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THE SHAPES OF POLYHEDRA

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by

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I. Introduction

Let A be a real matrix with n+d+1 rows, numbered 0,1,...,n+d, and with $n > 1$ columns. We assume that all nxn submatrices of A are non-singular and define the condition number $C = C(A)$ to be the ratio of the largest n×n subdeterminant to the smallest n×n subdeterminant of A in absolute value. In addition we assume that there is a positive vector π such that $\pi A = 0$. This implies that for any b, the body $K_b = \{x | Ax \le b\}$ is bounded.

Two such bodies K_b and K_c are said to have the same shape if one of them is a translation and expansion of the other, i.e. if there exists a vector ξ and a positive scalar λ such that $K_b = \lambda K_c + \xi$. We shall use a variant of the well known Banach-Mazur metric $\rho(K_b, K_c)$ on the set of non-empty, full dimensional bodies which is unchanged if either body is translated by an arbitrary vector or expanded by a positive factor; the metric gives a distance of zero if and only if the bodies have the same

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shape. The distance between two bodies will be small if their shapes are roughly the same. Our main result shows that for an arbitrary positive ϵ, there will be a finite subset, of small cardinality, of these bodies such that every body is within ϵ of some member of the subset. The result will then be applied to a version of Lenstra's algorithm to determine whether a convex body contains a lattice point, to study those bodies K_b which are free of lattice points and to obtain some new conclusions about neighborhood systems associated with the matrix A and Minkowski's successive minima for the family of symmetric bodies $(K_b - K_b)$.

We define a set S of the rows of A to be <u>dual feasible</u> if there is a vector π with $\pi_i > 0$ for $i \in S$, $\pi_i = 0$ for the remaining rows and $\pi A = 0$. Given our assumption that all nxn minors of A are non-singular, the cardinality of a set of dual feasible rows is at least n+1. The body K_b is a simplex if it is defined by a subset of n+1 of the inequalities whose corresponding rows are dual feasible; it is easy to see that two simplices defined by the same subset of n+1 rows are identical aside from translation and scaling. We denote the number of distinct simplices, or equivalently the number of minimal dual feasible sets of rows, by $f = f(A)$. Obviously $f \leq \binom{n+d+1}{n+1} = O(n^d)$ for fixed d; a more refined analysis, based on the Upper Bound theorem of McMullen (McMullen and Shepard,1971), would permit us to assert that $f = O(n^{d/2})$. In addition, we define $r = r(A)$ to be the number of rows of A whose complementary sets of n+d rows are dual feasible.

Our variant of the <u>Banach-Mazur distance</u> $\rho(K_b,K_c)$ for an arbitrary pair of non-empty full dimensional bodies K_b and K_c is defined as follows:

1.1 (Definition) Let λ_1 be the smallest λ for which $K_c \subseteq \lambda K_b + \xi^1$ for some ξ^1 and λ_2 the smallest λ for which $K_b \subseteq \lambda K_c + \xi^2$ for some ξ^2. Then

$\rho(K_b,K_c) = \log(\lambda_1 . \lambda_2)$.

It is easy to see that the distance function satisfies the following elementary properties:

1.2 (Lemma) 1. $\rho(K_b,K_c) \geq 0$ and is equal to 0 if and only if the two bodies are identical aside from translation and scaling.

2. $\rho(K_b,K_c) = \rho(\mu K_b + \xi, K_c)$ for positive μ and arbitrary ξ.

3. the triangle inequality: $\rho(K_b,K_d) \leq \rho(K_b,K_c) + \rho(K_c,K_d)$.

Proof: 1. Since $K_c \subseteq \lambda_1 K_b + \xi^1$ for some ξ^1 and $K_b \subseteq \lambda_2 K_c + \xi^2$ we see that $K_c \subseteq \lambda_1\lambda_2 K_c + (\xi^1 + \lambda_1\xi^2)$. Therefore $\lambda_1\lambda_2 \geq 1$ and is equal to 1 if and only if both inclusions are equalities.

2. Using the same notation we see that $K_c \subseteq (\lambda_1/\mu)(\mu K_b + \xi) + (\xi^1 - (\lambda_1/\mu)\xi)$ and $\mu K_b + \xi \subseteq \mu\lambda_1 K_c + (\mu\xi^1 + \xi)$ and therefore $\rho(\mu K_b + \xi, K_c) \leq \log(\lambda_1 . \lambda_2) = \rho(K_b,K_c)$. Equality is obtained by reversing the argument.

3. Let

$$K_b \subseteq \lambda_{b,c} K_c + \xi^{b,c},\ K_c \subseteq \lambda_{c,b} K_b + \xi^{c,b} \text{ and}$$
$$K_c \subseteq \lambda_{c,d} K_d + \xi^{c,d},\ K_d \subseteq \lambda_{d,c} K_c + \xi^{d,c}, \text{ so that}$$
$$\rho(K_b,K_c) = \log(\lambda_{b,c} . \lambda_{c,b}) \text{ and } \rho(K_c,K_d) = \log(\lambda_{c,d} . \lambda_{d,c}).$$

But then

$$K_b \subseteq \lambda_{b,c}(\lambda_{c,d} K_d + \xi^{c,d}) + \xi^{b,c} \text{ and}$$
$$K_d \subseteq \lambda_{d,c}(\lambda_{c,b} K_b + \xi^{c,b}) + \xi^{d,c}, \text{ so that}$$
$$\rho(K_b,K_d) \leq \log((\lambda_{b,c} . \lambda_{c,d}) . (\lambda_{d,c} . \lambda_{c,b})) = \rho(K_b,K_c) + \rho(K_c,K_d). \quad \square$$

Two bodies in the family K_b are identical, aside from translation and scaling, if their Banach-Mazur distance is 0; they are similar in shape if the distance is small. The major result of the paper will be to show that for any $\varepsilon > 0$, there exists a finite subset of the bodies K_b of cardinality not larger than

$$f(A)\lceil 2\log_2(nC)/\epsilon \rceil^d$$

such that every body is within ϵ of some member of the subset, using the Banach–Mazur measure of distance.

II. The Cone B

Let C be the set of vectors b in R^{n+d+1} for which the body K_b is non-empty. C is a polyhedral cone of dimension n+d+1 which contains the linear space L_A, of dimension n, spanned by the columns of A. Any two vectors in C whose difference lies in L_A yield bodies which are translates of each other. We define $B = C/L_A$, i.e., the set of equivalence classes of those b for which K_b is non empty, with two vectors identified if they differ by a linear combination of the columns of A. Since n dimensions are removed by this identification, B is a closed polyhedral cone of dimension d+1. A single representative from each equivalence class may be chosen in several ways: we may, for example, translate the bodies K_b so that n particular coordinates of b are equal to zero, or alternatively so that b is orthogonal to the columns of A.

The cone B lies in the vector space R^{n+d+1}/L_A. The dual of this sub-space can be thought of as the set of those linear functions, $\pi.b$, which are constant on each equivalence class, i.e., those linear functions with $\pi A = 0$. The dual cone of B, denoted B*, is the set of such linear functions with $\pi.b \geq 0$ for all b in B. It is easy to see that $B^* = \{\pi | \pi A = 0$ and $\pi \geq 0\}$. For if π, with $\pi A = 0$, is non negative, then certainly $\pi.b \geq 0$ for any b such that the inequalities $Ax \leq b$ yield a non empty set, i.e. for any equivalence class of vectors in B. Conversely for any such b, if $\pi.b \geq 0$ for all non negative π for which $\pi A = 0$ then, by the duality theorem, there is an x for which $Ax \le b$ and b belongs to B.

