COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 825

SEQUENTIAL GAMES OF RESOURCE EXTRACTION:

EXISTENCE OF NASH EQUILIBRIA

Rabah Amir

March, 1987

Sequential Games of Resource Extraction: Existence of Nash Equilibria

Rabah Amir

Department of Applied Mathematics and Statistics

SUNY at Stony Brook

ABSTRACT

A general model for noncooperative extraction of common-property resource is considered. The main result is that this sequential game has a Nash equilibrium in stationary strategies. The proof is based on an infinite dimensional fixed-point theorem, and relies crucially on the topology of epi-convergence. A byproduct of the analysis is that Nash equilibrium strategies may be selected such that marginal propensities of consumption are bounded above by one.

Key Words and Phrases: sequential games, stationary dynamic programming, Tychonoff's fixed-point theorem, topology of epi-convergence.

The author would like to thank the following people for helpful discussions: T. Basar, L. Mirman, A. Neyman, N. Papageorgiou, W. Perkins and R. Wets.

The research described in this paper was partially supported by ONR Grant No. N00014-86-K-0220.

1. Introduction

Sequential games are increasingly recognized to constitute an appropriate approach for the modelling and analysis of a wide variety of problems in economics, and related fields. Indeed, situations with several decision makers pursuing non identical goals, whose actions jointly determine the future states of the economy are abundant in economic theory. Such games, also referred to as dynamic games in the engineering literature (Basar-Olsder [1982]), are the discrete-time counterparts of differential games. They may also be regarded as deterministic analogs of stochastic games (Shapley [1953]) with uncountable state and action spaces.

Among the most common applications of sequential games in economics is the study of strategic intertemporal resource extraction. The case of common property resources was the object of several studies: Levhari-Mirman [1980] and Mirman [1979] consider the sum of discounted utilities as the objective for each player while Reinganum [1981] and Reinganum-Stokey [1981] consider profits. It should be pointed out that the model with utility maximization could also be used to study equilibrium growth in an aggregated economy with two or more classes, each concerned with the maximization of the sum of discounted utilities from its own consumption. In this sense, sequential games of resource extraction constitute a merging of two separate lines of research: optimal growth theory and optimal exploitation of natural resources on the one hand, and dynamic game theory on the other.

In all the above referenced studies and in Shubik-Whitt [1973], the adoption of the Markovian information structure is coupled with a lack of generality in the game formulation. Namely, particular functional forms are selected for the utility and growth functions. In fact, Reinganum [1981] argues that writers have faced the dilemma of choosing between the generality of the results which required the use of open-loop strategies, and the appropriateness of the information structure (i.e. Markovian) which imposed a degree of specificity in the growth and objective functions.

The present paper consists of a generalization of the model in Levhari-Mirman [1980] in that it considers the same sequential game with stationary strategies and general unspecified concave utility and growth functions. The difficulty with this general analysis arises from the fact that nonconvexities are inherent to such games. Clearly, the properties of the optimization problem faced by a player, given the other player's strategy (as a function of the state variable) depend on the properties of this strategy. Indeed such a problem amounts generally to maximizing a concave functional subject to a nonconvex feasible set, when formulated in the appropriate sequence spaces. The nonclassical optimal growth model provides such an example (Skiba [1978], Dechert-Nishimura [1983], Amir [1985]).

A related dynamic game, referred to as an altruistic growth model, has been studied in full generality (including a Nash equilibrium existence proof) in Bernheim-Ray [1983] and Leininger [1983].

The paper is organized as follows: Section 2 contains the statement of the problem under consideration. Section 3 studies the properties of the best response optimization problem, and Section 4 provides the Nash equilibrium existence theorem and its proof.

2. The Model

At any period t, the resource stock x_t may either be consumed (c_t^1 by agent 1 and c_t^2 by agent 2) or jointly invested ($x_t - c_t^1 - c_t^2$). Letting f denote the natural growth function of the resource, then $x_0 = s$ (historically given), and

$$x_{t+1} = f(x_t - c_t^1 - c_t^2), \quad t = 0, 1, 2, \ldots$$

The properties of f are

f is differentiable and $f' > 0$.

f is concave

$f(0) = 0, \lim_{x \downarrow 0} f'(x) > 1.$

It can easily be shown that, given the properties of f, there exists a unique $\bar{x} > 0$ such that $\bar{x} = f(\bar{x})$.

The preferences of agent i (i = 1, 2) are expressed by a utility function u_i satisfying the standard properties from optimal growth theory:

u is differentiable and $u' > 0$.

u is strictly concave

$\lim_{x \downarrow 0} u'(0) < +\infty$.

Furthermore, the objective of agent i, whose time rate of preference is given by the discount factor $0 < \delta_i < 1$, is to maximize the present (discounted) value of utility over an infinite horizon while taking into consideration the actions of the other agent. Specifically, the payoff to agent i is:

$$J_i(\{c_t^i\}) = \sum_{t=0}^{\infty} \delta_i^t u_i(k_t^i), \quad t = 0, 1, 2, \ldots \tag{2.1}$$

where $$k_t^i = \begin{cases} c_t^i & \text{if } c_t^1 + c_t^2 \leq x_t \\ x_{t/2} & \text{if } c_t^1 + c_t^2 > x_t \end{cases}$$

The state transition law is given by

$$x_{t+1} - f(x_t - c_t^1 - c_t^2) \leq 0, \quad x_0 = s \tag{2.2}$$

with $$x_t \geq c_t^1 \geq 0, \quad x_t \geq c_t^2 \geq 0, \quad t = 0, 1, 2, \ldots \tag{2.3}$$

Constraint (2.3) is the feasibility condition which forces individual consumptions to be positive and at most equal to the available stock. A pair $(\{c^1\}, \{c^2\})$ is referred to as feasible if it satisfies constraints (2.2)-(2.3)

and $$c_t^1 + c_t^2 \leq x_t, \quad t = 0, 1, \ldots \tag{2.4}$$

It is interior if, in addition, constraints (2.3) and (2.4) hold with all strict inequalities, and as a corner or boundary pair if (2.4) and (2.3) are satisfied with at least one equality.

