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TESTING FOR COINTEGRATION USING PRINCIPAL COMPONENT METHODS

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0. ABSTRACT

This paper studies cointegrated systems of multiple time series which are individually well described as integrated processes (with or without a drift). Necessary and sufficient conditions for cointegration are given. These conditions form the basis for a new class of statistical procedures designed to test for cointegration. The new procedures rely on principal components methods. They are simple to employ and they involve only the standard normal distribution. Monte Carlo simulations reported in the paper indicate that the new procedures provide simple and apparently rather powerful diagnostics for the detection of cointegration. Some empirical applications to macroeconomic data are conducted.

Key Words: Cointegration; Latent root; Principal components; Spectral density matrix; Time series.

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1. INTRODUCTION

A recent development that seems likely to be of lasting importance to the statistical analysis of economic time series is the theory of cointegration. The idea of cointegrated variables was introduced by Granger (1981, 1983) and Granger and Weiss (1983) and has been more systematically studied in the recent paper by Engle and Granger (1987). Cointegrated systems allow individual time series to be integrated of order one (I(1)) but require certain linear combinations of the series to be stationary or I(0) . This framework accommodates rather well the empirical observation that individual economic time series often exhibit nonstationary characteristics but that certain combinations of the series tend to move together over time. The notion may also be regarded as a statistical embodiment of ideas from economic theory concerning long run regularities or steady state behavior among economic variables. Examples now include modern theories of asset prices and the term structure of interest rates (see Campbell (1985) and Campbell and Shiller (1986)) as well as steady state theories of aggregate variables. The hypothesis of cointegration is therefore important in terms of its underlying economic ideas of long run equilibrium and in terms of its statistical implications for applied research. Useful overviews of the subject have recently been written by Granger (1986) and by Hendry (1986).

Evidence that a cointegrating vector exists provides strong support for a long run relationship amongst a group of variables whose short run behavior may be very much more complex. However, detecting the existence of cointegration in a multiple time series seems to give rise to nonstandard testing procedures. This is because the asymptotic theory of regression in cointegrated systems is very different from conventional theory for stationary time series, as is clear from earlier work by Phillips (1986b) and Phillips and Durlauf (1986). These authors provide a detailed study of regression theory in the presence and absence of cointegration. In both cases the limiting distribution theory is nonstandard. Conventional significance tests and regression diagnostics have nuisance parameter dependencies even asymptotically and this complicates the use of the asymptotic theory for inference. Moreover, in the absence of cointegration the parameters of the system are unidentified and the estimated regression coefficients have non degenerate limiting distributions. This further complicates the asymptotic theory for residual based diagnostic tests.

All of these complications present obstacles to the development of statistical tests of cointegration. Engle and Granger (1987) recommend the use of tests based on the residuals of cointegrating regressions. Under the null hypothesis of no cointegration all linear combinations of the variables are nonstationary. Under the alternative of cointegration, at least one linear combination (which may be consistently estimated by the cointegrating regression) is stationary. Test statistics for nonstationarity (or the presence of a unit root) in the residuals of the cointegrating regression might therefore be expected to provide discriminatory power against the (alternative) hypothesis of cointegration. However, because of the complications discussed in the last paragraph, all of the statistics considered by Engle and Granger have nonstandard limiting distributions and these distributions are different from the usual limit distributions of simple tests for unit roots. The asymptotic properties of these residual based tests for cointegration have recently been studied in another paper by the authors

(Phillips and Ouliaris (1987)).

It is possible to test for cointegration without using the residuals of a cointegrating regression. One alternative arises from the work of Phillips and Durlauf (1986). These authors explored multivariate tests for the presence of unit roots in multiple time series and gave a limiting distribution theory for Wald and modified Wald statistics under the null of no cointegration. They also constructed some general specification tests whose asymptotic distributions are χ^2, again under the null of no cointegration. Both procedures may be used to test against the alternative of a cointegrated system and both yield consistent tests. The Phillips-Durlauf tests rely in a simple way on the estimated coefficient matrix in a first order vector autoregression (VAR). Under the null of no cointegration this estimator is O(T)-consistent for the unit matrix. But, in a cointegrated system its probability limit is no longer the unit matrix. Hence, multivariate unit root tests may be expected to provide discriminatory power in the presence of cointegration. Stock and Watson (1986) have subsequently pursued this approach to the subject. Their paper explains the prefiltering of the data and the serial correlation corrections that are needed to remove parameter dependencies; and they recommend that attention be focussed on the modulus of the smallest latent root of the regression coefficient matrix (of the VAR) in mounting a test of cointegration. In the scalar case their procedure reduces to the unit root test introduced in Phillips (1987).

The main purpose of the present paper is to suggest an entirely new approach to testing for cointegration. The intuition behind the procedures we develop is simple and compelling. In effect, our approach is to perform a form of principal components analysis for time series. When a multiple time series is cointegrated, the cointegrating vector effectively reduces the variability in the original series (which are taken to be $I(1)$ processes) by an order of magnitude (from $I(1)$ to $I(0)$) . This reduction in variance should be detectable by principal components methods.

More specifically, if multiple time series which are individually $I(1)$ move together over time so that some combination of the series is $I(0)$, then this implies restrictions on the innovations that drive the full system. Phillips (1986b) showed that a necessary condition for cointegration is that the spectral density matrix of the innovation sequence (which we take to be weakly stationary) has deficient rank at the origin. We call this matrix Σ . Moreover, the number of zero latent roots of Σ is the number of cointegrating vectors and the associated latent vectors of Σ are the cointegrating vectors themselves. This result suggests that we can test for cointegration by assessing whether or not Σ has a negligible latent root. Moreover, since nonparametric consistent estimates of Σ are easily obtained it is possible to develop simple asymptotic tests of the cointegration hypothesis which apply for a very wide class of underlying innovations. To make matters even simpler, under very general conditions consistent estimates of Σ and hence its latent roots are asymptotically normal. Thus, the procedures we develop involve only the standard normal distribution!

This paper suggests two new procedures for detecting the presence of cointegration. The first test involves computing one sided confidence intervals for the smallest latent root of Σ (or an associated correlation matrix P). The second test is similar but relies on the ratio of the smallest latent root to the sum of the latent roots of Σ (or P). We also conduct a Monte Carlo study to assess the properties of the new tests. The new procedures, which we call bounds tests seem, to possess good power in the presence of cointegration and yet they are very conservative in size. On the basis of the Monte Carlo simulations, we recommend a simple rule for detecting cointegration amongst a group of integrated time series. This rule works well in our experiments for models of different dimensions and for a wide variety of data generating mechanisms.

In the scalar case our procedures may be used as (autoregressive) unit root tests. But they are more directly interpretable as tests for the presence of a unit root in the moving average (MA) representation of a stationary time series. In effect, our tests may be regarded as tests for the invertibility of an MA representation. This hypothesis is itself of independent interest. We therefore hope that our test procedures will have useful applications in this context as well as that of cointegrated systems.

