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AT YALE UNIVERSITY

Box 2125, Yale University
New Haven, Connecticut 06520

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BIMODAL t-RATIOS

by

P. C. B. Phillips

and

V. A. Hajivassiliou

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Cowles Foundation for Research in Economics
Yale University

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0. EXTENDED ABSTRACT

This paper studies the sampling distribution of the conventional t-ratio when the sample comprises independent draws from a standard Cauchy (0,1) population. It is shown that this distribution displays a striking bimodality for all sample sizes and that the bimodality persists asymptotically. An asymptotic theory is developed in terms of bivariate stable variates and the bimodality is explained by the statistical dependence between the numerator and denominator statistics of the t-ratio. This dependence also persists asymptotically. These results are in contrast to the classical t statistic constructed from a normal population, for which the numerator and denominator statistics are independent and the denominator, when suitably scaled, is a constant asymptotically.

Our results are also in contrast to those that are known to apply for multivariate spherical populations. In particular, data from an n dimensional Cauchy population are well known to lead to a t-ratio statistic whose distribution is classical t with $n-1$ degrees of freedom. In this case the univariate marginals of the population are all standard Cauchy (0,1) but the sample data involves a special form of dependence associated with the multivariate spherical assumption. Our results therefore serve to highlight the effects of the dependence in component variates that is induced by a multivariate spherical population.

Some extensions to symmetric stable populations with exponent parameter $\alpha \neq 1$ are also indicated. Simulation results suggest that the sampling distributions are well approximated by the asymptotic theory even for samples as small as $n = 20$.

1. INTRODUCTION

It is well known that ratios of random variables frequently give rise to bimodal distributions. Perhaps the simplest example is the ratio

$$(1) \qquad R = \frac{a+x}{b+y}$$

where x and y are independent $N(0,1)$ variates and a and b are constants. The distribution of R was found by Fieiller (1932) and its density may be represented in series form in terms of a confluent hypergeometric function (see Phillips (1982), equation (3.35)). It turns out, however, that the mathematical form of the density of R is not the most helpful instrument in analyzing or explaining the bimodality of the distribution that occurs for various combinations of the parameters (a,b). Instead, the joint normal distribution of the numerator and denominator statistics, $(a+x, b+y)$, provides the most convenient and direct source of information about the bimodality. An interesting numerical analysis of situations where bimodality arises in this example is given by Marsaglia (1965). Marsaglia shows that the density of R is unimodal or bimodal according to the region of the plane in which the mean (a,b) of the joint distribution lies. Thus, when (a,b) lies in the positive quadrant the distribution is bimodal whenever a is large (essentially $a > 2.257$).

Similar examples arise with simple posterior densities in Bayesian analysis and certain structural equation estimators in econometric models of simultaneous equations. Zellner (1978) provides an interesting example of the former, involving the posterior density of the reciprocal of a mean with a diffuse prior. An important, although to our knowledge previously unnoticed example of the latter, is the simple indirect least squares estimator in just identified structural equations (as studied, for instance, by Bergstrom (1962)).

The present paper shows that the phenomenon of bimodality can also occur with the classical t-ratio test statistic for populations with undefined second moments. The case of primary interest to us in this paper is the standard Cauchy (0,1) with density

$$\frac{1}{\pi(1 + x^2)} . \tag{2}$$

When the t-ratio test statistic is constructed from a random sample of n draws from this population the distribution is bimodal, even in the limit as $n \to \infty$.

This case of a Cauchy (0,1) population is especially important because it highlights the effects of statistical dependence in multivariate spherical populations. To explain why this is so, suppose $(X_1, \ldots, X_n)$ is multivariate Cauchy with density

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}(1 + x'x)^{(n+1)/2}} . \tag{3}$$

This distribution belongs to the multivariate spherical family and may be written in terms of a variance mixture of a multivariate $N(0, \sigma^2 I_n)$ as

$$\int_0^\infty N(0, \sigma^2 I_n) dG(\sigma^2) \tag{4}$$

where $1/\sigma^2$ is distributed as χ_1^2 and $G(\sigma^2)$ is the distribution function of σ^2 . Note that the marginal distributions of (3) are all Cauchy. In particular, the distribution of X_i is univariate Cauchy with density as in (2) for each i . However, the components of $(X_1, \ldots, X_n)$ are statistically dependent, in contrast to the case of a random sample from a Cauchy (0,1) population. The effect of this dependence, which is all that distinguishes (3) from the random sample Cauchy case, is dramatically illustrated by the distribution of the classical t-statistic:

$$t_X = \frac{\bar{X}}{S_X} = \frac{n^{-1}\Sigma_1^n X_i}{\left\{n^{-2}\Sigma_1^n (X_i - \bar{X})^2\right\}^{1/2}} . \tag{5}$$

Under (3), t_X is distributed as t with $n-1$ degrees of freedom, just as in the classical case of a random sample from a $N(0, \sigma^2)$ population. This was pointed out by Zellner (1976) and is an immediate consequence of (4) and the fact that t_X is scale invariant. However, the spherical assumption that underlies (3) and (4) and the dependence that it induces in the sample $(X_1, \ldots, X_n)$ is very restrictive. When it is removed and $(X_1, \ldots, X_n)$ comprise a random sample from a Cauchy (0,1) population, the distribution of t_X is very different. The new distribution is symmetric about the origin but it has distinct modes around ± 1. This bimodality persists even in the limiting distribution of t_X , so that both asymptotic and small sample theory are quite different from the classical case.

We know that the numerator and denominator statistics in the classical t-ratio are independent. Moreover, as $n \to \infty$ the denominator, upon suitable scaling, converges in probability to a constant. By contrast, in the Cauchy case, the numerator and denominator statistics of t_X converge weakly to random variables which are dependent, so that as $n \to \infty$ the t-statistic is a ratio of random variables. Moreover, it is the dependence between the numerator and denominator statistics (even in the limit) which induces the bimodality in the distribution. These differences are important and, as we shall show, they explain the contrasting shapes of the distributions in the two cases.

2. ANALYTIC RESULTS

Let $X_1, \ldots, X_n$ be a random sample from a Cauchy (0,1) distribution with density (2). Define

$$S^2 = n^{-2}\Sigma_1^n X_i^2 \ , \quad S_X^2 = n^{-2}\Sigma_1^n (X_i - \bar{X})^2 \tag{6}$$

$$t = \bar{X}/S \quad , \quad t_X = \bar{X}/S_X \ . \tag{7}$$

Throughout the paper, we will use the symbol " $\Rightarrow$ " to signify weak convergence as $n \to \infty$ and the symbol " $\equiv$ " to signify equality in distribution.

As is well known $\bar{X} \equiv$ Cauchy (0,1) for all n and, of course,

$$\bar{X} \Rightarrow X \equiv \text{Cauchy } (0,1)$$

as $n \to \infty$. Our attention will concentrate on the joint distribution of $(\bar{X}, S^2)$ and the associated statistic t given in (7). In fact, the distributions of t and t_X are asymptotically equivalent. More specifically, we have:

LEMMA 1

$$s^2 - s_X^2 = O_p(n^{-1}) ,$$

$$t - t_X = O_p(n^{-1}) .$$

Note that X_i^2 has density

$$\text{pdf}(y) = \frac{1}{\pi y^{1/2}(1+y)} , \quad y > 0 . \tag{8}$$

In fact, X_i^2 belongs to the domain of attraction of a stable law with exponent $\alpha = 1/2$. To see this we need only verify (Feller (1971, p. 313)) that if $F(y)$ is the distribution function of X_i^2 then

$$1 - F(y) + F(-y) \sim 2/\pi y^{1/2} , \quad y \to \infty$$

which is immediate from (8); and that the tails are well balanced. Here we have:

$$\frac{1 - F(y)}{1 - F(y) + F(-y)} \to 1 , \quad \frac{F(-y)}{1 - F(y) + F(-y)} \to 0 .$$

LEMMA 2

$$s^2 \Rightarrow Y ,$$

where Y *is a stable random variate with exponent* $\alpha = 1/2$ *and characteristic function given by*

$$cf_Y(v) = E(e^{ivY}) = \exp\left\{- \frac{2}{\pi^{1/2}}\cos\left(\frac{\pi}{4}\right)|v|^{1/2}\left[1 - i\ \text{sgn}(v)\tan\left(\frac{\pi}{4}\right)\right]\right\} . \tag{9}$$

Note that the characteristic function of the limiting variate Y given by (9) belongs to the general stable family, whose characteristic function (see Ibragimov and Linnik (1971, p. 43)) has the following form:

(10) $$\varphi(v) = \exp\left\{i\gamma v - c|v|^{\alpha}\left[1 - i\beta \; \mathrm{sgn}(v)\tan\left(\frac{\pi\alpha}{2}\right)\right]\right\} .$$

In the case of (9) the exponent parameter $\alpha = 1/2$, the location parameter $\gamma = 0$, the scale parameter $c = 2\pi^{-1/2}\cos(\pi/4)$ and the symmetry parameter $\beta = 1$.

