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THE CHARACTERISTIC FUNCTION OF THE DIRICHLET AND MULTIVARIATE *F* DISTRIBUTIONS

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by

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0. SUMMARY

Formulae are derived for the characteristic function of the inverted Dirichlet distribution and hence the multivariate F. The analysis involves a new function with multiple arguments that extends the confluent hypergeometric function of the second kind. This function and its properties are studied in the paper and a simple integral representation is given which is useful for numerical work. A special case connected with the multivariate t distribution is also explored.

Some key words: confluent hypergeometric function, contour integral, multivariate t, partial differential equation system.

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1. INTRODUCTION

Let X_j $(j = 0, 1, \ldots, m)$ be independent chi-squared variates with n_j degrees of freedom. The joint density of the ratios $Y_j = X_j/X_0$ $(j = 1, \ldots, m)$ is given by

$$(1) \qquad \mathrm{pdf}(y_1, \ldots, y_m) = \frac{\Gamma(n/2)}{\Pi_{j=0}^{m}\Gamma(n_j/2)} \frac{\Pi_{j=0}^{m} y_j^{n_j/2-1}}{\left(1 + \Sigma_{j=1}^{m} y_j\right)^{n/2}} , \quad n = \Sigma_{j=0}^{m} n_j$$

and this is known as an inverted Dirichlet distribution (see Tiao and Guttman (1965), Johnson and Kotz (1972, pp. 238-239)). The rescaled variates $F_j = (n_0/n_j)Y_j$ $(j = 1, \ldots, m)$ are multivariate F and arise in analysis of variance problems with m independent "effect" sums of squares X_j $(j = 1, \ldots, m)$ and an overall residual sum of squares X_0. See Finney (1941) and Johnson and Kotz (1972, pp. 240-243) for a brief review. Other applications occur in econometrics where some simultaneous equations estimators can be written quite simply as ratios of linear combinations of independent χ^2 variates leading to distributions that are of the form (1) in certain cases (see Anderson and Sawa (1973)).

This paper is concerned with the characteristic function (c.f.) of the multivariate distribution (1). As pointed out by Ifram (1970), the c.f. of the F distribution tends to be avoided in the literature even in the scalar case. This is because correct formulae for the c.f. of the F distribution are not well known and it is often assumed that no simple expression is available. Indeed, until very recently formulae for the c.f. of the F distribution that have been given in the literature are incorrect. This was pointed out in independent work by Pestana (1977) and Phillips (1982). Awad (1980) attempted to correct the earlier incorrect formulae by Ifram (1970) and Johnson and Kotz (1970, p. 190). Unfortunately, Awad's expression is also incorrect. In fact, his series representation (equation (2), p. 128 of Awad (1980)) is nonconvergent, as is easily verified. Indeed, if his series expression were valid, it would imply that the F distribution had finite moments of all orders, which of course is not so.

As shown in Phillips (1982) the true c.f. of the (scalar) F distribution has a rather simple expression in terms of a confluent hypergeometric function of the second kind (see equation (6) of Phillips (1982)). When $m = 1$ above, this result implies that the c.f. of (1) is simply

$$\text{(2)} \qquad cf(s) = E(e^{isY_1}) = \frac{\Gamma(n/2)}{\Gamma(n_0/2)} \Psi(n_1/2, 1 - n_0/2; -is)$$

where Ψ denotes the confluent function of the second kind. The special function Ψ has been extensively studied in the applied mathematics literature; see Erdéyli (1953, Ch. 6) for a detailed review. Its properties enable us to characterize the behavior of (2) in the vicinity of the origin, to extract moment formulae and to provide series representations for computation. These and other applications of (2) are discussed in Phillips (1982).

The present paper shows that the joint characteristic function of the inverted Dirichlet distribution (and, hence the multivariate F distribution) can be written in terms of a confluent hypergeometric function of the second kind with multiple arguments. The Appendix studies this function in detail and records some of its properties including formulae that are useful in computation. A simple integral representation of the function that arises directly from the multivariate t distribution in a special case is also explored. This representation is extended to apply in the general case.