Our organization of the paper is as follows. Necessary and sufficient conditions for cointegration are established in Section 2. Our new bounds tests for cointegration are developed in Section 3. We present several possible procedures, all centered on the same basic idea; and in Section 4 we show how these results may be interpreted as tests of invertibility. The new procedures are examined and compared in simulations that we report in Section 5. Size and power comparisons are given for models of different dimensions and various plausible data generating mechanisms. On the basis of our simulation experiments (which allow for 50 different data generating mechanisms in the ARIMA family) we also develop two explicit latent root tests of cointegration. These tests involve the minimum latent root and the ratio of the minimum latent root to the average of the roots of a consistent estimator of P. They are designed to deal with situations where the bounds test is inconclusive and they provide sharp criteria for acceptance or rejection of the null hypothesis in the latent root space. Section 6 reports some empirical applications of our methods to macroeconomic data. Concluding remarks are made in Section 7.

2. CONDITIONS FOR COINTEGRATION

Let $\{y_t\}_0^\infty$ be a multiple $(n \times 1)$ time series that is generated in discrete time according to:

(1) $$y_t = Ay_{t-1} + u_t \; ; \quad t = 1, 2, \ldots$$

with

$$A = I_n$$

and where y_0 may be any random vector, including a constant. In (1) $\{u_t\}$ is a zero mean, weakly stationary innovation sequence with spectral density matrix $f_{uu}(\lambda)$. Throughout the paper we shall require that:

(2) $$E|u_{i0}|^\beta < \infty \quad (i = 1, \ldots, n) \quad \text{for some} \quad \beta > 2 \; ;$$

and

(3) $\{u_t\}_0^\infty$ is strong mixing with mixing numbers α_m that satisfy

$$\Sigma_{m=1}^\infty \alpha_m^{1-2/\beta} < \infty \; .$$

Under these conditions

(4) $$\Sigma = 2\pi f_{uu}(0) = E(u_0u_0') + \Sigma_{k=1}^{\infty} E(u_0u_k' + u_ku_0')$$

(Phillips and Durlauf (1986, Corollary 2.2)). In fact, the series defining Σ is absolutely summable in view of (2) and (3). Under these conditions, therefore, $f_{uu}(\lambda)$ is bounded and (uniformly) continuous on $[-\pi, \pi]$.

We now make explicit the hypothesis of cointegration. The variables of y_t are said to be cointegrated if there exists an n-vector $\gamma \neq 0$ for which $\gamma' y_t$ is stationary. More specifically, we shall define y_t to be cointegrated if there exists a vector $\gamma \neq 0$ for which $v_t = \gamma' y_t$ is weakly stationary with continuous spectral density. This ensures that the action of the cointegrating vector γ reduces the integrated process y_t to a stationary time series with properties analogous to those of the innovations driving the mechanism (1). It follows directly from (1) that:

(5) $$\gamma' u_t = v_t - v_{t-1} .$$

Thus, some combination of the innovations in (1) has an MA representation with a unit root. (Note that we are not asserting that v_t is white noise). We deduce:

<u>THEOREM 2.1</u> *The system* (1) *is cointegrated with cointegrating vector* $\gamma \neq 0$ *iff*

(6) $$\gamma' f_{uu}(\lambda)\gamma = c\lambda^2 + o(\lambda^2) \quad \text{as} \quad \lambda \to 0$$

for some constant c *(possibly zero).*

PROOF. In view of (5) the spectrum of $\gamma' u_t$ is zero at the origin. To prove the necessity of (6) we observe that

$$\gamma' f_{uu}(\lambda)\gamma = |1 - e^{i\lambda}|^2 f_v(\lambda)$$

where $f_v(\lambda)$ is the spectrum of $v_t = \gamma' y_t$. Since $f_v(\lambda)$ is continuous under the hypothesis of cointegration and since $|1 - e^{i\lambda}|^2 = \lambda^2 + O(\lambda^4)$ as $\lambda \to 0$ we deduce that for some constant c

$$\gamma' f_{uu}(\lambda)\gamma = c\lambda^2 + o(\lambda^2) ,$$

proving the necessity of (6). This condition is also sufficient because $f_v(\lambda)$ is continuous and bounded on every interval $\varepsilon \le \lambda \le 2\pi$, $\varepsilon > 0$ (since $f_{uu}(\lambda)$ is continuous and bounded); and, in view of (6), $f_v(\lambda) \to c$ as $\lambda \to 0$. Hence, v_t has continuous and bounded spectrum and is weakly stationary. □

Note that (6) implies the necessary condition

(7) $$\gamma' \Sigma \gamma = 0$$

(that is, Σ is singular and γ lies in its null space). This necessary condition was given and discussed earlier in Phillips (1986b). When there are several distinct cointegrating vectors γ_i $(i = 1, \ldots, k < n)$ we have $\Sigma\gamma_i = 0$ and Σ has k zero latent roots.

Condition (6) is necessary and sufficient. It is important in what follows because it more completely characterizes the properties of the spectrum $f_{uu}(\lambda)$ under the hypothesis of cointegration. In particular, it tells us that $\gamma' f_{uu}(\lambda)\gamma$ is not only zero at $\lambda = 0$ but flat at the origin as well. This means that for cointegrated systems such components of the spectrum should be well estimated by an average of the periodogram ordinates in a band centered on the origin.

3. NEW TESTS OF COINTEGRATION

We shall develop tests based on the latent roots of a consistent estimate of the covariance matrix Σ given in (4). These tests may be regarded as performing a form of principal component analysis in the frequency domain, a subject on which there is a large literature (see, for example, Brillinger (1981, Ch. 9) and the references therein). The latter methods are concerned with approximating a given multiple time series by another that is of lower dimension and yet contains much of the information of the original series. When a multiple time series is cointegrated there clearly exists a linear filter of the series which retains much of the variability of the original series but which is of lower dimension. The issue of practical importance is whether the variability that is lost by this reduction is small enough to be negligible. In a cointegrated system the lost variability is smaller by an order of magnitude (of integration) so it should be possible to make an empirical assessment of the existence of cointegration by principal component methods. The tests we now develop are inspired by this line of reasoning.

We first consider the following estimator of Σ :

$$S_{Tk} = 2\pi\hat{f}_{uu}(0) = 2\pi\left[\frac{1}{2k+1}\Sigma_{s=-k}^{k} I_{uu}\left(\frac{2\pi s}{T}\right)\right] \tag{8}$$

$$= \frac{2\pi}{2k+1}\left[I_{uu}(0) + \Sigma_{s=1}^{k}\mathrm{Re}\left\{I_{uu}\left(\frac{2\pi s}{T}\right)\right\}\right]$$

where $I_{uu}(\lambda) = w_u(\lambda)w_u(\lambda)^*$ is the periodogram and $w_u(\lambda) = (2\pi T)^{-1/2}\Sigma_1^T u_t e^{i\lambda t}$ is the finite Fourier transform. In practical work $I_{uu}(\lambda)$ can be computed using the fast Fourier transform for highly composite T, although the computation of (8) is in no way burdensome for typical sample sizes in economics.