2.1 (Theorem) 1. B has f(A) facets. Each facet is defined by a subset S of n+1 rows of A which are dual feasible. The bodies corresponding to vectors b on this facet F_S are precisely those bodies K_b which may be translated so as to satisfy $b_i = 0$ for $i \in S$ and $b_i \ge 0$ for the remaining rows.

2. B has r(A) extreme rays. Each extreme ray corresponds to a row of A, say ℓ, such that the remaining n+d rows are dual feasible. The bodies corresponding to vectors b on the ray R_ℓ are precisely those which may be translated so as to satisfy $b_i = 0$ for all i different from ℓ and $b_\ell \ge 0$.

Proof: 1. For each S, the vectors in F_S, as defined above, form a subcone of B of dimension d. To show that they all lie on the boundary of B we argue as follows: By the definition of S, there exists a π with $\pi A = 0$, $\pi_i > 0$ for $i \in S$ and $\pi_i = 0$ otherwise. Any vector x satisfying $a_i x \le 0$ for $i \in S$ must then satisfy $a_i x = 0$ for these same rows and – by the nondegeneracy assumption – is therefore the zero vector. It follows that for b in F_S, K_b is the degenerate simplex defined by the n+1 rows of S and satisfying the remaining d inequalities, some of them strictly. It is on the boundary of B since a small perturbation will make this body empty. F_S is therefore a facet of B.

Conversely, any body K_b with b on the boundary of B must consist of a single point, since otherwise arbitrary small perturbations of b would retain feasibility of the system of inequalities $Ax \le b$. Let this point be translated to the origin. But then at least n+1 hyperplanes, defined by a dual feasible set of the rows of A, must pass through the origin and the body is contained in at least one facet F_S. This demonstrates that the union of the facets F_S is the entire boundary of B.

2. An extreme ray $\{\lambda b | \lambda \geq 0\}$ must lie on the boundary of B and, by the previous argument, must be contained in a facet F_S defined by n+1 rows of A which are dual feasible. After translation of K_b we may assume that $b_i = 0$ for $i \in S$ and $b_i \geq 0$ for the remaining coordinates. But if more than one of these remaining coordinates is strictly positive, b can be written as a convex combination of two non proportional vectors in the same facet and is not an extreme ray of B. It follows that only one of these remaining coordinates is strictly positive and the ray is as asserted in the statement of Theorem 2.1. □

2.2 (Theorem) B^* has r(A) facets and f(A) extreme rays. Each facet of B^* corresponds to a row of A, say ℓ, whose complementary set of rows is dual feasible. The facet consists of those non negative π's with $\pi_\ell = 0$ and $\pi A = 0$. Each extreme ray of B^* corresponds to a set of n+1 rows of A which are dual feasible. The vectors on the ray are those π with $\pi A = 0$, $\pi_i > 0$ for $i \in S$ and $\pi_i = 0$ for i not in S.

Theorem 2.2 follows from standard arguments relating the facets and extreme rays of a polyhedral cone to those of its dual.

III. An Example

Let us consider, as an example, the following matrix with 4 rows, numbered 0,1,2,3, and 2 columns:

$$A = \begin{bmatrix} -p & -q \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix},$$

with $p > q > 0$. The matrix has two sets of 3 rows which are dual feasible: rows 0, 2 and 3 with the dual vector $\pi^1 = (1,0,p-q,p)$ and rows 1,2 and 3 with the dual vector $\pi^2 = (0,1,1,1)$. By Theorem 2.2 the dual cone B^* has two extreme rays generated by non-negative multiples of these two vectors. The cone B consists of all b for which $\pi^1.b$ and $\pi^2.b$ are both greater than or equal to zero. If we translate the bodies K_b so that $b_2 = b_3 = 0$, the vectors in B consist of all pairs (b_0,b_1) in the non-negative quadrant.

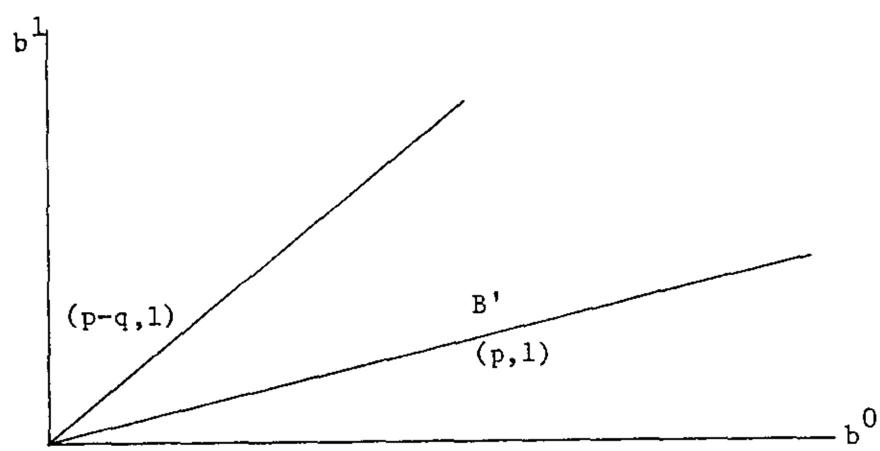

Figure 3.1

Figure 3.2 exhibits four examples of the bodies K_b, with the vectors $b = (b_0,1,0,0)$. If b_1 is taken to be a positive number different from 1, the bodies differ only by a scale factor.

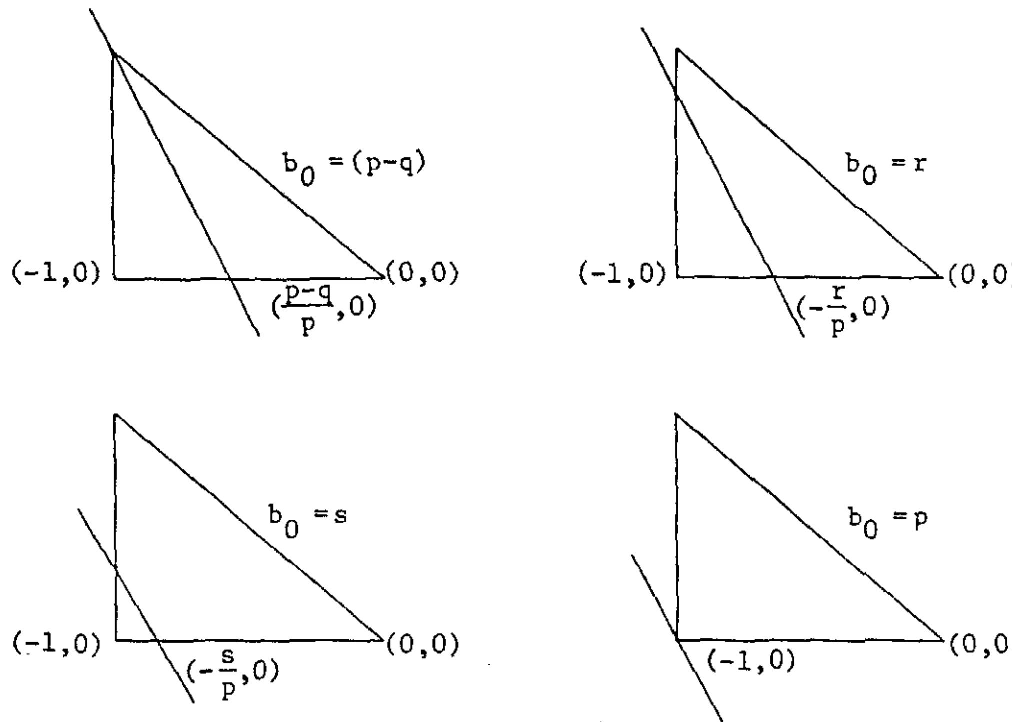

Figure 3.2

If b_0 is greater than p, inequality 0 is irrelevant; all of the corresponding bodies K_b will be precisely the same triangle. The bodies will be similar triangles, different from the former one, if b_0 is less than p–q. The full set of bodies K_b are, therefore, obtained by considering all non-negative multiples of the vectors $b = (b_0, 1, 0, 0)$ with $p-q \le b_0 \le p$: the subcone B′ of Figure 3.1.

