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EDGEWORTH EQUILIBRIA IN PRODUCTION ECONOMIES

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An *Edgeworth equilibrium* is an allocation that belongs to the core of every n-fold replica of the economy. In [2] we studied in the setting of Riesz spaces the properties of Edgeworth equilibria for pure exchange economies with infinite dimensional commodity spaces. In this work, we study the same problem for economies with production. Under some relatively mild conditions we establish (among other things) that:

1. Edgeworth equilibria exist;
2. Every Edgeworth equilibrium is a quasiequilibrium; and
3. An allocation is an Edgeworth equilibrium if and only if it can be "decentralized" by a price system.

1. INTRODUCTION

The problem of the existence of a competitive equilibrium in pure exchange economies with infinite dimensional commodity spaces has been extensively investigated in recent years, see [1,2,7,10,11,12,13,15,16,17,18,20,21,26,27]. Consequently, this problem is now well understood in contrast to the problem of existence of competitive equilibrium in production economies.

The seminal paper in this area is due to Bewley [6], where the commodity space is L_∞. An essential feature of the model examined by Bewley (which extends the classical Arrow-Debreu finite dimensional model [5]) is that the positive cone has a nonempty interior with respect to the norm topology. The latter property does not hold true for many important commodity spaces which are currently under investigation, e.g., L_1, L_2 or the space $ca(\Omega)$ of all countably additive measures

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on a compact Hausdorff topological space Ω. Consequently, in these spaces some additional restrictions must be placed on preferences and technologies to bound the marginal rates of substitution and the marginal rates of transformation in production.

Zame's paper [27] gives several examples of non-existence of equilibria where either the marginal rate of substitution or marginal rate of transformation is unbounded. In addition, Zame proves the existence of competitive equilibria in production economies for a rich class of normed vector lattices. The idea of Zame's proof is the same as that in Bewley, i.e., to prove existence on a suitable class of subeconomies and then go to the limit; where Bewley uses economies based on finite dimensional subspaces, Zame uses economies based on principal ideals. It is a clever argument and uses the lattice-theoretic structure of the commodity space in a nontrivial fashion. Bewley makes no use of the lattice-theoretic structure of L_∞.

Mas-Colell [17], stimulated by the work of Zame and building on his previous contribution to the literature on exchange economies with infinite dimensional commodity spaces, has investigated the existence and supportability of Pareto optima in production economies with infinite dimensional commodity spaces where the positive cone has an empty interior. This extends the work of Debreu [8] who assumed that the positive cone had a nonempty interior. Mas-Colell's approach is to extend his notion of properness of preferences to "properness" of technologies. There are two intuitions one should have about properness: (1) it bounds the marginal rates of substitution and transformation, and (2) for a utility function (production function) which is proper on the positive cone, the utility function (production function) can be viewed as the restriction of a function defined on a neighborhood of the positive cone, see Richard and Zame [22].

Finally, we mention the work of Kahn and Vohra [13]. In this paper the authors prove the existence of approximate or "ε" competitive equilibria in production economies where the commodity space is an ordered space with a semi-normed predual. They do not assume that preferences are proper or that the positive cone has a nonempty interior.

In all of the work on production economies with the exception of Bewley, the authors have assumed that each agent's consumption set is the positive cone. This is clearly an unacceptable assumption, but one that we shall also be forced to make. In this paper, we continue our investigation of Edgeworth equilibria, see [2]. Our first major result (Theorem 4.4) proves the existence of a core allocation for compact economies. Compact economies satisfy quite weak conditions comparable to those in Bewley, but for economies modeled on Riesz dual systems. In particular, we do not assume that the positive cone of the commodity space has a nonempty interior. We also mention that Yannelis [25] recently established the existence of core allocations in economies without ordered preferences.

To demonstrate the existence of Edgeworth equilibria, we must assume that each agent's consumption set is the positive cone and that preferences are strongly monotone. These assumptions are needed to show that in every replica there exists a core allocation with the equal treatment property. This is Theorem 4.7 in the paper.

Using Mas-Colell's notion of a proper economy, our next major result (Theorem 5.9) is that in a proper economy every Edgeworth equilibrium is a quasiequilibrium. Finally, we show for proper compact economies that Walrasian equilibria exist. The rest of the paper considers a special but important model of production economy, i.e., where the aggregate production set is a cone. In section seven of the paper we give an existence theorem for ε-Walrasian equilibria for this class of production economies.

In sum, our paper is concerned with the existence and relationship of the following equilibrium notions in a production economy: Edgeworth equilibria; quasiequilibria; Walrasian equilibria; and ε-Walrasian equilibria. Our research is most closely related to the work of Zame, Mas-Colell, and Khan and Vohra and we have benefited a great deal from seeing their unpublished research in this area.

2. MATHEMATICAL PRELIMINARIES

This work will be based upon the mathematical framework of Riesz spaces and Banach lattices. For extensive treatments of Riesz spaces and Banach lattices we refer the reader to [3,4,14,23].

Recall that a partially ordered vector space E is said to be a *Riesz space* (or a *vector lattice*) whenever for each $x,y \in E$ the least upper bound of the set $\{x,y\}$ (denoted by $x \vee y$) and the greatest lower bound of $\{x,y\}$ (denoted by $x \wedge y$) both exist in E. For an element x in a Riesz space, we put

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \text{ and } \quad |x| = x \vee (-x).$$

If E is a partially ordered vector space, then the set $E^+ = \{x \in E: x \geqslant 0\}$ is referred to as the positive cone of E and its elements are called positive elements.

The following useful property, known as the *Riesz Decomposition Property*, will be employed quite often in our proofs. It asserts that if in a Riesz space three positive elements x,y, and z satisfy $0 \leqslant x \leqslant y+z$, then there exist positive elements x_1 and x_2 with $x = x_1 + x_2$ such that $0 \leqslant x_1 \leqslant y$ and $0 \leqslant x_2 \leqslant z$.

Let E be a Riesz space. A subset A of E is said to be a *solid set* whenever $|x| \leqslant |y|$ and $y \in A$ imply $x \in A$. Every subset A of E is contained in a smallest solid set, called the solid hull of A and is denoted by $\mathrm{sol}(A)$. Clearly, $\mathrm{sol}(A) = \{x \in E: \exists\ y \in A \text{ with } |x| \leqslant |y|\}$. A solid vector subspace of E is referred to as an *ideal*.

An *order interval* is any set of the form $[a,b] = \{x \in E: a \leqslant x \leqslant b\}$. A subset of a Riesz space E is *order bounded* if it is contained in an order interval. A linear functional $f:E \longrightarrow \mathbb{R}$ is said to be order bounded whenever it carries order bounded subsets of E onto bounded subsets of $\mathbb{R}$. The vector space of all order bounded linear functionals of E is called the *order dual* of E and is denoted by $E^\sim$. Under the ordering $f \leqslant g$ whenever $f(x) \leqslant g(x)$ for all $x \in E^+$ the order dual $E^\sim$ is a Riesz space.

A Hausdorff locally convex topology τ on a Riesz space E is said to be a locally convex-solid (and (E,τ) is called a locally convex-solid Riesz space) whenever τ has a basis at zero consisting of convex and solid sets. The topological dual E' of a locally convex-solid Riesz space (E,τ) is always an ideal of the order dual $E^\sim$.

Regarding locally convex-solid Riesz spaces the following result will play an important role in our study.

THEOREM 2.1. *Let* (E,τ) *be a locally convex-solid Riesz space and let two nets* $\{x_\alpha\}$ *and* $\{y_\alpha\}$ *satisfy* $0 \leqslant x_\alpha \leqslant y_\alpha$ *for all* α. *If* $y_\alpha \xrightarrow{\tau} y$ *holds in* E *and the order interval* $[0,y]$ *is weakly copmact, then the net* $\{x_\alpha\}$ *has a weakly convergent subnet.*

PROOF. From the lattice identity $a = (a-b)^+ + a \wedge b$, we see that

$$0 \leqslant x_\alpha = (x_\alpha - y)^+ + x_\alpha \wedge y \leqslant (y_\alpha - y)^+ + y.$$

Using the Riesz Decomposition Property, we can write $x_\alpha = z_\alpha + v_\alpha$ with $0 \leqslant z_\alpha \leqslant (y_\alpha - y)^+$ and $0 \leqslant v_\alpha \leqslant y$. From $(y_\alpha - y)^+ \xrightarrow{\tau} 0$, we get $z_\alpha \xrightarrow{\tau} 0$. Also, from the weak compactness of $[0,y]$, we see that $\{v_\alpha\}$ has a weakly convergent subnet, and so from $x_\alpha = z_\alpha + v_\alpha$, we infer that $\{x_\alpha\}$ has a weakly convergent subnet.∎

In our economic model the basic concept describing the commodity-price duality will be that of a Riesz dual system. A *Riesz dual system* is a dual system $\langle E,E'\rangle$ such that

1. E is a Riesz space;
2. E' is an ideal of the order dual $E^\sim$ that separates the points of E; and
3. the duality of the system is the natural one, i.e.,

$$\langle x,x'\rangle = x'(x)$$

holds for all $x \in E$ and all $x' \in E'$.

A Riesz dual system $\langle E,E'\rangle$ is said to be *symmetric* whenever the order intervals of E are weakly compact (i.e., $\sigma(E,E')$-compact). A Riesz dual system $\langle E,E'\rangle$ is symmetric if and only if E is an ideal of $(E')^\sim$ (where E is identified in the usual manner as a vector subspace of $(E')^\sim$).

Regarding symmetric Riesz dual systems, the following result will be very important.

THEOREM 2.2. *Assume that* $\langle E,E'\rangle$ *is a symmetric Riesz dual system. If* A *is a relatively weakly compact subset of* E^+, *then* sol(A) *(the solid hull of* A) *is also a relatively weakly compact subset of* E.

PROOF. See the proof of [4, Theorem 13.8, p. 206].■

3. THE ECONOMIC MODEL

The characteristics of our economic model are described as follows.

A. *The commodity-price duality*

The commodity-price duality is given by a Riesz dual system $\langle E,E'\rangle$; E is the commodity space and E' is the price space.

B. *Consumers*

There are m consumers indexed by i such that:

1. Each consumer i has an initial endowment $\omega_i > 0$ and his consumption set X_i is a weakly closed convex subset of E^+ with $\omega_i \in X_i$.
2. The total endowment of the consumers (or simply the total endowment) will be denoted by ω, i.e., $\omega = \omega_1 + \cdots + \omega_m$.
3. The preference $\succ_i$ of each consumer i is represented by a quasi-concave utility function $u_i: X_i \longrightarrow \mathbb{R}^+$.
4. There is a locally convex-solid topology τ on E consistent with $\langle E,E'\rangle$ such that each utility function $u_i:(X_i,\tau) \longrightarrow \mathbb{R}^+$ is continuous.

C. *Producers*

We assume that there are k production firms indexed by j. The production of each producer j is described by its production possibility set Y_j, the elements of which are referred to as the *production plans* for the j producer. For a production plan $y = y^+ - y^- \in Y_j$, the negative part y^- of y is interpreted as the input and the positive part y^+ as the output. The production sets are assumed to satisfy the following properties.

1. Each Y_j is a weakly closed convex subset of E containing zero; and
2. For each j we have $Y_j \cap E^+ = \{0\}$.

The convex set $Y = Y_1 + \cdots + Y_k$ is known as the *aggregate production* set of the economy.

D. *Private Ownership*

Our economy is a private ownership economy. That is, we shall assume that each consumer i has a share θ_{ij} $(0 \leq \theta_{ij} \leq 1)$ of the profit of producer's j production plan; of course, $\sum_{i=1}^{m} \theta_{ij} = 1$ for each j. In other words, if each producer j chooses a production plan $y_j \in Y_j$ and the prevailing price vector is p, then the wealth w_i of the ith consumer is

$$w_i = p \cdot \omega_i + \sum_{j=1}^{k} \theta_{ij} p \cdot y_j \,.$$

Our economy is now defined as follows.

DEFINITION 3.1. *An economy* $\mathcal{E}$ *is a 4-tuple*

$$\mathcal{E} = (\langle E, E' \rangle, \{(X_i, \omega_i, \succ_i) : i=1,\ldots,m\}, \{Y_j : j=1,\ldots,k\}, \{\theta_{ij} : i=1,\ldots,m; j=1,\ldots,k\}),$$

where the agents' characteristics satisfy properties (A),(B),(C) and (D) above.

4. EDGEWORTH EQUILIBRIA

An (m+k)-tuple $(x_1,\ldots,x_m,y_1,\ldots,y_k)$, where $x_i \in X_i$ $(i=1,\ldots,m)$ and $y_j \in Y_j$ $(j=1,\ldots,k)$, is said to be an *allocation* whenever

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \omega_i + \sum_{j=1}^{k} y_j \,.$$

The set of all allocations will be denoted by $\mathcal{A}$. That is,

$$\mathcal{A} = \{(x_1,\ldots,x_m,y_1,\ldots,y_k) : x_i \in X_i,\ y_j \in Y_j \text{ and } \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \omega_i + \sum_{j=1}^{k} y_j\}.$$

It should be noted that the set $\mathcal{A}$ of all allocations is a weakly closed subset of E^{m+k}.

A production plan $y \in Y_j$ is said to be *feasible* for the jth producer whenever there exists an allocation $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ such that $y_j = y$. Similarly, a bundle $x \in X_i$ is said to be *feasible* for the ith consumer whenever there exists an allocation $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ with $x_i = x$.

The *feasible production set* $\hat{Y}_j$ for the jth producer is the set of all of its feasible production plans, i.e.,

$$\hat{Y}_j = \{y \in Y_j : \exists \ (x_1, \ldots, x_m, y_1, \ldots, y_k) \in \mathcal{A} \ \text{ with } \ y_j = y\} .$$

Similarly, the *feasible consumption set* $\hat{X}_i$ of the ith consumer is the set of all of its feasible consumption bundles, i.e.,

$$\hat{X}_i = \{x \in X_i : \exists \ (x_1, \ldots, x_m, y_1, \ldots, y_k) \in \mathcal{A} \ \text{ with } \ x_i = x\}.$$

Some basic properties of the sets $\hat{X}_i$ and $\hat{Y}_j$ are described in the next result.

THEOREM 4.1. *For an economy with a symmetric Riesz dual system and aggregate production set* $Y = Y_1 + \cdots + Y_k$ *the following statements hold.*

1. *If all production sets are order bounded from above, then each feasible production set* $\hat{Y}_j$ *is weakly compact.*
2. *If each feasible production set* $\hat{Y}_j$ *is weakly compact and* $X_i = E^+$ *holds for each* i*, then* $(Y + \omega) \cap E^+$ *is a weakly compact set.*
3. *If* $(Y + \omega) \cap E^+$ *is weakly compact, then the feasible consumption sets* $\hat{X}_i$ *are all weakly compact subsets of* E^+.