S_{Tk} is a smoothed periodogram estimate of $\Sigma = 2\pi f_{uu}(0)$. It is consistent as $T \uparrow \infty$ provided $k \uparrow \infty$ in such a manner that $k/T \downarrow 0$. The associated matrix $\hat{f}_{uu}(0)$ in (8) is the Daniell estimate of the spectral density matrix at the origin (see, for example, Priestley (1981, pp. 440-441)) and this involves a rectangular spectral window. Of course, other choices of spectral window may be used, leading to alternative estimators of Σ. However, the flat behavior of $\gamma' f_{uu}(\lambda)\gamma$ in the vicinity of the origin in the presence of cointegration (see (6) above) indicates that the choice of a rectangular spectral window may be rather appropriate for the purpose we intend.

To make our approach as general as possible we shall often wish to allow for a drift in the generating mechanism (1). In this case (1) is replaced by

$$(1)' \qquad y_t = \mu + y_{t-1} + u_t \; ; \quad t = 1, \ldots$$

with some constant n-vector μ. When there is cointegration in the system we have $\gamma'\mu = 0$, so that the cointegrating vector γ now annihilates the drift as well as the spectrum of u_t at the origin.

It is easy to accommodate (1)′ in our approach. We simply remove the zero frequency periodogram ordinate from (8) and continue to compute the finite Fourier transforms using first differences Δy_t. This is equivalent to computing (8) using first differences about their fitted mean (i.e. $\Delta y_t - \overline{\Delta y}$). Adjusting degrees of freedom in (8), we therefore recommend the use of the following estimate of Σ :

$$(9) \qquad S_{Tk} = \frac{\pi}{k} \Sigma_{s=1}^{k} \text{Re}\left\{ I_{uu}\left(\frac{2\pi s}{T}\right)\right\}$$

which we compute using measured first differences $u_t = \Delta y_t$.

As $T \uparrow \infty$ with k fixed we know from standard spectral theory (for example, Brillinger (1981), Theorem 7.3.3) that:

$$(10) \qquad S_{Tk} \Rightarrow \frac{\pi}{k} W_n(2k, f_{uu}(0)) \equiv \frac{1}{2k} W_n(2k, \Sigma)$$

where $W_n(\cdot,\cdot)$ signifies a Wishart matrix of dimension $n \times n$ with degrees of freedom and covariance matrix given by the first and second arguments of W_n , respectively. The latent roots of S_{Tk} are correspondingly distributed as the latent roots of the scaled Wishart matrix in (10) when $T \uparrow \infty$ for k fixed (Brillinger (1981), Theorem 9.4.4). Note that we use the symbol " $\equiv$ " in (10) to signify equality in distribution.

In view of Theorem 2.1 our main concern in testing for cointegration is naturally with the smallest latent roots of S_{Tk} . In particular, we need to assess whether these roots are negligible or statistically insignificant. This is, of course, a central element in principal components theory in multivariate analysis. Here the relevant distribution theory for the latent roots and extreme latent roots of a Wishart matrix has been fully developed in recent years. Muirhead (1982, Ch. 9) provides an extensive review of these developments. Unfortunately, the distributions of the extreme roots depend on the full eigenstructure of the covariance matrix Σ and involve zonal polynomial representations which make computational work difficult. The conventional approach in principal components theory has therefore been to work with large sample approximations to the Wishart distribution $W_n(2k, \Sigma)$ for large k. Much of this theory was originally developed by Anderson (1963). When the latent roots of Σ are distinct the results are particularly simple. If ℓ_i and λ_i $(i = 1, \ldots, n)$ are the latent roots of S_{Tk} and Σ then ℓ_i is asymptotically independent of ℓ_j $(i \neq j)$ and the standardized variates

$$k^{1/2}(\ell_i - \lambda_i)/\lambda_i \quad (i = 1, \ldots, n)$$

are asymptotically $(k \uparrow \infty)$ $N(0,1)$ (see, for example, Muirhead (1982), p. 403). Now order the roots as $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ and $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_n$. We deduce that

$$(11) \qquad k^{1/2}(\ell_n - \lambda_n)/\lambda_n \sim N(0,1) .$$

If z_α is the one-tailed $100(1-\alpha)\%$ upper significance point of the $N(0,1)$ distribution then (11) implies the following (approximate) $100(1-\alpha)\%$ one-sided confidence bound for λ_n :

$$(12) \qquad \lambda_n \leq \ell_n/(1 - z_\alpha/k^{1/2}) \sim \ell_n + \ell_n z_\alpha/k^{1/2} .$$

Similarly, a $100(1-\alpha)\%$ upper confidence bound for the sum of the $m' = m+1$ smallest latent roots of Σ is

$$(13) \qquad \Sigma_{j=n-m}^{n}\lambda_j < \Sigma_{j=n-m}^{n}\ell_j + \left(\Sigma_{j=n-m}^{n}\ell_j^2\right)^{1/2} z_\alpha/k^{1/2} .$$

Anderson (1963) suggested the following test. If the upper bound in (12) is sufficiently small then the smallest root λ_n may be taken to be negligible. In the same way if the upper bound in (13) is sufficiently small then the m' smallest latent roots of Σ may be deemed negligible.

These inferential procedures based on (12) and (13) may be used in the present context of tests for cointegration. However, the latent rows of Σ and S_{Tk} depend on the units of measurement and this presents difficulty in the selection of a "sufficiently small" criterion for the upper bounds (12) and (13). Anderson (1963) suggests an alternative procedure based on the ratio of the smallest latent roots to the sum of all the latent roots. In this case we obtain by simple manipulations the following $100(1-\alpha)\%$ upper confidence bound:

$$(14) \qquad \frac{\Sigma_{j=n-m}^{n}\lambda_j}{\Sigma_{j=1}^{n}\lambda_j} \leq \frac{\Sigma_{j=n-m}^{n}\ell_j}{\Sigma_{j=1}^{n}\ell_j} + z_\alpha B/k^{1/2}$$

where

$$B = \left[\left(\Sigma_{j=n-m}^{n}\ell_j\right)^2 (\Sigma_{j=1}^{n-m-1}\ell_j^2) + \left(\Sigma_{j=1}^{n-m-1}\ell_j\right)^2 \Sigma_{j=n-m}^{n}\ell_j^2\right]^{1/2} \Big/ \left(\Sigma_{j=1}^{n}\ell_j\right)^2$$

(see Anderson (1984), p. 475). Once again, if the upper bound given by (14) is sufficiently small then the smallest latent roots $\lambda_n, \ldots, \lambda_{n-m}$ may be deemed negligible relative to the sum of all the roots.