Let us calculate the Banach–Mazur distance between the pair of bodies B_r and B_s, the first of which is given by $b_0 = r$ and the second by $b_0 = s$, with $p-q \le r < s \le p$. If λ_1 is the smallest λ for which $B_r \subseteq \lambda B_s + \xi^1$ for some ξ^1 and λ_2 the smallest λ for which $B_s \subseteq \lambda B_r + \xi^2$ for some ξ^2, then $\rho(K_b, K_c) = \log(\lambda_1 . \lambda_2)$. Clearly B_r is contained in B_s, but not in any translated, scaled down version of the latter body; therefore $\lambda_1 = 1$. Also B_s is contained in $(s/r)B_r$, but not in any smaller multiple of this body, even if translations are allowed. It follows that $\rho(K_r, K_s) = \log(s/r)$.

We remark that the maximum distance between any pair of bodies is $\log(p/(p-q))$. Moreover, if $\varepsilon > 0$ is given, two successive bodies in the sequence for which

$$\log(b_0/(p-q)) - j\varepsilon \text{ for } j = 0,1,\ldots,\lceil \log(p/(p-q))/\varepsilon \rceil - 1$$

will be within ε of eachother, and any particular K_b will certainly be within ε of at least one member of this subset of bodies. This conclusion exemplifies the main result of the paper.

IV. The Cone of Shapes

A body K_b with b on the boundary of B consists of a single point which may be taken to be the origin of R^n. At least n+1 hyperplanes, defined by a dual feasible set of rows of A, pass through the origin and the remaining linear inequalities are satisfied by the origin. Aside from the vertex of the cone, every vector on the boundary of B will therefore correspond to a system of inequalities some of which are redundant in the sense that they are implied by the remaining inequalities. Vectors close to the boundary will give rise to bodies which are full dimensional, but which are also defined by proper subsets of the rows of A. The distinct bodies K_b, for which none of the inequalities are <u>redundant</u>, are determined by the vectors b in the following subcone of B:

4.1 (Definition) B′ is defined to be the set of b ϵ B for which

$$\max\{a_i x | x \in K_b\} = b_i \text{ for } i = 0,1,\ldots,n+d.$$

The set B′ is clearly a convex cone, for if b and c are both in B′ then for $\lambda,\mu \geq 0$, $\lambda b_i+\mu c_i \geq \max\{a_i.x | x \in K_{\lambda b+\mu c}\} \geq \lambda\max\{a_i.x | x \in K_b\} + \mu\max\{a_i.x | x \in K_c\} = \lambda b_i + \mu c_i$. It follows that $\lambda b + \mu c$ is also in B′.

An important property of this cone is that for two vectors b and c, which are selections from a pair of equivalence classes in B', $K_b \subseteq K_c$ if and only if $b \le c$.

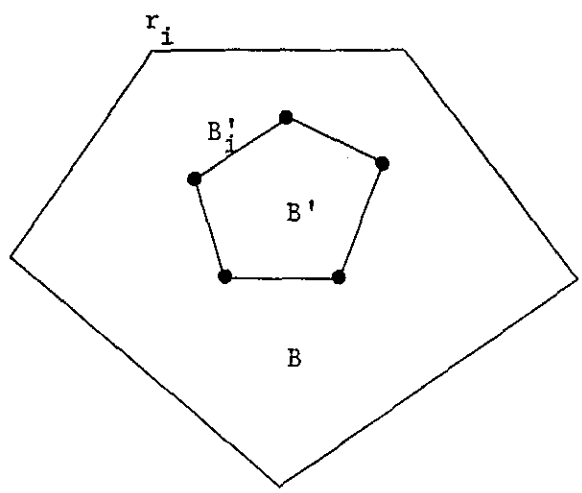

Figure 4.1

Figure 4.1 is an illustration, with d = 2, of the intersection of the cones B and B' with a hyperplane whose normal is interior to B^*.

A body K_b on the boundary of B' will be arbitrarily close to bodies for which some of the inequalities are redundant; it follows that a body on the boundary of B' will necessarily be described by a dual feasible proper subset of the inequalities with the remaining inequalities defining supporting hyperplanes to the body. The boundary of B' is therefore the union of r(A) subsets, B'_i, the ith of which consists of those bodies for which the ith inequality yields a supporting hyperplane. Except for simple examples each of these subsets may contain interior vertices and not be a facet of B'. It should be clear that a segment connecting an arbitrary point in B' with a point on the ith extreme ray of B will intersect B'_i.

Of particular interest are the simplices defined by a subset of n+1 dual feasible inequalities with the remaining inequalities representing hyperplanes which support the simplex at various of its vertices.

4.2 (Theorem) Let S be a dual feasible set of n+1 rows of A, and let b_i, for $i \in S$, be selected so that the simplex K_S defined by $a_i x \leq b_i$ for $i \in S$ and $b_\ell = \max\{a_\ell x | \; a_i x \leq b_i$ for $i \in S\}$ for ℓ not in S, is non-empty. These bodies are scaled and translated versions of a single simplex; the corresponding vectors b form an extreme ray of B′, which we term a <u>simplicial ray</u>.

Proof: Suppose, to the contrary, that $b = \lambda c + \mu d$ with λ and μ both positive and $c, d \in B'$. Translate and scale each of the bodies K_c and K_d so that $b_i = c_i = d_i$ for $i \in S$. If K_c is different from K_S then there must be at least one ℓ, not in S, for which $c_\ell < b_\ell$. But this implies that $d_\ell > b_\ell = \max\{a_\ell x | \; a_i x \leq d_i$ for $i \in S\}$ and therefore K_d is not in B′. □

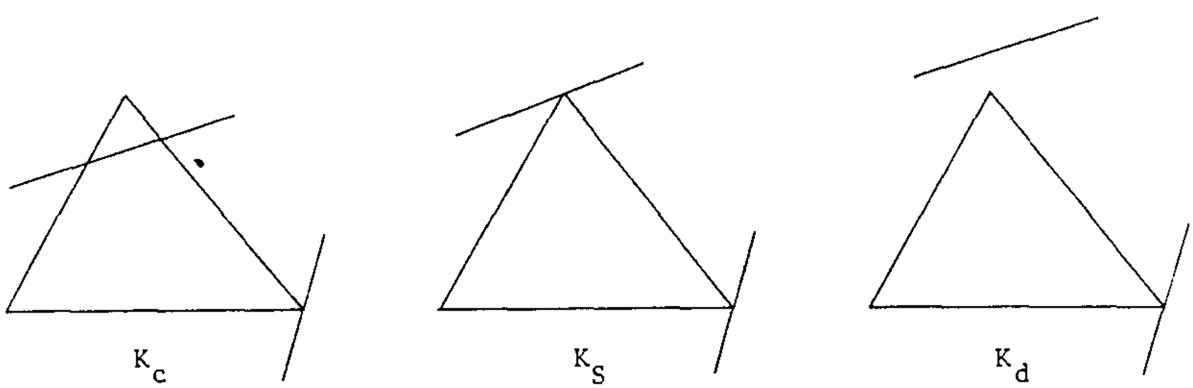

Figure 4.2

4.3 (Theorem) The maximum distance, according to the Banach-Mazur distance, between any two points in B′ is attained at a pair of simplicial rays.