PROOF. (1) Pick some $a \in E^+$ such that $z \in Y_j$ $(j=1, \ldots, k)$ implies $z \leqslant a$. Let $y \in \hat{Y}_j$. Choose an allocation $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ with $y_j = y$. Then we have

$$0 \leqslant y^- \leqslant \sum_{j=1}^{k} y_j^- + \sum_{i=1}^{m} x_i = \sum_{j=1}^{k} y_j^+ + \omega \leqslant ka + \omega = b \in E^+,$$

and so

$$-b \leqslant -y^- \leqslant y^+ - y^- = y \leqslant a \leqslant b .$$

Therefore, $\hat{Y}_j \subseteq [-b,b]$. Since $[-b,b]$ is weakly compact, we infer that $\hat{Y}_j$ is relatively weakly compact. Thus, in order to establish that $\hat{Y}_j$ is weakly compact, it suffices to show that $\hat{Y}_j$ is weakly closed.

To this end, let $\{y^\alpha\} \subseteq \hat{Y}_j$ satisfy $y^\alpha \xrightarrow{w} y$ in E. For each α pick an allocation $(x_1^\alpha, \ldots, x_m^\alpha, y_1^\alpha, \ldots, y_k^\alpha)$ with $y_j^\alpha = y^\alpha$. Since $\hat{Y}_j$ is relatively weakly compact, by passing to an appropriate subnet, we can assume that $y_j^\alpha \xrightarrow{w} y_j \in Y_j$ holds for each j. From

$$0 \leqslant x_i^\alpha \leqslant x_1^\alpha + \cdots + x_m^\alpha = \sum_{j=1}^{k} y_j^\alpha + \omega \in \hat{Y}_1 + \cdots + \hat{Y}_k + \omega,$$

we see that x_i^α belongs to the relatively weakly compact set $\mathrm{sol}[(\hat{Y}_1 + \cdots + \hat{Y}_k + \omega) \cap E^+]$ (Theorem 2.2). Thus, each net $\{x_i^\alpha\}$ has a weakly convergent subnet, and so (by passing to an appropriate subnet again) we can assume that $x_i^\alpha \xrightarrow{w} x_i \in X_i$ holds for all i. From

$$\sum_{i=1}^{m} x_i^\alpha = \sum_{i=1}^{m} \omega_i + \sum_{j=1}^{k} y_j^\alpha,$$

we get $\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \omega_i + \sum_{j=1}^{k} y_j$. This implies $y \in \hat{Y}_j$, and so $\hat{Y}_j$ is a weakly closed set, as desired.

(2) Since $X_i = E^+$ holds for each i, it is easy to see that

$$(Y + \omega) \cap E^+ \subseteq (\hat{Y}_1 + \cdots + \hat{Y}_k + \omega) \cap E^+.$$

Therefore, $(Y + \omega) \cap E^+$ is a relatively weakly compact set.

Now assume that a net $\{(y_1^\alpha + \cdots + y_k^\alpha + \omega)\}$ of $(Y + \omega) \cap E^+$ satisfies $y_1^\alpha + \cdots + y_k^\alpha + \omega \xrightarrow{w} z$. Since $\{y_j^\alpha\} \subseteq \hat{Y}_j$ holds for all j, we can assume that $y_j^\alpha \xrightarrow{w} y_j \in Y_j$ holds for all j. This implies $z = y_1 + \cdots + y_k + \omega \in (Y + \omega) \cap E^+$, and so $(Y + \omega) \cap E^+$ is weakly closed. Hence, $(Y + \omega) \cap E^+$ is weakly compact.

(3) Fix some i, and let $x \in \hat{X}_i$. Pick an allocation $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ with $x_i = x$. From

$$0 \leqslant x \leqslant x_1 + \cdots + x_m = \omega + \sum_{j=1}^{k} y_j \in (Y + \omega) \cap E^+,$$

we see that $x \in \mathrm{sol}[(Y + \omega) \cap E^+]$, and so $\hat{X}_i \subseteq \mathrm{sol}[(Y + \omega) \cap E^+]$. Since $\mathrm{sol}[(Y + \omega) \cap E^+]$ is a relatively weakly compact subset of E (Theorem 2.2), it follows that each $\hat{X}_i$ is a relatively weakly compact subset of E^+.

Next, assume that a net $\{x^\alpha\}$ of $\hat{X}_i$ satisfies $x^\alpha \xrightarrow{w} x_i$. For each α pick an allocation $(x_1^\alpha, \ldots, x_m^\alpha, y_1^\alpha, \ldots, y_k^\alpha)$ with $x_i^\alpha = x^\alpha$. By the preceding conclusion, we can assume (by passing to a subnet) that $x_i^\alpha \xrightarrow{w} x_i$ holds for each i. Let $y_\alpha = \sum_{j=1}^{k} y_j^\alpha$. From $y_\alpha + \omega = \sum_{j=1}^{k} y_j^\alpha + \omega \in (Y+\omega) \cap E^+$, we can assume (by passing to a subnet again) that $y_j^\alpha \xrightarrow{w} z \in Y$ holds. If $z = z_1 + \cdots + z_k \in Y$, then $(x_1,\ldots,x_m,z_1,\ldots,z_k)$ is an allocation, and so $x_i \in \hat{X}_i$. Thus, $\hat{X}_i$ is weakly closed, and hence each $\hat{X}_i$ is a weakly compact subset of E^+.■

We now come to the concept of a compact economy.

DEFINITION 4.2. *An economy is said to be a* compact economy *whenever*

1. *its Riesz dual system is symmetric; and*
2. *if* $Y = Y_1 + \cdots + Y_k$ *is its aggregate production set, then* $(Y+\omega) \cap E^+$ *is a weakly compact set.*

It should be noted that the weak compactness of $(Y+\omega) \cap E^+$ does not imply the weak compactness of the feasible production sets.

Now let S be a coalition of consumers (i.e., let S be a non-empty subset of $\{1,\ldots,m\}$). A subset $\{z_i\colon i \in S\}$ of E^+ is said to be a *feasible assignment for the coalition* S whenever

a) $z_i \in X_i$ for each $i \in S$; and

b) there exist production plans $h_j \in Y_j$ $(j=1,\ldots,k)$ such that

$$\sum_{i\in S} z_i = \sum_{i\in S} \omega_i + \sum_{j=1}^{k} \Big(\sum_{i\in S} \theta_{ij}\Big) h_j \,.$$

A coalition S *blocks* an allocation $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ whenever there exists a feasible assignment set $\{z_i\colon i \in S\}$ for S such that $z_i \succ_i x_i$ holds for all $i \in S$.

If $X_i = X_i + E^+$ holds for each i and preferences are monotone (i.e., $x > y$ in X_i implies $x \succ_i y$), then it should be clear that a coalition S blocks an allocation $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ if and only if there exist consumption bundles $z_i \in X_i$ $(i \in S)$ and production plans $h_j \in Y_j$ $(j=1,\ldots,k)$ such that

i) $z_i \succ_i x_i$ for each $i \in S$; and

ii) $\sum_{i\in S} z_i \leqslant \sum_{i\in S} \omega_i + \sum_{j=1}^{k} \Big(\sum_{i\in S} \theta_{ij}\Big) h_j$.

A *core allocation* is any allocation that cannot be blocked by any coalition. The *core* $\mathrm{Core}(\mathcal{E})$ of an economy $\mathcal{E}$ is the set of all core allocations.

LEMMA 4.3. *If in an economy with a symmetric Riesz dual system each production set is order bounded from above, then its core is non-empty.*

PROOF. Fix some $a \in E^+$ such that $y \in Y_j$ $(j=1,\ldots,k)$ implies $y \leqslant a$. For each coalition S of consumers define the set

$$V(S) = \{(z_1,\ldots,z_m) \in \mathbb{R}^m : \exists \text{ a feasible assignment } \{x_i : i \in S\} \text{ with } u_i(x_i) \geqslant z_i \ \forall\ i \in S\}.$$

The subsets $V(S)$ of $\mathbb{R}^m$ have the following properties.

1. For each coalition S, the set $V(S)$ is bounded from above with respect to $\mathbb{R}^S$. In particular, the non-empty set $V(S) \cap \mathbb{R}^m_+$ is bounded relative to $\mathbb{R}^S$.

To see that $V(S)$ is bounded from above in $\mathbb{R}^S$, assume by way of contradiction that there exists a sequence $\{(z_1^n,\ldots,z_m^n)\}$ of $V(S)$ and some $r \in S$ such that $z_r^n \geqslant n$ holds for all n. Pick $x_i^n \in X_i$ $(i \in S)$ and $y_j^n \in Y_j$ $(j=1,\ldots,k)$ such that

$$\sum_{i\in S} x_i^n = \sum_{i\in S} \omega_i + \sum_{j=1}^{k} \Big(\sum_{i\in S} \theta_{ij}\Big) y_j^n \quad \text{and} \quad u_r(x_r^n) \geqslant z_r^n \geqslant n \quad \text{for all } n.$$

Clearly, $\{x_r^n\} \subseteq \hat{X}_r$ holds. Since $\hat{X}_r \subseteq [0,\omega+ka]$, the sequence $\{x_r^n\}$ has a weak accumulation point, say x. Now for each natural number ℓ, the element x belongs to the weak closure of the set $\mathrm{co}\{x_r^n : n \geqslant \ell\}$, and hence x belongs to the τ-closure of $\mathrm{co}\{x_r^n : n \geqslant \ell\}$. Thus, by the τ-continuity of u_r, there exists a convex combination $\sum_{i=\ell}^{\ell+\mu} \lambda_i x_r^i$ with

$$\Big|u_r(x) - u_r\Big(\sum_{i=\ell}^{\ell+\mu} \lambda_i x_r^i\Big)\Big| < 1.$$

If ν is an integer among $\{\ell,\ldots,\ell+\mu\}$ satisfying

$$u_r(x_r^\nu) = \min\{u_r(x_r^i) : i = \ell,\ldots,\ell+\mu\},$$

then by the quasi-concavity of u_r, we see that

$$\ell \leqslant \nu \leqslant u_r(x_r^\nu) \leqslant u_r\Big(\sum_{i=\ell}^{\ell+\mu} \lambda_i x_r^i\Big) \leqslant u_r(x) + 1 < \infty .$$

Since ℓ is arbitrary, the latter inequality is impossible. Therefore, $V(S)$ is bounded from above relative to $\mathbb{R}^S$.

2. Each $V(S)$ is a non-empty proper closed subset of $\mathbb{R}^m$.

Let S be a coalition of consumers. Since $(u_1(\omega_1),\dots,u_m(\omega_m)) \in V(S)$, we see that $V(S) \neq \emptyset$. By part (1), we know that $V(S)$ is bounded from above relative to $\mathbb{R}^S$, and this implies that $V(S)$ is a proper subset of $\mathbb{R}^m$.

To see that $V(S)$ is closed, assume that a net $\{(z_1^\alpha,\dots,z_m^\alpha)\}$ of $V(S)$ satisfies $(z_1^\alpha,\dots,z_m^\alpha) \longrightarrow (z_1,\dots,z_m)$ in $\mathbb{R}^m$. For each α pick $x_i^\alpha \in X_i$ $(i \in S)$ and $y_j^\alpha \in Y_j$ $(j=1,\dots,k)$ such that

$$\sum_{i\in S} x_i^\alpha = \sum_{i\in S} \omega_i + \sum_{j=1}^{k} \Big(\sum_{i\in S} \theta_{ij}\Big) y_j^\alpha , \qquad (\star)$$

and

$$z_i^\alpha \leqslant u_i(x_i^\alpha) \quad \text{for all } i \in S.$$

In case $\sum_{i\in S} \theta_{ij} = 0$, we can assume without loss of generality that $y_j^\alpha = 0$.

Since for each j we have $(\sum_{i\in S} \theta_{ij}) y_j^\alpha \in \hat{Y}_j$ and $\hat{Y}_j$ is weakly compact (Theorem 4.1), it follows (by passing to a subnet if necessary) that $y_j^\alpha \xrightarrow{w} y_j \in Y_j$ holds for all $j=1,\dots,k$. Also, from $(\star)$ we see that $x_i^\alpha \in \hat{X}_i$ holds for all $i \in S$, and so from Theorem 4.1(3) (by passing to a subnet again) we can assume that $x_i^\alpha \xrightarrow{w} x_i \in X_i$ holds for all $i \in S$. From $(\star)$, we infer that

$$\sum_{i\in S} x_i = \sum_{i\in S} \omega_i + \sum_{j=1}^{k} \Big(\sum_{i\in S} \theta_{ij}\Big) y_j .$$

To complete the proof of part (2), it suffices to show that $z_i \leqslant u_i(x_i)$ holds for each $i \in S$. To this end, fix $i \in S$ and let $\varepsilon > 0$. Pick some β with

$$z_i - \varepsilon < z_i^\alpha \quad \text{for all } \alpha \geqslant \beta.$$

Since x_i is in the weak closure of the set $\mathrm{co}\{x_i^\alpha : \alpha \geqslant \beta\}$, it follows that x_i is also in the τ-closure of $\mathrm{co}\{x_i^\alpha : \alpha \geqslant \beta\}$. Thus, by the τ-continuity of u_i, there exists a convex combination $\sum_{s=1}^{t} \lambda_s x_i^{\alpha_s}$ with $\alpha_s \geqslant \beta$ such that

$$u_i\Big(\sum_{s=1}^{t} \lambda_s x_i^{\alpha_s}\Big) < u_i(x_i) + \varepsilon .$$

If γ is an index among $\{\alpha_1,\dots,\alpha_t\}$ with

$$u_i(x_i^\gamma) = \min\{u_i(x_i^{\alpha_s}) : s=1,\dots,t\},$$

then $\gamma \geqslant \beta$ holds, and by the quasi-concavity of u_i we see that

$$z_i - \varepsilon < z_i^\gamma \leqslant u_i(x_i^\gamma) \leqslant u_i\Big(\sum_{s=1}^{t} \lambda_s x_i^{\alpha_s}\Big) < u_i(x_i) + \varepsilon .$$

Thus, $z_i < u_i(x_i) + 2\varepsilon$ holds for all $\varepsilon > 0$, from which it follows that $z_i \leqslant u_i(x_i)$ holds for all $i \in S$, as desired.