2. THE CHARACTERISTIC FUNCTION IN THE GENERAL CASE

We shall work with the general inverted Dirichlet distribution whose density function is given by

$$(3) \qquad \mathrm{pdf}(y_1, \ldots, y_m) = \frac{\Gamma(\beta)}{\Pi_{j=0}^{m}\Gamma(\beta_j)} \frac{\Pi_{j=1}^{m} y_j^{\beta_j - 1}}{\left(1 + \Sigma_{j=1}^{m} y_j\right)^{\beta}} , \quad \beta = \Sigma_{j=0}^{m}\beta_j .$$

with $\beta_j > 0$ $(j = 0, \ldots, m)$ (see Tiao and Guttman (1965) equation (2.10)). This is related to the Dirichlet distribution by the transformation

$$x_j = y_j/(1 + y_1 + \ldots + y_m) , \quad j = 1, \ldots, m$$

leading to the joint density

$$(4) \qquad \mathrm{pdf}(x_1, \ldots, x_m) = \left[\Pi_{j=0}^{m}\Gamma(\beta_j)\right]^{-1} \Gamma(\beta)\Pi_{j=1}^{m} x_j^{\beta_j - 1} \left(1 - \Sigma_{j=1}^{m} x_j\right)^{\beta_0 - 1}$$

for $x_j > 0$, $\Sigma_{j=1}^{m} x_j < 1$ (again, see Tiao and Guttman (1965)).

The c.f. of (4) is known to be given by a confluent function of the first kind with many arguments. In fact, as shown by Exton (1976, pp. 232-233), we have

$$\text{(5)}\quad cf(s_1, \ldots, s_m) = E\left\{e^{i(s_1x_1+\ldots+s_mx_m)}\right\} = \Phi^{[m]}(\beta_1, \ldots, \beta_m;\ \beta;\ is_1, \ldots, is_m)$$

where

$$\text{(6)}\quad \Phi^{[m]}(b_1, \ldots, b_m;\ c;\ z_1, \ldots, z_m) = \Sigma \frac{(b_1)_{k_1} \cdots (b_m)_{k_m} z_1^{b_1} \cdots z_m^{b_m}}{(c)_k \ k_1! \cdots k_m!} .$$

The summation in (6) is over all integers $k_j = 0, 1, \ldots, \infty$ $(j = 1, \ldots, m)$. This series is a confluent form of a Lauricella function (see Exton (1976), p. 42) and when $m = 1$ it reduces to the well known confluent hypergeometric function of the first kind with scalar argument. However, (6) is inconvenient for numerical calculation when $m > 2$. Instead a single integral representation is available:

$$\text{(7)}\qquad \Phi^{[m]} = \frac{\Gamma(c)}{2\pi i} \int_L e^t\ t^{\Sigma_1^m b_j - c} \prod_{j=1}^{m} (t - z_j)^{-b_j} dt$$

where L denotes any path in the complex t-plane originating at $-\infty$ encircling in the positive direction all finite singularities of the integrand and returning to $-\infty$. Such a contour is illustrated in Figure 1. The loop integral (7) was given by Erdéyli (1939, equation (7)). It may be computed by complex numerical integration along a convenient and appropriate contour in the t-plane.