Proof: Let c be in B′. Theorem 4.3 will follow immediately if we show that the maximum distance between c and any vector in B′ is attained at a vector b whose corresponding body K_b is a simplex defined by some dual feasible set of n+1 rows S. For any b in B′, let λ_1 be the smallest λ for which $K_c \subseteq \lambda K_b + \xi^1$ for some ξ^1 and λ_2 the smallest λ for which $K_b \subseteq \lambda K_c + \xi^2$ for some ξ^2 so that $\rho(K_b,K_c) = \log(\lambda_1 . \lambda_2)$. Since both bodies are in B′, the inclusion $K_c \subseteq \lambda K_b + \xi^1$ is equivalent to $c \le \lambda b + A\xi^1$ and λ_1 is the solution to the linear program

$$\min \lambda \quad \text{subject to}$$
$$c \le \lambda b + A\xi^1.$$

Assume that K_b is scaled and translated so that $\lambda_1 = 1$ and $\xi^1 = 0$. Let π be an optimal basic feasible solution to the dual linear program, i.e., $\underline{\pi A = 0,\ \pi b = 1,\ \pi \ge 0,\ \max \pi c}$, with $\pi_0, \ldots \pi_n$ being basic variables and $\pi_{n+1}, \ldots \pi_{n+d}$ nonbasic. Then $\pi_{n+1} = \ldots = \pi_{n+d} = 0$ and $c_i \le b_i$ with equality for $i = 0,1,\ldots,n$.

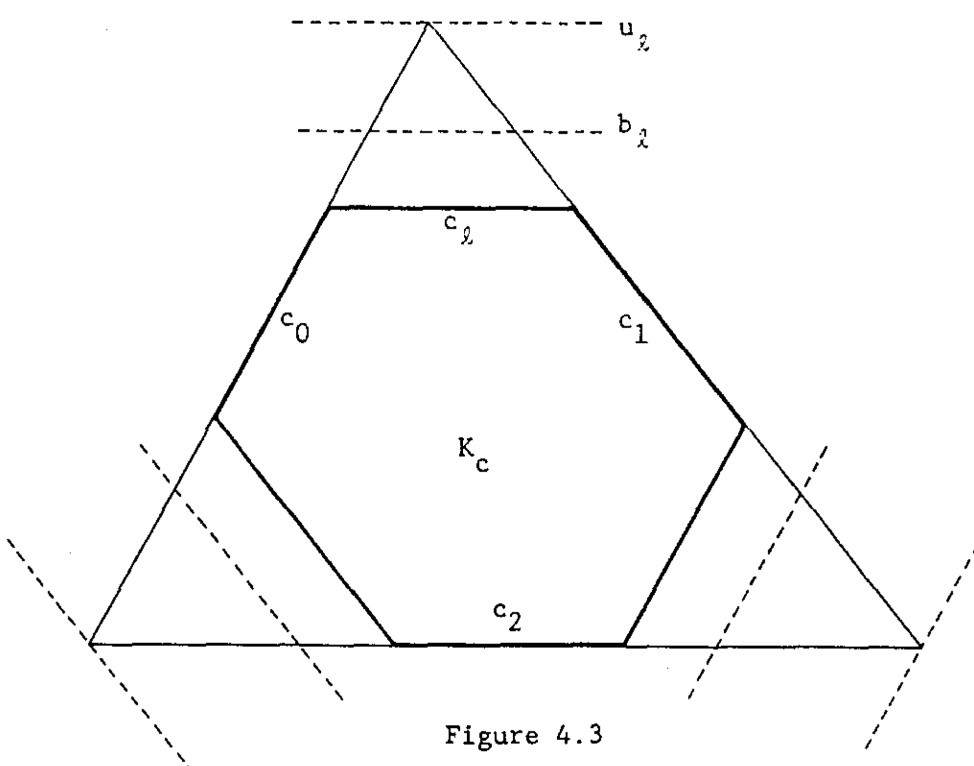

Figure 4.3

Define K_u to be the simplex given by $u_i = c_i$ for $i = 0,1,\ldots,n$ and $u_\ell = \max\{a_\ell x | a_i x \le c_i$ for $i = 0,1,\ldots,n\}$ for $\ell = n+1,\ldots,n+d$. Let us calculate the distance between K_c and K_u. The linear program

$$\min \lambda \quad \text{subject to}$$
$$c \le \lambda u + A\xi^1.$$

has a feasible solution with $\lambda = 1$. Using the same dual variables π as before we see that the dual linear program

$$\max \pi c \quad \text{subject to}$$
$$\pi u = 1$$
$$\pi A = 0$$
$$\pi \ge 0$$

also has a feasible solution with its objective function equal to 1, since $\pi_i = 0$ for i not in S. It follows that the minimizing λ in the first linear program is actually equal to 1. In order to calculate $\rho(K_c, K_u)$ we determine λ_2 from the linear program

$$\min \lambda \quad \text{subject to}$$
$$u \le \lambda c + A\xi^2.$$

But since $u \ge b$, the minimizing value of λ_2 is not less than the value of λ_2 used in defining $\rho(K_c, K_b)$. Therefore $\rho(K_c, K_u) \ge \rho(K_c, K_b)$. This concludes the proof of Theorem 4.3. □

Theorem 4.3 enables us to estimate the distance between an arbitrary pair of vectors in B′ by considering the distances between pairs of simplices each obtained by selecting a dual feasible set of n+1 rows of A. The following lemma will be used to give an estimate of the Banach-Mazur distance between an arbitrary pair of simplicial vertices in B′:

4.4 (Lemma) Let A^1 and A^2 be two dual feasible (n+1)xn submatrices of

A. Consider the two simplicies $K^i = \{x A^i x \le b^i\}$. Define λ_1 to be the smallest λ such that $K^2 \subseteq \lambda K^1 + x^1$ for some x^1 and similarly for λ_2. Then $\lambda_1 \lambda_2 \le n^2 C^2$ with $C = C(A)$.

Proof: For each row i, maximize the ith linear function in $A^1 x$ for x in K^2 and let c^1 be the vector of maxima as i ranges from 0 to n. Then $A^1 x \le c^1$ is the smallest simplex, obtained by translating the hyperplanes defined by the rows of A^1, which covers K^2. Since all such simplices are similiar, this is also the smallest multiple of K^1, which when translated, covers K^2, and therefore $c^1 = \lambda_1 b^1 + A^1 x^1$ for some x^1.

By the duality theorem the ith row of A^1 is a non-negative linear combination of n rows of A^2 and the ith entry in c^1 is that same linear combination of the entries in b^2. We may therefore write $(A^1, c^1) = U^1(A^2, b^2)$ with U^1 a non-negative (n+1)x(n+1) matrix with the property that each row of U^1 contains a single entry equal to zero. By Cramer's rule the entries in the ith row of U^1 (the dual variables in the ith linear programming problem) may be written as the ratio of two determinants; the denominator an nxn submatrix of A^2, and the numerator an nxn matrix composed of n-1 rows of A^2 and one row of A^1. The entries in U^1 are therefore bounded by C in absolute value.