3. Each $V(S)$ is comprehensive, i.e., $(x_1,\ldots,x_m) \in V(S)$ and $(z_1,\ldots,z_m) \leqslant (x_1,\ldots,x_m)$ imply $(z_1,\ldots,z_m) \in V(S)$.

4. If $(x_1,\ldots,x_m) \in V(S)$ and $(z_1,\ldots,z_m)$ satisfy $z_i = x_i$ for all $i \in S$, then $(z_1,\ldots,z_m) \in V(S)$.

5. The market game derived from the economy is balanced.

To see this, consider a balanced family $\mathcal{B}$ of coalitions with weights $\{w_S\colon S \in \mathcal{B}\}$. That is, $\sum_{S \in \mathcal{B}_i} w_S = 1$ holds for all i, where as usual $\mathcal{B}_i = \{S \in \mathcal{B} : i \in S\}$. Now let $(z_1,\ldots,z_m) \in \bigcap_{S \in \mathcal{B}} V(S)$. We have to show that $(z_1,\ldots,z_m) \in V(\{1,\ldots,m\})$.

Let $S \in \mathcal{B}$. Since $(z_1,\ldots,z_m) \in V(S)$, there exist $x_i^S \in X_i$ $(i \in S)$ and $y_j^S \in Y_j$ $(j=1,\ldots,k)$ with

$$\sum_{i \in S} x_i^S = \sum_{i \in S} \omega_i + \sum_{j=1}^{k} \Big(\sum_{i \in S} \theta_{ij}\Big) y_j^S$$

and $u_i(x_i^S) \geqslant z_i$ for all $i \in S$. Now put

$$x_i = \sum_{S \in \mathcal{B}_i} w_S x_i^S \in X_i,\ i = 1,\ldots,m, \text{ and}$$

$$y_j = \sum_{S \in \mathcal{B}} \sum_{i \in S} w_S \theta_{ij} y_j^S = \sum_{i=1}^{m} \theta_{ij}\Big(\sum_{S \in \mathcal{B}_i} w_S y_j^S\Big) \in Y_j,\ j = 1,\ldots,k.$$

Since each x_i is a convex combination, it follows from the quasi-concavity of u_i that $z_i \leqslant u_i(x_i)$ holds for all $i = 1,\ldots,m$. Moreover, we have

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \sum_{S \in \mathcal{B}_i} w_S x_i^S = \sum_{S \in \mathcal{B}} w_S\Big(\sum_{i \in S} x_i^S\Big) = \sum_{S \in \mathcal{B}} w_S\Big[\sum_{i \in S} \omega_i + \sum_{j=1}^{k}\Big(\sum_{i \in S} \theta_{ij}\Big) y_j^S\Big]$$

$$= \sum_{i=1}^{m} \sum_{S \in \mathcal{B}_i} w_S \omega_i + \sum_{j=1}^{k} \sum_{S \in \mathcal{B}} \sum_{i \in S} w_S \theta_{ij} y_j^S = \sum_{i=1}^{m} \omega_i + \sum_{j=1}^{k} y_j ,$$

which proves that $(z_1,\ldots,z_m) \in V(\{1,\ldots,m\})$, as desired.

Next, by Scarf's classical result [24], the market game derived from the economy has a non-empty core (i.e., the set $V(\{1,\ldots,m\}) \setminus \bigcup_{S\in\mathcal{C}} \operatorname{Int}V(S)$ is non-empty, where $\mathcal{C}$ denotes the set of all coalitions). Let $(z_1,\ldots,z_m)$ be a core vector. Pick $x_i \in X_i$ $(i=1,\ldots,m)$ and $y_j \in Y_j$ $(j=1,\ldots,k)$ such that

a) $\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \omega_i + \sum_{j=1}^{k} y_j$, and

b) $u_i(x_i) \geqslant z_i$ for $i=1,\ldots,m$.

Clearly, $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ is an allocation, and we claim that it is a core allocation. To see the latter, assume by way of contradiction that there exists an allocation S and a feasible assignment $\{h_i\colon i\in S\}$ satisfying $h_i \succ_i x_i$ for all $i\in S$. Then $u_i(h_i) > u_i(x_i) \geqslant z_i$ holds for all $i\in S$, and from this we see that $(z_1,\ldots,z_m)\in \operatorname{Int}V(S)$, which is a contradiction. Hence $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ is a core allocation, and therefore the economy has a non-empty core.■

We are now in the position to establish that a compact economy has always a non-empty core.

THEOREM 4.4. *If the economy is compact, then its core is a non-empty weakly closed subset of the set of all allocations.*

PROOF. Put $\hat{Y} = [(Y+\omega)\cap E^+] - \omega$, where $Y = Y_1 + \cdots + Y_k$, and note that $\hat{Y}$ is a weakly compact set. Also, we shall consider the set

$$\hat{\mathcal{A}} = \{(x_1,\ldots,x_m,y)\in E^{m+1}\colon \quad y = \sum_{j=1}^{k} y_j \quad \text{with} \quad (x_1,\ldots,x_m,y_1,\ldots,y_k)\in\mathcal{A}\}.$$

Clearly, $\hat{\mathcal{A}} \subseteq \hat{X}_1 \times\cdots\times \hat{X}_m \times \hat{Y}$, and from this and Theorem 4.1 it is easy to see that $\hat{\mathcal{A}}$ is a weakly compact subset of E^{m+1}. The proof of the theorem has two steps.

I. The core is non-empty.

For each $a\in E^+$ we shall denote by $\mathcal{E}_a$ the economy which comes from our original economy $\mathcal{E}$ by replacing each Y_j by $Y_j^a = \{y\in Y_j\colon\ y \leqslant a\}$. By Lemma 4.3, we know that $\operatorname{Core}(\mathcal{E}_a) \neq \emptyset$. For each $a\in E^+$ pick some $(x_1^a,\ldots,x_m^a,y_1^a,\ldots,y_k^a)$ in the core of $\mathcal{E}_a$ and let $y^a = \sum_{j=1}^{k} y_j^a$. Then $(x_1^a,\ldots,x_m^a,y^a)\in\hat{\mathcal{A}}$ for each $a\in E^+$.

Since $\hat{\mathscr{A}}$ is weakly compact, the net $\{(x_1^a,\ldots,x_m^a,y^a): a\in E^+\}$ has a weak accumulation point in $\hat{\mathscr{A}}$, say $(x_1,\ldots,x_m,y)$. Then $x_1+\cdots+x_m=\omega+y$ and $y\in Y$. Pick $y_j\in Y_j$ $(j=1,\ldots,k)$ with $y=y_1+\cdots+y_k$, and we claim that the allocation $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ is a core allocation for our original economy.

To see this, assume by way of contradiction that there exist a coalition S of consumers, consumption bundles $h_i\in X_i$ $(i\in S)$, and production plans $z_j\in Y_j$ $(j=1,\ldots,k)$ such that

a) $h_i \succ_i x_i$ for all $i\in S$, and

b) $\sum_{i\in S} h_i = \sum_{i\in S}\omega_i + \sum_{j=1}^{k}(\sum_{i\in S}\theta_{ij})z_j$.

Now note that for each $i\in S$ the set

$$V_i=\{(f_1,\ldots,f_m,g)\in\hat{\mathscr{A}}:\ f_i \succcurlyeq_i h_i\}$$

is a weakly closed subset of E^{m+1}, and so $V=\bigcup_{i\in S} V_i$ is also weakly closed. Thus, its complement V^c is weakly open. Since $(x_1,\ldots,x_m,y)\in V^c$ and $(x_1,\ldots,x_m,y)$ is a weak accumulation point of the net $\{(x_1^a,\ldots,x_m^a,y^a): a\in E^+\}$, there exists some $a\geqslant |z_1|+\cdots+|z_k|$ such that $(x_1^a,\ldots,x_m^a,y^a)\in V^c$. Clearly, $z_j\in Y_j^a$ for each j. Also, $h_i\succ_i x_i^a$ holds for all $i\in S$, and so in view of (b) we have $(x_1^a,\ldots,x_m^a,y_1^a,\ldots,y_k^a)\notin \mathrm{Core}(\mathscr{E}_a)$, a contradiction. Therefore, $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ is a core allocation for our original economy.

II. The core is a weakly closed set.

Denote by C the (non-empty) set of all core allocations, and let $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ be an allocation lying in the weak closure of C. Assume by way of contradiction that there exist a coalition S, consumption bundles $z_i\in X_i$ $(i\in S)$ and production plans $y_j\in Y_j$ $(j=1,\ldots,k)$ such that

$$z_i \succ_i x_i \text{ for all } i\in S \text{ and } \sum_{i\in S} z_i = \sum_{i\in S}\omega_i + \sum_{j=1}^{k}(\sum_{i\in S}\theta_{ij})y_j.$$

For each $i\in S$ the set of allocations

$$W_i=\{(h_1,\ldots,h_m,g_1,\ldots,g_k)\in\mathscr{A}:\ h_i\succcurlyeq_i z_i\}$$

is a weakly closed subset of E^{m+k}. Thus the set $W=\bigcup_{i\in S} W_i$ is weakly closed in E^{m+k}, and so its complement W^c is weakly open. From $(x_1,\ldots,x_m,y_1,\ldots,y_k)\in W^c$, we infer that $W^c \cap C \neq \emptyset$. If $(h_1,\ldots,h_m,g_1,\ldots,g_k) \in W^c \cap C$, then we have

$$z_i \succ_i h_i \quad \text{for all} \quad i \in S \quad \text{and} \quad \sum_{i\in S} z_i = \sum_{i\in S} \omega_i + \sum_{j=1}^{k} \Big(\sum_{i\in S} \theta_{ij}\Big) y_j ,$$

which contradicts the fact that $(h_1,\ldots,h_m,g_1,\ldots,g_m)$ is a core allocation. Hence, $(x_1,\ldots,x_m,y_1,\ldots,y_k) \in C$, and so C is a weakly closed set.■

Next, let us briefly recall the replication concept of an economy with production as it was introduced by H. Nikaido in [19, p. 288]. If n is a natural number, then the n-fold replica of the economy is a new economy with the following characteristics.

1. The new economy has the same Riesz dual system $\langle E,E'\rangle$.
2. There are mn consumers indexed by (i,s) $(i=1,\ldots,m;\ s=1,\ldots,n)$ such that the consumers (i,s) $(s=1,\ldots,n)$ are of the "same type" as the consumer i of the original economy. That is, each consumer (i,s) has:
   a) X_i as his consumption set, i.e., $X_{is} = X_i$;
   b) an initial endowment ω_{is} equal to ω_i, i.e., $\omega_{is} = \omega_i$ (and so the total endowment of the new economy is $\sum_{i=1}^{m}\sum_{j=1}^{k} \omega_{is} = n\omega$); and
   c) a utility function u_{is} equal to u_i, i.e., $u_{is} = u_i$.
3. There are kn producers indexed by (j,t) $(j=1,\ldots,k;\ t=1,\ldots,n)$ with the the following properties.
   i) The production possibility set of the (j,t) producer is Y_j, i.e., $Y_{jt} = Y_j$; and
   ii) The share θ_{isjt} of the (i,s) consumer to the profit of the (j,t) producer is given by

$$\theta_{isjt} = \begin{cases} 0 & \text{if } s \neq t \\ \theta_{ij} & \text{if } s = t \end{cases} .$$

THEOREM 4.5. *Every replication of a compact economy is itself compact.*

PROOF. Consider the n-fold replica $\mathcal{E}_n$ of a compact economy $\mathcal{E}$. Since $\mathcal{E}_n$ has the same Riesz dual system as $\mathcal{E}$, we see that $\mathcal{E}_n$ has a symmetric Riesz dual system.

On the other hand, the aggregate production set of $\mathcal{E}_n$ satisfies

$$\Big(\sum_{j=1}^{k}\sum_{t=1}^{n} Y_{jt} + n\omega\Big) \cap E^+ = \Big[\sum_{t=1}^{n} (Y_1 + \cdots + Y_k) + n\omega\Big] \cap E^+ = (nY + n\omega) \cap E^+ = n[(Y+\omega) \cap E^+].$$

Since $n[(Y+\omega)\cap E^+]$ is weakly compact, we infer that $\mathcal{E}_n$ is a compact economy.■

Now let $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ be an allocation of the original economy. If n is a natural number, then by assigning the consumption bundle x_i to each consumer (i,s) (i.e., $x_{is}=x_i$ for $s=1,\ldots,n$) and the production plan y_j to each producer (j,t) (i.e., $y_{jt}=y_j$ for $t=1,\ldots,n$), it is easy to see that this assignment defines an allocation for the n-fold replica economy. Thus, every allocation of the original economy can be considered (in the above manner) as an allocation for every n-fold replica of the original economy.

DEFINITION 4.6. *An allocation of an economy is said to be an Edgeworth Equilibrium whenever it belongs to the core of every n-fold replica of the economy.*

Do Edgeworth equilibria exist? Before presenting an affirmative answer, let us review a few facts about preferences. Recall that a preference $\succeq$ on a convex set X is said to be

a) *strongly monotone*, whenever $x,y\in X$ and $x>y$ imply $x\succ y$; and
b) *convex*, whenever $x\succ y$ in X implies $\alpha x+(1-\alpha)y\succ y$ for all $0<\alpha<1$.

The following two basic properties about preferences will be employed in the proof of the next theorem.

1. *If a preference $\succeq$ defined on E^+ is weakly convex (i.e., $\{x\in E^+: x\succeq y\}$ is convex for all $y\in E^+$), continuous for some linear topology on E and strongly monotone, then $\succeq$ is also convex.*

To see this, assume $x\succ y$ and let $0<\alpha<1$. Then $x\succ 0$, and since $\lim_{\varepsilon\uparrow 1}\varepsilon x=x$ holds for every linear topology on E, it follows that there exists some $0<\varepsilon<1$ with $\varepsilon x\succ y$. By the weak convexity, we have $\alpha(\varepsilon x)+(1-\alpha)y\succeq y$. On the other hand, from $\alpha x+(1-\alpha)y>\alpha(\varepsilon x)+(1-\alpha)y$ and the strong monotonicity of $\succeq$, we infer that $\alpha x+(1-\alpha)y\succ\alpha(\varepsilon x)+(1-\alpha)y$. Thus, $\alpha x+(1-\alpha)y\succ y$.