It is easy to assess when the upper bound of (14) is small. Take for example, the case where $m = 1$ and our focus of attention is the smallest root of Σ. If the upper bound given in (14) is less than $0.10/n$ then we can say with (approximate) $100(1-\alpha)\%$ confidence that the smallest latent root of Σ is less than 10% of the average of the roots, i.e. $\Sigma_1^n \lambda_j/n$. This might be interpreted as strong evidence in favor of cointegration (our alternative hypothesis).

Note that we can apply the test in the opposite direction to provide confirmation of the null hypothesis of no cointegration. Thus, a lower $100(1-\alpha)$% confidence bound for the ratio of the roots is given by:

$$(15) \qquad \frac{\Sigma_{j=n-m}^n \ell_j}{\Sigma_{j=1}^n \ell_j} - z_\alpha B/k^{1/2} \le \frac{\Sigma_{j=n-m}^n \lambda_j}{\Sigma_{j=1}^n \lambda_j} .$$

In this case if the lower bound (15) is greater than $0.10/n$ then we have $100(1-\alpha)$% confidence that the smallest root of Σ is greater than 10% of the average of the roots, $\Sigma_1^n \lambda_j/n$. This might be interpreted as substantial support for the absence of cointegration.

The bounds tests based on (14) and (15) relate different latent roots of Σ . If the units of measurement of the variables that comprise y_t in (1) or (1)' are all the same this procedure seems justified. However, we may often be interested in situations where the component variables involve different units of measurement. This is most likely to be the case when the long run equilibrium relationship of interest relates real and monetary aggregates (as in the quantity theory of money). In such situations it seems preferable to work with dimensionless quantities. These may be constructed as follows.

First we define the variance of the innovation sequence $\{u_t\}$ in (1) or (1)', viz.

$$\Sigma_0 = E(u_t u_t') .$$

We may require Σ_0 to be nonsingular. Otherwise the support of the distribution of u_t has dimension less than n and there is an exact linear dependence in the series which could be removed prior to the analysis. Now define the matrix

$$(16) \qquad P = \Sigma_0^{-1/2}\Sigma\Sigma_0^{-1/2} = I + \Sigma_{m=1}^{\infty}(\Gamma_m + \Gamma_m')$$

where

$$\Gamma_m = \Sigma_0^{-1/2}E(u_0u_m')\Sigma_0^{-1/2}$$

and $\Sigma_0^{1/2}$ is the positive definite square root of Σ_0 . Next, we introduce the sample estimate of Σ_0

$$S = T^{-1}\Sigma_1^T\hat{u}_t\hat{u}_t'$$

where $\hat{u}_t = \Delta y_t - \overline{\Delta y}$ = measured first differences about the mean. The corresponding sample estimate of P is

$$(17) \qquad R = S^{-1/2}S_{Tk}S^{-1/2} .$$

Since $S = \Sigma_0 + O_p(T^{-1/2})$ we now find in place of (10):

$$R \Rightarrow \frac{1}{2k}W\ (2k,\ P)$$

as $T \uparrow \infty$ with k fixed. Define $r_1 \geq r_2 \geq \ldots \geq r_n$ to be the latent roots of R and $\rho_1 > \rho_2 > \ldots > \rho_n$ to be the latent roots of P . In the same way as before, the latent roots r_i are (approximately) independently distributed as $N(\rho_i,\ \rho_i^2/k)$ for large k . We may therefore deduce confidence bounds for the roots of P and ratios of the roots of P as we did before for Σ .

Thus, for the smallest root ρ_n of P , we have the upper $100(1-\alpha)\%$ confidence limit:

$$\rho_n \le r_n + r_n z_\alpha / k^{1/2} \; ; \tag{18}$$

and the corresponding lower $100(1-\alpha)\%$ confidence limit:

$$r_n - r_n z_\alpha / k^{1/2} \le \rho_n \, , \tag{19}$$

where, as before, $\Phi(z_\alpha) = 1-\alpha$ and $\Phi(\;)$ is the standard $N(0,1)$ c.d.f. Similarly, upper and lower $100(1-\alpha)\%$ confidence bounds for the ratio of the sum of the m smallest roots to the sum of all of the roots of P are given by:

$$\frac{\Sigma_{j=n-m}^{n} \rho_j}{\Sigma_{j=1}^{n} \rho_j} \le \frac{\Sigma_{j=n-m}^{n} r_j}{\Sigma_{j=1}^{n} r_j} + z_\alpha D / k^{1/2} \tag{20}$$

and

$$\frac{\Sigma_{j=n-m}^{n} r_j}{\Sigma_{j=1}^{n} r_j} - z_\alpha D / k^{1/2} \le \frac{\Sigma_{j=n-m}^{n} \rho_j}{\Sigma_{j=1}^{n} \rho_j} \tag{21}$$

where

$$D = \left[\left(\Sigma_{j=n-m}^{n} r_j \right)^2 \left(\Sigma_{j=1}^{n-m-1} r_j^2 \right) + \left(\Sigma_{j=1}^{n-m-1} r_j \right)^2 \Sigma_{j=n-m}^{n} r_j^2 \right]^{1/2} \Big/ \left(\Sigma_{j=1}^{n} r_j \right)^2 \, .$$

In applying these confidence bounds to tests of cointegration some general guidelines will once again be useful. Since we are now working in terms of dimensionless quantities like correlation coefficients some broadly applicable rules are possible. Thus, for the smallest root bounds given by (18) and (19) we could suggest the following: if the upper bound (18) is less than 0.05 then there is strong evidence in favor of cointegration; if the upper bound is less than 0.01 then the evidence might be taken as being very strong. Conversely, if the lower bound is above 0.05 (respectively, 0.10) then the evidence favors (respectively, strongly favors) the null of no cointegration. For the ratio bounds given by (20) and (21) we might continue to work with the earlier rule based on the value $0.10/n$. For example, if the upper bound (19) is less than $0.10/n$ then the evidence supports the existence of m cointegrating vectors.

The adequacy of these broadly defined criteria for the bounds tests has been investigated by simulation methods. The results, which are very encouraging in terms of both size and power, are reported below in Section 5.

4. TESTS FOR INVERTIBILITY AND UNIT ROOTS

The bounds tests developed in the preceding section may be interpreted as simple tests of the invertibility of the moving average representation of a stationary time series. Note that $\rho_n = 0$ iff $\lambda_n = 0$ (i.e. the smallest latent root of Σ is zero). This is true iff there is a degeneracy in the MA representation of the stationary process u_t. In particular, $\lambda_{min}(\Sigma) = 0$ iff we can write

$$(22) \qquad u_t = D(L)\varepsilon_t = \{AC(L) + (1-L)B(L)\}\varepsilon_t$$

where A is $n \times n$ of rank $< n$, $C(L)$ and $B(L)$ are matrices of polynomials in the lag operator L and ε_t is an $iid(0,\Omega)$ sequence of primitive innovations with nonsingular covariance matrix Ω . The degeneracy in D(L) occurs for any vector γ in the null space of the matrix A . Then $\gamma' D(1) = 0$ and $\gamma' u_t = (1-L)\gamma' B(L)\varepsilon_t = v_t - v_{t-1}$ for v_t stationary with spectral density $\gamma' B(e^{i\lambda})\Omega B(e^{i\lambda})^* \gamma$. In such cases the MA representation (22) is noninvertible. Test of the invertibility of (22) may therefore be mounted using the bounds test procedures developed in the previous section. Note that the null hypothesis in this case is invertibility and the alternative is noninvertibility (corresponding to Σ and P being of deficient rank). We should remark, in addition, that invertibility may fail due to the existence of a degeneracy in the spectrum $f_{uu}(\lambda) = g(z) = D(z)\Omega D(z)^*$ at a point on the unit circle $z = e^{i\lambda}$ other than $\lambda = 0$. Obviously the bounds tests described earlier are constructed to focus attention on the frequency $\lambda = 0$. Analogous procedures may be developed to explore possible degeneracies at other frequencies.