One could also impose the constraint $c_t^1 + c_t^2 \leq x_t$, $t = 0, 1, \ldots$, thus making the actions available to one player dependent on those taken by the other player, and obtaining a generalized game formation (Cf. Debreu [1952]). Furthermore, it turns out that the assumptions $u'(0) = +\infty$ and $f(0) = 0$ are sufficient to guarantee interior Nash equilibria; in other words, the constraints (2.3) are never binding at equilibrium.

The Nash equilibrium is commonly considered as the appropriate solution concept for non-cooperative resource extraction. Based on the criterion that neither player can improve on his payoff by unilaterally departing from the equilibrium, it is defined as follows, for the problem at hand:

<u>Definition 2.1</u>

A Nash pair $(\{c_t^{1*}\}, \{c_t^{2*}\})$ is said to constitute a Nash equilibrium for the game problem (2.1)-(2.3) if: -$(\{c_t^{1*}\}, \{c_t^{2*}\})$ is a feasible pair.

-$J_1(\{c_t^{1*}\})\Big|_{\{c_t^{2*}\}} \geqq J_1(\{c_t^1\})\Big|_{\{c_t^{2*}\}}$, for all $\{c_t^1\}$ such that the pair $(\{c_t^1\}, \{c_t^{2*}\})$ is feasible.

-$J_2(\{c_t^{2*}\})\Big|_{\{c_t^{1*}\}} \geqq J_2(\{c_t^2\})\Big|_{\{c_t^{1*}\}}$, for all $\{c_t^2\}$ such that the pair $(\{c_t^{1*}\}, \{c_t^2\})$ is feasible.

We restrict the information available to each player, at any given stage of the sequential game (2.1)-(2.3), to the knowledge of the state x_t at that period. We will thus be concerned with Nash equilibria in pure stationary strategies: policies which take the same action whenever the system is in the state, regardless of date. Such equilibria satisfy a strong version of subgame perfection. They are also referred to as closed-loop no-memory (Basar-Olsder [1982] or as recursive equilibria (Cave [1984]). They rule out the possibility of threats or retalitory behavior, and are insensitive to the history through which a given state is arrived at.

This information structure is thus somewhat restrictive and does not capture some important strategic interactions that one might associate with non-cooperative extraction of common-property resources. Nevertheless, such equilibria have considerable appeal in dynamic models of strategic competition. Furthermore, they remain equilibria even when strategies are allowed to depend on (all or part of) the history of the game.

3. Properties of the Best Response Map

This section provides some intermediate results needed for the characterization of Nash equilibria, the subject of the next section. We will be concerned with the dynamic optimization problem faced by Player I when Player II's stationary strategy is given as a single-valued function $\gamma_2(\cdot)$ satisfying $0 < \gamma_2(x) < x$, for $x > 0$, namely:

$$\max_{\{c_t^1\}} J_1(\{c_t^1\}) = \sum_{t=0}^{\infty} \delta_1^t u_1(c_t^1) \tag{3.1}$$

subject to

$$x_{t+1} - f(x_t - c_t^1 - \gamma_2(x_t)) \leqq 0, \quad x_0 = s \text{ (fixed)} \tag{3.2}$$

and

$$0 \leqq c_t^1 \leqq x_t - \gamma_2(x_t), \quad t = 0, 1, \ldots \tag{3.3}$$

The maximizer in problem (3.1)-(3.3), to be represented by $\gamma_1(\cdot)$, is the best response of Player 1 to $\gamma_2(\cdot)$. It is in general a multivalued function or correspondence. Clearly, the properties of $\gamma_1(\cdot)$ will depend on those of $\gamma_2(\cdot)$. First, observe that since the objective functional is concave, $\gamma_1(\cdot)$ will be a (single-valued) function if the constraint set given by the intersection of the two sets represented by inequalities

(3.2) and (3.3) is convex. This will be the case if and only if $\gamma_2(\cdot)$ is a convex function, as is easily seen. However, there is no justification for restricting the players to use only convex strategies, particularly since the best response to a convex strategy is continuous but not necessarily convex.

Define a value function V_{γ_2} to Player I of responding optimally to $\gamma_2(\cdot)$ by

$$V_{\gamma_2}(x) = \max_{\{c_t^1\}} J_1(\{c_t^1\}), \quad x_0 = x .$$

Let $x_m = \max\{x_0, \overline{x}\}$, where $\overline{x}$ is the unique fixed-point of f. The properties of f imply that $x_t \leqq x_m$, for all t = 0, 1, ... , and hence

$$V_{\gamma_2}(x) \leqq \frac{1}{1-\delta_1} u_1(x_m) .$$

It is easily shown that V_{γ_2} is the unique fixed-point of the contraction mapping R: $B[0,x_m] \to B[0,x_m]$ (bounded functions on $[0,x_m]$ with sup norm) $v \to \sup_{0 \leqq c \leqq x-\gamma_2(x)} \{u_1(c) + \delta v[f(x-c-\gamma_2(x))]\}$.

Hence,
$$V_{\gamma_2}(x) = \max_{0 \leqq c^1 \leqq x-\gamma_2(x)} M_1(c^1;x), \tag{3.4}$$

where
$$M_1(c^1;x) = u_1(c^1) + \delta_1 V_{\gamma_2}[f(x-c^1-\gamma_2(x))]. \tag{3.5}$$

Before proceeding to the statement of results, we set some notations and abbreviations: For a real-valued function $F(\cdot)$, we denote the left and right-hand limits of F at a point x by $F(x^-)$ and $F(x^+)$, the left and right hand (one-sided) derivatives by $F'(x^-)$ and $F'(x^+)$. Semicontinuity for single-valued functions is abbreviated u.s.c. (uppersemi-continuous) or l.s.c. (lowersemi-continuous). The corresponding notions for set-valued functions are abbreviated u.h.c. (uppersemi-continuous) and l.h.c. (lowersemi-continuous), in accordance with Hildenbrand-Kirman [1976]. Finally, an ε-neighborhood of a point x will be referred to as $N(x;\varepsilon)$.

The following result gives a sufficient condition for $M_1(c_1;x)$ to achieve its maximum (as a function of c_1, for c_1 in $[0,x]$), or in other words for the optimization problem (3.1)-(3.3) to be well-defined.