2. *If $X_i=E^+$ holds for each consumer i and each preference $\succeq_i$ is in addition strongly monotone, then a coalition S blocks an allocation $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ if and only if there exists a feasible assignment $\{h_i: i\in S\}$ for S such that*
   - i) *$h_i\succeq_i x_i$ for all $i\in S$, and*
   - ii) *$h_i\succ_i x_i$ holds for at least one $i\in S$.*

To see this, assume that (i) and (ii) above are true. Fix some $r \in S$ with $h_r \succ_r x_r$. Since $\tau\text{-}\lim_{\varepsilon \uparrow 1} \varepsilon h_r = h_r$, it follows from $X_r = E^+$ and the τ-continuity of $\succcurlyeq_r$ that there exists some $0 < \varepsilon < 1$ with $\varepsilon h_r \succ_r x_r$. If $\ell > 1$ is the number of elements of S, then put $f_r = \varepsilon h_r$ and $f_i = h_i + [(1-\varepsilon)/(\ell - 1)]h_r \in E^+$ for $i \in S$ and $i \neq r$. From the strong monotonicity of preferences, we infer that $f_i \succ_i x_i$ for all $i \in S$, and moreover $\sum_{i \in S} f_i = \sum_{i \in S} h_i$. The above show that S blocks the allocation $(x_1,\ldots,x_m,y_1,\ldots,y_k)$.

We are now in the position to present an existence theorem for Edgeworth equilibria.

THEOREM 4.7. *If the economy is compact, preferences are in addition strongly monotone and* $X_i = E^+$ *holds for all* i*, then the set of all Edgeworth equilibria is a non-empty weakly closed subset of* E^{m+k}.

PROOF. Let $\mathcal{E}_n$ denote the n-fold replica of our original economy. For each n, let

$$C_n = \mathcal{A} \cap \mathrm{Core}(\mathcal{E}_n).$$

It should be clear that the set of all Edgeworth equilibria is precisely the set $\bigcap_{n=1}^{\infty} C_n$. The proof will be based upon the following properties of the sets C_n.

1. Each C_n is non-empty.

Note first that (by Theorem 4.5) the economy $\mathcal{E}_n$ is a compact economy. By Theorem 4.4, we know that $\mathrm{Core}(\mathcal{E}_n) \neq \emptyset$. Let

$$(x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},\ldots,x_{m1},\ldots,x_{mn},y_{11},\ldots,y_{1n},y_{21},\ldots,y_{2n},\ldots,y_{k1},\ldots,y_{kn})$$

be a core allocation for $\mathcal{E}_n$. Then we claim that

$$x_{ir} \sim_i x_{is} \quad \text{for } r,s = 1,\ldots,n \text{ and } i = 1,\ldots,m,$$

i.e., no consumer prefers his bundle to that of another consumer of the same type.

To see this, note first that (by rearranging the consumers of each type), we can suppose that $x_{ir} \succcurlyeq_i x_{i1}$ holds for all i and r. Put

$$z_i = \frac{1}{n} \sum_{r=1}^{n} x_{ir} \geqslant 0, \; i = 1,\ldots,m, \text{ and}$$

$$y_j = \frac{1}{n} \sum_{t=1}^{m} y_{jt} \in Y_j, \; j = 1,\ldots,k.$$

Then we have

$$\sum_{i=1}^{m} z_i = \frac{1}{n} \sum_{i=1}^{m} \sum_{r=1}^{n} x_{ir} = \frac{1}{n}\left(\sum_{i=1}^{m} \sum_{r=1}^{n} \omega_{ir} + \sum_{j=1}^{k} \sum_{t=1}^{n} y_{jt} \right) = \omega + \sum_{j=1}^{k} y_j ,$$

and so $(z_1,\ldots,z_m,y_1,\ldots,y_k) \in \mathcal{A}$. Also, by the quasi-concavity of the utility functions, we have $z_i \succeq_i x_{i1}$ for each $i=1,\ldots,m$. Now assume by way of contradiction that there exists some (i,r) such that $x_{ir} \succ_i x_{i1}$. The latter, in view of the convexity of $\succeq_i$, implies $z_i \succ_i x_{i1}$. Now if each consumer $(i,1)$ is assigned the bundle z_i and each producer (j,t) chooses the production plan y_j (i.e., $y_{jt} = y_j$), then it is easy to see that $\{z_i\colon i=1,\ldots,m\}$ is a feasible assignment for the coalition $\{(i,1)\colon i=1,\ldots,m\}$ that blocks the original core allocation, which is impossible. This contradiction establishes the validity of our claim.

Next, note that by the quasi-concavity of the utility functions we have $z_i \succeq_i x_{ir}$ for $r=1,\ldots,n$ and $i=1,\ldots,m$. An easy argument now shows that $(z_1,\ldots,z_m,y_1,\ldots,y_k) \in C_n$, and thus C_n is non-empty.

2. For each n we have $C_{n+1} \subseteq C_n$.

This follows easily from the fact that if a coalition S of consumers of $\mathcal{E}_n$ blocks an allocation of $\mathcal{A}$, then S also blocks the same allocation in $\mathcal{E}_{n+1}$.

3. Each C_n is weakly closed.

Let the net $\{(x_1^\alpha,\ldots,x_m^\alpha,y_1^\alpha,\ldots,y_k^\alpha)\}$ of C_n converge weakly to $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ in E^{m+k}. By Theorem 4.4 we know that $\mathrm{Core}(\mathcal{E}_n)$ is a weakly compact subset of $E^{(m+k)n}$, and so if we consider each $(x_1^\alpha,\ldots,x_m^\alpha,y_1^\alpha,\ldots,y_k^\alpha)$ in $\mathrm{Core}(\mathcal{E}_n)$, then it follows easily that $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ must be also in $\mathrm{Core}(\mathcal{E}_n)$. That is, $(x_1,\ldots,x_m,y_1,\ldots,y_k) \in C_n$, and so C_n is a weakly closed subset of E^{m+k}.

4. The set of all Edgeworth equilibria is weakly closed.

This follows from (3) by observing that the set of all Edgeworth equilibria is precisely the set $\bigcap_{n=1}^{\infty} C_n$.

5. The economy has an Edgeworth equilibrium.

For each n let

$$\hat{C}_n = \{(x_1,\ldots,x_m,y) \in E^{m+1}\colon\ y = \sum_{j=1}^{k} y_j \ \text{ with } \ (x_1,\ldots,x_m,y_1,\ldots,y_k) \in C_n\}.$$

Since each C_n is non-empty, we see that each $\hat{C}_n$ is likewise non-empty. From $C_{n+1} \subseteq C_n$, it follows that $\hat{C}_{n+1} \subseteq \hat{C}_n$. In addition, we claim that each $\hat{C}_n$ is a weakly compact subset of E^{m+1}. To see the latter, note first that from

$$\hat{C}_n \subseteq \hat{X}_1 \times \cdots \times \hat{X}_m \times [(Y+\omega) \cap E^+ - \omega]$$

and Theorem 4.1(3) we see that each $\hat{C}_n$ is a relatively weakly compact subset of E^{m+1}. Now let $\{(x_1^\alpha,\ldots,x_m^\alpha,y^\alpha)\}$ be a net of some $\hat{C}_n$ satisfying $(x_1^\alpha,\ldots,x_m^\alpha,y^\alpha) \xrightarrow{w} (x_1,\ldots,x_m,y)$. Pick $y_j^\alpha \in Y_j$ $(j=1,\ldots,k)$ with $y^\alpha = \sum_{j=1}^{k} y_j^\alpha$ and $(x_1^\alpha,\ldots,x_m^\alpha,y_1^\alpha,\ldots,y_k^\alpha) \in C_n$. An easy argument shows that there exist $y_j \in Y_j$ $(j=1,\ldots,k)$ such that $y = \sum_{j=1}^{k} y_j$ and $(x_1,\ldots,x_m,y_1,\ldots,y_k) \in \mathcal{A}$. If $(x_1,\ldots,x_m,y_1,\ldots,y_k) \notin C_n$, then some coalition S of the n-fold replica economy $\mathcal{E}_n$ blocks $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ in $\mathcal{E}_n$. Since

$$(x_1^\alpha,\ldots,x_m^\alpha,y^\alpha) \xrightarrow{w} (x_1,\ldots,x_m,y)$$

and each set $\{z \in E^+ : z \succ_i x_i\}$ is weakly open relative to E^+, it is easy to see that S blocks $(x_1^\alpha,\ldots,x_m^\alpha,y_1^\alpha,\ldots,y_k^\alpha)$ in $\mathcal{E}_n$ for some α, which is a contradiction. Hence, $(x_1,\ldots,x_m,y) \in \hat{C}_n$. This implies that $\hat{C}_n$ is weakly closed, and hence weakly compact.

Now from the finite intersection property we have $\bigcap_{n=1}^{\infty} \hat{C}_n \neq \emptyset$. Fix some $(x_1,\ldots,x_m,y) \in \bigcap_{n=1}^{\infty} \hat{C}_n$, and then pick $y_j \in Y_j$ $(j=1,\ldots,k)$ with $y = \sum_{j=1}^{k} y_j$. We claim that $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ is an Edgeworth equilibrium. To see this, assume by way of contradiction that $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ can be blocked by a coalition S in the r-fold replica of the economy. Since $(x_1,\ldots,x_m,y) \in \hat{C}_r$, there exist $z_j \in Y_j$ $(j=1,\ldots,k)$ such that $(x_1,\ldots,x_m,z_1,\ldots,z_k) \in C_r$, and an easy argument shows that $(x_1,\ldots,x_m,z_1,\ldots,z_k)$ can be blocked by the coalition S in the r-fold replica of the economy, which is impossible. The proof of the theorem is now complete.■

It should be noted that when $Y_j = \{0\}$ for each j, then our production economy reduces to the pure exchange case.

5. QUASIEQUILIBRIA AND EDGEWORTH EQUILIBRIA

In this section we shall study the relationships between Edgeworth equilibria and quasiequilibria.

DEFINITION 5.1. *An allocation* $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ *is said to be a* ***Walrasian (or a competitive) equilibrium*** *whenever there exists a price* $p \neq 0$ *such that:*

a) For each consumer i *the bundle* x_i *is a maximal element in the budget set*

$$\mathcal{B}_i(p) = \{x \in X_i: \ p\cdot x \leqslant p\cdot\omega_i + \sum_{j=1}^{k} \theta_{ij} p\cdot y_j\},$$

i.e., $x_i \in \mathcal{B}_i(p)$ *and* $x_i \succcurlyeq_i x$ *holds for all* $x \in \mathcal{B}_i(p)$*; and*

b) For each j *the production plan* y_j *maximizes profit at prices* p *over* Y_j*, i.e.,*

$$p\cdot y_j = \max\{p\cdot z: \ z \in Y_j\}, \ j = 1,\ldots,k.$$

It should be clear that an allocation $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ is a Walrasian equilibrium if and only if there exists a price $p \neq 0$ such that:

1. $p\cdot x_i \leqslant p\cdot\omega_i + \sum_{j=1}^{k} \theta_{ij} p\cdot y_j$;

2. $x \succ_i x_i$ in X_i implies $p\cdot x > p\cdot\omega_i + \sum_{j=1}^{k} \theta_{ij} p\cdot y_j$; and

3. $p\cdot y_j = \max\{p\cdot z: \ z \in Y_j\}$ for $j = 1,\ldots,k$.

DEFINITION 5.2. *An allocation* $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ *is said to be a* ***quasiequilibrium*** *whenever there exists a price* $p \neq 0$ *such that:*

α) $p\cdot x_i \leqslant p\cdot\omega_i + \sum_{j=1}^{k} \theta_{ij} p\cdot y_j$;

β) $x \succ_i x_i$ *in* X_i *implies* $p\cdot x \geqslant p\cdot\omega_i + \sum_{j=1}^{k} \theta_{ij} p\cdot y_j$; *and*

γ) $p\cdot y_j = \max\{p\cdot z: \ z \in Y_j\}$ *for* $j = 1,\ldots,k$.

Any price p that satisfies the properties of Definitions 5.1 or 5.2 is known as a price *supporting the allocation*.

Clearly, a Walrasian equilibrium is a quasiequilibrium. Also, the next result tells us that a competitive equilibrium is always an Edgeworth equilibrium.

THEOREM 5.3. *Every Walrasian equilibrium is an Edgeworth equilibrium.*

PROOF. Since a Walrasian equilibrium remains a Walrasian equilibrium in every n-fold replica, it suffices to show that a Walrasian equilibrium is a core allocation.

To this end, let $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ be a Walrasian equilibrium supported by a price p. Assume by way of contradiction that there exist a coalition S, consumption bundles $z_i \in X_i$ $(i \in S)$ and production plans $h_j \in Y_j$ $(j=1,\ldots,k)$ such that

a) $\sum_{i\in S} z_i = \sum_{i\in S} \omega_i + \sum_{j=1}^{k} (\sum_{i\in S} \theta_{ij})h_j$, and

b) $z_i \succ_i x_i$ for each $i \in S$.

Now note that

$$p\cdot z_i > p\cdot\omega_i + \sum_{j=1}^{k} \theta_{ij} p\cdot y_j \geqslant p\cdot\omega_i + \sum_{j=1}^{k} \theta_{ij} p\cdot h_j$$

holds for all $i \in S$, and so

$$p\cdot(\sum_{i\in S} z_i) = \sum_{i\in S} p\cdot z_i > \sum_{i\in S} p\cdot\omega_i + \sum_{i\in S}\sum_{j=1}^{k} \theta_{ij} p\cdot h_j = p\cdot[\sum_{i\in S}\omega_i + \sum_{j=1}^{k}(\sum_{i\in S}\theta_{ij})h_j],$$

which contradicts (a). Hence, $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ is a core allocation, as desired.■

In the pure exchange case, A. Mas-Colell [16] proved that quasieqilibria exist, and the authors generalized this result in [2] by proving that every Edgeworth equilibrium is a quasiequilibrium. In the infinite dimensional setting, W. Zame [27] was the first to establish the existence of quasiequilibria in economies with production.

Our next objective is to show that under certain conditions an Edgeworth equilibrium is a quasiequilibrium. To do this, we need some preliminary discussion.

We start by introducing some useful convex sets. For each consumer i we define his "share set" by

$$Z_i = \{\sum_{j=1}^{k} \theta_{ij} z_j \colon z_j \in Y_j\} = \sum_{j=1}^{k} \theta_{ij} Y_j .$$

Now consider m consumption bundles $x_i \in X_i$ $(i=1,\ldots,m)$. For each i, we shall denote by F_i^* the "strictly better set" of x_i, i.e., F_i^* is the convex set defined by

$$F_i^* = \{x \in X_i \colon x \succ_i x_i\}.$$

With the above convex sets, we shall also associate the following important convex set

$$H^* = co[\bigcup_{i=1}^{m}(F_i^* - Z_i - \omega_i)]$$
$$= \{\sum_{i=1}^{m}\lambda_i(v_i - z_i - \omega_i): \lambda_i \geq 0,\ v_i \succ_i x_i,\ z_i \in Z_i \text{ and } \sum_{i=1}^{m}\lambda_i = 1\}.$$

In order to insure that the sets F_i^* are non-empty, we shall assume for the rest of this section that each preference relation satisfies the following non-satiation property.