Lemma 2 tells us that the denominator of the t ratio (7) is the square root of a stable random variate in the limit as $n \to \infty$. This is to be contrasted with the classical case where $nS_X^2 \underset{p}{\to} \sigma^2 = E(X_i^2)$ under general conditions.

The density of S^2 is graphed for various values of n in Figure 1. We see that S^2 is unimodal with mode lying in the interval (0,1) for all n. The distribution is very well approximated by the asymptotic even for small values of n $(n \geq 10)$.

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Figure 1 about here
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Note that when $n = 1$, the numerator and denominator of t are identical up to sign. In this case we have $t = \pm 1$ and the distribution assigns probability mass of 1/2 at +1 and −1. When $n > 1$ the numerator and denominator statistics of t continue to be statistically dependent. This dependence persists as $n \to \infty$. We have:

LEMMA 3

$$(\bar{X},\ S^2) \Rightarrow (X,Y)$$

where (X,Y) *are jointly stable variates with characteristic function given by*

$$\text{(11)} \qquad cf_{X,Y}(u,v) = \exp\left\{-2\pi^{-1/2}(-iv)^{-1/2}{}_1F_1\left(-\tfrac{1}{2},\ \tfrac{1}{2};\ u^2/4iv\right)\right\}$$

where ${}_1F_1$ *denotes the confluent hypergeometric function. An equivalent form is*

$$\text{(12)} \qquad cf_{X,Y}(u,v) = \exp\left\{-|u| - \pi^{-1/2}e^{-iu^2/4v}\Psi(3/2,\ 3/2;\ iu^2/4v)\right\}$$

where Ψ *denotes the confluent hypergeometric function of the second kind.*

For the definition of the hypergeometric functions that appear in (11) and (12) see Lebedev (1972, Ch. 9). Note that when $u = 0$ (11) reduces to

$$\text{(13)} \qquad \exp\left\{-2\pi^{-1/2}(-iv)^{1/2}\right\} .$$

We now write $-iv$ in polar form as

$$-iv = |v|e^{-i\,\mathrm{sgn}(v)\pi/2}$$

so that

$$(-iv)^{1/2} = |v|^{1/2}e^{-i\,\mathrm{sgn}(v)\pi/4}$$

$$= |v|^{1/2}\cos(\pi/4)\{1 - i\ \mathrm{sgn}(v)\tan(\pi/4)\}$$

from which it is apparent that (11) reduces to the marginal characteristic function of the stable variate Y given earlier in (9). When $v = 0$ the representation (12) reduces immediately to the marginal characteristic function, $\exp(-|u|)$, of the Cauchy variate X . In the general case the joint characteristic function $cf_{X,Y}(u,v)$ does not factorize and X and Y are dependent stable variates.

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Figure 2 about here
- - - - - - - - - - -

Figures 2a-d show Monte Carlo estimates (by smoothed kernel methods) of the joint probability surface of $(\bar{X}, S^2)$ for various values of n . As is apparent from the pictures the density involves a long curving ridge that follows roughly a parabolic shape

$$Y = a + bX^2 , \quad a \geq 0 , \quad b > 0$$

in the $(\bar{X}, S^2)$ plane. Simple estimates by ordinary least squares of such quadratic relations are presented in Figure 3 for seven values of n . The point estimates obtained are given in Table 1, part A. The prominent ridge in the joint density of $(\bar{X}, S^2)$ is a manifestation of the dependence between the two statistics $\bar{X}$ and S^2 . As is also apparent from these figures the joint distribution of $(\bar{X}, S^2)$ seems to be well approximated by the asymptotic distribution of (X,Y) . Indeed the probability surfaces appear to stabilize quite rapidly (from $n \geq 10$).

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Figure 3 about here
- - - - - - - - - - -

TABLE 1

OLS estimates of the Ridge in $pdf(\bar{X}, S^2)$

Estimated Relation: $y = a + bx^2$

A'. Cauchy Draws

Value of n	$\hat{a}$ estimate	$\hat{b}$ estimate
2	.169	.570
5	.288	.451
10	.330	.411
30	.364	.390
50	.364	.398
100	.376	.382
200	.375	.376

B'. Stable Density Draws (with exponent parameter α and $n = 10$)

Value of α	$\hat{a}$ estimate	$\hat{b}$ estimate
1/3	.484	.485
2/3	.475	.420
1	.329	.406
4/3	.252	.322
5/3	.218	.171

Note that the ridge in the joint density surfaces of Figures 2a-d is symmetric about the Y axis. The ridge is associated with clusters of probability mass for various values of Y on either side of the Y axis and equidistant from it. These clusters of mass along the ridge produce a clear bimodality in the conditional distribution of X given S^2 for all moderate to large S^2. For small S^2 the probability mass is concentrated in the vicinity of the origin in view of the dependence between $\bar{X}$ and S^2. The clusters of probability mass along the ridge in the (X,Y) plane are also responsible for the bimodality in the distribution of certain ratios of the statistics $(\bar{X}, S^2)$ such as the t ratio statistics $t = \bar{X}/S$ and $t_X = \bar{X}/S_X$. These distributions are investigated by simulation in the following section.

3. SIMULATION EVIDENCE IN THE CAUCHY CASE

The empirical distributions reported here were obtained as follows: For a given value of n, 25000 random samples of size n were drawn from the standard Cauchy distribution with density given by (2) and corresponding cumulative distribution function

$$(14) \qquad F(x) = \frac{1}{\pi} \arctan(x) , \quad -\infty < x < \infty .$$

Since (14) has a closed form inverse, the probability integral transform method was used in generating the draws. To estimate the probability density functions, the kernel method was employed (see Tapia and Thompson (1978)). For the univariate distributions (Figures 1 and 4 above and 6 below) the kernel estimate at point x is

$$\hat{f}(x) = \frac{1}{R} \sum_{r=1}^{R} \varphi\left(\frac{x - x_r}{h}\right) / h \ , \quad R = 25{,}000 \ , \tag{15}$$

where $\varphi(\cdot)$ is the standard $N(0,1)$ density and the window width h was chosen to be equal to 0.2. For the bivariate distributions in Figures 2 and 4 above and 5 below, the estimate at point (x,y) is

$$\hat{f}(x,y) = \frac{1}{R^2} \sum_{r=1}^{R} \sum_{s=1}^{R} \varphi\left(\frac{x - x_r}{h_x}\right) \varphi\left(\frac{y - y_s}{h_y}\right) / h_x \cdot h_y \tag{16}$$

with $h_x = h_y = 0.2$.

We now investigate the sampling behavior of the t-ratio statistics t and t_X . These are shown in Figures 4a and 4b. Note that the bimodality is quite striking and persists for all sample sizes.