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Figure 1 about here
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The c.f. of the density (3) is also a confluent function with many arguments. But, as in the scalar case considered in Phillips (1982), the appropriate function is a confluent function of the second kind. To our knowledge this function does not appear in the applied mathematics literature and we have therefore devoted the Appendix to study its properties. We define the function

$$(8)\quad \Psi^{[m]}(b_1, \ldots, b_m; c; z_1, \ldots, z_m) = K\int_0^\infty \cdots \int_0^\infty e^{-u_1 z_1 - \cdots - u_m z_m} u_1^{b_1 - 1} \cdots u_m^{b_m - 1} \left(1 + u_1 + \cdots + u_m\right)^{c - \Sigma_1^m b_i - 1} du_1 \cdots du_m$$

where

$$K = \left[\prod_{j=1}^{m} \Gamma(b_j)\right]^{-1}$$

for $\operatorname{Re}(b_j) > 0$ $(j = 1, \ldots, m)$ and $\operatorname{Re}(z_j) > 0$ $(j = 1, \ldots, m)$. The domain of definition of $\Psi^{[m]}$ can be extended beyond $\operatorname{Re}(z_j) > 0$ $(j = 1, \ldots, m)$ and this is useful in determining the required c.f. In fact, if

$$(9)\qquad \operatorname{Re}(c) < 1$$

then the integral representation (8) continues to hold for z_j on the imaginary axis i.e. for $\operatorname{Re}(z_j) \geq 0$ $(j = 1, \ldots, m)$.

Setting $z_j = -is_j$ in (8) we deduce that the c.f. of the inverted Dirichlet distribution (3) is:

$$(10)\qquad \operatorname{cf}(s_1, \ldots, s_m) = \frac{\Gamma(\beta)}{\Gamma(\beta_0)} \Psi^{[m]}(\beta_1, \ldots, \beta_m; 1 - \beta_0; -is_1, \ldots, -is_m)$$

Observe that condition (9) is satisfied in the present case since $c = 1 -$

β_0 and $\beta_j > 0$ $(j = 0, \ldots, m)$ in (3).

A simple integral representation of (10) is also available. The following is based on the analysis of $\Psi^{[m]}$ in the Appendix.

THEOREM

$$(11) \qquad cf(s_1, \ldots, s_m) = \{\Gamma(\beta_0)\}^{-1}\int_0^\infty e^{-x} x^{\Sigma_0^m \beta_j - 1} \prod_{j=1}^m (x - is_j)^{-\beta_j} dx .$$

The integral representation (11) follows directly from (10) and the lemma in the Appendix. It is clearly more convenient for numerical purposes than the multi dimensional integral based on (8). When $m = 1$ we find from (11) and (A17) that

$$(12) \qquad cf(s_1) = \frac{\Gamma(\beta_0 + \beta_1)}{\Gamma(\beta_0)} \Psi(\beta_1, 1 - \beta_0; -is_1)$$

which is the characteristic function of the one dimensional inverted Dirichlet distribution (3). Setting $\beta_0 = n_0/2$, $\beta_1 = n_1/2$ (12) reduces to the earlier formula (2) given in Phillips (1982).

3. CONNECTION WITH THE MULTIVARIATE t DISTRIBUTION

Let $y = k^{-1/2}t_k$ where t_k is an m-vector multivariate t variate with k degrees of freedom. We may write $y \equiv Y/w$ where $Y \equiv N(0, I_n)$, $w^2 \equiv \chi_k^2$ and Y is independent of w . Here the symbol " $\equiv$ " signifies equality in distribution.

We shall consider the transformed vector variate

$$f = (f_1, f_2, \ldots, f_m) = (y_1^2, y_2^2, \ldots, y_m^2) .$$

Clearly, f has the distribution (1) with $n_j = 1$ $(j = 1, \ldots, m)$ and $n_0 = k$. Using the representation $y \equiv Y/w$ it is now easy to obtain the c.f. by iterated expectations

$$\begin{aligned} cf(s_1, \ldots, s_m) &= E_w\left\{E_Y\left\{e^{(is_1Y_1^2+\ldots+is_mY_m^2)/w^2} \middle| w\right\}\right\} \\ &= E_w\left\{\prod_{j=1}^{m} (1 - 2is_j/w^2)^{-1/2}\right\} \\ &= \left\{\Gamma(k/2)2^{k/2}\right\}^{-1}\int_0^\infty e^{-u/2}u^{k/2-1} \prod_{j-1}^{m} (1-2is_j/u)^{-1/2}du \\ &= \{\Gamma(k/2)\}^{-1}\int_0^\infty e^{-v}v^{(k+m)/2-1} \prod_{j=1}^{m} (v - is_j)^{-1/2}dv . \end{aligned}$$

This expression is, of course, a special case of (11) with $\beta_0 = k/2$ and $\beta_j = 1/2$ $(j = 1, \ldots, m)$.