Lemma 3.1. The best response $\gamma_1(\cdot)$ to the strategy $\gamma_2(\cdot)$ is well-defined if $\gamma_2(\cdot)$ is a l.s.c. function. Furthermore, $\gamma_1(\cdot)$ is a u.h.c. set-valued function.

Proof of Lemma 3.1

If $\gamma_2(\cdot)$ is a l.s.c. function, then by Theorem 4.2.2 in Bank et al. [1983], $V_{\gamma_2}(\cdot)$ is a u.s.c. function. Hence,
$M_1(c_1;x) = u_1(c_1) + \delta_1 V_{\gamma_2}[f(x-c_1-\gamma_2(x))]$ is also a u.s.c. function of c, and thus achieves its maximum on $[0,x]$.

That the best response correspondence $\gamma_1(\cdot)$ is u.h.c. follows directly from Theorem 4.2.1 in Bank et al. [1983].

On the other hand, it is apparent from the proof of Lemma 3.1 that if $\gamma_2(\cdot)$ is a u.s.c. function, $M_1(\cdot;x)$ will be a l.s.c. function of c, and thus may or may not achieve its maximum on $[0, x_m]$. Likewise, Player 1 may choose a u.s.c. selection from the u.h.c. correspondence $\gamma_1(\cdot)$, thus possibly making Player 2's subsequent best response optimization ill-defined. However, if Player 1 also picks a l.s.c. selection out of $\gamma_1(\cdot)$ (such a selection will be shown to always exist), then this potential difficulty does not arise.

In the corresponding one-person dynamic program, it is well-known that the value function is an increasing continuous function (see, e.g. Amir [1985a]). In the present context, however, the properties of V_1 will clearly depend on those of γ_2. We show in particular that V_1 and γ_2 are differentiable and continuous at the same points, and that V_1 increases between two given points if and only if the effective stock level (i.e. the stock level after Player 2's consumption) increases between those two points.

<u>Lemma 3.2.</u> <u>The value function</u> V_{γ_2} <u>satisfies:</u> $V_{\gamma_2}(x) \gtreqless V_{\gamma_2}(y)$ <u>if and only if</u> $\frac{\gamma_2(x) - \gamma_2(y)}{x - y} \lesseqgtr 1$, <u>for all distinct</u> x, y <u>in</u> $[0, x_m]$.

<u>Proof of Lemma 3.2</u>

Let $\gamma_2(\cdot)$ be any (single-valued) strategy by Player 2 and $\gamma_1(\cdot)$ be the (multi-valued) best response of Player 1. Consider two different values of the initial resource stock, say $x > 0$ and $x + \alpha$, $\alpha > 0$.

By the principle of optimality, if starting at initial stock x Player 1 alters his consumption level in the first period, $\gamma_1(x)$, to $\{\gamma_1(x+\alpha -\alpha + \gamma_2(x+\alpha) - \gamma_2(x)\}$[1] while Player 2 maintains his first-period consumption at $\gamma_2(x)$, it follows that

$$V_1(x) = u_1[\gamma_1(x)] + \delta_1 V_1[f(x-\gamma_1(x)-\gamma_2(x))] \tag{A.1}$$

$$\geq u_1[\gamma_1(x+\alpha)-\alpha+\gamma_2(x+\alpha)-\gamma_2(x)]+\delta_1 V_1[f(x+\alpha-\gamma_1(x+\alpha)-\gamma_2(x+\alpha))]. \tag{A.2}$$

Likewise, starting at $x+\alpha$:

---

[1] $\gamma_1(x)$ should be considered as any point in the set $\{\gamma_1(x)\}$ (The same interpretation is attached to $\gamma_1(x+\alpha)$), provided the same point is taken throughout the argument.

$$V_1(x+\alpha) = u_1[\gamma_1(x+\alpha)]+\delta_1 V_1[f(x+\alpha-\gamma_1(x+\alpha)-\gamma_2(x+\alpha))] \quad (A.3)$$

$$\geqq u_1[\gamma_1(x)+\alpha-\gamma_2(x+\alpha)+\gamma_2(x)]+\delta_1 V_1[f(x-\gamma_1(x)-\gamma_2(x))]. \quad (A.4)$$

Comparing (A.2) and (A.3), then (A.4) and (A.1), we get, respectively:

$$V_1(x) \geqq V_1(x+\alpha) \quad \text{if} \quad \gamma_2(x+\alpha) - \gamma_2(x) - \alpha \geqq 0$$

and

$$V_1(x+\alpha) \geqq V_1(x) \quad \text{if} \quad -\gamma_2(x+\alpha) + \gamma_2(x) + \alpha \geqq 0,$$

so that $V_1(x+\alpha) = V_1(x)$ if and only if $\gamma_2(x+\alpha) - \gamma_2(x) - \alpha = 0$.

The desired conclusion clearly follows by setting $y = x+\alpha$.

<u>Lemma 3.3</u>. <u>A necessary condition for</u> $\gamma_1(x)$ <u>to be the best response correspondence to the single valued strategy</u> $\gamma_2(x)$ <u>is that:</u>

$$\left[1 - \frac{\gamma_2(x)-\gamma_2(y)}{x-y}\right]\left[1 - \frac{\gamma_1(x)-\gamma_1(y)}{x-y} - \frac{\gamma_2(x)-\gamma_2(y)}{x-y}\right] \geqq 0,$$ <u>for all distinct</u> x, y <u>in</u> $[0, x_m]$.

<u>Proof of Lemma 3.3</u>

The following preliminary result is needed:

<u>Lemma A.1</u>. <u>Suppose that</u> $u : \mathbb{R}^+ \to \mathbb{R}^+$ <u>is a strictly concave</u>, <u>increasing function</u>, <u>and that</u> a, b, c, d <u>are distinct real numbers such that</u> $a+d = b+c$. <u>Then</u> $a \leqq b \leqq c \leqq d$ <u>implies that</u> $u(a) + u(d) \leqq u(b) + u(c)$. <u>Conversely if</u> $u(a) + u(d) \leqq u(b) + u(c)$, <u>then</u>

$$\min\{a,d\} \leqq \min\{b,c\} \leqq \max\{b,c\} \leqq \max\{a,d\} .$$

<u>Proof</u>. It is straightforward.