If $x \in X_i$, then there exists some $z \in X_i$ with $z \succ_i x$.

An important property of the convex set H^* is described in the next theorem.

THEOREM 5.4. *Assume that $X_i = E^+$ holds for all i. If $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ is an Edgeworth equilibrium, then for each $h \geq 0$ we have $0 \notin h + H^*$.*

PROOF. Let $h \geq 0$, and assume by way of contradiction that $0 \in h + H^*$. Thus, there exist $v_i \in F_i^*$, $z_i \in Z_i$ and $\lambda_i \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$ such that $h + \sum_{i=1}^{m}\lambda_i(v_i - z_i - \omega_i) = 0$, and so

$$\sum_{i=1}^{m}\lambda_i(v_i - z_i - \omega_i) \leq 0\,. \qquad (\star)$$

Next, let $S = \{i: \lambda_i > 0\}$, and note that from $(\star)$ it follows that

$$\sum_{i\in S}\lambda_i v_i \leq \sum_{i\in S}\lambda_i z_i + \sum_{i\in S}\lambda_i\omega_i\,. \qquad (\star\star)$$

Now if n is a positive integer and $i \in S$, let n_i be the smallest integer greater or equal than $n\lambda_i$ (i.e., $0 \leq n_i - n\lambda_i \leq 1$). Since $v_i \succ_i x_i$ and $\lim_{n\to\infty} n\lambda_i/n_i = 1$ for each $i \in S$, we can choose (by the continuity of the utility functions) n large enough so that

$$f_i = (n\lambda_i/n_i)v_i \succ_i x_i \quad \text{for all } i \in S. \qquad (\star\star\star)$$

(Here we use the fact that $X_i = E^+$ so that $f_i \in X_i$.) Taking into account $(\star\star)$, we infer that

$$\sum_{i\in S} n_i f_i = \sum_{i\in S} n\lambda_i v_i \leq \sum_{i\in S} n\lambda_i z_i + \sum_{i\in S} n\lambda_i\omega_i$$
$$\leq \sum_{i\in S} n_i(n\lambda_i/n_i)z_i + \sum_{i\in S} n_i\omega_i\,.$$

Since $0 \leqslant n\lambda_i/n_i \leqslant 1$, we see that $h_i = (n\lambda_i/n_i)z_i \in Z_i$, and so from the preceding inequality, we conclude that

$$\sum_{i\in S} n_i f_i \leqslant \sum_{i\in S} n_i h_i + \sum_{i\in S} n_i \omega_i .$$

By rearranging the consumers, we can also assume that $S = \{1,\ldots,\ell\}$, where $1 \leqslant \ell \leqslant m$. For each $i \in S$ pick $h_{ij} \in Y_j$ $(j = 1,\ldots,k)$ such that

$$h_i = \sum_{j=1}^{k} \theta_{ij} h_{ij} .$$

Let $n = n_1 + \cdots + n_\ell$, and let $\mathscr{E}_n$ denote the n-fold replica of our economy. For each $i \in S$, let T_i be the set of consumers of $\mathscr{E}_n$ defined by

$$T_i = \{(i,s)\colon n_0 + n_1 + \cdots + n_{i-1} + 1 \leqslant s \leqslant n_1 + \cdots + n_i\},$$

where $n_0 = 0$. Clearly, $T_i \cap T_r = \emptyset$ for $i \neq r$. Now consider the coalition T of $\mathscr{E}_n$ given by $T = \bigcup_{i\in S} T_i$. Next, for each consumer $(i,s) \in T_i$ we assign the bundle

$$\xi_{is} = f_i,$$

and to each producer (j,t) $(j=1,\ldots,k;\ n_0 + n_1 + \cdots + n_{i-1} + 1 \leqslant t \leqslant n_1 + \cdots + n_i)$ we assign the production plan

$$\zeta_{jt} = h_{ij} .$$

Figure 1 clarifies the situation.

Now note that

$$\xi_{is} \succ_{(i,s)} x_{is} \quad \text{for all} \quad (i,s) \in T,$$

and moreover,

$$\begin{aligned}
\sum_{(i,s)\in T} \xi_{is} &= \sum_{i\in S} n_i f_i \\
&\leqslant \sum_{i\in S} n_i \omega_i + \sum_{i\in S} n_i h_i \\
&= \sum_{(i,s)\in T} \omega_{is} + \sum_{i\in S} n_i \sum_{j=1}^{k} \theta_{ij} h_{ij} \\
&= \sum_{(i,s)\in T} \omega_{is} + \sum_{j=1}^{k} \sum_{t=1}^{n} \Big(\sum_{(i,s)\in T} \theta_{isjt} \Big) \zeta_{jt} .
\end{aligned}$$

The above show that the coalition T blocks $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ in the n-fold replica of the economy, which is impossible. Hence, $0 \notin h + H^*$ must hold, as desired.■

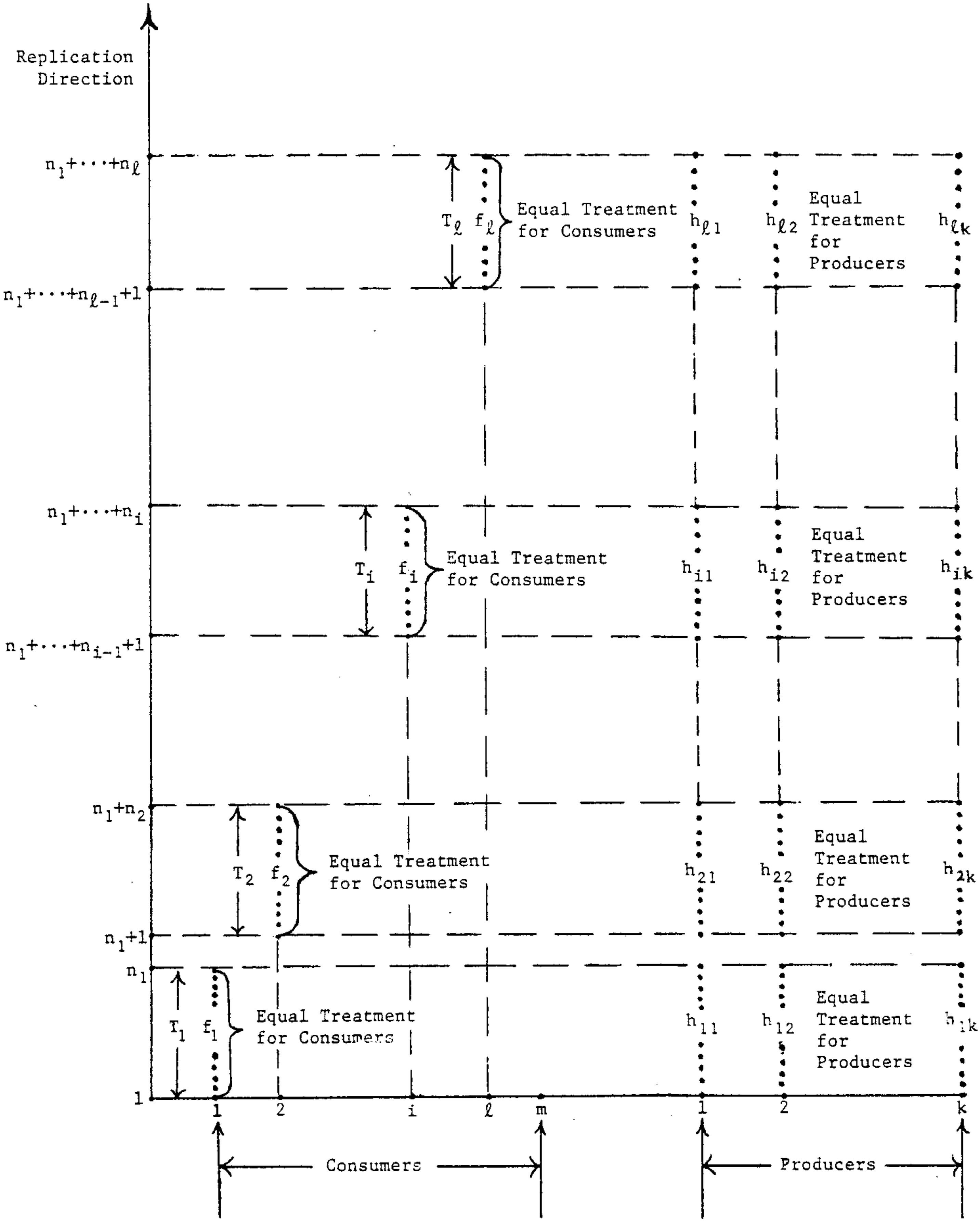
Replication Direction
$n_1+\cdots+n_\ell$
$n_1+\cdots+n_{\ell-1}+1$
$n_1+\cdots+n_i$
$n_1+\cdots+n_{i-1}+1$
n_1+n_2
n_1+1
n_1
1
T_ℓ f_ℓ Equal Treatment for Consumers
$h_{\ell 1}$ $h_{\ell 2}$ Equal Treatment for Producers $h_{\ell k}$
T_i f_i Equal Treatment for Consumers
h_{i1} h_{i2} Equal Treatment for Producers h_{ik}
T_2 f_2 Equal Treatment for Consumers
h_{21} h_{22} Equal Treatment for Producers h_{2k}
T_1 f_1 Equal Treatment for Consumers
h_{11} h_{12} Equal Treatment for Producers h_{1k}
1 2 i ℓ m
1 2 k
Consumers
Producers

FIGURE 1

To continue our discussion we need the concept of uniform properness for preferences and production sets as it was introduced by A. Mas-Colell in [16] and [17]. The uniform properness for preferences is defined as follows.

DEFINITION 5.5. (A. Mas-Colell) *A preference relation* $\succcurlyeq$ *on a convex set* X *is said to be uniformly proper whenever there exist a vector* $a > 0$ *and some* τ*-neighborhood* V *of zero such that* $x - \alpha a + z \succcurlyeq x$ *in* X *with* $\alpha > 0$ *imply* $z \notin \alpha V$.

Recall that if a preference relation $\succcurlyeq$ is defined on E^+, then a commodity bundle $v > 0$ is said to be *strongly desirable* whenever $x + \alpha v \succ x$ holds for all $x \in E^+$ and all $\alpha > 0$. In case $\succcurlyeq$ is a uniformly proper preference relation defined on E^+, then the vector $a > 0$ in the definition of properness is a strongly desirable commodity bundle. Indeed, in this case, if $x \in E^+$ and $\alpha > 0$ satisfy $x \succcurlyeq x + \alpha a$, then from $(x + \alpha a) - \alpha a + 0 = x \succcurlyeq x + \alpha a$ and the uniform properness, it follows that $0 \notin \alpha V$, which is impossible. Hence, $x + \alpha a \succ x$ holds for all $x \in E^+$ and all $\alpha > 0$.

In [17] A. Mas-Colell also introduced the concept of properness for production sets. Following A. Mas-Colell's ideas in [17], we shall say that a set T is *a pre-technology set* for a production set Z whenever

1) $Z \subseteq T$;
2) $x \in T$ implies $x^+ = x \vee 0 \in T$; and
3) $T - E^+ = T$, i.e., T satisfies the free disposal condition.

Now the corresponding notion of uniform properness for production sets is as follows.

DEFINITION 5.6. (A. Mas-Colell) *A production set* Z *is said to be uniformly proper whenever there exist a pre-technology set* T *for* Z *a vector* $b > 0$ *and a* τ*-neighborhood* V *of zero such that* $y \in T \setminus Z$ *and* $y + \alpha b + z \in Z$ *with* $\alpha > 0$ *imply* $z \notin \alpha V$.

Concerning uniformly production sets we have the following useful property.

LEMMA 5.7. *Suppose that a production set* Z *is uniformly proper, and let* T, $b > 0$ *and* V *be as in Definition 5.6. If* $z \in Z$, $y \in E^+$ *and* $\alpha > 0$ *satisfy* $y \in \alpha V$ *and* $y \leqslant z^- + \alpha b$, *then* $z - \alpha b + y \in Z$.

PROOF. Assume that $z \in Z$, $y \in E^+$ and $\alpha > 0$ satisfy $y \in \alpha V$ and $y \leqslant z^- + \alpha b$. We can also assume that V is a symmetric neighborhood. Put $x = z - \alpha b + y$, and note that

$$x = z^+ - (z^- + \alpha b - y) \leqslant z^+. \qquad (\star)$$

Since $z \in Z \subseteq T$, it follows that $z^+ \in T$, and so from $(\star)$ we see that $x = z - \alpha b + y \in T$.

Now assume by way of contradiction that $x \notin Z$. Then we have $z = x + \alpha b - y \in Z$. From $x \in T \setminus Z$ and the properness condition, we infer that $-y \notin \alpha V$, i.e., $y \notin \alpha V$, which is impossible. The proof of the lemma is now complete.■

We now come to the concept of a proper economy.

DEFINITION 5.8. *An economy is said to be a* <u>*proper economy*</u> *whenever all preferences and all production sets are uniformly proper. That is, an economy is proper whenever there exist* $a_i > 0$ *(i=1,...,m),* $b_j > 0$ *(j=1,...,k), pre-technology sets* T_j *for* Y_j *(j=1,...,k) and an open convex solid* τ*-neighborhood* V *of zero such that*

1) $x - \alpha a_i + z \succcurlyeq_i x$ *in* X_i *and* $\alpha > 0$ *imply* $z \notin \alpha V$*; and*

2) $y + \alpha b_j + z \in Y_j$ *with* $y \in T_j \setminus Y_j$ *and* $\alpha > 0$ *imply* $z \notin \alpha V$.

A. Mas-Colell [17] used uniform properness on preferences and production sets to show that any Pareto optimal allocation can be supported by a non-zero price. Next, we use properness to show that every Edgeworth equilibrium is a quasiequilibrium.

THEOREM 5.9. *If* $X_i = E^+$ *holds for each* i *and the economy is proper, then every Edgeworth equilibrium is a quasiequilibrium.*

PRROF. This proof is an adaptation of A. Mas-Colell's proof of Theorem 1 in [17].