APPENDIX:

THE CONFLUENT HYPERGEOMETRIC FUNCTION OF THE SECOND KIND WITH MULTIPLE ARGUMENTS

The confluent hypergeometric function

$$\text{(A1)} \qquad \Phi(b,c;z) = \Sigma_{k=0}^{\infty} \frac{(b)_k z^k}{(c)_k k!} = w_1 \ , \quad \text{say} \ .$$

is known to satisfy the linear differential equation

$$\text{(A2)} \qquad zw'' + (c-z)w' - bw = 0 \ , \quad w = w(z) \ .$$

The function

$$w_2 = z^{1-c}\Phi(1+b-c,\ 2-c;\ z)$$

is also a solution of (A2). The confluent hypergeometric function of the second kind is a linear combination of these particular solutions w_1 and w_2. In fact,

$$\text{(A3)} \qquad \Psi(b,c;z) = \frac{\Gamma(1-c)}{\Gamma(1+b-c)} \Phi(b,c;z) + \frac{\Gamma(c-1)}{\Gamma(b)} z^{1-c}\Phi(1+b-c,\ 2-c;\ z)$$

$$= [\Gamma(b)]^{-1}\int_0^\infty e^{-zu}u^{b-1}(1+u)^{c-b-1}du$$

with the latter integral representation holding for $\text{Re}(b) > 0$ and $\text{Re}(z) > 0$; see Erdéyli (1953, pp. 255-257).

The confluent function $\Phi^{[m]}$ that is defined in the text by (6) satisfies a system of m partial differential equations that extend (A2) to m variables. Indeed, $\Phi^{[m]}$ satisfies

(A4) $$\Sigma_{k=1}^{m} z_k \partial_{ij} w + (c - z_j)\partial_j w - b_j w = 0 \ , \quad j = 1, \ \ldots, \ m$$

where $\partial_{ij} = \partial^2/\partial k_i \partial z_k$, $\partial_j = \partial/\partial z_j$. The system (A4) was studied by Erdéyli (1939) and, more recently, by Exton (1976). There are n+1 linearly independent (particular) solutions of (A4) and the general solution may be written as a linear combination of these particular integrals. Exton (1976, pp. 173-178) shows that

$$\int_g^h e^t \, t^{b_1 + \ldots b_n - c} (t - z_1)^{-b_1} \ldots (t - z_n)^{-b_n} dt$$

is a solution of (A4) provided the contour of integration is a closed path on the Riemann surface of the integrand or a simple path in the t-plane connecting any two zeroes of the integrand. Let $z_1, \ \ldots, \ z_m$ be distinct points and suppose $z_j \neq 0$ for all j . Then m+1 linearly independent solutions of (A4) are given by the contour integrals

(A5) $$w_i = \int_0^{(z_i+)} e^t \, t^{\Sigma_1^m b_j - c} \prod_{j=1}^{m} (t - z_j)^{-b_j} dt \ , \quad i = 1, \ \ldots, \ m$$

(A6) $$w_0 = \Phi^{[m]} = \frac{\Gamma(c)}{2\pi i} \int_L e^t \, t^{\Sigma_1^m b_j - c} \prod_{j=1}^{m} (t - z_j)^{-b_j} dt \ .$$

The contour in (A5) is a loop that starts and ends at $t = 0$ and encircles the point z_i once in the positive sense. In (A6) L is a loop that starts and ends at $-\infty$ and encircles all finite singularities of the integrand in the positive direction. These contours are illustrated in Figure 1.