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Figure 4 about here
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4. EXTENSIONS TO THE STABLE FAMILY

Our attention has concentrated on the sampling and asymptotic behavior of statistics based on a random sample from an underlying Cauchy (0,1) population. This has helped to achieve a sharp contrast between our results and those that are known to apply with Cauchy (0,1) populations under the special type of dependence implied by spherical symmetry. However, many of qualitative results given here, such as the bimodality of the t ratios, continue to apply for a much wider class of underlying populations. In particular, if $(X_1, \ldots, X_n)$ is a random sample from a symmetric stable population with characteristic function

$$\text{(17)} \qquad cf(s) = e^{-|s|^{\alpha}}$$

and exponent parameter $\alpha < 2$ then the t-ratios t and t_X have bimodal distributions similar in form to those shown in Figures 4a and 4b above for the special case $\alpha = 1$. To generate random variates characterized by (17) a procedure described in Section 1 of Kanter and Steiger (1974) was used. We show some examples of the new distributions which apply in these cases for various values of α in Figures 5 through 7. Note how in Figures 5a and 5b the bimodality is accentuated for $\alpha < 1$ and attenuated as $\alpha \to 2$. When $\alpha = 2$, of course, the distribution is classical t with $n-1$ degrees of freedom. These effects are also evident from Figures 6a-d and 7, which show that the ridge in the joint distribution is most pronounced for $\alpha = 1/3$ but withers as α rises to 5/3. See also Table 1, part B.

5. LACK OF CORRELATION VERSUS INDEPENDENCE

Data from an n dimensional spherical population with finite second moments have zero correlation, but are independent only when normally distributed. The standard multivariate Cauchy (with density given by (3)) has no finite integer moments but its spherical characteristic may be interpreted as the natural analogue of uncorrelated components in multivariate families with thicker tails. Our results contrast the distributional effects of lack of correlation and statistical independence in such cases. When there is only "lack of correlation" as in the spherical Cauchy case, we know that the distribution of inferential statistics such as the t-ratio reproduce the behavior that they have under independent normal draws. When there are independent draws from a Cauchy population, the statistical behavior of the t-ratio is very different. It no longer mimics behavior under a normal population but has characteristics, such as a random denominator in the limit, which distinguish its distribution from the classical t-ratio and induce the bimodality studied in this article. This example highlights the statistical implications of the differences between lack of correlation and independence in nonnormal populations.

APPENDIX: PROOFS

Proof of Lemma 1

$$S_X^2 = S^2 - n^{-1}\bar{X}^2 = S^2 + O_p(n^{-1})$$

since $\bar{X} \Rightarrow$ Cauchy (0,1) . Similarly,

$$t_X = \bar{X}\left[S^2 + O_p(n^{-1})\right]^{-1/2} = t + O_p(n^{-1})$$

as required.

Proof of Lemma 2. We start by finding the characteristic function of X_i^2 . This is

$$E(e^{ivX_i^2}) = \int_{-\infty}^{\infty} \frac{e^{ivx^2}dx}{\pi(1 + x^2)}$$

$$= \int_0^{\infty} \frac{e^{ivr}dr}{\pi r^{1/2}(1+r)}$$

$$= \left(\Gamma\left(\frac{1}{2}\right)\right)^{-1} \Psi\left(\frac{1}{2}, \frac{1}{2}; -iv\right)$$

where Ψ is a confluent hypergeometric function of the second kind. It follows that the characteristic function of $S^2 = n^{-2}\Sigma_1^n X_i^2$ is:

$$
\text{(A1)}\quad
\begin{aligned}
E(e^{ivS^2}) &= \prod_{i=1}^{n} E\left(e^{ivX_i^2/n^2}\right) \\
&= \left[\left(\Gamma\left(\tfrac{1}{2}\right)\right)^{-1} \Psi\left(\tfrac{1}{2},\ \tfrac{1}{2};\ -iv/n^2\right)\right]^n .
\end{aligned}
$$

We now use the following asymptotic expansion of the Ψ function (see Erdeyli (1953), p. 262)

$$
\Psi\left(\frac{1}{2},\ \frac{1}{2};\ \frac{-iv}{n^2}\right) = \Gamma\left(\frac{1}{2}\right) + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{-iv}{n^2}\right)^{1/2} + o\left(\frac{1}{n}\right)
$$

so that (A1) tends as $n \to \infty$ to:

$$
\exp\left\{\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)^2}(-iv)^{1/2}\right\} = \exp\left\{\frac{-2}{\pi^{1/2}}(-iv)^{1/2}\right\}
$$

Using the argument given in the text from equations (13) to (14) we deduce (9) as stated.

<u>Proof of Lemma 3</u>. We take the joint Laplace transform

$$
L(z,w) = \int_{-\infty}^{\infty} \frac{e^{zx+wx^2}}{\pi(1 + x^2)}\, dx
$$

and transform $x \to (r,h)$ according to the decomposition $x = r^{1/2}h$ where $r = x^2$ and $h = \operatorname{sgn}(x) = \pm 1$. Using the Bessel function integral

$$
\int_h e^{zrh/2}(dh) = {}_0F_1\left(\frac{1}{2},\ \frac{1}{4}z^2 r\right) = \Sigma_{k=0}^{\infty} \frac{(z^2 4)^k r^k}{k!\left(\frac{1}{2}\right)_k}
$$

we obtain

$$L(z,w) = \frac{1}{\pi}\Sigma_{k=0}^{\infty}\frac{(z^2/4)^k}{k!\left(\frac{1}{2}\right)_k} \int_0^{\infty} \frac{e^{wr}r^{k-1/2}}{(1+r)}dr$$

(A2)

$$= \frac{1}{\pi}\Sigma_{k=0}^{\infty}\frac{(z^2/4)^k\,\Gamma\left(k+\frac{1}{2}\right)}{k!\left(\frac{1}{2}\right)_k}\,\Psi\left(k+\frac{1}{2},\ k+\frac{1}{2},\ -w\right)$$

from the integral representation of the Ψ function (Erdeyli (1953), p. 255). We now use the fact that

(A3)

$$\Psi\left(k+\frac{1}{2},\ k+\frac{1}{2};\ -w\right) = \Gamma\left(\frac{1}{2}-k\right){}_1F_1\left(k+\frac{1}{2},\ k+\frac{1}{2};\ -w\right)$$

$$+ \frac{\Gamma\left(k-\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}(-w)^{1/2-k}{}_1F_1\left(1,\ \frac{3}{2}-k;\ -w\right)$$

(see Erdeyli (1953), p. 257)

$$\Gamma\left(\frac{1}{2}-k\right) = \frac{\pi}{(-1)^k\Gamma\left(k+\frac{1}{2}\right)}$$

and

$${}_1F_1\left(k+\frac{1}{2},\ k+\frac{1}{2};\ -w\right) = e^{-w}\ .$$

Combining (A2) and (A3) we have:

(A4) $$L(z,w) = \Sigma_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!\left(\frac{1}{2}\right)_k} e^{-w}$$

$$+ \frac{1}{\pi}\Sigma_{k=0}^{\infty} \frac{(z^2/4)^k \Gamma\left(k - \frac{1}{2}\right)}{k!\left(\frac{1}{2}\right)_k} (-w)^{1/2-k} {}_1F_1\left(1, \frac{3}{2} - k; -w\right)$$

Let

$$z = \frac{iu}{T}, \quad w = \frac{iv}{T^2}.$$

It follows from (A4) that

$$L\left(\frac{iu}{T}, \frac{iv}{T^2}\right) = 1 + \left\{\frac{\Gamma\left(-\frac{1}{2}\right)}{\pi} \Sigma_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_k (u/4iv)^k}{k!\left(\frac{1}{2}\right)_k}\right\} \left(\frac{-iv}{T^2}\right)^{1/2} + o\left(\frac{1}{T}\right)$$

and thus

$$\left[L\left(\frac{iu}{T}, \frac{iv}{T^2}\right)\right]^T \to \exp\left\{\frac{\Gamma\left(-\frac{1}{2}\right)}{\pi} {}_1F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{u^2}{4iv}\right)(-iv)^{1/2}\right\}.$$

Since

$$cf_{\overline{X},S^2}(u,v) = \left[L\left(\frac{iu}{T}, \frac{iv}{T^2}\right)\right]^T$$

and

$$\Gamma\left(-\frac{1}{2}\right) = -2\pi^{1/2}.$$

We deduce that

$$\text{(A5)} \qquad cf_{X,Y}(u,v) = \exp\left\{-\frac{2}{\pi^{1/2}}\ {}_1F_1\left(-\frac{1}{2}, \frac{1}{2}, \frac{u^2}{4iv}\right)(-iv)^{1/2}\right\}$$

as required for (11).