In the same way $(A^2, c^2) = U^2(A^1, b^1)$ with $c^2 = \lambda_2 b^2 + A^2 x^2$ for some x^2, with the same upper bound on the entries of U^2 and again with the property that each row of U^2 contains a single entry equal to zero. It follows that if $\xi = (\lambda_1 x^2 + x^1)/(\lambda_1 \lambda_2 - 1)$ and $b = b^2 + A^2 \xi$ then

$$\begin{aligned} U^2 U^1 b &= U^2 U^1 (b^2 + A^2 \xi) \\ &= U^2 (c^1 + A^1 \xi) \end{aligned}$$

$$= U^2(\lambda_1 b^1 + A^1[x^1+\xi])$$
$$= \lambda_1 c^2 + A^2(x^1+\xi)$$
$$= \lambda_1\lambda_2 b^2 + A^2(\lambda_1 x^2 + x^1 + \xi)$$
$$= \lambda_1\lambda_2 b.$$

Therefore $\lambda_1\lambda_2\ b_1 \leq n^2C^2.b_1$. This demonstrates Lemma 4.4. □

The following estimate of the distance between an arbitrary pair of bodies is an immediate consequence of the arguments of this section.

4.5 (Theorem) $\rho(K_b,K_c) \leq 2\log(nC)$ for any pair of bodies K_b and K_c.

Proof: We simply apply Lemma 4.4 to a pair of simplices which maximize the distance in B'; A^1 is the subset of n+1 rows of A corresponding to the first of these simplices and A^2 to the second. □

We conclude this section with another result which describes a measure of similarity of the bodies K_b. Let $a = (a_1,\ldots,a_n)$; we define the <u>width</u> of the body K_b in the direction "a" to be

$$w(a,K_b) = \max(ax|x \in K_b) - \min(ax|x \in K_b).$$

The width is invariant under translations of the body and satisfies $w(\lambda a,K_b) = \lambda w(a,K_b)$.

4.6 (Theorem) Let a_i and a_j be an arbitrary pair of rows of A. Then

$$w(a_i,K_b) \leq 2C.w(a_j,K_b)$$

Proof: Without loss of generality we demonstrate the theorem for $j = 0$. We may assume that $b \in B'$ so that $b_0 = \max(a_0x|x \in K_b)$. The minimum of a_0 over K_b is found by solving the linear program

$$\max -a_0x \text{ subject to}$$
$$a_ix \leq b_i \text{ for } i = 1,\ldots,n+d,$$

and from the duality theorem, $-a_0$ is a positive linear combination of n of the rows of A for which equality holds in the linear program. Let us take these rows to be 1,...,n so that 0,1,...,n forms a dual feasible set. Moreover, let us translate the body so that this minimum is achieved at the origin; after this translation we have $w(a_0, K_b) = b_0 > 0$ and $b_i = 0$ for $i = 1,\dots,n$.

The other widths are not decreased if K_b is replaced by the simplex K defined by inequalities $0,\dots,n$, with the remaining inequalities relaxed so as to take on their maximum values in K. We will show that at each vertex v of K, we have $|a_\ell . v| \le Cb_0$, for every ℓ, which is sufficient for our argument. Consider, without loss of generality, the vertex defined by $a_i . x = b_i$ for $i = 0,1,\dots,n-1$. Then for any row ℓ we have $a_\ell = \mu_0 a_0 + \dots + \mu_{n-1} a_{n-1}$ with μ_i equal to the ratio of two nxn subdeterminants of A and therefore not larger than C, in absolute value. It follows that

$$|a_\ell . v| = |\mu_0 a_0 . v + \dots + \mu_{n-1} a_{n-1} . v| = |\mu_0 a_0 . v| \le Cb_0. \quad \square$$

V. The Hilbert Metric

The major conclusion of the previous section is an upper bound for the diameter of the cone B′, using the Banach-Mazur distance between pairs of rays in this subcone. This same distance function can be used to define balls of size ϵ: the set of rays which are within ϵ of a given ray. Our primary goal is to describe an upper bound for the number of ϵ-balls required to cover the cone B'. This is essentially an estimate of the Hausdorff volume of B′.

It will be useful for us to introduce the classic Hilbert metric, which is a distance function for rays in the larger cone B. (see Kohlberg and Pratt(1982) for a discussion of the Hilbert metric.)

5.1 (Definition) We define the <u>Hilbert distance</u> between a pair of interior vectors b and c in B to be

$$h(b,c) = \max \log[(\pi^1.b/\pi^1.c).(\pi^2.c/\pi^2.b)]$$

with the maximum taken over all $\pi^1, \pi^2 \in B^*$.

The Hilbert distance clearly satisfies $h(\lambda b, \mu c) = h(b,c)$ for $\lambda, \mu > 0$, so that it is defined on rays in the cone B. It is a distance function; for if π^1 and π^3 maximize the above expression for b and d then

$$h(b,d) = \log[(\pi^1.b/\pi^1.d).(\pi^3.d/\pi^3.b)] =$$

$$\log[(\pi^1.b/\pi^1.c).(\pi^3.c/\pi^3.b)] + \log[(\pi^1.c/\pi^1.d).(\pi^3.d/\pi^3.c)]$$

$$\leq h(b,c)+h(c,d).$$

The Hilbert distance has an elementary interpretation in terms of the

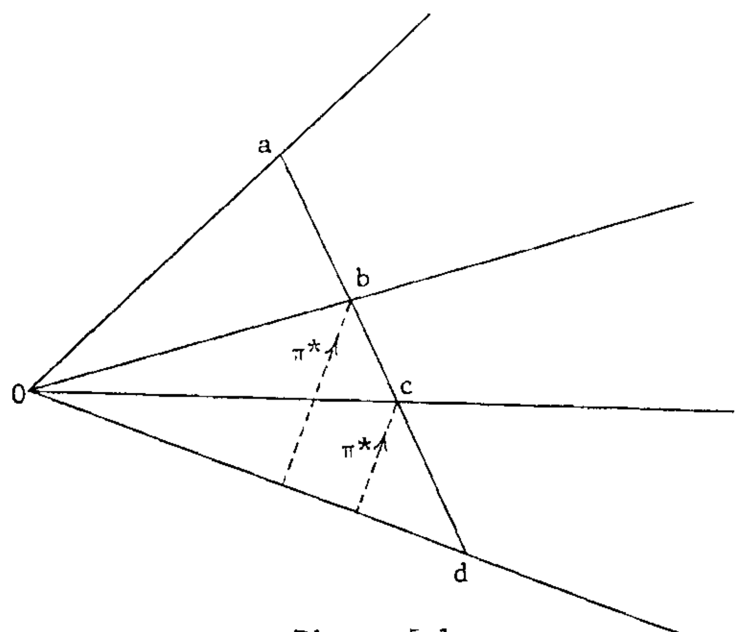

Figure 5.1

projective cross-ratio. Draw the intersection, with the cone B, of the two dimensional plane containing b, c and the origin, as in Figure 5.1. Assume that the two points have been scaled so that $\pi b = \pi c$ for some π interior to the dual cone, i.e., so that the line joining them intersects the boundary of B at the two vectors a and d. Then

$$h(b,c) = \log(([a,c]/[a,b]).([d,b]/[d,c])),$$

with $[x,y]$ the length of the line connecting x and y.

To see that the logarithm of the cross-ratio is indeed the Hilbert distance, we first remark that the dual vector π^* which maximizes $(\pi.b/\pi.c)$, or equivalently minimizes $(\pi.c/\pi.b)$, will be the normal to a supporting hyperplane to B at the intersection of the line segment from b through c with the boundary of B. For if π^* is the dual vector which minimizes $(\pi.c/\pi.b)$ it will also minimize $\pi[(1+\lambda)c-\lambda b]/\pi b$ for any $\lambda > -1$. If we select $\lambda > 0$, so that $d = [(1+\lambda)c-\lambda b]$ is on the boundary of B, we see that the minimizing vector must satisfy $\pi^* d = 0$ and therefore be the normal to a supporting hyperplane to B at d. But then $(\pi^*.b/\pi^*.c) = [d,b]/[d,c]$ using the similar right triangles.