The confluent function of the second kind $\Psi^{[m]}$ defined by (8) is also a solution of (A4). To see this we write $\Psi^{[m]}$ in the alternative form

(A7) $$\Psi^{[m]}(b_1, \ldots, b_m; c; z_1, \ldots, z_m)$$

$$= [\Gamma(b_m)]^{-1} \int_0^\infty e^{-u_m z_m} u_m^{b_m - 1} (1 + u_m)^{c-b_m-1}$$

$$\Psi^{[m-1]}(b_1, \ldots, b_{m-1}; c - b_m; (1 + u_m)z_1, \ldots, (1 + u_m)z_{m-1})du_m .$$

Substituting $w = \Psi^{[m]}$ in (A4) and writing partial derivatives of $\Psi^{[m-1]}$ as $\partial_k \Psi^{[m-1]} = \Psi_k^{[m-1]}$, we find that the left side of the m'th equation of (A4) is proportional to:

$$\int_0^\infty \Big\{ -\Sigma_{k=1}^{m-1} e^{-u_m z_m} u_m^{b_m}(1+u_m)^{c-b_m} \Psi_k^{[m-1]} + z_m e^{-u_m z_m} u_m^{b_m+1} (1+u_m)^{c-b_m-1} \Psi^{[m-1]}$$

$$- (c-z_m) e^{-u_m z_m} u_m^{b_m}(1+u_m)^{c-b_m-1} \Psi^{[m-1]} - b_m e^{-u_m z_m} u_m^{b_m-1} (1+u_m)^{c-b_m-1} \Psi^{[m-1]} \Big\} du_m$$

$$= -\int_0^\infty \frac{d}{du_m} \Big[e^{-u_m z_m} u_m^{b_m}(1+u_m)^{c-b_m} \Psi^{[m-1]}(b_1, \ldots, b_{m-1}; c; (1+u_m)z_1, \ldots, (1+u_m)z_{m-1}) \Big] du_m$$

$$= 0 .$$

To verify that $\Psi^{[m]}$ is a solution of the j'th equation of (A4) we write

$$\Psi^m = [\Gamma(b_j)]^{-1} \int_0^\infty e^{-u_j z_j} u_j^{b_j-1} (1+u_j)^{c-b_j-1} \Psi^{[m-1]}(b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_m; c-b_j;$$

$$(1+u_j)z_1, \ldots, (1+u_j)z_{j-1}, (1+u_j)z_{j+1}, \ldots, (1+u_j)z_m)$$

and, in an analogous way, $\Psi^{[m]}$ satisfies the j'th equation of (A4) by substitution.

Since $\Psi^{[m]}$ is a particular integral of (A4) it may be written as a linear combination of the solutions w_j $(j = 0, \ldots, m)$. Let

(A8) $$\Psi^{[m]} = \Sigma_{j=0}^{m} A_j w_j \; .$$

The constants A_j $(j = 0, \ldots, m)$ in this expression may be found by taking certain limits as $z_j \to 0$. From the definition of $\Psi^{[m]}$ in (8) and in view of the intended application to (3) we require $\mathrm{Re}(b_j) > 0$ $(j = 1, \ldots, m)$. It is also convenient to assume temporarily that $0 < \mathrm{Re}(c) < 1$ and that $z_j \neq 0$ for all j. Then, by allowing $z_j \to 0$ so that $\mathrm{Re}(z_j) \to 0+$ for all j, we obtain

$$A_0 = \lim_{z_1, \ldots, z_m \to 0+} K \int_0^\infty \cdots \int_0^\infty e^{-u_1 z_1 + \ldots - u_m z_m} u_1^{b_1 - 1} \cdots u_m^{b_m - 1}$$

$$(1 + u_1 + \ldots + u_m)^{c - \Sigma_1^m b_i - 1} du_1 \ldots du_m$$

$$= K \int_0^\infty \cdots \int_0^\infty u_1^{b_1 - 1} \cdots u_m^{b_m - 1} (1 + u_1 + \ldots + u_m)^{c - \Sigma_1^m b_i - 1} du_1 \ldots du_m$$