The second representation in the Lemma is obtained by noting that

$$a^{-1}x^a{}_1F_1(a,\ a+1;\ -x) = \Gamma(a) - e^{-x}\Psi(1-a,\ 1-a,\ x)$$

(Erdeyli (1953), p. 266). Using this result we find

$$\text{(A6)} \quad \left(-\frac{1}{2}\right)^{-1}(-iv)^{1/2}{}_1F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{u^2}{4iv}\right) = \frac{1}{2}|u|\left\{\Gamma\left(-\frac{1}{2}\right) - e^{u^2/4iv}\Psi\left(\frac{3}{2}, \frac{3}{2}; \frac{-u^2}{4iv}\right)\right\}.$$

Using (A6) in (A5) we obtain (12) as stated.

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Figure 1

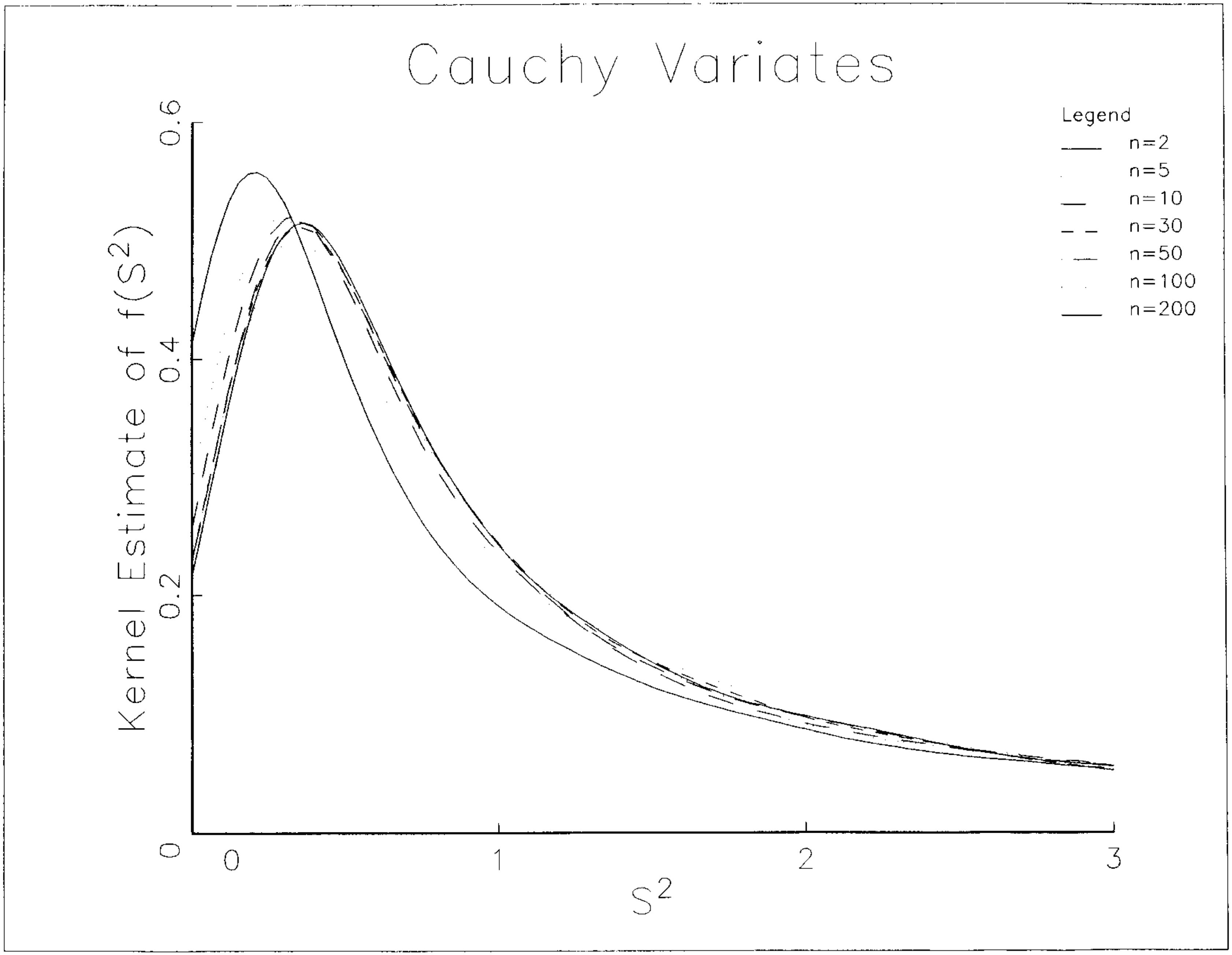

Cauchy Variates
Kernel Estimate of f(S2)
0.6
0.4
0.2
0
0
1
2
3
S2
Legend
n=2
n=5
n=10
n=30
n=50
n=100
n=200