We have the following relation between the Hilbert metric $h(b,c)$ and the Banach-Mazur distance:

5.2 (Theorem) If b and c are both in B', $h(b,c) = \rho(K_b,K_c)$

Proof: Define λ_1 to be the smallest λ such that $K_c \subseteq \lambda K_b + x^1$ for some x^1, and similarly for λ_2, so that $\rho(b,c) = \log \lambda_1\lambda_2$. If all of the constraints are binding for both K_b and K_c then a necessary and sufficient condition that $K_c \subseteq \lambda K_b + x^1$ is that $c \leq \lambda b + Ax^1$. But then $\pi c \leq \lambda \pi b$ for all π in B^*. It follows that $\lambda_1 \geq \pi^2 c/\pi^2 b$ for any π^2 in B^*. A similiar inequality for λ_2 tells us that $\rho(K_b,K_c) = \log \lambda_1\lambda_2 \geq h(b,c)$.

To obtain the converse inequality, let

$$h(b,c) = \log[(\pi^1.b/\pi^1.c).(\pi^2.c/\pi^2.b)],$$

with π^1 and π^2 selected so as to maximize the cross-ratio. Define $t_1 = \pi^2c/\pi^2b$ and $t_2 = \pi^1b/\pi^1c$, so that $h(b,c) = \log t_1t_2$. By definition, $t_1 \geq \pi c/\pi b$, or $\pi(t_1b - c) \geq 0$, for all π in B^*. Using the duality theorem we see that there is an x^1 such that $-Ax^1 \leq t_1b - c$ and therefore $\lambda_1 \leq t_1$. In the same fashion $\lambda_2 \leq t_2$. This demonstrates Theorem 5.2. □

VI. Approximation of the Bodies K_b

We are now prepared to demonstrate the major conclusion of the paper.

6.1 (Theorem) Let $\varepsilon > 0$ be given. Then there is a subset of the bodies K_b of cardinality not larger than

$$f(A)\lceil 2\log(nC)/\varepsilon \rceil^d$$

such that every body with a non empty interior has a Banach-Mazur distance less than or equal to ε from at least one member of this subset.

Proof: Let c be an vector in B' which is not a vertex of B' and which will be kept fixed during the argument. Let b be an arbitrary vector in $B'$and consider the programming problem:

$$\min \lambda \quad \text{such that } c \leq \lambda b + Ax^1 .$$

As in the proof of Theorem 4.3, we assume that K_b has been scaled and translated so that $x^1 = 0$ and $\lambda = 1$. We shall cover the cone B' by $f(A)$ regions, each consisting of those vectors b for which a given (n+1)-tuple of the rows of A is an optimal dual basis for the above linear program, i.e., $\underline{\pi A = 0,\ \pi b = 1,\ \pi \geq 0,\ \max \pi c}$ has an optimum solution vanishing outside these rows. For each such region, we shall construct a "dense" set separately.

Consider, for example, the region corresponding to the (n+1)-tuple 0,1,...,n. We then have $c_i \leq b_i$ with equality for $i = 0,1,\ldots,n$. (see Figure 4.3.) We shall now select a specific translate of K_c which depends on the the particular region and not on any other property of the vector b. For each $i = 0,1,\ldots,n+d$, let $u_i = \max\{a_i x | a_\ell x \leq c_\ell \text{ for } \ell = 0,1,\ldots,n\}$. Then $u_i \geq c_i$ with equality for $i = 0,1,\ldots,n$ and with strict inequality for at least one of the other coordinates, since c is not a vertex of B'. Let $\lambda^* > 1$ be the largest value of λ such that the body $\{x | Ax \leq u-\lambda(u-c)\}$ is non empty. This limiting body will consist of a single point, which we translate to the origin, so that the origin is contained in each of the bodies $K_{u-\lambda(u-c)}$, for $0 \leq \lambda \leq \lambda^*$. From this point on, the coordinates of b,c and u refer to this translation.

We have $b_i = c_i$ for $i = 0,1,\ldots,n$ and $0 < c_i \leq b_i \leq u_i$ for $i = n+1,\ldots,n+d$. This provides us with a d dimensional coordinate system for those bodies in B' for which the programming problem selects this particular set of dual feasible rows. Since $u-\lambda^*(u-c) \geq 0$, it follows that $u_i/c_i \leq \lambda^*/(\lambda^*-1)$.

The vector (u_i-c_i) is different from zero and is on the boundary of B; it is in that face determined by the particular dual feasible rows (0,1,...,n). It is therefore easy to see from the following figure

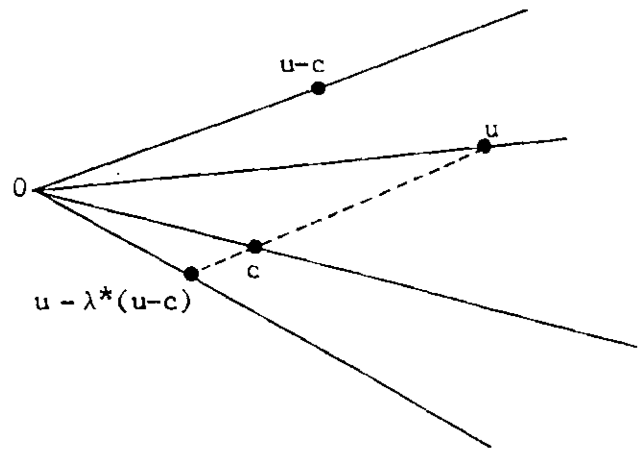

Figure 6.1

that $\log \lambda^{*}/(\lambda^{*}-1) = h(c,u) \leq 2\log(nC)$, so that $u_i/c_i \leq n^2C^2$.

Consider the family of bodies with right hand sides $f_i = c_i$ for $i = 0,1,\ldots n$ and $\log f_i/c_i = j_i\epsilon$ with $j_i = 0,\ldots,\lceil 2\log(nC)/\epsilon \rceil -1$, for $i = n+1,\ldots n+d$. Given the vector b, let K_f be the particular member of the family for which

$$\log(f_i/c_i) = j_i\epsilon \leq \log(b_i/c_i) < (j_i+1)\epsilon \text{ for } i = n+1,\ldots n+d.$$

For this f we have $f_i \leq b_i \leq f_i 2^{\epsilon}$ for all i. If we then calculate the Hilbert distance between b and f, we see that

$$1 \leq \pi b/\pi f \leq 2^{\epsilon}$$

for any π in the dual cone B^{*}. It follows that the Hilbert distance between b and f is $\leq \epsilon$. Since f is in B′, the Hilbert distance is equal to the Banach-Mazur distance and therefore at least one body in the family is within ϵ of K_b, using the Banach-Mazur metric.

The number of bodies in the family is

$$f(A)\lceil 2\log(nC)/\epsilon \rceil^{d}.$$

This demonstrates Theorem 6.1. □

VII. Lattice Free Bodies

In this section we shall apply our result on the approximation of convex bodies K_b to those bodies which are free of lattice points. For our purposes it is sufficient to restrict our attention to the ordinary lattice of integers Z^n; a general lattice in R^n can be dealt with by an appropriate linear transformation. If a convex body contains no non-zero lattice points and is also symmetric about the origin, Minkowski's Theorem asserts that its volume is not larger than 2^n. There is, however, no corresponding bound on the volume of a convex body if it contains no lattice points but is not symmetric about the origin; it can have arbitrarily high volume and yet be flat in some direction so as to avoid all lattice points.