In the univariate case $(n = 1)$ the bounds tests may also be interpreted as tests for the presence of a unit root. We now write $\sigma^2 = \Sigma$, $\sigma_0^2 = \Sigma_0$, $s_{Tk} = S_{Tk}$, $p^2 = P$ and $r^2 = R$. Our interest is in the (alternative) hypothesis

$$H : p^2 = \sigma^2/\sigma_0^2 = 0 .$$

We accept H if the upper limit

$$r^2 + r^2 z_\alpha / k^{1/2}$$

is sufficiently small (< 0.05 say) for a preassigned significance level of α (one-tailed 5%, say). We reject H on the other hand if the lower limit

$$r^2 - r^2 z_\alpha / k^{1/2}$$

is above a preassigned point such as 0.05. Note that when H is true we necessarily have the MA representation

$$u_t = v_t - v_{t-1}$$

(cf. (22) above). It follows from (1) that $y_t = v_t$ and y_t is stationary. Thus, the alternative hypothesis in this test corresponds to a stationary alternative to (1). Thus, the bounds test based on r^2 here corresponds to a test for the presence of a unit root in the autoregressive representation (1) against a stationary alternative. We remark that this idea has recently been pursued in interesting empirical research by Cochrane (1986). Cochrane's approach is to empirically estimate the variance ratio σ^2/σ_0^2 for aggregate time series like GNP and assess its magnitude. Our bounds test formalizes this notion into a statistical test.

5. MONTE CARLO RESULTS

(a) Bounds Tests

This section reports the results of a Monte Carlo experiment designed to assess the performance of the latent root bounds tests for no cointegration. We are primarily interested in two issues: (a) the stability of the empirical distribution functions of the bounds statistics to the form of the data generation process (DGP) of the innovation sequence $\{u_t\}$; and (b) the power of the test procedures under the alternative of cointegration.

The simulations were carried out for model sizes ranging from two to five integrated variables, and for three different assumptions about the data generation process:

(a) vector MA(1) : $u_t = \xi_t + \psi\xi_{t-1}$

(b) vector AR(1) : $u_t = \phi u_{t-1} + \xi_t$

(c) vector ARMA(1,1) : $u_t = \phi^* u_{t-1} + \psi^*\xi_{t-1} + \xi_t$

where ψ , ϕ , ϕ^* and ψ^* are the parameter matrices (each $n \times n$) of the process and ξ_t is n-vector white noise.

Estimates of the asymptotic distributions of the various test criteria under the null hypothesis of no cointegration were generated using 500 observations and 5000 repetitions. The innovation sequences for the integrated processes were assumed to be independent so as to ensure that the multiple correlation coefficient between the innovation sequences was zero under the null (see Phillips and Ouliaris (1987), Section 2). Thus, the parameter matrices ψ , ψ^* , and ϕ and ϕ^* were all set to diagonal matrices. Specific values for the diagonal entries are given in the notes to Tables 1-2.

Simulations under the alternative of cointegration were conducted for sample sizes equal to 100, 150 and 200 observations, using 2500 iterations. Note that under the alternative of cointegration, the innovation sequences are generated to be linearly dependent with multiple correlation coefficient, ρ , equal to unity (compared to zero under the null)--again see Phillips and Ouliaris (1987). In order to center the cointegrating regression, it was assumed that there was a single cointegrating vector $(m = 1)$ of the form $\gamma' = (1, -i')$, where i is the $(n-1) \times 1$ sum vector and n is (as before) the dimension of the system. Note that cointegrating vectors of this form appear quite frequently in modern economic models of long-run equilibrium behavior. Examples include the term structure of interest rates, purchasing power parity, and the monetary equation, $MV = PY$. (See Campbell and Shiller (1986) and Engle and Granger (1987) for additional examples). Finally, the spectral density matrix was estimated by (18), using $k = T^{0.6}$ ordinates of the second-order periodogram, which was computed using the IMSL Fast Fourier transform routine FTRCC.

Tables 1 and 2 and Figures 1 and 2 summarize the results for the critical values and power of the latent root procedures for testing the null hypothesis of no cointegration. These procedures are based on the confidence bounds given by (13), (14), (18) and (20). Note that the critical values of the ratio test are expressed as a proportion of the mean of the latent roots. The simulations show that the critical values of the upper bounds for the minimum latent root (see (13) and (18)) are sensitive to the assumptions made about the form of the DGP (see Tables 1a and 1b). This, of course, makes it difficult to design simple decision rules for rejecting the null hypothesis of no cointegration. However, the simulations indicate that the probability of obtaining a latent root that is near zero under the null hypothesis of no cointegration is essentially zero. Thus, the *minimum* upper bound for the smallest latent root is much greater than zero for all the models considered. Moreover, the tabulations reported in Table 2 show that the power of the minimum latent root procedure based on S_{Tk} and inequality (13) is high and is seldom below 90% at the 5% level of significance. In the case of the unit free bounds test for the minimum latent root, power was found to be 100%, irrespective of the form the DGP, the number of observations, and the dimension of the model. This feature of the bounds test is shown clearly in Figure 1 which plots the cumulative distribution function obtained under the null and the alternative for Models 5(a)-5(c). As this figure demonstrates, the location of the distributions under the null and the alternative are so clearly differentiated that their respective supports are quite disjoint.

In the case of the ratio bounds procedures based on S_{Tk} and inequality (14), we see even stronger evidence of the absence of a zero latent root under the null hypothesis of no cointegration. The *minimum* upper bound for the smallest latent root as a percentage of the overall mean is less than 10% *only* in the case of Model 5(c) (see Table 1b), and is typically much higher. This, of course, implies that a 10% decision rule for rejecting the null hypothesis would be too conservative, since the true size of the test, at least for the DGP's considered in the tables, would be zero. However, the power of the ratio test, although high for models involving two integrated variables, is low for Models 5(b) and 5(c) (see Table 2).