$$= \frac{K \Pi_1^m \Gamma(b_i) \Gamma(1-c)}{\Gamma(\Sigma_1^m b_i + 1 - c)}$$

(A9) $$= \frac{\Gamma(1-c)}{\Gamma(\Sigma_1^m b_i + 1 - c)} \; .$$

Next, to find A_1 we let $z_j \to 0$ with $\mathrm{Re}(z_j) \to 0+$ for $j = 2, \ldots, m$ giving

$$\text{(A10)}\qquad K_1\int_0^\infty e^{-u_1 z_1} u_1^{b_1-1}(1+u_1)^{c-b_1-1}du_1 = A_0 w_0(z_1, 0, \ldots, 0)$$

$$+ A_1\int_0^{(z_1+)} e^t t^{b_1-c}(t-z_1)^{-b_1}dt$$

where

$$K_1 = [\Gamma(b_1)]^{-1}\Psi^{[m-1]}(b_2, \ldots, b_m; c-b_1; 0, \ldots, 0)$$

$$= \frac{\Gamma(1+b_1-c)}{\Gamma(b_1)\Gamma(\Sigma_1^m b_j + 1 - c)} .$$

The second term on the right side of (A10) can be written

$$A_1 z_1^{1-c}\int_0^{(1+)} e^{z_1 r} r^{b_1-c}(r-1)^{-b_1}dr$$

$$= A_1 z_1^{1-c}\,\frac{2\pi i\Gamma(b_1-c+1)}{\Gamma(2-c)\Gamma(b_1)}\,\Phi(b_1-c+1, 2-c; z_1)$$

(see Erdéyli (1953), p. 272, equation (2)). We now differentiate (A10) with respect to z_1 , multiply the equation through by z_1^c and allow $z_1 \to 0$ with $\mathrm{Re}(z_1) \to 0+$. The left side is

$$-K \lim_{z_1\to 0} z_1^c \int_0^\infty e^{-u_1 z_1} u_1^{b_1} (1+u_1)^{c-b_1-1} du_1$$

$$= -K_1 \lim_{z_1\to 0} \int_0^\infty e^{-v} v^{b_1} (v + z_1)^{c-b_1-1} dv$$

$$= -K_1 \int_0^\infty e^{-v} v^{c-1} dv$$

$$= -K_1 \Gamma(c) \ .$$

The first term on the right side of (A10) tends to zero with these operations. The second term tends to

$$\frac{(1-c)2\pi i A_1 \Gamma(b_1 - c + 1)}{\Gamma(2-c)\Gamma(b_1)} \ .$$

We deduce that

$$A_1 = -\frac{K_1 \Gamma(c)\Gamma(2-c)\Gamma(b_1)}{(1-c)2\pi i \Gamma(b_1 - c + 1)}$$

$$= \frac{\Gamma(c-1)\Gamma(2-c)}{2\pi i \Gamma(\Sigma_1^m b_j + 1 - c)}$$

In a similar way we find

$$\text{(A11)} \qquad A_j = \frac{\Gamma(c-1)\Gamma(2-c)}{2\pi i \Gamma(\Sigma_1^m b_j + 1 - c)} \ , \quad j = 2, \ldots, m \ .$$

From (A8)-(A11) it follows that

$$\text{(A12)} \qquad \Psi^{[m]} = \frac{\Gamma(1-c)}{\Gamma(\Sigma_1^m b_j + 1 - c)} \Phi^{[m]}(b_1, \ldots, b_m; c; z_1, \ldots, z_m)$$

$$+ \frac{\Gamma(c-1)\Gamma(2-c)}{2\pi i\Gamma(\Sigma_1^m b_j + 1 - c)} \Sigma_{k=1}^m \int_0^{(z_k+)} e^t \, t^{\Sigma_1^m b_j - c} \prod_{j=1}^m (t - z_j)^{-b_j} dt$$