Figure 2a

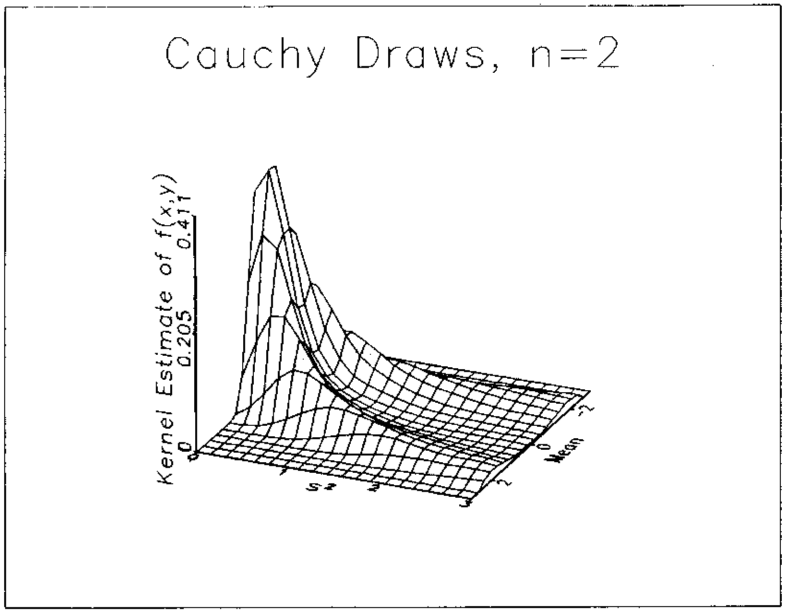


Figure 2b

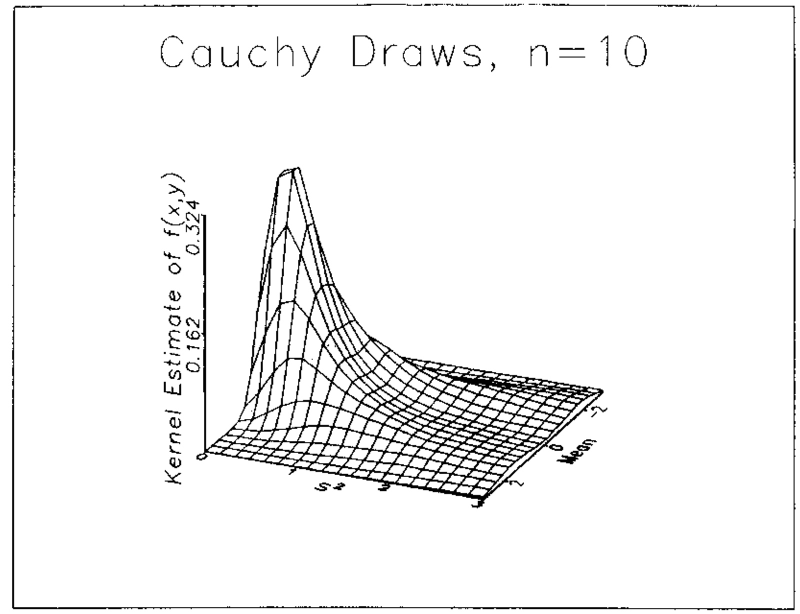


Figure 2c

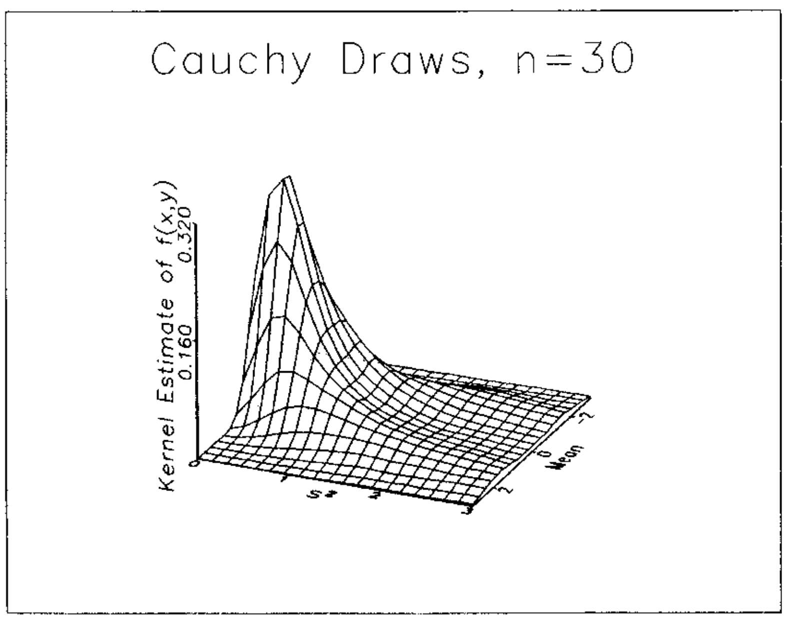


Figure 2d

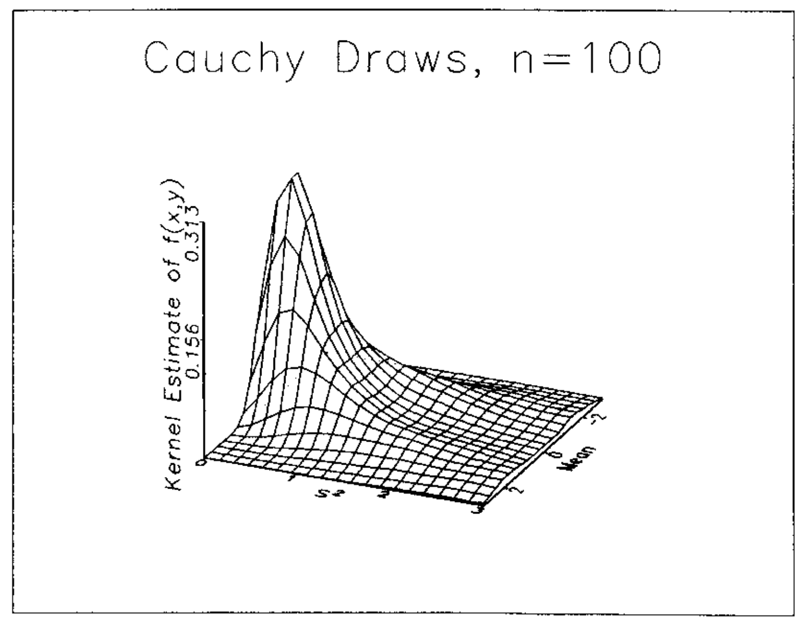
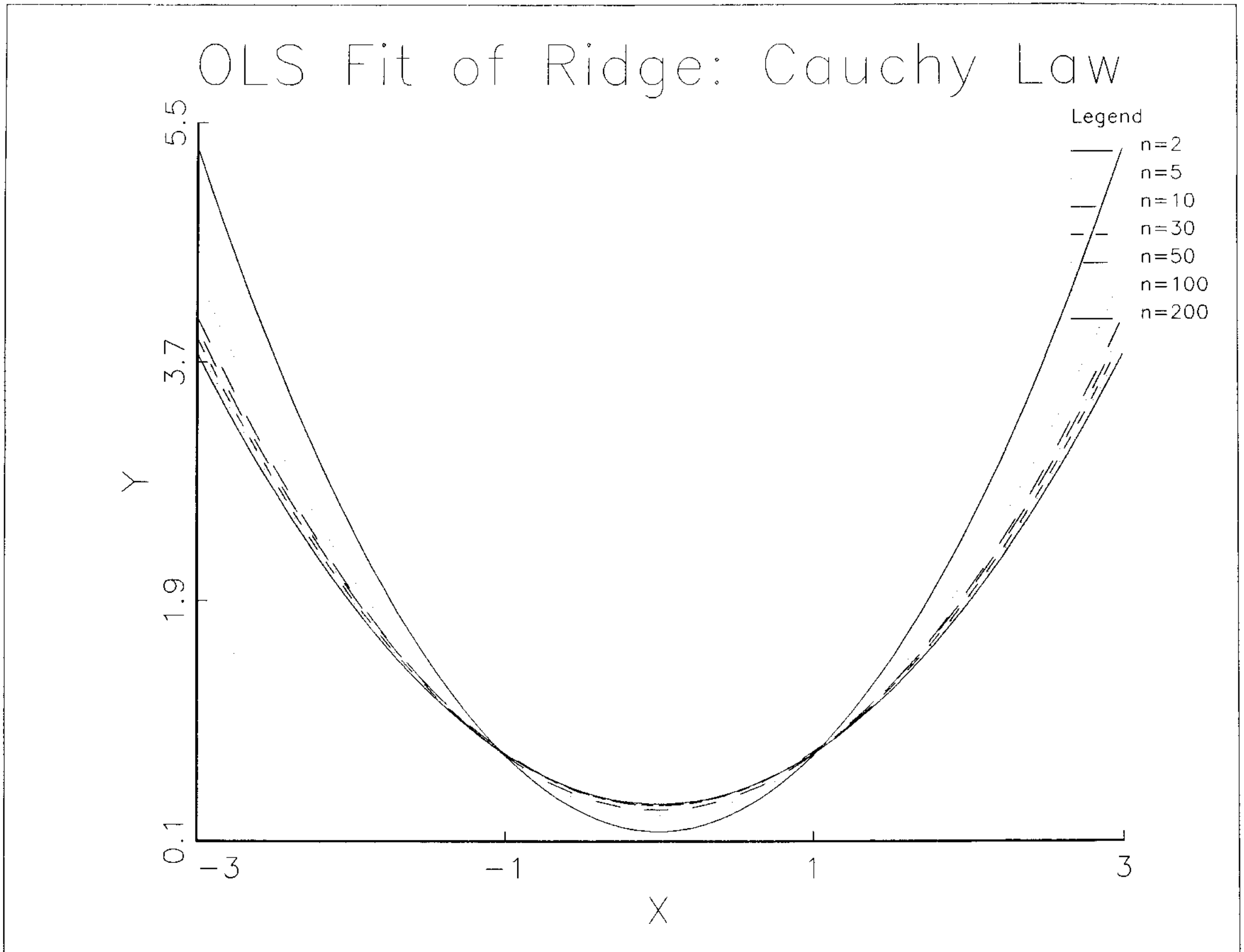