Let a_i (i=1,...,m), b_j (j=1,...,k) and V be as in Definition 5.8 of a proper economy. Put $a = a_1 + \cdots + a_m + b_1 + \cdots + b_k$, and let Γ be the (non-empty) open convex cone generated by $-a + \frac{1}{m}V$, i.e.,

$$\Gamma = \{\alpha(-a + \tfrac{1}{m}v)\colon\ \alpha > 0 \text{ and } v \in V\}.$$

Now let $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ be an Edgeworth equilibrium. If H^* is the convex set associated with $(x_1,\ldots,x_m)$ as in Theorem 5.4, then we claim that $H^* \cap \Gamma = \emptyset$.

To see this, assume by way of contradiction that $H^* \cap \Gamma \neq \emptyset$. Then there exist $f_i \geqslant 0$ with $f_i \succ_i x_i$, $\lambda_i \geqslant 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, $y_{ij} \in Y_j$ (i=1,...,m;j=1,...,k) and some $\varepsilon > 0$ such that

$$\sum_{i=1}^{m} \lambda_i (f_i - \sum_{j=1} \theta_{ij} y_{ij} - \omega_i) + \varepsilon a \in \frac{\varepsilon}{m} V . \tag{1}$$

Note that the set $S = \{i: \lambda_i > 0\}$ is non-empty.

Now consider the positive elements

$$y = \sum_{i=1}^{m} \lambda_i [\omega_i + \sum_{j=1}^{k} \theta_{ij} (y_{ij})^+], \quad \text{and}$$

$$z = \sum_{i=1}^{m} \lambda_i [f_i + \sum_{j=1}^{k} \theta_{ij} (y_{ij})^-] + \varepsilon a = \sum_{i=1}^{m} (\lambda_i f_i + \varepsilon a_i) + \sum_{i=1}^{m} \sum_{j=1}^{k} [\lambda_i \theta_{ij} (y_{ij})^- + \frac{\varepsilon}{m} b_j].$$

From (1), we see that

$$z - y = \sum_{i=1}^{m} \lambda_i (f_i - \sum_{j=1}^{k} \theta_{ij} y_{ij} - \omega_i) + \varepsilon a \in \frac{\varepsilon}{m} V . \tag{2}$$

Moreover, we have

$$0 \leqslant (z - y)^+ \leqslant z = \sum_{i=1}^{m} (\lambda_i f_i + \varepsilon a_i) + \sum_{i=1}^{m} \sum_{j=1}^{k} [\lambda_i \theta_{ij} (y_{ij})^- + \frac{\varepsilon}{m} b_j]. \tag{3}$$

From $z - y \in \frac{\varepsilon}{m} V$ and the solidness of V we see that

$$(z - y)^+ \in \frac{\varepsilon}{m} V . \tag{4}$$

Applying the Riesz Decomposition Property to (3), we can write

$$(z - y)^+ = s + t, \tag{5}$$

where,

$$0 \leqslant s \leqslant \sum_{i=1}^{m} (\lambda_i f_i + \varepsilon a_i), \quad \text{and} \tag{6}$$

$$0 \leqslant t \leqslant \sum_{i=1}^{m} \sum_{j=1}^{k} [\lambda_i \theta_{ij} (y_{ij})^- + \frac{\varepsilon}{m} b_j] . \tag{7}$$

Now applying the Riesz Decomposition Property to (6), we can write $s = \sum_{i=1}^{m} s_i$ with $0 \leqslant s_i \leqslant \lambda_i f_i + \varepsilon a_i$ for each i. From $0 \leqslant s_i \leqslant s \leqslant (z - y)^+ \in \frac{\varepsilon}{m} V$ and the solidness of V, we see that

$$s_i \in \frac{\varepsilon}{m} V, \quad i=1,\ldots,m. \tag{8}$$

Let

$$g_i = \begin{cases} f_i & \text{if } i \notin S \\ f_i + \frac{\varepsilon}{\lambda_i} a_i - \frac{1}{\lambda_i} s_i \geqslant 0 & \text{if } i \in S \end{cases} .$$

Clearly, $g_i \succcurlyeq_i f_i$ holds for all $i \notin S$, and we claim that $g_i \succ_i f_i$ for each $i \in S$. Indeed, if in the latter case we have

$$f_i = g_i - \frac{\varepsilon}{\lambda_i} a_i + \frac{1}{\lambda_i} s_i \succcurlyeq_i g_i ,$$

then by the properness we must have $\frac{1}{\lambda_i} s_i \notin \frac{\varepsilon}{\lambda_i} V_i$, i.e., $s_i \notin \varepsilon V$, which contradicts (8).

Next, using (7) and the Riez Decomposition Property we can write $t = \sum_{i=1}^{m} \sum_{j=1}^{k} t_{ij}$ with $0 \leqslant t_{ij} \leqslant \lambda_i \theta_{ij} (y_{ij})^- + \frac{\varepsilon}{m} b_j$. Let $T = \{(i,j): \lambda_i \theta_{ij} > 0$, and define

$$z_{ij} = \begin{cases} y_{ij} - \varepsilon(m\lambda_i\theta_{ij})^{-1} b_j + (\lambda_i\theta_{ij})^{-1} t_{ij} & \text{if } (i,j) \in T \\ 0 \quad \text{if } (i,j) \notin T \end{cases} .$$

Fix $(i,j) \in T$. From $0 \leqslant t_{ij} \leqslant t \leqslant (z-y)^+ \in \frac{\varepsilon}{m} V$ and the solidness of V, we infer that $t_{ij} \in \frac{\varepsilon}{m} V$, and so $(\lambda_i\theta_{ij})^{-1} t_{ij} \in (m\lambda_i\theta_{ij})^{-1} V$. Now from the inequality

$$0 \leqslant (\lambda_i\theta_{ij})^{-1} t_{ij} \leqslant (y_{ij})^- + \varepsilon(m\lambda_i\theta_{ij})^{-1} b_j$$

and Lemma 5.7, we conclude that $z_{ij} \in Y_j$. Hence, $z_{ij} \in Y_j$ holds for all (i,j).

Now for $\lambda_i = 0$, we have $s_i \leqslant \varepsilon a_i$, and so $\varepsilon a_i - s_i \geqslant 0$ for all $i \notin S$. Similarly, for $\lambda_i\theta_{ij} = 0$, we have $t_{ij} \leqslant \frac{\varepsilon}{m} b_j$, and so $\frac{\varepsilon}{m} b_j - t_{ij} \geqslant 0$ for all $(i,j) \notin T$. Taking into account these observations, we see that

$$\sum_{i=1} \lambda_i (g_i - \sum_{j=1}^{k} \theta_{ij} z_{ij} - \omega_i)$$

$$= \sum_{i=1}^{m} \lambda_i (f_i - \sum_{j=1}^{k} \theta_{ij} y_{ij} - \omega_i) + \varepsilon \sum_{i \in S} a_i - \sum_{i \in S} s_i + \frac{\varepsilon}{m} \sum_{(i,s) \in T} b_j - \sum_{(i,s) \in T} t_{ij}$$

$$\leqslant \sum_{i=1}^{m} \lambda_i (f_i - \sum_{j=1}^{k} \theta_{ij} y_{ij} - \omega_i) + \varepsilon \sum_{i \in S} a_i - \sum_{i \in S} s_i + \sum_{i \notin S} (\varepsilon a_i - s_i) + \frac{\varepsilon}{m} \sum_{(i,s) \in T} b_j + \sum_{(i,s) \in T} t_{ij} + \sum_{(i,s) \notin T} (\frac{\varepsilon}{m} b_j - t_{ij})$$

$$= \sum_{i=1}^{m} \lambda_i (f_i - \sum_{j=1}^{k} \theta_{ij} y_{ij} - \omega_i) + \varepsilon \sum_{i=1}^{m} a_i - \sum_{i=1}^{m} s_i + \frac{\varepsilon}{m} \sum_{i=1}^{m} \sum_{j=1}^{k} b_j - \sum_{i=1}^{m} \sum_{j=1}^{k} t_{ij}$$

$$= \sum_{i=1}^{m} \lambda_i (f_i - \sum_{j=1} \theta_{ij} y_{ij} - \omega_i) + \varepsilon a - (s+t)$$

$$= z - y - (s+t) = z - y - (z-y)^+ = -(z-y)^- \leqslant 0 .$$

Clearly, the element

$$g = \sum_{i=1}^{m} \lambda_i (g_i - \sum_{j=1} \theta_{ij} z_{ij} - \omega_i) \in H^*$$

satisfies $g \leqslant 0$. Now let $h = -g \geqslant 0$. Then from $h + g = 0$, we see that $0 \in h + H^*$, which contradicts Theorem 5.4. Thus $H^* \cap \Gamma = \emptyset$ holds, as claimed.

Finally, by the classical separation theorem there exist a non-zero price p and some constant c such that

$$p \cdot h \geqslant c \geqslant p \cdot g$$

holds for all $h \in H^*$ and all $g \in \Gamma$. Since Γ is a cone, we see that $c \geqslant 0$. Now if $x \succ_i x_i$ holds in E^+, then $x - \sum_{i=1}^{k} \theta_{ij} y_j - \omega_i \in H^*$, and so $p \cdot x \geqslant p \cdot \omega_i + \sum_{i=1}^{k} \theta_{ij} p \cdot y_j$. On the other hand, we know that each $a_i > 0$ is a strongly desirable commodity for $\succ_i$. If $z \in Y_r$, then put $v = a_1 + \cdots + a_m$, $z_j = y_j$ for $j \neq r$ and $z_r = z$, and note that

$$y_r - z + \frac{\alpha}{m} v = \sum_{i=1}^{m} \frac{1}{m}[(x_i + \alpha v_i) - \sum_{j=1}^{k} \theta_{ij} z_j - \omega_i] \in H^*$$

holds for all $\alpha > 0$. Hence, $p \cdot y_r - p \cdot z + \frac{\alpha}{m} p \cdot v \geqslant 0$ holds for all $\alpha > 0$, and so $p \cdot y_r \geqslant p \cdot z$ for all $z \in Y_r$. Therefore, $p \cdot y_r = \max\{p \cdot z : z \in Y_r\}$ holds for all $r = 1, \ldots, k$, and this completes the proof of the theorem.∎

Recall that a positive element $x > 0$ is said to be *strictly positive* (in symbols, $x \gg 0$) whenever $p \cdot x > 0$ holds for all $0 < p \in E'$. If $\omega \gg 0$ and preferences are strongly monotone, then it is easy to see that the concepts of quasiequilibrium and Walrasian equilibrium coincide. Therefore, the next theorem that generalizes the classical theorem of G. Debreu and H. Scarf [9] is an immediate consequence of the preceding result.

THEOREM 5.10. *Assume that each consumption set satisfies $X_i = E^+$ and that preferences are in addition strongly monotone. If the economy is proper and $\omega \gg 0$, then an allocation is an Edgeworth equilibrium if and only if it is a Walrasian equilibrium.*

In particular, in this case, if the economy is also compact, then Walrasian equilibria exist.

6. DECENTRALIZING EDGEWORTH EQUILIBRIA

Recall that the "share set" of each consumer i is defined by

$$Z_i = \{\sum_{j=1}^{k} \theta_{ij} z_j : z_j \in Y_j\} = \sum_{j=1}^{k} \theta_{ij} Y_j .$$

Also, for each fixed $a \in E^+$ and each consumer i, we define the convex set

$$Z_i^a = \{\sum_{j=1}^{k} \theta_{ij} z_j : z_j \in Y_j \text{ and } z_j \leqslant a\}.$$

In case the economy has a symmetric Riesz dual system the convex sets Z_i^a are weakly closed. The details follow.

LEMMA 6.1. *If the economy has a symmetric Riesz dual system, then for each i and each $a \in E^+$ the convex set Z_i^a is weakly closed.*

PROOF. Fix i and $a \in E^+$, and let f be an element in the weak closure of Z_i^a. Then f is also in the τ-closure of Z_i^a. Pick a net $\{f_\alpha\}$ of Z_i^a with $f_\alpha \xrightarrow{\tau} f$. For each α choose $y_j^\alpha \in Y_j$ with $y_j^\alpha \leqslant a$ and $f_\alpha = \sum_{j=1}^{k} \theta_{ij} y_j^\alpha$. (In case $\theta_{ij} = 0$, we shall assume that $y_j^\alpha = 0$.)

Since $0 \leqslant (y_j^\alpha)^+ \leqslant a$ holds for all α and j and the order interval $[0,a]$ is weakly compact, we can suppose (by passing to an appropriate subnet) that for each j we have

$$(y_j^\alpha)^+ \xrightarrow{w} y_j^1. \qquad (\star)$$

Also, from

$$0 \leqslant \theta_{ij}(y_j^\alpha)^- \leqslant \sum_{t=1}^{k} \theta_{it}(y_t^\alpha)^- = -\sum_{t=1}^{k} \theta_{it} y_t^\alpha + \sum_{t=1}^{k} \theta_{it}(y_t^\alpha)^+$$

$$\leqslant -f_\alpha + a \leqslant (f - f_\alpha)^+ + f^+ + a \xrightarrow{\tau} f^+ + a$$

and Theorem 2.1, it follows (by passing to a subnet again) that for each j we have

$$(y_j^\alpha)^- \xrightarrow{w} y_j^2. \qquad (\star\star)$$

From $(\star)$ and $(\star\star)$, we infer that

$$y_j^\alpha = (y_j^\alpha)^+ - (y_j^\alpha)^- \xrightarrow{w} y_j^1 - y_j^2 = y_j$$

for each j. Since each Y_j is weakly closed, we see that $y_j \in Y_j$, and moreover, from $y_j^\alpha \leqslant a$, we infer that $y_j \leqslant a$. Finally, note that

$$f = \text{w-lim}\, f_\alpha = \text{w-lim} \sum_{j=1}^{k} \theta_{ij} y_j^\alpha = \sum_{j=1}^{k} \theta_{ij} y_j \in Z_i^a ,$$

and the proof is finished.■

Consider m consumption bundles $x_i \in X_i$ $(i=1,\ldots,m)$. For each i we shall denote by F_i the "better set" of x_i, i.e., F_i is the weakly closed convex set defined by

$$F_i = \{x \in X_i \colon x \succcurlyeq_i x_i\}.$$

With the above convex sets we shall associate the important convex set

$$H_a = \text{co}[\bigcup_{i=1}^{m} (F_i - Z_i^a - \omega_i)]$$

$$= \{\sum_{i=1}^{m} \lambda_i(v_i - z_i - \omega_i) \colon \lambda_i \geqslant 0,\ v_i \succcurlyeq_i x_i,\ z_i \in Z_i^a \text{ and } \sum_{i=1}^{m} \lambda_i = 1\}.$$

THEOREM 6.2. *Assume that the economy has a symmetric Riesz dual system that preferences are monotone and that each consumption set satisfies* $X_i + E^+ = X_i$. *If* $x_i \in X_i$ *(i=1,...,m) are consumption bundles, then for each* $a \in E^+$ *the convex set*

$$H_a = co[\bigcup_{i=1}^{m} (F_i - Z_i^a - \omega_i)]$$

is a weakly closed subset of E.