Turning to the unit free ratio bounds procedure (based on inequality (20)), we observe that the critical values of the upper bounds are relatively stable across DGPs for the innovation sequence, though they do appear to be sensitive to the dimension of the system (see the Ratio Test panel of Table 1(b)). Power was found to be 100%, once again irrespective of the DGP for the innovation sequence, the number of observations being used, and the dimension of the model. The stability of the upper bounds and the power of the unit free bounds test is shown clearly in Figure 2. Again we see that the distribution of the bounds test under the null and the alternative are clearly differentiated. The figure also highlights the conservative nature of a 10% rejection rule for the unit free ratio test.

In summary, the Monte Carlo results indicate that latent root procedures are apparently very powerful tools for detecting the presence or absence of a cointegrating vector. On the basis of our results we feel able to recommend the following diagnostic procedure for testing the null hypothesis of no cointegration:

(1) Evaluate the upper and lower bound confidence bounds given in (20) and (21) for the ratio of the minimum latent root to the overall mean of the roots of R ;

(2) Reject the null hypothesis of no cointegration if the <u>upper</u> bound is less than 0.10;

(3) Accept the null hypothesis of no cointegration if the <u>lower</u> bound is greater than 0.10.

Our simulations indicate that this is a simple, powerful and yet conservative procedure for testing for cointegration.

(b) <u>Explicit Latent Root Tests</u>

The bounds procedure recommended above involves a region in which the test is inconclusive, just as in the case of the Durbin-Watson bounds tests of conventional regression theory. The inconclusive region arises when the 10% decision rule lies between the computed lower and upper bounds for the minimum latent root. In order to overcome this problem, we need to revert to a more traditional significance testing approach which partitions the support of the null distribution into acceptance and rejection regions. The tests we develop in this section are of this type. They are suggested by the Monte Carlo simulations for the unit free versions of the bounds tests. These simulations demonstrated that the empirical distributions of the bounds tests under the null and the alternative hypotheses are typically quite disjoint. This feature of the empirical distributions, together with their apparent stability to variations in the form of the DGP for the innovation sequence, makes it possible to select critical values for the minimum latent root and its ratio to the overall mean which fall in the region between the empirical domains of the null and alternative distributions. Relative to the 10% rule for the bounds tests, these critical values would yield tests which are more powerful, but compromise on size.

We therefore derived critical values for the unit free minimum latent root and its ratio to the overall mean of the latent roots. Table 3 contains the critical values for the tests statistic at various levels of significance for systems involving 2 to 5 integrated variables. The critical values are for the unit free estimator, which showed the least sensitivity to the form of the DGP in the simulations for the bounds procedure reported above.

The critical values presented in Table 3 were obtained by averaging the corresponding upper and lower percentiles values of the null and alternative distributions, respectively, across 50 DGP's for the innovation sequence. The empirical distributions were generated using 2000 iterations and 250 observations. The form of the DGP was assumed to be ARMA(1,1). The parameters of the process were selected randomly from a uniform distribution over the interval $[a,b]$, restricting draws to ensure a stationary and invertible ARMA(1,1) by setting $a = -0.60$, $b = 0.80$. This delimitation of parameter constellations avoids potential problems arising from ARMA coefficients in the region [-1, -0.6]. This region provides a buffer zone between the null and alternative hypotheses. By eliminating the region we help to keep power high and yet retain very conservative size in our test procedure for a wide class of potential innovation sequences. Note that as $a \to -1$ the system becomes trivially cointegrated since it involves variables with different orders of integration. Thus, when the autoregressive coefficient $\to -1$ we have a system involving I(1) and I(2) variables, whereas when the moving average coefficient $\to -1$ we end up with a system of I(1) and I(0) variables. Our test is designed to be powerful against these forms of cointegration as well as those that involve just I(1) system variables.

The critical values presented in Table 3 are easy to use. The null hypothesis of no cointegration is rejected if the computed test statistic is less than the critical value at a given level of significance. In the case of systems involving 2 integrated variables, for example, we reject the null hypothesis of no cointegration at the 99% level if the minimum latent root is less than 0.3397, or the ratio of the minimum latent root to the overall mean is less than 0.2521. Note that relative to the 10% decision rule for the upper bounds, these statistics are more liberal in terms of size, but involve greater power.

6. EMPIRICAL APPLICATIONS

In this section we apply the unit free bounds procedure and the exact latent root tests for testing the null hypothesis of no cointegration to five models of potentially cointegrated systems:

(1) Total and Non-Durable Consumption and Disposable Income;

(2) Nominal Money and Income;

(3) The Quantity Equation: $MV = PY$;

(4) Real Stock Prices and Dividends; and

(5) The Term Structure of Interest Rates.

Models (1)-(3) were originally formulated as cointegrated systems by Engle and Granger (1987). They found evidence in favor of a cointegrating vector only in the case of real non-durable consumption and disposable income. Their analysis was based on the residuals of the cointegrating regression, using the DW and ADF statistics to detect nonstationarity in this vector. The critical values for the DW and ADF were generated by a small Monte Carlo experiment for an assumed, but arbitrary, DGP for the innovation sequence. Campbell and Shiller (1986) employed these critical values to accept the null hypothesis of no cointegration between real stock prices and dividends, and accept cointegration between short- and long-term yields on bonds (the one month and 20 year yields respectively).

Table 4 presents the results of applying the unit free bounds procedures and the exact latent root tests to the above models. The innovations for the system were estimated by the measured differences of the original series. Using the 10% decision rule for the ratio test, we find that the null hypothesis of no cointegration cannot be rejected for Models 3-5. Indeed, there is strong evidence against cointegration. For example, in the case of the real quantity equation, the lower bounds for the minimum latent root, expressed either in absolute terms or as a proportion of the average root, are not close to zero. This result holds true for all definitions of the money supply--M1, M2, M3 and total liquid asset holdings. The results for consumption and disposable income (Model 1) are inconclusive, since the lower bound of the minimum latent root as a ratio of the overall mean is less than 10%. In the case of Model 2, which is testing the long run relationship between nominal income and money, the bounds test is inconclusive using M2 and M3.

Additional evidence is brought to bear when we apply the explicit latent root tests. Using either the exact minimum latent root test or the exact ratio test we may reject the null hypothesis that consumption (however defined) and disposable income is not cointegrated. At the 90% level, nominal income and M2, M3, and aggregate liquid assets are found to be cointegrated using either decision rule. In the case of Model 3, which is testing the stationarity of velocity over the sample period, we cannot accept the null hypothesis of no cointegration using the explicit minimum latent root test. Using the ratio test, we cannot reject the null hypothesis when money is defined as M1, but reject for broader definitions of the money supply at all significance levels. However, the minimum latent roots for these models are too large relative to the critical value of the minimum eigenvalue to strongly reject no cointegration in these cases. Finally, at the 90% level, we cannot reject the null hypothesis of no cointegration in the case of the term structure of interest rates and stock prices and dividends.