The lattice width of the body K_b is defined to be the minimum of

$$w(v,K_b) = \max\{vx \mid x \in K_b\} - \min\{vx \mid x \in K_b\}$$

as v varies over all non-zero integral vectors in Z^n. Khinchine (1948) demonstrated the existence of a universal function f(n) so that the lattice width of a lattice-point-free convex body in R^n is bounded by f(n). This idea was exploited by H. W. Lenstra, Jr.(1983) in his polynomial algorithm for integer programming with a fixed number of variables. In order to determine whether a convex body K_b contains a lattice point, Lenstra constructs a non-zero integral vector v, and a particular function f(n), with the property that if $w(v,K_b) > f(n)$ the body contains a lattice point which can easily be determined. If, on the other hand, the width with respect to v is less than or equal to f(n), the problem can be reduced to at most f(n)+1 similar problems involving n-1 variables, each obtained by intersecting K_b with the hyperplanes $v.x = v_0$, where v_0 takes on all integral values between the maximum and the minimum of v.x in K_b. The process continues, reducing the number of variables at each step, until a lattice point is obtained or the lattice free character of K_b is verified.

Lenstra's original construction of the integral vector v was based on an algorithm which is polynomial in the data of the problem if the number of variables was fixed; his estimate of f(n) was on the order of c^{n^2}. Grötschel, Lovász and Schrijver (1983) showed that the same order of magnitude could be obtained by an algorithm which was polynomial in n as well, and Babai (1985) improved the estimate of f(n) to a linear exponential, again achievable in time polynomial in n.

A considerable sharpening is available if the requirement of polynomiality in n is relaxed. Based on the work of Lagarias, Lenstra and Schnorr (1987), Hastad (1986) demonstrated the existence of an integral vector v such that the width of a lattice free convex body, in the direction v, is less than $n^{5/2}$; Kannan and Lovász (1987) improved this to $c_0 n^2$. In both of these latter arguments, a lattice point in K_b can be found in polynomial time if the width is greater than the corresponding value of f(n), but finding v can be done in polynomial time only if the number of variables n is fixed.

Lenstra's algorithm, and its variants, can be cast in the form of a decision tree. The data defining the body K_b are entered at the upper node of the tree and a calculation is performed which is relayed along each of the f(n) branches connecting the upper node to the second level of nodes. Each of these nodes is connected to f(n-1) branches leading to a third level; the tree continues for n levels. The construction of the tree

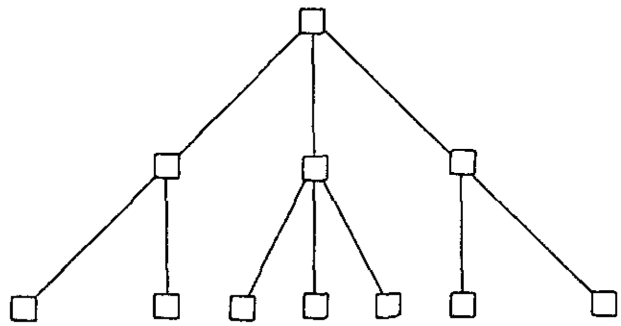

Figure 7.1

requires an amount of work which is exponential in n, but the number of branches emanating from each node is polynomial in n and once the tree has been constructed, the computation at each node – the solution of a pair of linear programs, and the determination of a lattice point in a body of large width – is polynomial in n as well.

Theorem 6.1 permits us to assert that a single decision tree with these properties can be constructed for <u>all</u> of the bodies K_b arising from the same matrix A as long as d+1, the difference between the number of rows and columns of A, is fixed. This is an immediate consequence of the following theorem:

7.1 (Theorem) There exists a set V of non-zero integral vectors, of cardinality not larger than $f(A)\lceil 2\log(nC)\rceil^{d}$, such that for every lattice free body K_b,

$$w(v,K_b) = \max\{vx \mid x \in K_b\} - \min\{vx \mid x \in K_b\} \le 2c_0n^2$$

for at least one $v \in V$.

Proof: Suppose that K_b and K_c are bodies with $\rho(K_b,K_c) \le 1$ and let v and u be the non-zero integral vectors which minimize the lattice width for

K_b and K_c respectively. Then there are λ_1 and λ_2 with $\lambda_1\lambda_2 \le 2$, $K_c \subseteq \lambda_1 K_b + \xi^1$ for some ξ^1 and $K_b \subseteq \lambda_2 K_c + \xi^2$ for some ξ^2. It follows that

$$w(v,K_c) \le \lambda_1 w(v,K_b) \le \lambda_1 w(u,K_b) \le \lambda_1\lambda_2 w(u,K_c) \le 2w(u,K_c)$$

so that v yields a lattice width for K_c which is not more than twice the minimal lattice width.

Now let us consider a set of bodies T of cardinality not larger than $f(A)|\bar{~}2\log(nC)\bar{~}|^d$ such that every K_b has a distance less than or equal to unity from at least one member of the set, and let V be the set of non-zero integral vectors which minimize the lattice width for the bodies in T. It follows that for every K_b there is a v in V such that $w(v,K_b)$ is not more than twice the minimal lattice width of K_b. In particular, if K_b is free of lattice points, then $w(v,K_b) \le 2c_0n^2$. This demonstrates Theorem 7.1. □

In the construction of the single decision tree for all of the bodies K_b we associate with the top node $2c_0n^2$ branches for each $v \in V$, for a total of $2c_on^2.f(A)|\bar{~}2\log(nC)\bar{~}|^d$ branches; and similarly for the nodes at lower levels. If the matrix A consists of integers, log(C) is polynomial in the bit size of A. The number of branches emanating from each node as well as the computational work at each node is therefore polynomial in the data, including n, as long as d is fixed. Of course, the tree is difficult to construct.

VIII. Neighborhood Systems and Successive Minima

In order to study the the family of integer programming problems

$$\min \Sigma a_{0j}h_j \quad \text{subject to}$$
$$\Sigma a_{ij}h_j \le b_i \text{ for } i = 1,\ldots,n+d$$
$$h_j \text{ integral,}$$

Scarf (1981,1981,1986) introduced the concept of a neighborhood system $\{N(h)\}$. Each lattice point $h = (h_1, \ldots, h_n)$ has associated with it a set of neighbors $N(h)$ which is arbitrary aside from the two conditions

1. $N(h) = N(0) + h$, and
2. if $k \in N(h)$ then $h \in N(k)$.

Given a particular neighborhood system, a lattice point h is defined to be a local minimum for the integer program if it is feasible and if all of its neighbors are either infeasible or yield a strictly larger value of the objective function. The following construction provides a neighborhood system depending on the matrix A alone, and which has the property that a local minimum is a global minimum for all b.

Let h be a lattice point and define $K^* = \{x | \Sigma a_{ij}x_j \leq \max(0, \Sigma a_{ij}h_j)$ for $i = 0, \ldots, n+d\}$. K^* is the smallest body K_b which contains both h and the origin. h is then defined to be a neighbor of the origin if K^* contains no lattice points in its interior. If the matrix is in general position - in the sense that for each row a_i of A the only lattice point satisfying $a_i h = 0$ is the origin - this system is the unique, minimal neighborhood system with the property that a local minimum is a global minimum for all values of the right hand side b. If this condition is not satisfied some of the neighbors defined in this fashion may be superfluous. (Scarf,1986)

Two special cases have been examined by Scarf (1981,1985). If A is a 4×2 matrix of integers, the neighbors of the origin N(0) are contained in the union of a polynomial number of lattice lines; and if A is a 4×3 matrix the neighbors of the origin lie in the union of three adjacent lattice planes, one of which passes through the origin. We have the following generalization:

8.1 (Theorem) The neighbors of the origin N(0) are contained in a set of n-1 dimensional lattice hyperplanes of cardinality not larger than

$$2c_o n^2 . f(A) |^{-} 2\log(nC)^{-}|^d .$$

The proof is an immediate application of Theorem 7.1, since the smallest body K_b containing 0 and h contains no lattice points in its interior. Again it should be remarked that if A is a matrix of integers the number of lattice hyperplanes is polynomial in the data, including the number of variables n, for fixed d.