Now, to prove Lemma 3.1, add inequalities (A.2) and (A.4):

$$u_1[\gamma_1(x)] + u_1[\gamma_1(x+\alpha)] \geqq u_1[\gamma_1(x+\alpha)-\alpha+\gamma_2(x+\alpha)-\gamma_2(x)] + u_1[\gamma_1(x)+\alpha-\gamma_2(x+\alpha)+\gamma_2(x)]. \quad (A.5)$$

Let

$$\begin{cases} G_1(x,\alpha) = \gamma_1(x+\alpha) - \alpha + \gamma_2(x+\alpha) - \gamma_2(x) \\ G_2(x,\alpha) = \gamma_1(x) + \alpha - \gamma_2(x+\alpha) + \gamma_2(x). \end{cases}$$

Then, by Lemma A.1, since $G_1 + G_2 = \gamma_1(x) + \gamma_1(x+\alpha)$, we have $\max\{G_1, G_2\} \geqq \max\{\gamma_1(x), \gamma_1(x+\alpha)\} \geqq \min\{\gamma_1(x), \gamma_1(x+\alpha)\} \geqq \min\{G_1, G_2\}$. This, in turn, implies that, either

$$\begin{aligned} G_1(x,\alpha) &\geqq \gamma_1(x+\alpha) \\ G_1(x,\alpha) &\geqq \gamma_1(x) \end{aligned} \quad (A.6)$$

or

$$\begin{aligned} G_1(x,\alpha) &\leqq \gamma_1(x+\alpha) \\ G_1(x,\alpha) &\leqq \gamma_1(x) \end{aligned} \quad (A.7)$$

Equivalently,

$$\begin{cases} \dfrac{\gamma_2(x+\alpha) - \gamma_2(x)}{\alpha} \geqq 1 \\ \dfrac{\gamma_1(x+\alpha) - \gamma_1(x)}{\alpha} + \dfrac{\gamma_2(x+\alpha) - \gamma_2(x)}{\alpha} \geqq 1 \end{cases} \quad (A.6a)$$

or

$$\begin{cases} \dfrac{\gamma_2(x+\alpha) - \gamma_2(x)}{\alpha} \geqq 1 \\ \dfrac{\gamma_1(x+\alpha) - \gamma_1(x)}{\alpha} + \dfrac{\gamma_2(x+\alpha) - \gamma_2(x)}{\alpha} \leqq 1 \end{cases} \quad (A.7a)$$

The argument given above is valid only if the altered consumption levels via inequalities (A.2) and (A.4) are feasible, that is, if

$$\begin{cases} 0 \leqq G_1(x,\alpha) \leqq x & \text{(A.8a)} \\ 0 \leqq G_2(x,\alpha) \leqq x+\alpha. & \text{(A.8b)} \end{cases}$$

The upper bounds are easily seen to be satisfied. One of the two lower bounds must also be satisfied since $G_1(x,\alpha) + G_2(x,\alpha) = \gamma_1(x) + \gamma_2(x+\alpha) > 0$. Assume that the lower bound in (A.8a) is not satisfied, i.e., that $G_1(x,\alpha) < 0$, then since $G_1 + G_2 = \gamma_1(x) + \gamma_1(x+\alpha)$,

$$\begin{cases} G_2(x,\alpha) > \gamma_1(x) \\ G_2(x,\alpha) > \gamma_1(x+\alpha) \ . \end{cases}$$

Solving this system of equations yields (A.7a). Similarly if $G_2(x,\alpha) < 0$, it can be shown that (A.6a) follows. Hence, the same conclusion follows without using changes in paths if feasibility is violated in (A.2) and (A.4).

Finally, since x and α are arbitrary, we can express (A.6a) and (A.7a) in the more compact fashion given in the statement of the Lemma.

The condition given in Lemma 3.3 has a clear strategic implication: If Player II'2 strategy γ_2 is "greedy" (resp. "helpful") between stock levels x and y, i.e. has slopes larger (resp. smaller) than one, then Player I will respond in such a way that total consumption is "greedy" (resp. "helpful").

The following result may be referred to as the envelope theorem for problem (3.1)-(3.3):

Lemma 3.4. The value function V_{γ_2} is continuous (resp. differentiable) at a point x in $[0, x_m]$ if and only if γ_2 is continuous at x (resp. if and only if γ_1 is continuous at x and γ_2 is differentiable at x). Furthermore, at points of differentiability of V_{γ_2}, V'_{γ_2} is given by

$$V'_{\gamma_2}(x) = u'[\gamma_1(x)](1-\gamma_2'(x)),$$

and at points of discontinuity of V_{γ_2},[1]

$$V_{\gamma_2}(x^s) = \begin{cases} +\infty, & \text{if } \gamma_2'(x^2) = -\infty, \quad s = + \text{ or } - \\ -\infty, & \text{if } \gamma_2'(x^s) = +\infty, \quad s = + \text{ or } - . \end{cases}$$

Proof of Lemma 3.4

From (A.4) and (A.1), it follows that, for any sequence $(\alpha_n) \downarrow 0$

$$V_1(x+\alpha_n) - V_1(x) \geq u_1[\gamma_1(x) + \alpha - \gamma_2(x+\alpha_n) + \gamma_2(x)] - u_1[\gamma_1(x)] \qquad (A.9)$$

or equivalently,

$$\frac{V_{\gamma_2}(x+\alpha_n) - V_{\gamma_2}(x)}{\alpha_n} \geq \frac{u_1\left[\gamma_1(x)+ 1 - \frac{\gamma_2(x+\alpha_n)\ \gamma_2(x)}{n}\right] - u_1[\gamma_1(x)]}{\epsilon\left[1 - \frac{\gamma_2(x+\alpha_n)\ \gamma_2(x)}{\alpha_n}\right]}\left[1 - \frac{\gamma_2(x+\alpha_n)-\gamma_2(x)}{\alpha_n}\right] \qquad (A.10)$$

Likewise, (A.2) and (A.3) imply that

$$V_{\gamma_2}(x+\alpha_n)-V_{\gamma_2}(x) \leq u_1[\gamma_1(x+\alpha)]-u_1[\gamma_1(x+\alpha_n)-\alpha+\gamma_2(x+\alpha_n)-\gamma_2(x)] \qquad (A.11)$$

$$\frac{V_{\gamma_2}(x+\alpha_n)-V_{\gamma_2}(x)}{\alpha_n} \leq \frac{u_1[\gamma_1(x+\alpha_n)-u_1\left[\gamma_1(x+\alpha_n)- \epsilon\left(1 - \frac{\gamma_2(x+\alpha_n)\ \gamma_2(x)}{\alpha_n}\right)\right]}{\epsilon\left[1 - \frac{\gamma_2(x+\alpha_n)-\gamma_2(x)}{\alpha_n}\right]}\left[1-\frac{\gamma_2(x+\alpha_n)-\gamma_2(x)}{\alpha_n}\right] \qquad (A.12)$$

[1] If x is a point of jump discontinuity of a real-value function F, then we adopt the convention that $F'(x^s) = +\infty$, for s = + or − .