PROOF. Fix $a \in E^+$, and let f be in the weak closure of H_a. Then f is in the τ-closure of H_a, and so there exists a net $\{f_\alpha\}$ of H_a with $f_\alpha \xrightarrow{\tau} f$. For each α let $v_i^\alpha \succcurlyeq_i x_i$, $z_i^\alpha \in Z_i^a$, $\lambda_i^\alpha \geqslant 0$ with $\sum_{i=1}^{m} \lambda_i^\alpha = 1$ such that

$$f_\alpha = \sum_{i=1}^{m} \lambda_i^\alpha (v_i^\alpha - z_i^\alpha - \omega_i).$$

By passing to a subnet, we can assume that $\lambda_i^\alpha \longrightarrow \lambda_i$ holds in $\mathbb{R}$ for each i. Clearly, $\lambda_1 + \cdots + \lambda_m = 1$. Let $S = \{i \colon \lambda_i > 0\}$, and note that $S \neq \emptyset$. From

$$0 \leqslant \sum_{i=1}^{m} \lambda_i^\alpha v_i^\alpha + \sum_{i=1}^{m} \lambda_i^\alpha (z_i^\alpha)^- = \sum_{i=1}^{m} \lambda_i^\alpha (v_i^\alpha - z_i^\alpha - \omega_i) + \sum_{i=1}^{m} \lambda_i^\alpha \omega_i + \sum_{i=1}^{m} \lambda_i^\alpha (z_i^\alpha)^+ \leqslant f_\alpha + \omega + a,$$

we see that

$$0 \leqslant \lambda_i^\alpha v_i^\alpha \leqslant f_\alpha + \omega + a \quad \text{and} \quad 0 \leqslant \lambda_i^\alpha (z_i^\alpha)^- \leqslant f_\alpha + \omega + a$$

hold for all i and all α. Thus, by Theorem 2.1, we can assume (by passing to a subnet again) that for each $i \in S$ we have

$$v_i^\alpha \xrightarrow{w} v_i \in F_i \quad \text{and} \quad (z_i^\alpha)^- \xrightarrow{w} z_i^1 .$$

From $0 \leqslant (z_i^\alpha)^+ \leqslant a$ and the weak compactness of $[0,a]$, we can assume (by passing to a subnet once more) that for each i we have $(z_i^\alpha)^+ \xrightarrow{w} z_i^2$. Thus, taking into consideration that Z_i^a is weakly closed (Lemma 6.1), we see that

$$z_i^\alpha = (z_i^\alpha)^+ - (z_i^\alpha)^- \xrightarrow{w} z_i^2 - z_i^1 = z_i \in Z_i^a .$$

In addition, from $0 \leqslant \lambda_i^\alpha (z_i^\alpha)^+ \leqslant \lambda_i^\alpha a$, it follows that $\lambda_i^\alpha (z_i^\alpha)^+ \xrightarrow{w} 0$ for all $i \notin S$.

Now from the weak closedness of E^+ and the inequality

$$\begin{aligned} f_\alpha &= \sum_{i=1}^m \lambda_i^\alpha [v_i^\alpha - (z_i^\alpha)^+ + (z_i^\alpha)^- - \omega_i] \\ &\geqslant \sum_{i\in S} \lambda_i^\alpha v_i^\alpha - \sum_{i=1}^m \lambda_i^\alpha (z_i^\alpha)^+ + \sum_{i\in S} \lambda_i^\alpha (z_i^\alpha)^- - \sum_{i=1}^m \lambda_i^\alpha \omega_i , \end{aligned}$$

we infer that

$$f = w\text{-}\lim f_\alpha \geqslant \sum_{i\in S} \lambda_i v_i - \sum_{i\in S} \lambda_i z_i^2 + \sum_{i\in S} \lambda_i z_i^1 - \sum_{i\in S} \lambda_i \omega_i = \sum_{i\in S} \lambda_i (v_i - z_i - \omega_i) = g .$$

For $i \notin S$, let $v_i = x_i$ and $z_i = 0$. Then $v_i + f - g \succcurlyeq_i v_i \succcurlyeq_i x_i$, and

$$f = (f-g) + g = \sum_{i=1}^m \lambda_i [(v_i + f - g) - z_i - \omega_i] \in H_a ,$$

and the proof of the theorem is finished.■

When preferences are strongly monotone the Edgeworth equilibria are characterized as follows.

THEOREM 6.3. *Assume that the economy has a symmetric Riesz dual system that preferences are strongly monotone and that* $X_i = E^+$ *holds for each consumer* i. *Then an allocation* $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ *is an Edgeworth equilibrium if and only if for each* $f > 0$, *each* $\varepsilon > 0$ *and each* $a \in E^+$ *with* $a \geqslant y_1 \vee \cdots \vee y_k$ *there exists a price* $p \in E'$ *such that:*

1. $p \cdot f = 1$;
2. $x \succ_i x_i$ *in* E^+ *implies* $p \cdot x \geqslant p \cdot \omega_i + \sum_{j=1}^k \theta_{ij} p \cdot y_j - \varepsilon$; *and*
3. $p \cdot y_j \geqslant \sup\{p \cdot z : z \in Y_j \text{ and } z \leqslant a\} - \varepsilon$ *for each* j.

PROOF. Assume that $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ is an Edgeworth equilibrium. Fix $f > 0$, $\varepsilon > 0$ and $a \in E^+$ with $a \geqslant y_1 \vee \cdots \vee y_k$. From

$$\frac{\varepsilon}{m}f + H_a = \frac{\varepsilon}{2m}f + (\frac{\varepsilon}{2m}f + H_a) \subseteq \frac{\varepsilon}{2m}f + H^*,$$

and Theorem 5.4, we see that $0 \notin \frac{\varepsilon}{m}f + H_a$. Since H_a is weakly closed (Theorem 6.2), it follows from the classical separation theorem that there exists some $p \in E'$ such that

$$p\cdot(\frac{\varepsilon}{m}f + g) > 0 \qquad (\star)$$

holds for all $g \in H_a$. Since $h_i = \sum_{j=1}^{k} \theta_{ij}y_j \in Z_j^a$, it follows that $0 = \sum_{i=1}^{m} \frac{1}{m}(x_i - h_i - \omega_i) \in H_a$, and from $(\star)$ we see that $p\cdot f > 0$. Thus, replacing p by $p/p\cdot f$, we can assume that $p\cdot f = 1$.

Now let $x \succcurlyeq_i x_i$ holds in E^+. Then $x - \sum_{j=1}^{k} \theta_{ij}y_j - \omega_i \in H_a$, and so $p\cdot(\frac{\varepsilon}{m}f + x - \sum_{j=1}^{k} \theta_{ij}y_j - \omega_i) > 0$. This implies

$$p\cdot x \geqslant p\cdot\omega_i + \sum_{j=1}^{k} \theta_{ij}p\cdot y_j - \frac{\varepsilon}{m} > p\cdot\omega_i + \sum_{j=1}^{k} \theta_{ij}p\cdot y_j - \varepsilon.$$

Next, let $z \in Y_j$ satisfy $z \leqslant a$. Put $h_t = y_t$ for $t \neq j$ and $h_j = z$. From

$$\frac{1}{m}(y_j - z) = \frac{1}{m}(\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} \omega_i - \sum_{t=1}^{k} h_t) = \sum_{i=1}^{m} \frac{1}{m}(x_i - \sum_{t=1}^{k} \theta_{it}h_t - \omega_i) \in H_a$$

and $(\star)$, we see that $p\cdot(\frac{\varepsilon}{m}f + \frac{1}{m}(y_j - z)) > 0$. Therefore, $p\cdot y_j \geqslant p\cdot z - \varepsilon$ holds for all $z \in Y_j$ with $z \leqslant a$, from which it follows that

$$p\cdot y_j \geqslant \sup\{p\cdot z\colon\ z \in Y_j \text{ and } z \leqslant a\} - \varepsilon\,.$$

For the converse, assume that the allocation $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ satisfies (1), (2) and (3). Also, assume by way of contradiction that there exist an n-fold replica of the economy, a coalition S of consumers of the n-fold replica, a subset $\{h_{is}\colon (i,s) \in S\}$ of E^+ and production plans $z_{jt} \in Y_{jt} = Y_j$ $(j=1,\ldots,k;$ $t=1,\ldots,n)$ such that

$$h_{is} \succ_{(i,s)} x_{is} \quad \text{for all } (i,s) \in S, \text{ and} \qquad (1)$$

$$\sum_{(i,s)\in S} h_{is} = \sum_{(i,s)\in S} \omega_{is} + \sum_{j=1}^{k}\sum_{t=1}^{n} (\sum_{(i,s)\in S} \theta_{isjt})z_{jt}\,. \qquad (2)$$

Now let $f = \sum_{(i,s)\in S} h_{is}$ and let $a = \sum_{j=1}^{k} |y_j| + \sum_{j=1}^{k}\sum_{t=1}^{n} |z_{jt}|$. Then for each ℓ there exists some $p_\ell \in E'$ such that

$$p_\ell \cdot f = 1, \tag{3}$$

$$x \succcurlyeq_i x_i \quad \text{in} \quad E^+ \quad \text{implies} \quad p_\ell \cdot x \geqslant p_\ell \cdot \omega_i + \sum_{j=1}^{k} \theta_{ij} p_\ell \cdot y_j - \frac{1}{\ell}, \quad \text{and} \tag{4}$$

$$p_\ell \cdot y_j \geqslant \sup\{p_\ell \cdot y \colon y \in Y_j \ \text{ and } \ y \leqslant a\} - \frac{1}{\ell}. \tag{5}$$

Choose $0 < \delta < 1$ such that $\delta h_{is} \succ_{(i,s)} x_{is} = x_i$ holds for all $(i,s) \in S$. By (4) for each $(i,s) \in S$ we have

$$\begin{aligned} p_\ell \cdot (\delta h_{is}) &\geqslant p_\ell \cdot \omega_{is} + p_\ell \cdot \Big(\sum_{j=1}^{k} \theta_{ij} y_j\Big) - \frac{1}{\ell} \\ &= p_\ell \cdot \omega_{is} + p_\ell \cdot \Big(\sum_{j=1}^{k} \sum_{t=1}^{n} \theta_{isjt} y_j\Big) - \frac{1}{\ell} \\ &\geqslant p_\ell \cdot \omega_{is} + p_\ell \cdot \Big(\sum_{j=1}^{k} \sum_{t=1}^{n} \theta_{isjt} z_{jt}\Big) - \frac{1}{\ell}, \end{aligned} \tag{6}$$

and so

$$p_\ell \cdot \Big(\delta \sum_{(i,s)\in S} h_{is}\Big) \geqslant p_\ell \cdot \Big(\sum_{(i,s)\in S} \omega_{is}\Big) + p_\ell \cdot \Big[\sum_{j=1}^{k} \sum_{t=1}^{n} \Big(\sum_{(i,s)\in S} \theta_{isjt}\Big) z_{jt}\Big] - \frac{mn}{\ell}. \tag{7}$$

Combining (2) and (7), we obtain

$$\delta = \delta p_\ell \cdot \Big(\sum_{(i,s)\in S} h_{is}\Big) \geqslant p_\ell \cdot \Big(\sum_{(i,s)\in S} h_{is}\Big) - \frac{mn}{\ell} = 1 - \frac{mn}{\ell}$$

for each ℓ, and so $\delta \geqslant 1$, which is a contradiction. Therefore, the allocation $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ is an Edgeworth equilibrium.■

7. ECONOMIES WHOSE AGGREGATE PRODUCTION SETS ARE CONES

In this section we shall study economies having a cone as an aggregate production set. We start with the following definition.

DEFINITION 7.1. *A subset Y of a vector space E is said to be continuous for a linear topology ξ on E (briefly, ξ-continuous) whenever $\{y_\alpha\} \subseteq Y$ and $y_\alpha^- \xrightarrow{\xi} 0$ imply $y_\alpha^+ \xrightarrow{\xi} 0$.*

The continuity property of the production set conveys the fact that small inputs produce small outputs. If the production set is a cone, then the continuity of the production set seems to be a natural condition. The next two results will clarify the situation. Recall that a *production set* is any weakly closed convex subset Y of E satisfying $Y \cap E^+ = \{0\}$.

THEOREM 7.2. *Assume that a production set Y is a cone. If $\langle E,E'\rangle = \langle ca(\Omega), ca'(\Omega)\rangle$ for some Hausdorff compact topological space Ω (in particular, if E is finite dimensional), then Y is norm continuous.*

PROOF. Let Y be a production set which is a cone, and let $\{z_n\} \subseteq Y$ satisfy $\|z_n^-\| \longrightarrow 0$. Assume by way of contradiction that $\{z_n^+\}$ does not converge in norm to zero. Then, by passing to a subsequence, we can assume that $\|z_n^+\| \geqslant \varepsilon > 0$ holds for all n and some $\varepsilon > 0$. Now let

$$x_n = z_n/\|z_n^+\| = z_n^+/\|z_n^+\| - z_n^-/\|z_n^+\| . \qquad (\star)$$

Since Y is a cone, we have $x_n \in Y$ for each n. From

$$\|z_n^-/\|z_n^+\|\| = \|z_n^-\|/\|z_n^+\| \leqslant \|z_n^-\|/\varepsilon \longrightarrow 0,$$

we see that $\lim z_n^-/\|z_n^+\| = 0$. On the other hand, we have $\|z_n^+/\|z_n^+\|\| = 1$ for each n. Since the set $\{y \in E^+ : \|y\| = 1\}$ is weakly compact, it follows that $\{z_n^+/\|z_n^+\|\}$ has a weak accumulation point $z > 0$. From $(\star)$, we conclude that $z \in Y \cap E^+ = \{0\}$, which is impossible. Hence, $\|z_n^+\| \longrightarrow 0$ must hold, and the proof of the theorem is finished.∎

THEOREM 7.3. *Assume that the economy has a symmetric Riesz dual system, preferences are also strongly monotone, $X_i = E^+$ for each i and that the aggregate production set $Y = Y_1 + \cdots + Y_k$ is a cone. If the economy has a Walrasian equilibrium, then*

$\{y_n\} \subseteq Y$ *and* $y_n^- \xrightarrow{\tau} 0$ *imply* $y_n^+ \wedge x \xrightarrow{\tau} 0$ *for each* $x \in E^+$.