In their previously cited paper, Kannan and Lovász demonstrate a sharper version of the theorem that a lattice free convex body K_b has a lattice width not larger than $c_0 n^2$. They show that such a body either has a lattice width less than 2, or there exist two linearly independant lattice hyperplanes with normals v_1, v_2 such that for j = 1,2,

$$w(v_j, K_b) \leq 2c_0(n+1)^3 \log^2(n+1).$$

If the first of these alternatives is applicable to the smallest body containing 0 and a neighbor of the origin h, the lattice hyperplane which minimizes the lattice width of K_b will yield v.h = 0 or 1. In the second case, these inequalities imply that K_b is contained in the union of no more than $[2c_0(n+1)^3 \log^2(n+1)]^2$ lattice hyperplanes of dimension n-2, each of them of the form $\{x: v_1.x = 1, v_2.x = 1\}$. Each of these hyperplanes can be extended to an n-1 dimensional hyperplane which passes through the origin. With some attention to detail this permits us to argue that the neighbors of the origin lie in the union of a polynomial number of lattice hyperplanes of the form v.h = 0 or 1. It is an interesting conjecture that the set of neighbors of the origin consists of those lattice points contained in the union of a polynomial number of polyhedra.

As a final application of our arguments, we turn our attention to Minkowski's concept of the successive minima of the lattice of integers with respect to a symmetric convex body. For each b in B' the convex body $(K_b - K_b)$ is symmetric about the origin, and we may associate with it a distance function

$$F_b(x) = \max\{\lambda | x/\lambda \in (K_b - K_b)\},$$

which is symmetric, convex and homogeneous of degree one. The <u>successive minima</u>, $\lambda_1(b),\dots,\lambda_n(b)$, with respect to this distance function, are defined as follows: $\lambda_i(b)$ is the smallest λ such that the body $\{x: F_b(x) \le \lambda\}$ contains i linearly independant lattice points. Alternatively, let $h^1(b)$ minimize $F_b(h)$ over all non-zero lattice points, $h^2(b)$ minimize $F_b(h)$ over all lattice points which are linearly independant of $h^1(b)$ and generally let $h^i(b)$ minimize $F_b(h)$ over all lattice points which are linearly independant of $h^1(b),\dots,h^{i-1}(b)$. Then $\lambda_i(b) = F_b(h^i(b))$. The successive minima depend on the particular vector b defining the distance function; the vectors $h^i(b)$ which realize the successive minima are not necessarily uniquely defined for a specific b, since there may be several different vectors h, independent of $h^1(b),\dots,h^{i-1}(b)$, which minimize $F_b(h)$. We have the following result which relates the vectors realizing the successive minima to neighbors of the origin for the matrix A.

8.2 (Theorem) For all b in B', the lattice points $h^i(b)$ which realize the successive minima are neighbors of the origin.

Proof: Let $h = h^i(b)$ and let $K^* = \{x | \Sigma a_{ij}x_j \le \max(0, \Sigma a_{ij}h_j)$ for $i = 0,\dots,n+d\}$. In order to demonstrate that h is a neighbor of the origin we need to argue that K* contains no lattice points in its interior. By definition h will lie on the boundary of the symmetric body

$S = \lambda_i(b)(K_b - K_b)$ and any lattice point contained in the interior of S will necessarily be linearly dependent on $h^1(b),\ldots,h^{i-1}(b)$.

Let us first argue that $K^* - K^* \subseteq S$. Since $h \in S$ it follows that x and x+h both lie in $\lambda_i(b)K_b$ for some x. If we translate K_b so that x = 0, the set S is unchanged. But then $\max(0,\Sigma a_{\ell j}h_j) \le \lambda_i(b).b_\ell$ for $\ell = 0,\ldots,n+d$, so that $K^* \subseteq \lambda_i(b)K_b$ and therefore $K^* - K^* \subseteq \lambda_i(b)(K_b - K_b) = S$.

If h is not a neighbor of the origin there is a lattice point k interior to K*. Since S contains (K* - K*) it follows that both k and h-k are interior to S; both of them must be linearly dependant on $h^1(b),\ldots,h^{i-1}(b)$ and as a consequence so is h. This contradicts the definition of $h = h^i(b)$ and demonstrates the theorem. □

It is an immediate consequence of Theorem 8.1 that all of the vectors $h^i(b)$ lie in the union of a set of n-1 dimensional lattice hyperplanes, the set having a cardinality not larger than $2c_o n^2.f(A)|^-2\log(nC)^-|^d$. Again, if A is a matrix of integers, the vectors representing the successive minima lie in the union of a set of lattice hyperplanes, whose cardinality is polynomial in the bit size of A, for fixed d. It is an intriguing conjecture that the set of $h^i(b)$ for $i \le j$ lie in the union of a polynomial number of j-1 dimensional hyperplanes.

REFERENCES

Babai, L., On Lovász's Lattice Reduction and the Nearest Lattice Point Problem, Combinatorica,5, (1985)

Grötschel, M., L. Lovász and A. Schrijver, Geometric Methods in Combinatorial Optimization, in : Progress in Combinatorial Optimization (W. R. Pullyblank, ed.), Proc. Silver Jubilee Conference on Comb. Opt. , Univ. of Waterloo, Vol. 1, (1982), Academic Press, N. Y. (1984)

Hastad, J.,private communication (1986)

Kannan, R. and L. Lovász, Covering Minima and Lattice Point Free Convex Bodies, Department of Computer Science, Princeton University (1987)

Khintchine, A.,A Quantitative Formulation of Kronecker's Theory of Approximation, Izv. Akad. Nauk. SSSR, Ser. Mat.,12, pp 113-122 (1948) (in Russian)

Kohlberg, E. and J. W. Pratt, The Contraction Mapping Approach to the Perron-Frobenius Theory: Why Hilbert's Metric?, Mathematics of Operations Research,7, pp 198-210 (1982)

Lagarias, J., H. W. Lenstra and C. P. Schnorr, Korkine-Zolatarev Bases and Successive Minima of a Lattice and its Reciprocal Lattice, to appear in Combinatorica (1987)

Lenstra, H. W., Integer Programming with a Fixed Number of Variables, Mathematics of Operations Research,8, pp 538-548 (1983)

Lenstra, A. K., H. W. Lenstra and L. Lovász, Factoring Polynomials with Rational Coefficients, Mathematische Annalen,261, pp 513-534 (1982)

McMullen P. and G.C. Shephard, Convex Polytopes and the Upper Bound Conjecture, Cambridge University Press, London (1971)

Scarf, H. E., Production Sets with Indivisibilities, Part I:

Generalities, Econometrica,49, pp 1-32 (1981)

______: Production Sets with Indivisibilities, Part II: the Case of Two Activities, Econometrica,49, pp 395-423 (1981)

______: Integral Polyhedra in Three Space, Mathematics of Operations Research,10, pp 403-438 (1985)

______: Neighborhood Systems for Production Sets with Indivisibilities, Econometrica,54, pp 507-532 (1986)