicating this distribution over $(\pi/2, 3\pi/2)$ and assigning probability mass 1/2 to each interval we obtain an induced distribution on the complete unit circle. Observe that since $X \equiv 1/X$ we have

$$\tan\theta \equiv \tan(\pi/2 - \theta) \tag{2}$$

Next, for any constant b we find by direct use of (i) that

$$Z = \frac{X + b}{1 - bX} \equiv -\frac{1 - bX}{X + b} = -\frac{1}{Z} .$$

This implies symmetry of Z and we have

$$Z \equiv 1/Z . \tag{3}$$

Let $b = \tan\beta$ for some $\beta \in (-\pi/2, \pi/2)$. Then (3) may be written in the equivalent form

$$\tan(\theta+\beta) \equiv \tan(\pi/2 - \theta - \beta) \ .$$

Since

$$\tan(\pi/2 - \theta - \beta) = \frac{\tan(\pi/2 - \theta) - \tan\beta}{1 + \tan(\pi/2)\tan\beta} \equiv \frac{\tan\theta - \tan\beta}{1 + \tan\theta\tan\beta} = \tan(\theta-\beta)$$

we deduce that

$$\tan(\theta+\beta) \equiv \tan(\theta-\beta) \ .$$

By reversing the transformation this implies that the distribution on the unit circle that is induced by θ is invariant under a rotation of 2β radians. Since β is arbitrary we deduce that the distribution on the unit circle is invariant to rotation and therefore θ is uniform on $(-\pi/2, \pi/2)$. Hence, (i) ⇒ (ii) and the proposition follows □.

2.2. REMARK

As observed in the proof, (i) implies that X is symmetrically distributed i.e. $X \equiv - X$. We may therefore write (i) in the equivalent form:

(i)′ $X \equiv Xk \equiv 1/X$, $k \in O(1)$.

Here, $O(n)$ represents the orthogonal group of order n so that since $n = 1$, $k = \pm 1$. This form is useful because it helps to suggest a multivariate generalization of the proposition.

3. A MULTIVARIATE EXTENSION

Let $X = (x_{ij})_{n\times m}$ be multivariate (matrix) Cauchy with density

$$f(X) = \left[\pi^{nm/2}\Gamma_n(n/2)\right]^{-1}\Gamma_n((n+m)/2)\,|I + XX'|^{-(n+m)/2} \ . \tag{5}$$

We shall write $X \equiv C_{n,m}(0,I)$. It is known that all submatrices of X

are distributed as matrix Cauchy (e.g. see Dawid (1)) and that X is spherically symmetric. These properties help us to characterize $C_{n,m}(0,I)$. We have:

3.1. PROPOSITION

$$X \equiv C_{n,m}(0,I) \quad \textit{iff}$$

(i) $X \equiv HXK$; $H \in O(n)$, $K \in O(m)$

(ii) $x_{ij} \equiv 1/x_{ij}$ *and has no atoms* $(i = 1,\ldots, n;\ j = 1,\ldots, m)$.

PROOF. If $X \equiv C_{n,m}(0,I)$ then (i) follows directly from the form of the density (5). Moreover, since all univariate marginals of X are $C(0,1)$, (ii) is a consequence of Theorem 1. To prove sufficiency, note that (i) implies X is spherically symmetric. This means that all of its marginal distributions are of the same type. This includes the univariate marginals. But, in view of (ii), we have

$$x_{ij} \equiv 1/x_{ij} \equiv -1/x_{ij} ,$$

the second distributional identity following from (i). Theorem 1 now implies that $x_{ij} \equiv C(0,1)$ (all i, j). It follows that X is of the Cauchy type and necessarily $X \equiv C_{n,m}(0,I)$ with density (5). □

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