Since $\gamma_1(\cdot)$ is u.h.c., there will always exist a selection $\tilde{\gamma}_1$ with the property that $\lim_{\alpha_n \downarrow 0} \tilde{\gamma}_1(x+\alpha_n)$ exists, for any given sequence $(\alpha_n) \downarrow 0$.

Taking the limit as $\alpha_n \downarrow 0$ in (A.10) and (A.12), noticing that $V_1'(x^+)$ exists if and only if $\gamma_2'(x^+)$ exists and γ_1 is continuous at x, in which case both RHS's have a common limit, it follows that for all such x's :

$$V'_{\gamma_2}(x^+) = \begin{cases} u_1'[\tilde{\gamma}_1(x)](1-\gamma_2'(x^+) & \text{if } \gamma_2'(x^+) \text{ exists (and is finite)} \\ +\infty & \text{if } \gamma_2'(x^+) = -\infty \\ -\infty & \text{if } \gamma_2'(x^+) = +\infty . \end{cases}$$

Similarly, starting at the stock level $x - \alpha_n$ and repeating the above argument yields:

$$V'_{\gamma_2}(x^-) = \begin{cases} u_1'[\tilde{\gamma}_1(x)](1-\gamma_2'(x\ )) & \text{if } \gamma_2'(x^-) \text{ exists (and is finite)} \\ +\infty & \text{if } \gamma_2'(x^-) = -\infty \\ -\infty & \text{if } \gamma_2'(x^+) = +\infty . \end{cases}$$

Now, to a given u.s.c. strategy $\gamma_1(\cdot)$ of Player 1, Player 2 will determine his best response correspondence $\gamma_2(\cdot)$ by solving an optimization problem similar to (3.1)-(3.3).

Let $\gamma_1(\cdot)$ be a selection from the maximizer in the optimization problem (3.1)-(3.3), and define

$$H_{\gamma_2}(x) = f(x-\gamma_1(x)-\gamma_2(x)), \quad x \in [0, x_m].$$

<u>Lemma 3.5</u>. <u>For all</u> x, y in $[0, x_m]$, V_{γ_2} <u>satisfies</u>

$$\frac{V_{\gamma_2}[H_{\gamma_2}(x) - V_{\gamma_2}[H_{\gamma_2}(y)]}{H_{\gamma_2}(x) - H_{\gamma_2}(y)} \geq 0.$$

Proof: First, consider the case where γ_1 is a corner maximizer at x, that is $\gamma_1(x) = 0$ or $\gamma_1(x) = x - \gamma_2(x)$. If $\gamma_1(x) = 0$, it follows from (A.5) that for any $y > 0$, $\gamma_2(y) - \gamma_2(x) = y-x$. Letting $y = 0$, and then $\gamma_2(y) = 0$, it follows that $\gamma_2(x) = x$, and $H_{\gamma_2}(n) = 0$, hence the desired inequality for any $y > 0$.

If $\gamma_1(x) = x - \gamma_2(x)$, then $H_{\gamma_2}(x) = 0$, and the conclusion follows.

Now, if γ_1 is interior at x, consider first the case where there exists $\varepsilon > 0$ such that H is constant on $N(x,\varepsilon)$. Then the desired relation clearly holds with equality for any two points in $N(x,\varepsilon)$.

Finally, if H is not constant on $N(x,\varepsilon)$ for all $\varepsilon > 0$, select a sequence (x_n), $x_n \to x$, such that $V'_{\gamma_2}[H_{\gamma_2}(x_n)] \to \overline{\lim_{y\to x}}\, V'_{\gamma_2}[H_{\gamma_2}(y)]$ and $V'[H(x_n)]$ exists for all n (this is possible since γ_2, being of bounded variation, implies V_{γ_2} is differentiable a.e., by Lemma 3.4). For all n, the first-order condition for maximization in (3.4) can be written as

$$V'_{\gamma_2}[H_{\gamma_2}(x_n)] = \frac{u_1'[\gamma_1(x_n)]}{\delta f'(x_n-\gamma_1(x_n)-\gamma_2(x_n))}$$

Taking the limit as $n \to \infty$ yields

$$\overline{\lim}\, V'_{\gamma_2}[H_{\gamma_2}(y)] = \frac{u_1'[\gamma_1(x)]}{\delta f'(x-\gamma_1(x)-\gamma_2(x\ ))} > 0,$$

where $\gamma_1(x)$ is an appropriate selection from the closed set $\{\gamma_1(x)\}$. Repeating this argument for each of the other three Dini derivates (Cf. Royden [1968] of V at H(x), the proof is completed.

Remark: The various conditions on the slopes of the functions under consideration can be similarly stated in terms of Dini derivates (i.e.

$\overline{\lim}$ and $\underline{\lim}$ of the left and right-hand incrementary ratios) of these functions, in an obvious manner, Cf. Titschmarch [1938].

Two useful corollaries follow immediately from the proof of Lemma 3.5.

Corollary 3.6: For all $x \in [0, x_m]$, the Dini derivates of V at H(x) are bounded by 0 below and by K above, where $K = \sup \dfrac{u_1'[\gamma_1(x)]}{\delta f'(x-\gamma_1(x)-\gamma_2(x))}$

Corollary 3.7: For all $x,y \in [0, x_m]$, H_{γ_2} satisfies

$$\frac{H_{\gamma_2}[H_{\gamma_2}(x)] - H_{\gamma_2}[H_{\gamma_2}(y)]}{H_{\gamma_2}(x) - H_{\gamma_2}(y)} \geq 0.$$

Proof: This is a direct consequence of Lemma 3.2, 3.3 and 3.5

The interpretation of this result is worth noting: Given a strategy γ_2 by Player II (l.s.c. and with bounded variation), Player I responds in such a way that the resulting state trajectory is monotonic. In particular, by Lemma 3.3 and Corollary 3.7, the generated states do not lie on the "greedy" portions of γ_2 (where the Dini derivates are larger than one). Thus, the state sequence resulting from the strategies $\gamma_2(\cdot)$ and a best response to it, $\gamma_1(\cdot)$, does not lie on portions of s at which γ_2 is "greedy," except possibly the initial state.