and since

$$\Gamma(c)\Gamma(1-c) = -\Gamma(c-1)\Gamma(2-c)$$

we deduce from (A6) and (A12) that

$$\text{(A13)} \qquad \Psi^{[m]}(b_1, \ldots, b_m; c; z_1, \ldots, z_m)$$

$$= \frac{\Gamma(c)\Gamma(1-c)}{2\pi i\Gamma(\Sigma_1^m b_j + 1 - c)} \left\{\int_L - \Sigma_{k=1}^m \int_0^{(z_k+)}\right\} \left[e^t \, t^{\Sigma_1^m b_j - c} \prod_{j=1}^m (t - z_j)^{-b_j}\right] dt \,.$$

It is now convenient to take L as the path displayed in Figure 2 with branch cuts from $-\infty$ to the origin and from the origin to each of the points z_k $(k = 1, \ldots, m)$. Let $C = C_1 + C_2 + \ldots + C_m$ be the part of this contour which is the curve centered at the origin in the Figure. Upon shrinking C to the origin we obtain

$$\text{(A14)} \qquad \int_L - \Sigma_{k=1}^m \int_0^{(z_k+)} = \int_{-\infty}^0 + \int_0^{-\infty}$$

Figure 2 about here

To evaluate the right side of (A14) we note that in passing from L_1 to L_2 in the Figure we have in effect encircled each of the branch points

0, z_1, ..., z_m . We make the multivalued functions in the integrand of (A13) definite by writing $t = xe^{-\pi i}$ on L_1 and choosing the branches

$$t^{\Sigma_1^m b_j - c} = x^{\Sigma_1^m b_j - c} e^{-i\pi(\Sigma_1^m b_j - c)}$$

$$(t - z_j)^{-b_j} = (-x - z_j)^{-b_j}$$

on L_1 with $0 < x < \infty$. Then on L_2 we have

$$t^{\Sigma_1^m b_j - c} = x^{\Sigma_1^m b_j - c} e^{i\pi(\Sigma_1^m b_j - c)}$$

$$(t - z_j)^{-b_j} = (-x - z_j)^{-b_j} e^{-i2\pi b_j}$$

again with $0 < x < \infty$. It follows that the integrals on the right side of (A14) have the form

$$e^{-i\pi(\Sigma_1^m b_j - c + 1)} \int_\infty^0 e^{-x} x^{\Sigma_1^m b_j - c} \prod_{j=1}^m (-x - z_j)^{-b_j} dx$$

$$+ e^{-i\pi(\Sigma_1^m b_j + c - 1)} \int_0^\infty e^{-x} x^{\Sigma_1^m b_j - c} \prod_{j=1}^m (-x - z_j)^{-b_j} dx$$

$$= \{e^{i\pi(1-c)} - c^{-i\pi(1-c)}\} \int_0^\infty e^{-x} x^{\Sigma_1^m b_j - c} \prod_{j=1}^m (x + z_j)^{-b_j} dx$$

$$= 2i \sin\{\pi(1-c)\} \int_0^\infty e^{-x} x^{\Sigma_1^m b_j - c} \prod_{j=1}^m (x + z_j)^{-b_j} dx .$$

Noting that

$$\Gamma(c)\Gamma(1-c) = \pi/\sin\{\pi(1-c)\}$$

we deduce the following result from (A13) and the above:

<u>LEMMA</u>. *If* $\mathrm{Re}(b_j) > 0$ $(j = 1, \ldots, m)$ *and* $\mathrm{Re}(c) < 1$ *then*

$$\Psi^{[m]}(b_1, \ldots, b_m; c; z_1, \ldots, z_m)$$

$$= \left\{\Gamma(\Sigma_1^m b_j + 1 - c)\right\}^{-1} \int_0^\infty e^{-x} x^{\Sigma_1^m b_j - c} \prod_{j=0}^{m} (x + z_j)^{-b_j} dx \; . \tag{A15}$$

The integral representation (A15) is much simpler than (8) and is also more convenient for numerical computation.