Figure 3



OLS Fit of Ridge: Cauchy Law
Legend
n=2
n=5
n=10
n=30
n=50
n=100
n=200
5.5
3.7
1.9
0.1
Y
-3
-1
1
3
X

Figure 4a

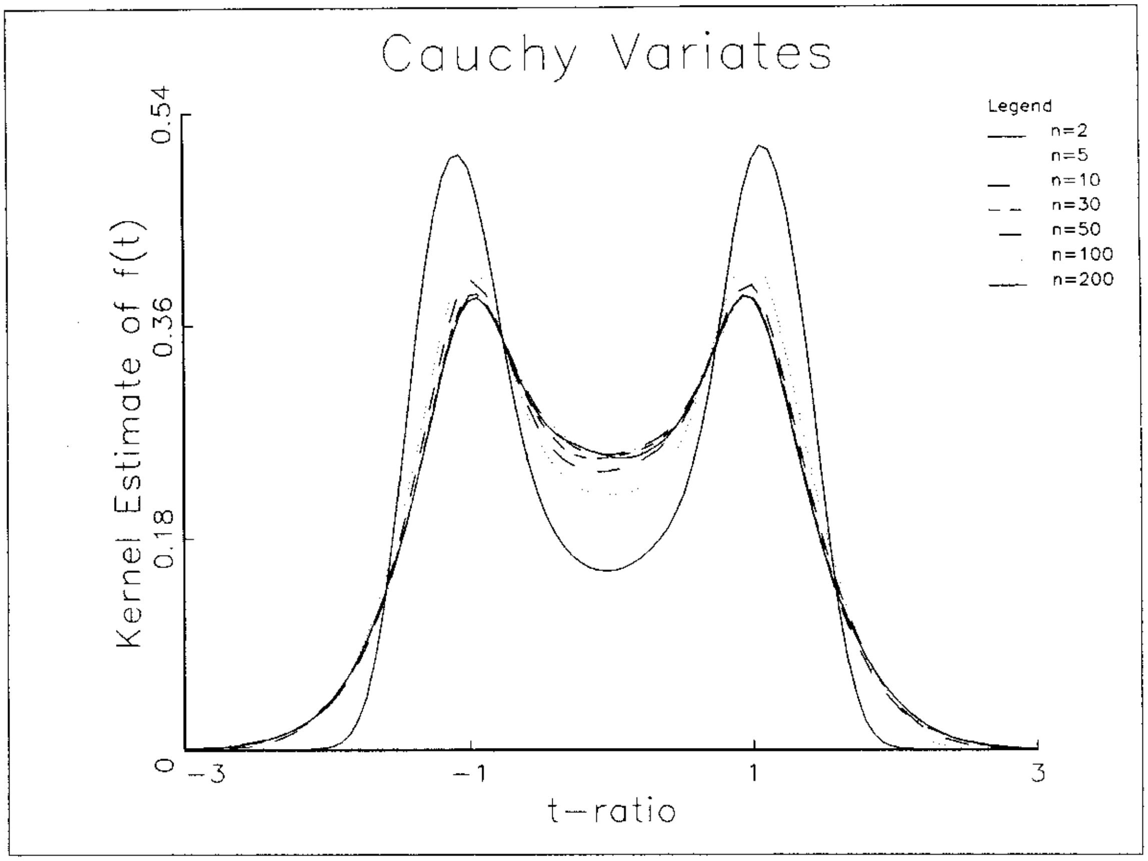


Figure 4b

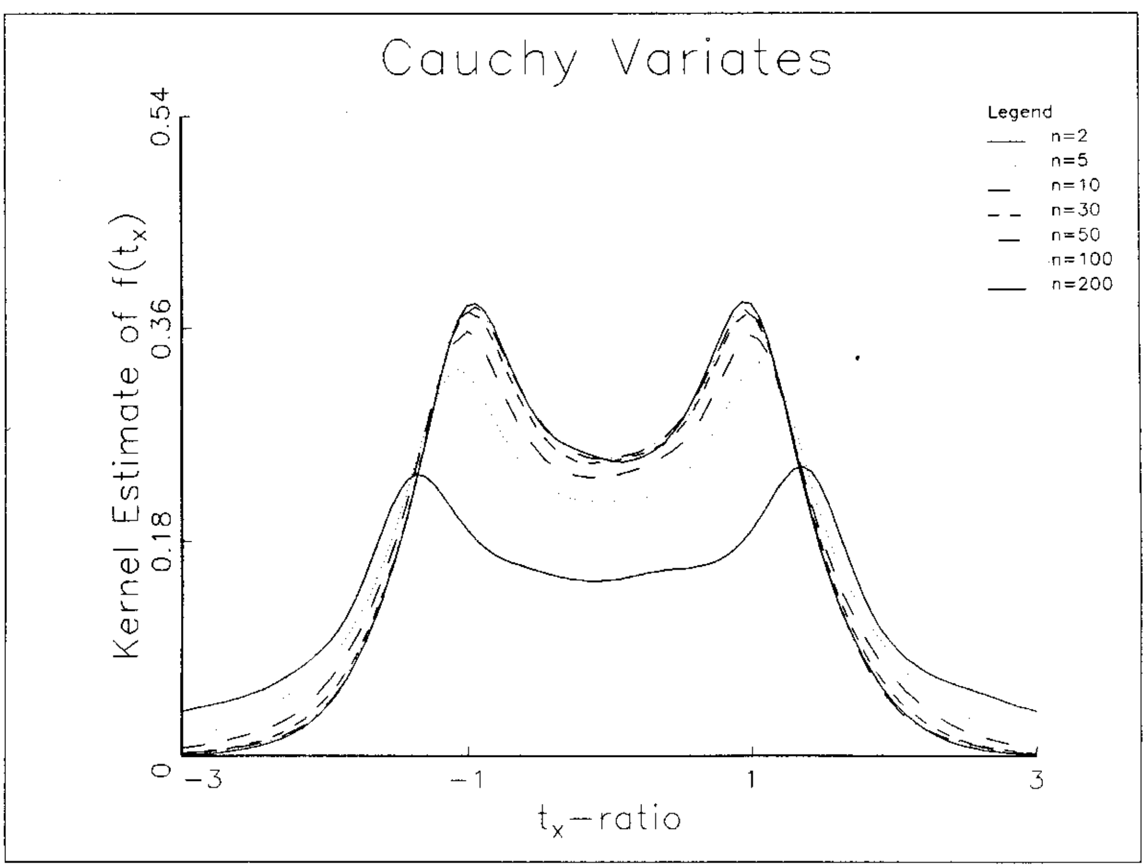


Figure 5a