4. Existence of Nash Equilibria.

This section contains the Main Theorem which states that a Nash equilibrium in stationary strategies exists for the game at hand. The proof is given in the sequel. We start by setting up useful notations and definitions.

For a given initial state x_0, let S_{x_0} denote the compact interval $[0, x_m]$. Define the following spaces of functions on S_{x_0}:

$BV = \{\gamma : S_{x_0} \to S_{x_0}$, such that γ is of bounded variation, l.s.c. and $0 \leq \gamma(x) \leq x, \quad \forall x \in S_{x_0}\}$.

$\Gamma_{x_0} = \{\gamma : S_{x_0} \to S_{x_0}$ such that $\gamma(0) = 0$, γ is l.s.c. and $\dfrac{\gamma(x) - \gamma(y)}{x - y} \leq 1$, $\forall x,y \in S_{x_0}/\{x_0\}\}$.

$\hat{\Gamma}_{x_0} = \{\gamma : S_{x_0} \to S_{x_0}$, $\gamma(0) = 0$, γ is l.s.c. and there exists $h \in \Gamma_{x_0}$ such that γ is the best response to $h\}$.

We first prove:

<u>Lemma 4.1</u>: <u>The space</u> BV <u>includes both</u> Γ_{x_0} <u>and</u> $\hat{\Gamma}_{x_0}$.

<u>Proof</u>: Let $x \in \Gamma_{x_0}$. Define $\varphi(x) = x - \gamma(x)$, $x \in S_{x_0}/\{x_0\}$. Then $\dfrac{\varphi(x) - \varphi(y)}{x - y} = 1 - \dfrac{\gamma(x) - \gamma(y)}{x - y} \geq 0$. Since $\gamma(x) = x - \varphi(x)$, f is of bounded variation (the difference of two increasing functions).

Let $\gamma \in \hat{\Gamma}_{x_0}$. Then

$V_h(x) = u_1[\gamma(x)] + \delta_1 V_h[H_h(x)]$, where $H_h(x) = f(x-\gamma(x)-h(x))$.

Clearly, $\gamma(x) = U_1^{-1}\{V_h(x) - \delta_1 V_h[H_h(x)]\}$.

By Lemmas 3.2, 3.4, V_h and H_h are nondecreasing, and hence $\gamma \in BV$.

Note also that the l.s.c. requirement in the definition of $\hat{\Gamma}_{x_0}$ is not vacuous; indeed, the pointwise minimum selection out of the u.h.c.

best response is easily seen to be l.s.c., in this case.

Let $L_1 : \Gamma_{x_0} \times \Gamma_{x_0} \to \hat{\Gamma}_{x_0} \times \hat{\Gamma}_{x_0}$ be the best response map, i.e.

$L_1(\gamma_1,\gamma_2) = (\hat{\gamma}_1,\hat{\gamma}_2)$ where $V_{\gamma_2}(x) = u_1[\hat{\gamma}_1(x)] + \delta_1 V_{\gamma_2}[H_{\gamma_2}(x))]$ and

$$V_{\gamma_1}(x) = u_2[\hat{\gamma}_2(x)] + \delta_2 V_{\gamma_1}[H_{\gamma_1}(x)], \quad \forall x \in S_{x_0}.$$

Let $T : \hat{\Gamma}_{x_0} \to \Gamma_{x_0}$ be defined by

$$T\hat{\gamma}(x) = \begin{cases} \hat{\gamma}(x_0) , & x = x_0 \\ x + \min\limits_{0 \leq y \leq x} [\hat{\gamma}(y)-y], & x \neq x_0 \end{cases}$$

In words, $T\hat{\gamma}$ is the largest function bounded above by $\hat{\gamma}$, and whose slopes are all less than one. The min is achieved since $[\hat{\gamma}(y)-y]$ is l.s.c. in y.

Define $\quad L_2 : \hat{\Gamma}_{x_0} \times \hat{\Gamma}_{x_0} \to \Gamma_{x_0} \times \Gamma_{x_0}$

$$(\hat{\gamma}_1,\hat{\gamma}_2) \to (T\hat{\gamma}_1, T\hat{\gamma}_2)$$

An important consequence of Corollary 3.7 is that any fixed-point of the operator $L = L_2 \circ L_1 : \Gamma_{x_0} \times \Gamma_{x_0} \to \Gamma_{x_0} \times \Gamma_{x_0}$ is a Nash equilibrium of the sequential game at hand. To see this, notice that for any $\gamma_2 \in BV$, Corollary 3.7 implies that if γ_1 is a best response to γ_2 (in the sense of solving Problem (3.1)-(3.3).), then γ_1 is also a best response to $T\gamma_2 \in \Gamma_{x_0}$.

We are now ready for

<u>Main Theorem</u>: <u>There exist a Nash equilibrium in stationary strategies for the sequential game</u> (2.1)-(2.3). <u>Furthermore, the equilibrium</u>

strategies are such that all their slopes are bounded above by one.

Before proceeding to the proof, some mathematical preliminaries are presented; more details on this material may be found in the survey by Kall [1986] and Rockafellar and Wets [1984]. Let LSC be the linear space of all real-valued l.s.c. functions on some compact subset D of R. Endow LSC with the topology of epi-convergence (to be denoted e) defined as follows:

$f_n \xrightarrow{e} f$, if $\forall x \in D$,

$\exists (y_n) \to x$ such that $\overline{\lim_n}\, f_n(y_n) \leqq f(x)$, and

$\forall (x_n) \to x$, $\underline{\lim}_n\, f_n(x_n) \geqq f(x)$.[1]

With the e topology, LSC is known to be Hausdorff, first countable linear topological space; convergence may thus be described in terms of sequences only. Also, bounded subsets of LSC are relatively compact.