Acceptance of no cointegration for real stock prices and dividends is consistent with the results of Shiller (1981), who rejected the present value model of stock prices using volatility bounds tests. The results for the term structure of interest rates are consistent with previous results for the rational expectations theory of the term structure (see Shiller (1986, Table 2) for a summary of these results). Given the observed empirical regularity of the relationship between consumption and income, strong rejection of cointegration between consumption and disposable income using the bounds procedure is somewhat surprising, but is probably a reflection of the conservative nature of the 10% bounds rule in terms of size. Accepting the null hypothesis of no cointegration between narrowly defined measures of money and income is consistent with the empirical observation that the velocity of money, for narrowly defined definitions of the money supply, has behaved erratically since the deregulation of the banking system in 1981.

7. CONCLUSION

Testing for the presence of cointegration amongst aggregate economic time series seems likely to become a standard method of assessing the empirical support for steady state theories of macroeconomic behavior. Cointegrated systems capture the idea that individual economic time series often exhibit nonstationary characteristics but that certain combinations of the series move together over time. Since linear combinations of nonstationary variables are typically also nonstationary, evidence that a cointegrating vector exists obviously provides strong support for the existence of a long-run relationship amongst a group of integrated variables.

This paper develops diagnostic procedures that are designed to detect the presence of cointegration in multiple time series. Drawing on earlier work, we provide necessary and sufficient conditions for cointegration. These conditions prescribe the behavior of $f_{uu}(\lambda)$, the spectral density matrix of the innovations, in a neighborhood of the origin. Under cointegration, $\Sigma = 2\pi f_{uu}(0)$ is singular and its smallest latent root is zero. These results motivate a new class of statistical tests for cointegration that are based on principal components methods. The tests place upper and lower bounds on the minimum latent root (and ratio of the minimum to the average latent root) of a unit free form of the matrix Σ. The resulting bounds tests are simple to construct, involve only the standard normal distribution and yield criteria for acceptance and rejection of cointegration. Monte Carlo simulations indicate that the bounds test that we recommend in Section 5 provides a powerful diagnostic procedure for testing the null hypothesis of no cointegration. We also develop exact latent root significance tests to deal with situations where the bounds test is inconclusive. These partition the latent root space into acceptance and rejection regions and thereby provide sharp criteria for the presence or absence of cointegration. The exact latent root tests appear from our simulation evidence to be more powerful than the bounds procedure, but less conservative in terms of test size.

We emphasize that our diagnostic procedures do not purport to test a null hypothesis of cointegration or, more specifically, the hypothesis $\lambda_{min}(\Sigma) = 0$. The pitfalls of developing classical Neyman Pearson tests of the null $\lambda_{min}(\Sigma) = 0$ have recently been considered at length in Phillips and Ouliaris (1987). In effect, no generally applicable theory is possible and classical tests of this null are inconsistent. Our approach in this paper works from a null hypothesis of no cointegration. This enables us to use conventional asymptotic methods to assess whether any latent roots of Σ are negligible. The methods we recommend have an established precedent in principal components theory. Indeed, they may be regarded as performing a form of principal components analysis for time series.

TABLE 1

(a) CRITICAL VALUES FOR BOUNDS TESTS OF
NO-COINTEGRATION BASED ON S_{Tk}

Size	Min	5%	10%	15%	20%	25%	Max
	<u>Minimum Latent Root Bound (Inequality (13))</u>						
2(a)	1.4503	2.2308	2.3796	2.4729	2.5523	2.6340	4.8233
(b)	2.9681	4.1828	4.4459	4.6194	4.7702	4.8923	9.9191
(c)	9.4610	13.4401	14.3726	15.0415	15.5502	16.0805	31.2651
5(a)	0.8148	1.2406	1.3186	1.3673	1.4111	1.4494	2.4358
(b)	1.8669	2.7031	2.8781	3.0009	3.0998	3.1865	5.6714
(c)	2.0251	2.9079	3.0986	3.2165	3.3194	3.4037	6.1329
	<u>Minimum Latent Bound: Ratio Test (Inequality (14))</u>						
2(a)	0.4555	0.7358	0.7758	0.8007	0.8225	0.8403	1.2109
(b)	0.4433	0.5709	0.6089	0.6325	0.6541	0.6721	1.1151
(c)	0.1185	0.2226	0.2448	0.2600	0.2731	0.2847	0.7244
5(a)	0.3477	0.4906	0.5190	0.5398	0.5569	0.5717	0.9770
(b)	0.1015	0.1645	0.1779	0.1869	0.1945	0.2016	0.4522
(c)	0.0567	0.1051	0.1140	0.1202	0.1250	0.1294	0.3031

(b) CRITICAL VALUES FOR BOUNDS TESTS OF
NO-COINTEGRATION BASED ON R

Size	Min	5%	10%	15%	20%	25%	Max
	<u>Minimum Latent Bound (Inequality (18))</u>						
2(a)	1.3669	1.9253	2.0145	2.0811	2.1312	2.1788	3.3051
(b)	2.5414	3.3412	3.9710	3.5695	3.6469	3.7084	5.3405
(c)	4.4783	5.6640	5.8338	5.9547	6.0406	6.1132	7.5020
5(a)	0.8633	1.2540	1.3259	1.3740	1.4068	1.4410	2.3076
(b)	1.6536	2.4808	2.5937	2.6782	2.7450	2.8064	4.0417
(c)	1.5223	2.2549	2.3617	2.4487	2.5090	2.5519	3.7517
	<u>Minimum Bound: Ratio Test (Inequality (20))</u>						
2(a)	0.7584	0.9671	0.9965	1.0165	1.0331	1.0466	1.2151
(b)	0.7688	0.8959	0.9217	0.9354	0.9473	0.9579	1.2090
(c)	0.9489	1.0422	1.0589	1.0693	1.0772	1.0842	1.2136
5(a)	0.5083	0.6804	0.7123	0.7364	0.7562	0.7702	1.1043
(b)	0.4066	0.5891	0.6158	0.6339	0.6476	0.6602	0.9332
(c)	0.4529	0.6084	0.6357	0.6539	0.6673	0.6800	0.9641