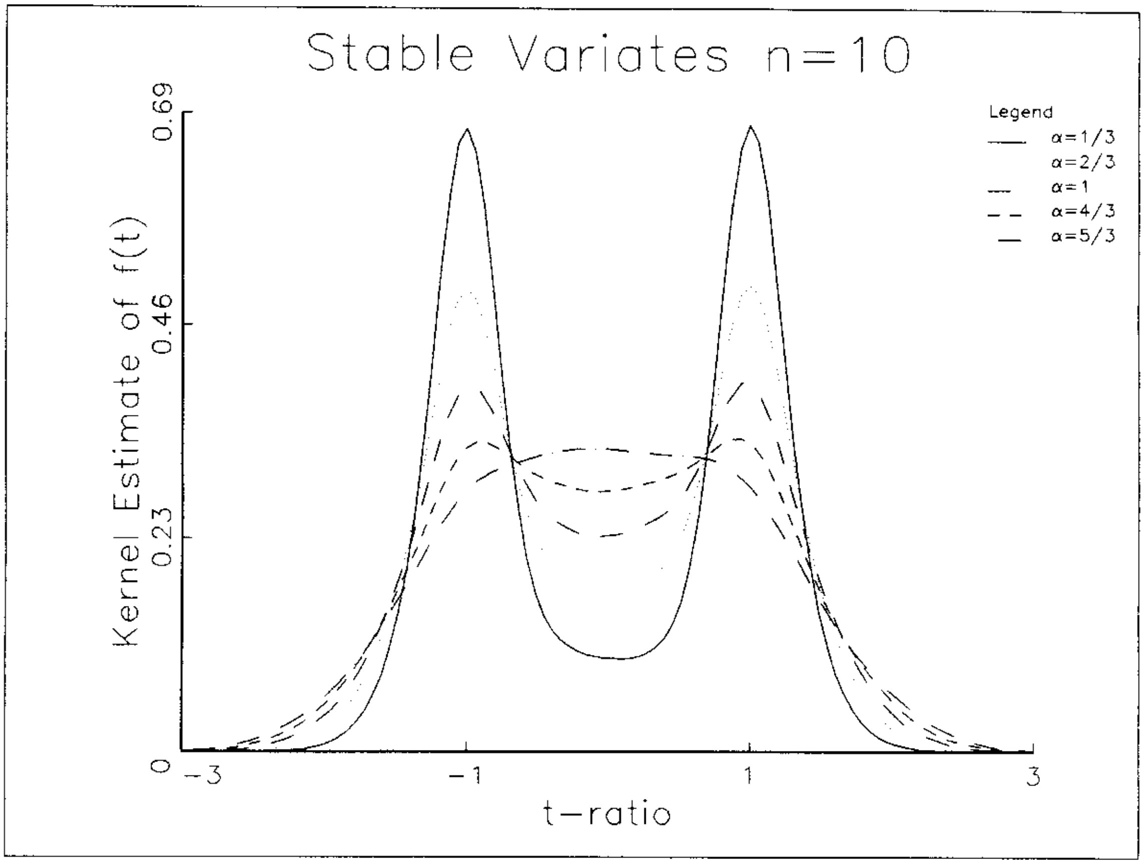


Figure 5b

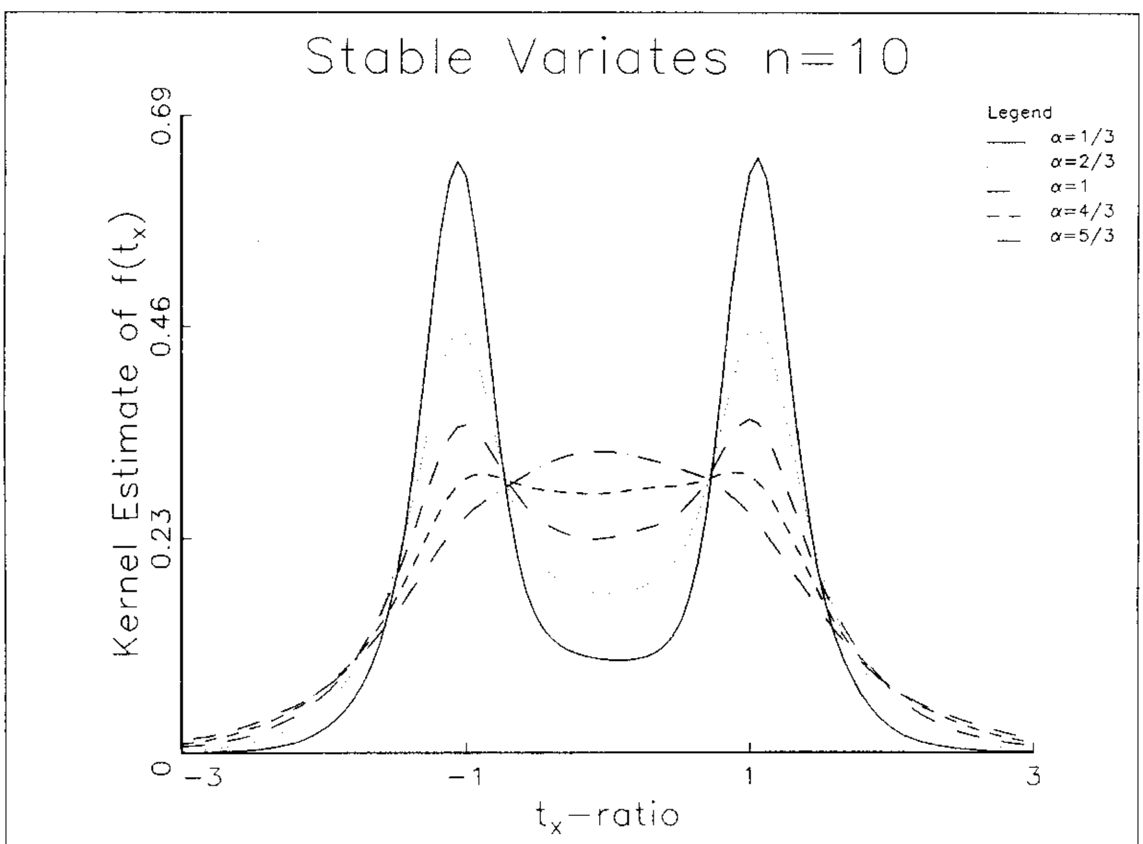


Figure 6a

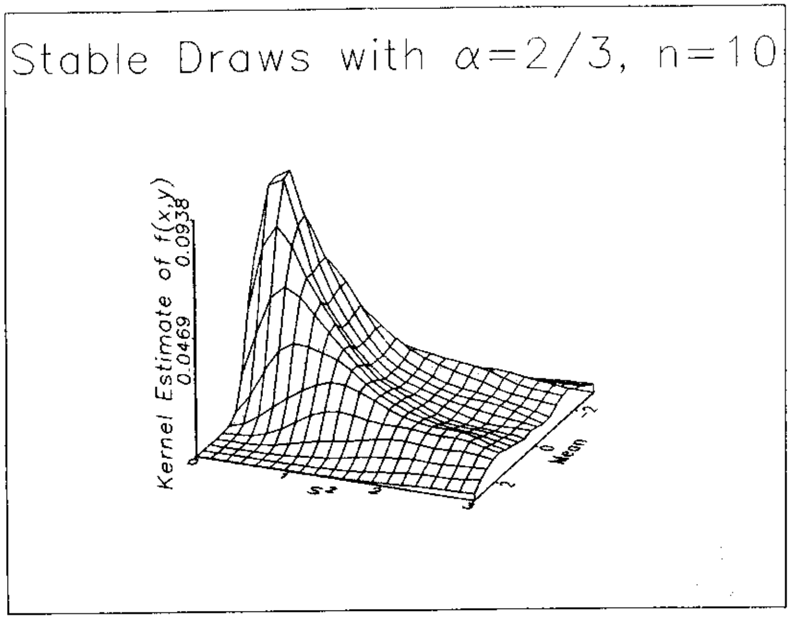


Figure 6b

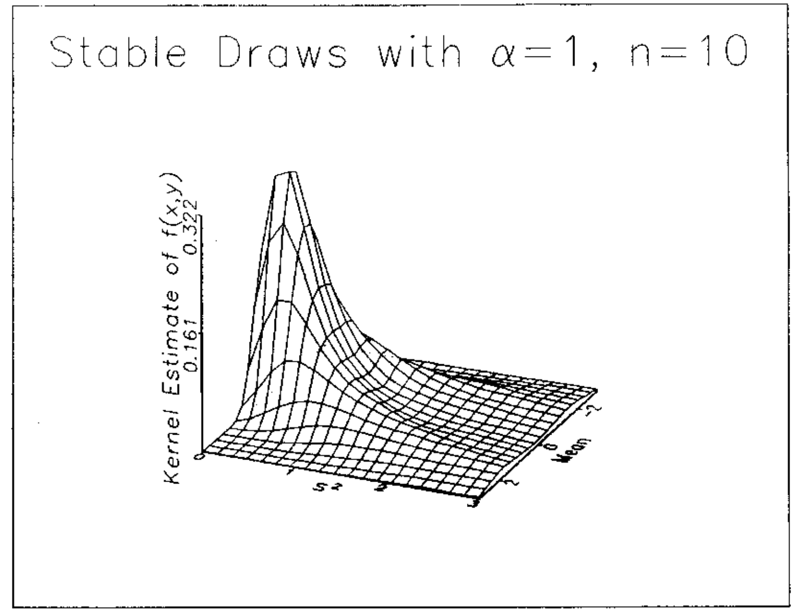


Figure 6c