Γ_{x_0} is clearly a closed subset of LSC. Hence Γ_{x_0} is also compact in the e-topology, and is further metrizable (Cf. Rockafellar and Wets [1984].

Uniform (or continuous) convergence implies e-convergence. However, pointwise and e-convergence are generally not comparable, nevertheless, if a sequence of functions is equi-l.s.c. (Cf. Kall [1986]), the latter two convergence modes are equivalent.

Finally, the importance of the topology of epi-convergence lies in the following fundamental fact in optimization theory:

---

[1] It follows then that for the sequence (y_n), $\lim_n f_n(y_n) = f(x)$.

$$f_n \xrightarrow{e} f \Rightarrow \inf f_n \longrightarrow \inf f.$$

Observe that pointwise convergence, and hence weak convergence (i.e. pointwise convergence at every point of continuity of the limit function, Cf. Billingsley [1968] do not enjoy this property, crucial to the analysis here.

Proof of Main Theorem: The aim is to use Tychonoff's fixed-point theorem for the map $L = L_2 \circ L_1$ (Cf. Istratesçu [1983]). The domain $\Gamma_{x_0} \times \Gamma_{x_0}$ is compact in the e-topology, since it is a closed subset of LSC.

We start with a preliminary result of a general nature (u = uniform).

Lemma 4.2: Let $f_n : D_1 \subset \mathbb{R} \to D_2 \subset \mathbb{R}$ and $h_n : D_2 \to \mathbb{R}$ be such that $f_n \xrightarrow{e} f$ and $h_n \xrightarrow{u} h$. Moreover, let h_n be a nondecreasing continuous function. Then $h_n f_n \xrightarrow{e} h \circ f$.

Proof: First, we show that for any sequence $y_n \to y$,

$h_n(y_n) \to h(y)$. To this end, consider

$$|h_n(y_n) - h(y)| \leqq |h_n(y_n) - h(y_n)| + |h(y_n) - h(y)|$$

$$\leqq \sup_{x \in D_2} |h_n(x) - h(x)| + |h(y_n) - h(y)| \xrightarrow{n\to\infty} 0$$

(the first term goes to 0 since $h_n \xrightarrow{u} h$, and the second since h is a continuous function).

Now, since $f_n \xrightarrow{e} f$, for any sequence $(x_n) \to x$, we have $\underline{\lim}_n f_n(x_n) \geqq f(x)$. By above result, and taking an appropriate subsequence

$$\underline{\lim}_n h_n[f_n(x_n)] = h\,[\underline{\lim}_n f_n(x_n)]$$

$$\geqq h[f(x)], \text{ since } h \text{ is nondecreasing.}$$

Also, since $f_n \xrightarrow{e} f$, there exists a sequence $(z_n) \to x$ such that $\overline{\lim_n}\, f_n(z_n) = f(x)$. Again,

$$\overline{\lim_n}\, h_n[f_n(z_n)] = h[\overline{\lim_n}\, f_n(z_n)] = h[f(x)].$$

We conclude that $h_n \circ f_n \xrightarrow{e} h \circ f$

Lemma 4.3: $L_1 : \Gamma_{x_0} \times \Gamma_{x_0} \to \hat{\Gamma}_{x_0} \times \hat{\Gamma}_{x_0}$ is a continuous map in the e-topology.

Proof: Let $(\gamma_1^n, \gamma_2^n) \xrightarrow{e} (\gamma_1, \gamma_2)$ in $\Gamma_{x_0} \times \Gamma_{x_0}$, and let $(\hat{\gamma}_1^n, \hat{\gamma}_2^n) = L_1(\gamma_1^n, \gamma_2^n)$. We must show that $(\hat{\gamma}_1^n, \hat{\gamma}_2^n) \xrightarrow{e} (\hat{\gamma}_1, \hat{\gamma}_2) = L_1(\gamma_1, \gamma_2)$. By definition of L, we have, $\forall x \in S_{x_0}$,

$$-\delta_1 V_{\gamma_2^n}[H_{\gamma_2^n}(x)] = u_1[\hat{\gamma}_1^n(x)] - V_{\gamma_2^n}(x) \qquad (4.1)$$

By Lemma 3.4, $-V_{\gamma_2^n}$ is a decreasing l.s.c function, for each n. Hence, the sequence $(-V_{\gamma_2^n})$ has an epi-convergent subsequence, appropriately relabeled $(-V_n)$, with e-limit $-V$. As $\gamma_1^n \xrightarrow{e} \gamma_1$ and $\gamma_2^n \xrightarrow{e} \gamma_2$, $H_{\gamma_2^n} \xrightarrow{e} H_{\gamma_2} = f(x-\hat{\gamma}_1(x)-\gamma_2(x))$. We now show: $V_n \circ H_n \xrightarrow{e} V \circ H_{\gamma_2}$.

To this end, first extend the domain of each V_n from $D_n = \{H_n(x), x \in S_{x_0}\}$ to $\bigcup_{n=1}^{\infty}(\text{Convex hull } D_n)$, by adding straight lines to the graph of V_n in such a way that V_n is continuous and nondecreasing. This is possible by Corollary 3.6, which also implies that the new sequence (V_n) is equicontinuous and hence also equi-l.s.c. This, in turn, yields the pointwise, and therefore the uniform convergence (by Arzela-Ascoli theorem), of (V_n) to V. Then, $V_n \circ H_n \xrightarrow{e} V \circ H_{\gamma_2}$ follows from Lemma 4.2.

Taking termwise e-limits in (4.1):

$$-\delta_1 V[H_{\gamma_2}(x))] = u_1[\hat{\gamma}_1(x)] - V(x) \qquad (4.2)$$

Since $V_{\gamma_2^n}(x) \geq u_1[h(x)] + \delta_1 V_{\gamma_2^n}[f(x-h(x)-\gamma_2^n(x))]$, $\forall h \in \hat{\Gamma}_{x_0}$, it follows that

$$V(x) = u_1[\hat{\gamma}_1(x)]+\delta_1 V[H_{\gamma_2}(x)]$$

$$\geq u_1[h(x)]+\delta_1 V[f(x-h(x)-\gamma_2(x))].$$

This establishes that $\hat{\gamma}_1 = L_1(\gamma_2)$. A similar argument for Player II's best response optimization finishes the proof.