TABLE 2

REJECTION RATES FOR BOUNDS TESTS OF NO-COINTEGRATION

BASED ON S_{Tk}

Model Size	Sample Size	Min	5%	10%	15%	20%	25%
		Minimum Latent Root Bound (Inequality (13))					
2(a)	100	84.95	99.20	99.55	99.70	99.75	99.85
(b)		96.05	99.55	99.75	99.90	99.90	99.90
(c)		100.00	100.00	100.00	100.00	100.00	100.00
(a)	150	95.60	99.85	99.95	100.00	100.00	100.00
(b)		92.10	98.60	99.15	99.60	99.75	99.75
(c)		100.00	100.00	100.00	100.00	100.00	100.00
(a)	200	98.15	100.00	100.00	100.00	100.00	100.00
(b)		98.65	100.00	100.00	100.00	100.00	100.00
(c)		100.00	100.00	100.00	100.00	100.00	100.00
5(a)	100	65.30	88.40	90.45	91.55	92.25	92.60
(b)		41.10	83.35	88.05	90.95	92.25	93.90
(c)		66.65	96.45	98.20	98.80	99.20	99.45
(a)	150	76.90	92.40	94.05	94.75	95.20	95.55
(b)		33.80	83.30	88.55	91.80	94.05	95.10
(c)		76.55	98.65	99.25	99.55	99.80	99.90
(a)	200	81.60	94.10	95.40	96.15	96.85	97.00
(b)		26.15	79.35	86.40	90.55	92.35	94.00
(c)		83.65	99.80	100.00	100.00	100.00	100.00
		Minimum Latent Root Ratio (Inequality (14))					
2(a)	100	80.75	99.30	99.65	99.65	99.75	99.75
(b)		82.05	95.20	96.65	97.40	97.90	98.15
(c)		96.25	99.90	99.95	99.95	99.95	100.00
(a)	150	89.90	99.90	99.95	99.95	99.95	99.95
(b)		92.15	98.60	99.15	99.60	99.75	99.75
(c)		74.55	98.80	99.25	99.50	99.60	99.75
(a)	200	94.25	99.95	100.00	100.00	100.00	100.00
(b)		96.45	99.60	99.85	100.00	100.00	100.00
(c)		82.00	99.50	99.70	99.80	99.90	99.95
5(a)	100	88.35	98.05	98.75	99.15	99.40	99.60
(b)		4.00	26.10	31.90	35.95	39.90	43.55
(c)		9.55	58.05	65.00	69.45	72.80	75.95
(a)	150	91.60	98.70	99.35	99.50	99.85	99.85
(b)		2.75	25.50	32.30	37.80	41.80	45.10
(c)		13.75	72.85	79.35	83.55	86.45	88.65
(a)	200	92.75	99.40	99.65	99.65	99.75	99.75
(b)		1.65	25.95	34.50	40.45	45.35	50.30
(c)		18.35	85.55	90.70	93.15	94.90	95.80

NOTES TO TABLES 1-2

(i) The empirical distribution under the null hypothesis of no cointegration was generated using 5000 iterations and 500 observations. Simulations under the alternative used 2000 iterations, and sample sizes were between 100 and 200 observations.

(ii) Model dimension includes the dependent variable.

(iii) The parameters of the DGP of u_t were set to:

Model Size	MA(1) Diagonal of ψ	AR(1) Diagonal of ϕ	ARMA(1,1) Diagonal of ϕ^*	
1	0.90	0.75	0.89	0.95
2	0.45	0.53	0.77	0.11
3	0.35	0.42	0.12	0.56
4	0.67	0.92	0.45	0.05
5	0.12	0.88	0.34	0.98

(iv) Remaining elements of ψ , ϕ , ϕ^* and ψ^* were set to zero.

(v) Thus, for a ARMA(1,1) model with 2 integrated variables, the data generation process for the innovation sequence would be as follows:

First process $u_{1t} = 0.89u_{1t-1} + 0.95\varepsilon_{1t-1} + \varepsilon_{1t}$

Second process $u_{2t} = 0.77u_{2t-1} + 0.11\varepsilon_{2t-1} + \varepsilon_{2t}$

(vi) The random numbers ε_t were drawn from N(0,1) distribution using the random number generator: "SUPER-DUPER."

TABLE 3

CRITICAL VALUES FOR EXPLICIT LATENT ROOT TESTS
BASED ON THE UNIT FREE ESTIMATOR R

	99%	95%	90%	85%	80%	75%
	Minimum Latent Root					
2	0.3397	0.3620	0.3763	0.3869	0.3958	0.4038
3	0.2782	0.2987	0.3124	0.3224	0.3304	0.3381
4	0.2106	0.2340	0.2476	0.2576	0.2660	0.2732
5	0.2018	0.2220	0.2349	0.2438	0.2514	0.2584
	Ratio of Minimum Latent Root to Mean					
2	0.2521	0.2554	0.2592	0.2624	0.2655	0.2685
3	0.1718	0.1803	0.1866	0.1914	0.1955	0.1994
4	0.1360	0.1487	0.1565	0.1620	0.1668	0.1709
5	0.1261	0.1371	0.1438	0.1490	0.1533	0.1572

Notes to Table 3

(i) Critical values were obtained by averaging, across all the data generation processes, the upper and lower percentile values of the empirical distributions obtained under the alternative of cointegration and the null of no cointegration, respectively.

(ii) The innovation sequences followed an ARMA(1,1) process, viz:

$$u_{t,i} = \theta_i u_{t-1,i} + \xi_{t,i} + \psi_i \xi_{t-1,i} \ , \quad i = 1, \ldots, 5$$

$$-0.70 \le \theta_i \le 0.60 \ , \quad -0.40 \le \psi_i \le 0.80 \ .$$

(iii) The critical values reported above are based on 50 random selections of $[\phi_i, \psi_i]$. For each selection of $[\phi_i, \psi_i]$, we generated the empirical distributions of the latent roots using 2000 iterations and 250 observations.

TABLE 4

EMPIRICAL RESULTS FOR COINTEGRATION MODELS:
UNIT FREE ESTIMATOR

Model	T	k	r_n	$r_n/\bar{r}$	Bounds upper r_n	lower r_n	upper $(r_n/\bar{r})$	lower $(r_n/\bar{r})$
1. Consumption and Disposable Income: (Real, per capita, 82$)								
1947(2)–1986(1)								
(a) Non-Durable	156	12	0.3120	0.1508	0.7187	0.1903	0.5066	0.0966
(b) Total			0.2814	0.1271	0.6482	0.1716	0.4319	0.0767
2. Nominal Money and Income: 1959(3)–1986(1)								
(a) M1	108	10	0.4504	0.1925	1.1846	0.2655	0.6567	0.1122
(b) M2			0.3663	0.1613	0.9634	0.2158	0.5598	0.0854
(c) M3			0.3921	0.1740	1.0314	0.2310	0.6000	0.0960
(d) Liquid Assets			0.3795	0.2119	0.9981	0.2234	0.7166	0.1310
3. Quantity Equation: MV = PY 1959(3)–1986(1)								
(a) M1	108	10	1.8257	0.2077	4.8019	1.0762	1.0092	0.2379
(b) M2			1.6355	0.1690	4.3009	0.9641	0.8299	0.1843
(c) M3			1.6027	0.1552	4.2154	0.9447	0.7657	0.1656
(d) Liquid Assets			1.3769	0.1350	3.6217	0.8116	0.6724	0.1378
4. Stock Prices and Dividends								
1872–1985	114	10	0.5556	0.2881	1.4561	0.3262	0.9359	0.2167
5. Term-Structure: One Month, 20 Year Yield								
1959(4)–1983(1)	296	17	0.4010	0.2894	0.7644	0.2595	0.8555	0.3024

Notes: Data for Models 1-3 are from the June edition of the 1986 Citibase databank. Data for Models 4 and 5 were kindly provided by J. Campbell and R. Shiller.

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