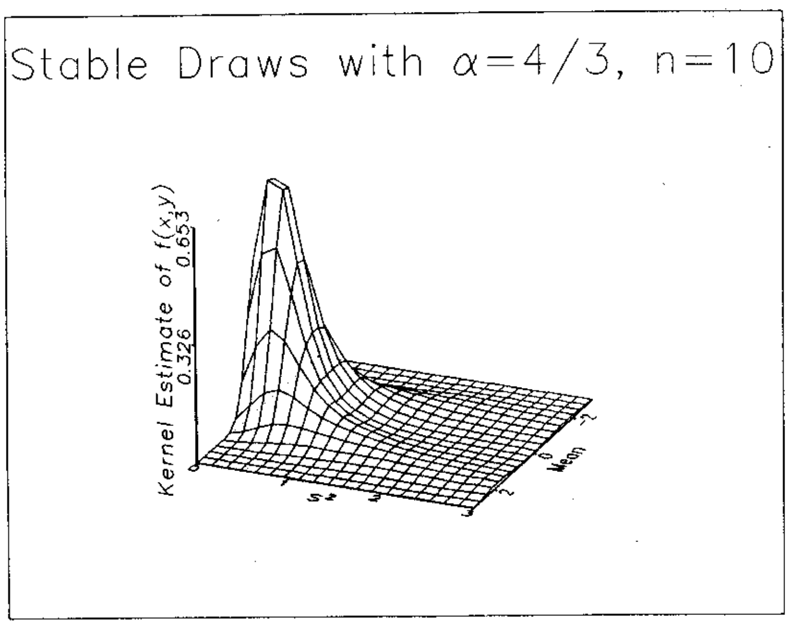


Figure 6d

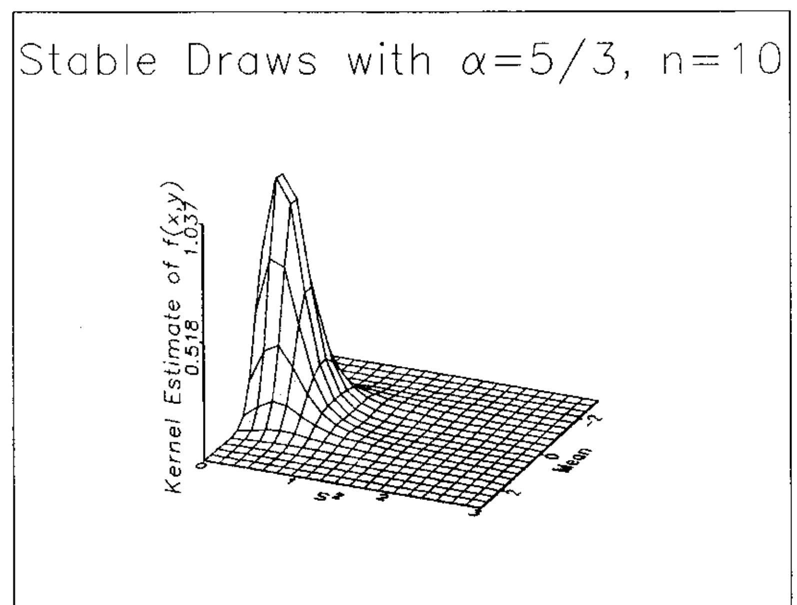
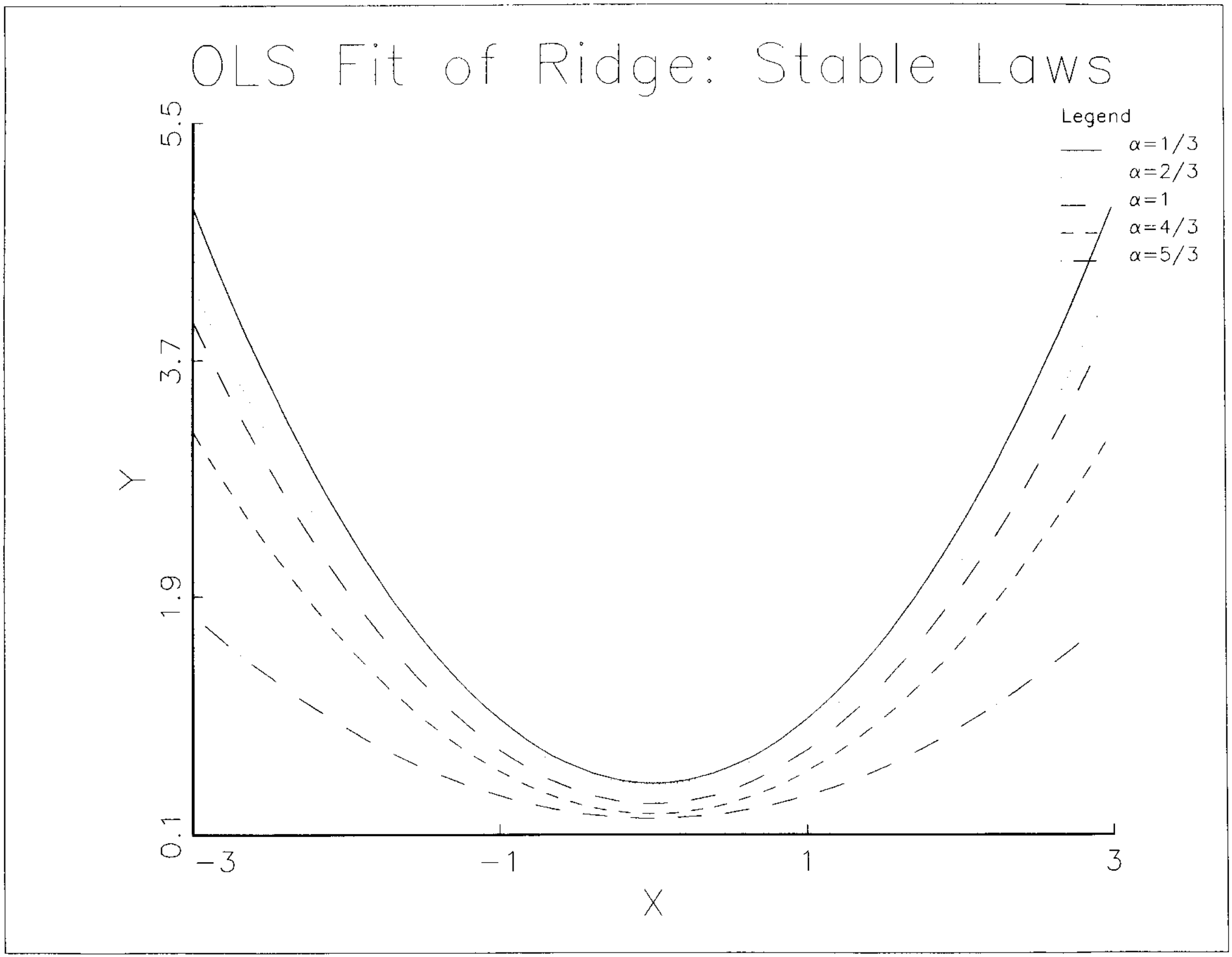


Figure 7



OLS Fit of Ridge: Stable Laws
Legend
α=1/3
α=2/3
α=1
α=4/3
α=5/3
Y
X
5.5
3.7
1.9
0.1
−3
−1
1
3