Now consider the case $m = 1$. Writing $z_1 = re^{i\varphi}$ we find that (A15) becomes upon transforming $x \to x/z_1 = y$

$$\{\Gamma(b_1 + 1 - c)\}^{-1} \int_0^\infty e^{-x} x^{b_1 - c} (x + z_1)^{-b_1} dx$$

$$= \{\Gamma(b_1 + 1 - c)\}^{-1} z_1^{1-c} \int_0^{\infty e^{-i\varphi}} e^{-yz_1} y^{b_1 - c} (1+y)^{-b_1} dy$$

$$= \{\Gamma(b_1 + 1 - c)\}^{-1} \Gamma(b_1 + 1 - c) z_1^{1-c} \Psi(b_1 - c + 1, 2-c; z_1) \tag{A16}$$

$$= \Psi(b_1, c; z_1) \; . \tag{A17}$$

Line (A16) follows from the integral representation of the Ψ function with scalar argument; see Erdéyli (1953) equation (3), p. 256. Line (A17) follows from equation (6), p. 257 of the same reference.

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FIGURE 1

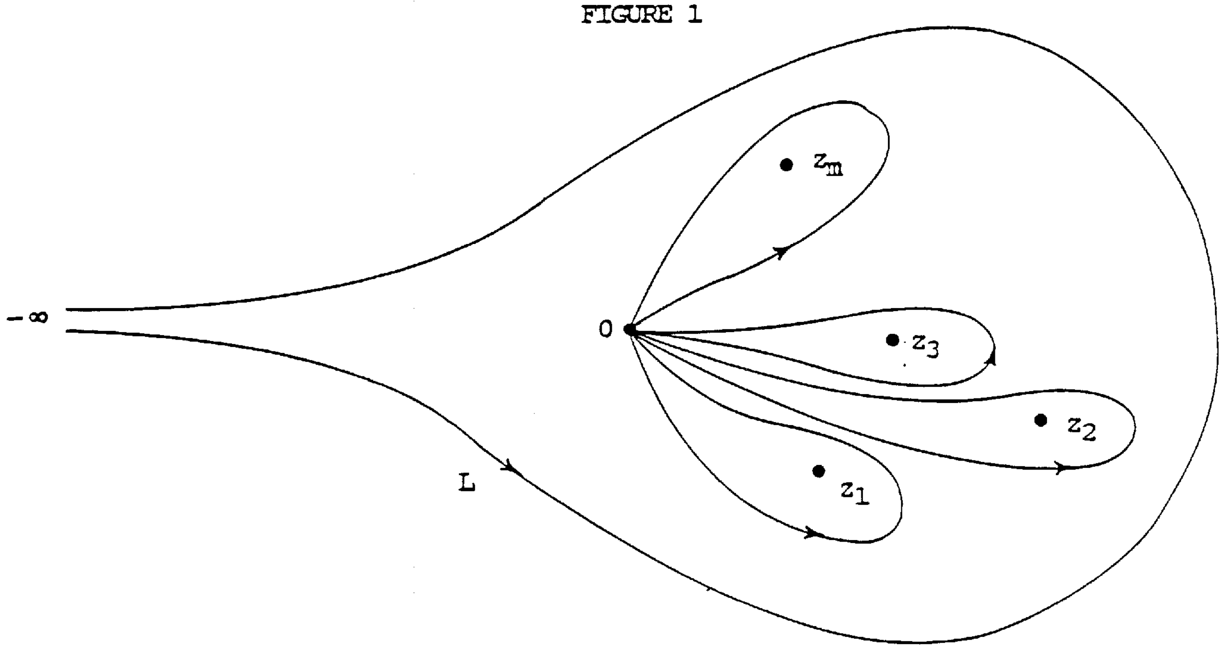

FIGURE 2