Lemma 4.4: $L_2 : \hat{\Gamma}_{x_0} \times \hat{\Gamma}_{x_0} \to \Gamma_{x_0} \times \Gamma_{x_0}$ is a continuous map in the e-topology.

Proof: Clearly, it suffices to show that $T : \hat{\Gamma}_{x_0} \to \Gamma_{x_0}$ is continuous.[1] Let $\hat{\gamma}_n \xrightarrow{e} \hat{\gamma}$ in $\hat{\Gamma}_{x_0}$. We have to show that $T\hat{\gamma}_n \xrightarrow{e} T\hat{\gamma}$ in Γ_{x_0}. Fix $x \in S_{x_0}$. If $T\hat{\gamma}(x) = \hat{\gamma}(x)$, i.e. the min in the definition of T is achieved at $y = x$, then clearly $T\hat{\gamma}_n$ e-converges to $T\hat{\gamma}$ at x. If the min is achieved at some $z_n < x$, then $T\hat{\gamma}_n(x) = x+\hat{\gamma}_n(z_n)-z_n = x+ \min_{0\leq y\leq x} [\hat{\gamma}_n(y)-y] \to x+ \min_{0\leq y\leq x} [\hat{\gamma}(y)-y] = T\hat{\gamma}(x)$, since $[\hat{\gamma}_n(y)-y] \xrightarrow{e} [\hat{\gamma}(y)-y]$. This establishes the pointwise convergence of $T\hat{\gamma}_n$ to $T\hat{\gamma}$ for those points x for which $T\hat{\gamma}(x) \neq \hat{\gamma}(x)$. Since for any such point x, there exists $N(x,\varepsilon)$ such that $T\hat{\gamma}$ and $T\gamma_n$, for sufficiently large n, are linear with slope 1 on $N(x,\varepsilon)$, the pointwise convergence is also uniform on $N(x,\varepsilon)$, and hence implies e-convergence.

Finally, if $x = x_0$, then $T\hat{\gamma}_n(x_0) = \hat{\gamma}_n(x_0)$ clearly e-converges to $T\hat{\gamma}(x_0) = \hat{\gamma}(x_0)$.

---

[1] This actually follows from more abstract general results on the convergence of Yosida estimates, Cf. Attouch [1984]. The reason for selecting this newly developed topology for the strategy spaces lies in this Lemma. It is well known that L_2 fails to be continuous in the more natural topology of weak convergence (on $\hat{\Gamma}_{x_0}$ and Γ_{x_0}).

By Tychonoff's theorem, the map $L = L_2 \circ L_1$ has a fixed point denoted (g_1, g_2) which is easily seen to be a Nash equilibrium.

Now, since the best response to g_2 is a u.h.c. correspondence, it admits another selection $\hat{g}_1$ which coincides with g_1 everywhere except possibly at x_0 and which is such that $\frac{\hat{g}_1(x) - \hat{g}_1(y)}{x - y} \leq 1$, for all x, y. Similarly, $\hat{g}_2$, with the same properties as $\hat{g}_1$, may be constructed. It is clear that $(\hat{g}_1, \hat{g}_2)$ is also a Nash equilibrium of the original game.

References

Amir, R. [1985], "A Characterization of Globally Optimal Paths in the Non classical Growth Model," Cowles Foundation Discussion Paper 754, Yale University.

Basar, T. and G.J. Olsder [1982], Dynamic Non cooperative Game Theory, Academic Press.

Bank, B., J. Guddat, D. Klatte, B.Kummer and K. Tammer [1983], Nonlinear Parametric Optimization, Birkhauser Verlag.

Bernheim, D. and D. Ray [1983], "Altruistic Growth Economics, IMSSS report, Stanford University.

Bertsekas, D. [1971], Dynamic Programming and Stochastic Control, Academic Press

Billingsley [1968], Weak Convergence of Probability Measures, John Wiley and Sons, New York.

Debreu, G. [1952], "A Social Equilibrium Existence Theorem," Proc. of Nat. Acad. of Science, U.S.A., Vol. 38, pp. 886 893.

Dechert, D. and K. Nishimura [1983], "A Complete Characterization of Optimal Growth Paths in an Aggregated Model with a Non Concave Production Function," Journal of Economic Theory, 3k (2), 332 354.

Hildenbrand, W. and A. Kirman [1976], Introduction to Equilibrium Analysis, North Holland/American Elsevier.

Kall, P. [1986], "Approximation to Optimization Problems: An Elementary Review," Math. of Oper. Res., Vol. 11, No. 1.

Leininger, W. [1983], "The Existence of Perfect Equilibria in a Model of Growth with Altruism Between Generation," ICERD Working Paper 84/83, London School of Economics.

Levhan, D. and L. Mirman [1980], "The Great Fish War: An Example Using a Dynamic Cournot Nash Solution," Bell Journal of Economics, pp. 322 344.

Mirman, L. [1979], "Dynamic Models of Fishing: A Heuristic Approach," in P.T. Liu and J.G. Sutinen, ets., Control Theory in Mathematical Economics, New York, Decker, 39 73.

Nash, J. [1951], "Noncooperative Games," Annals of Mathematics, 54, No. 2, pp. 286 295.

Reinganum, J. [1981], "Noncooperative Extraction of Nonrenewable Common Property Resources," California Institute of Technology.

Rockafeller, T. and R. Wets, "Variational Systems, an Introduction," in Lecture Notes in Math., Apringer Verlag, Vol.1091, pp. 1-54.

Royden, H. [1968], Real Analysis, MacMillan Company, New York.

Skiba, A.K. [1978], "Optimal Growth with a Convex Concave Production Function, Econometrica, Vol. 46, No. 3.

Shapley, L. [1953], "Stochastic Games," Proceedings of the National Academy of Science, USA, 39, pp. 1095-1100.

Shubik, M. and W. Whitt [1973], "Fiat Money in an Economy with one Nondurable Good and No Credit," in A. Bragmiere (ed.), Topics in Differential Games, Amsterdam: North Holand, 401-448.

Whitt, W. [1980], "Representation and Approximation of Non Cooperative Sequential Games," SIAM Journal on Control and Optimization, Vol. 18, pp. 33-48.