In particular, if in this case $E = \mathbb{R}^n$, *then* $\{y_n\} \subseteq Y$ *and* $\|y_n^-\| \longrightarrow 0$ *imply* $\|y_n^+\| \longrightarrow 0$.

PROOF. Let $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ be a Walrasian equilibrium supported by a price p, and let $y = y_1 + \cdots + y_k$. Since Y is a cone, we have

$$\max\{p\cdot z:\ z \in Y\} = p\cdot y = 0.$$

Now let $\{y_n\} \subseteq Y$ satisfy $y_n^- \longrightarrow 0$, and let $x \in E^+$. From $p\cdot y_n^+ - p\cdot y_n^- = p\cdot y_n \leqslant 0$, we see that $p\cdot y_n^+ \leqslant p\cdot y_n^-$, and so in view of $p\cdot(y_n^+ \wedge x) \leqslant p\cdot y_n^+ \leqslant p\cdot y_n^- \longrightarrow 0$, we conclude that

$$p\cdot(y_n^+ \wedge x) \longrightarrow 0. \qquad (\bigstar\bigstar)$$

Since preferences are strongly monotone, we have $p \gg 0$, and so the function

$$\|x\| = p\cdot|x|,\quad x \in E,$$

defines an order continuous norm on E. By [3, Theorem 12.9, p. 87], the topology generated by $\|\cdot\|$ and τ agree on the order interval $[0,x]$, and so in view of $(\bigstar\bigstar)$, we see that $y_n^+ \wedge x \xrightarrow{\tau} 0$. ■

If $x_i \in X_i$ $(i=1,\ldots,m)$, then we shall denote by G the convex set

$$G = \mathrm{co}\Big[\bigcup_{i=1}^{m} (F_i - \omega_i)\Big]$$
$$= \Big\{\sum_{i=1}^{m} \lambda_i(v_i - \omega_i):\ v_i \succ_i x_i,\ \lambda_i \geqslant 0 \ \text{ and } \ \sum_{i=1}^{m} \lambda_i = 1\Big\}.$$

LEMMA 7.4. *Assume that the economy has a symmetric Riesz dual system and that each consumption set satisfies* $X_i + E^+ = X_i$. *If* $x_i \in X_i$ $(i=1,\ldots,m)$, *then the convex set*

$$G = co\Big[\bigcup_{i=1}^{m} (F_i - \omega_i)\Big]$$

is weakly closed.

PROOF. Apply Theorem 6.2 by taking $Y_j = \{0\}$ $(j=1,\ldots,k)$ and $a = 0$. ■

THEOREM 7.5. *Assume that for an economy we have:*

a) Its Riesz dual system $\langle E,E'\rangle$ *is given by a reflexive Banach lattice, preferences are strongly monotone and* $X_i = E^+$ *for each* i;

b) There is only one producer whose production set Y *is a norm continuous cone; and* *c) The share θ_i of each consumer to the profit of the producer is positive, i.e., $\theta_i > 0$ for each $i=1,\ldots,m$.*

If $(x_1,\ldots,x_m,y)$ is an allocation, then the convex set

$$H = co[\bigcup_{i=1}^{m} (F_i - \theta_i Y - \omega_i)] = G - Y$$

is weakly closed.

PROOF. Since $\theta_i > 0$ and Y is a cone, we see that $\theta_i Y = Y$ for each i, and from this it easily follows that $H = G - Y$. By Lemma 7.4, we know that G is a weakly closed set. To see that $G - Y$ is also weakly closed, let f be in the weak closure of $G - Y$. Since $G - Y$ is convex, f belongs to the norm closure of $G - Y$. So, there exists a sequence $\{g_n - y_n\}$ of $G - Y$ with $\lim \|g_n - y_n - f\| = 0$.

For each n write $g_n = \sum_{i=1}^{m} \lambda_i^n (z_i^n - \omega_i)$, $z_i^n \succcurlyeq_i x_i$, $\lambda_i^n \geqslant 0$ with $\sum_{i=1}^{m} \lambda_i^n = 1$ for each n. Then we have

$$g_n + \omega \geqslant g_n + \sum_{i=1}^{m} \lambda_i^n \omega_i = \sum_{i=1}^{m} \lambda_i^n z_i^n \geqslant 0,$$

and so from

$$0 \leqslant y_n^- = (-y_n)^+ \leqslant (g_n - y_n + \omega)^+ \longrightarrow (f + \omega)^+ \text{ (norm)},$$

we see that $\{y_n^-\}$ is a norm bounded sequence.

Next, we claim that $\{y_n\}$ is a norm bounded sequence. Indeed, if this is not the case, then we can assume without loss of generality that $\lim \|y_n\| = \infty$. Since $\{y_n^-\}$ is norm bounded, we see that $\lim \|y_n^- / \|y_n\|\| = 0$. In view of $(y_n/\|y_n\|)^- = y_n^- / \|y_n\|$, the norm continuity of Y implies

$$\|(y_n / \|y_n\|)^+\| = \| y_n^+ / \|y_n\| \| \longrightarrow 0$$

and so

$$1 = \| y_n/\|y_n\| \| \leqslant \|y_n^+ / \|y_n\|\| + \|y_n^- / \|y_n\|\| \longrightarrow 0,$$

which is a contradiction. Hence, $\{y_n\}$ is norm bounded.

Since E is reflexive, $\{y_n\}$ has a weakly convergent subsequence. We can assume that $y_n \xrightarrow{w} y \in Y$. From $g_n = (g_n - y_n) + y_n \xrightarrow{w} f + y$ and the closedness of G, we see that $g = f + y \in G$. Hence, $f = g - y \in G - Y$, and thus $G - Y$ is a weakly closed set.■

Let $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ be an allocation. Then we have

$$\sum_{i=1}^{m} x_i = \omega + \sum_{j=1}^{k} y_j = \omega + \sum_{j=1}^{k} y_j^+ - \sum_{j=1}^{k} y_j^-,$$

and so

$$\sum_{i=1}^{m} x_i + \sum_{j=1}^{k} y_j^- = \omega + \sum_{j=1}^{k} y_j^+ .$$

The vector $\omega + \sum_{j=1}^{k} y_j^+$ represents the *total supply* in the economy under the allocation $(x_1,\ldots,x_m,y_1,\ldots,y_k)$. We shall use the letter e to designate the total supply of an allocation $(x_1,\ldots,x_m,y_1,\ldots,y_k)$, i.e., we shall write

$$e = \omega + \sum_{j=1}^{k} y_j^+ = \sum_{i=1}^{m} \omega_i + \sum_{j=1}^{k} y_j^+ .$$

DEFINITION 7.6. *An allocation* $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ *is said to be an* ε*-Walrasian equilibrium whenever for each* $\varepsilon > 0$ *there exists some price* p *such that:*

a) $p\cdot e = 1$ *(where* $e = \omega + \sum_{j=1}^{k} y_j^+$*);*

b) $x \succcurlyeq_i x_i$ *in* X_i *implies* $p\cdot x \geqslant p\cdot\omega_i + \sum_{j=1}^{k} \theta_{ij} p\cdot y_j - \varepsilon$*; and*

c) $p\cdot y_j \geqslant \sup\{p\cdot z\colon z \in Y_j\} - \varepsilon$ *for each* j.

Note that if the consumption sets satisfy $X_i + E^+ = X_i$, then any price p that satisfies property (b) of Definition 7.6 is necessarily a positive price. Indeed, if $x \geqslant 0$, then by the monotonicity of $\succcurlyeq_1$ we have $x_1 + \delta^{-1}x \succcurlyeq_1 x_1$ for all $\delta > 0$, and so

$$p\cdot x \geqslant \delta(p\cdot\omega_1 + \sum_{j=1}^{k} \theta_{1j} p\cdot y_j - p\cdot x_1 - \varepsilon)$$

for all $\delta \geqslant 0$, from which it follows that $p\cdot x \geqslant 0$.

Also, it should be noted that every Walrasian equilibrium is an ε-Walrasian equilibrium.

Finally, we close the paper by presenting an existence theorem for ε-Walrasian equilibria.

THEOREM 7.7. *Assume that for a compact economy we have*

1) *Its Riesz dual system* $\langle E,E'\rangle$ *is given by a reflexive Banach lattice;*
2) *Preferences are strongly monotone and* $X_i = E^+$ *holds for each* i*; and*
3) *Its aggregate production set* $Y = Y_1 + \cdots + Y_k$ *is a norm continuous weakly closed cone.*

Then the economy has ε*-Walrasian equilibria.*

PROOF. Consider a new economy with Riesz dual system $\langle E,E'\rangle$ having the same consumers, endowments and preferences but having one producer whose production set is Y. Also, assume that each consumer has the share $\theta_i = \frac{1}{m}$ $(i=1,\ldots,m)$ to the profit of the producer. It is easy to see that this new economy is compact, and so by Theorem 4.7 it has an Edgeworth equilibrium, say $(x_1,\ldots,x_m,y)$. If $y = y_1 + \cdots + y_k \in Y$, then we claim that $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ is an ε-Walrasian equilibrium.

To see this, let $\varepsilon > 0$. Clearly, $H = G - Y$ holds, and by Theorem 7.5 the convex set H is weakly closed. On the other hand, if $e = \sum_{i=1}^{m} \omega_i + \sum_{j=1}^{k} y_j^+ > 0$, then we have

$$\frac{\varepsilon}{m}e + G - Y = \frac{\varepsilon}{2m}e + (\frac{\varepsilon}{2m}e + H) \subseteq \frac{\varepsilon}{2m} + H^*,$$

and so from Theorem 5.4, we infer that $0 \notin \frac{\varepsilon}{m} + G - Y$. Thus, by the classical separation theorem, there exists some $p \in E'$ such that

$$p\cdot(\frac{\varepsilon}{m}e + g - y) > 0 \qquad (\star)$$

holds for all $g \in G$ and all $y \in Y$. Since $0 = \sum_{i=1}^{m} \frac{1}{m}(x_i - \omega_i) - \frac{1}{m}y \in G - Y$, we see that $p\cdot e > 0$, and so, replacing p by $p/p\cdot e$, we can assume that $p\cdot e = 1$.

Now assume that $x \succ_i x_i$ holds. Then $x - \omega_i \in G$ and $\sum_{j=1}^{k} \theta_{ij}y_j \in Y$. Thus, from $(\star)$, we see that

$$p\cdot x \geqslant p\cdot\omega_i + p\cdot(\sum_{j=1}^{k} \theta_{ij}y_j) - \frac{\varepsilon}{m} \geqslant p\cdot\omega_i + p\cdot(\sum_{j=1}^{k} \theta_{ij}y_j) - \varepsilon\,.$$

Next, fix some j, and note that

$$\frac{1}{m}\sum_{t=1}^{k} y_t = \sum_{i=1}^{m} \frac{1}{m}(x_i - \omega_i) \in G.$$

For $z \in Y_j$, put $z_t = y_t$ for $t \neq j$ and $z_j = z$, and note that

$$\frac{1}{m}(y_j - z) = \sum_{i=1}^{m} \frac{1}{m}(x_i - \omega_i) - \frac{1}{m}\sum_{t=1}^{k} z_t \in G - Y\,.$$

Therefore, from $(\star)$ we see that $p\cdot(\frac{\varepsilon}{m}e + \frac{1}{m}(y_j - z)) > 0$, from which it follows that

$$p\cdot y_j \geqslant p\cdot z - \varepsilon \quad \text{for all} \quad z \in Y_j,$$

and so,

$$p\cdot y_j \geqslant \sup\{p\cdot z\colon z \in Y_j\} - \varepsilon \quad \text{for all} \quad j.$$

The above show that $(x_1,\ldots,x_m,y_1,\ldots,y_k)$ is an ε-Walrasian equilibrium, and the proof of the theorem is finished.∎

REFERENCES

1. C. D. ALIPRANTIS AND D. J. BROWN, Equilibria in markets with a Riesz space of commodities, *J. Math. Economics* 11(1983), 189-207.
2. C. D. ALIPRANTIS, D. J. BROWN AND O. BURKINSHAW, Edgeworth Equilibria, Cowles Foundation Discussion Paper, No. 756-A, 1985.
3. C. D. ALIPRANTIS AND O. BURKINSHAW, *Locally Solid Riesz Spaces*, Academic Press, New York & London, 1978.
4. C. D. ALIPRANTIS AND O. BURKINSHAW, *Positive Operators*, Academic Press, New York & London, 1985.
5. K. J. ARROW AND G. DEBREU, Existence of an equilibrium for a competitive economy, *Econometrica* 22(1954), 265-290.
6. T. F. BEWLEY, Existence of equilibria in economies with infinitely many commodities, *J. Economic Theory* 4(1972), 514-540.
7. G. CHICHILNISKY AND G. HEAL, Competitive equilibria in L_p and Hilbert spaces with unbounded short sales, Columbia University Mimeograph, 1984.
8. G. DEBREU, Valuation equilibrium and Pareto optimum, *Proc. Nat. Acad. Sci. U.S.A.* 40(1954), 558-592.
9. G. DEBREU AND H. SCARF, A limit theorem on the core of an economy, *International Economic Review* 4(1963), 235-246.
10. D. DUFFIE, Competitive equilibria in general spaces, *J. Math. Economics,* forthcoming.
11. D. DUFFIE AND C. HUANG, Implementing Arrow-Debreu equilibria by continuous trading of few long-lived securities, *Econometrica* 53(1985), 1337-1356.
12. L. E. JONES, A competitive model of product differentiation, *Econometrica* 52(1984), 507-530.
13. M. A. KHAN AND R. VOHRA, Approximate equilibrium theory in economies with infinitely many commodities, Preprint, August 1985.
14. W. A. J. LUXEMBURG AND A. C. ZAANEN, *Riesz Spaces I*, North-Holland, Amsterdam, 1971.
15. A. MAS-COLELL, A model of equilibrium with differentiated commodities, *J. Math. Economics* 2(1975), 263-296.
16. A. MAS-COLELL, The price equilibrium problem in topological vector lattices, *Econometrica*, forthcoming.
17. A. MAS-COLELL, Valuation equilibrium and Pareto optimum revisited, Preprint, September 1985.
18. T. NEGISHI, Welfare economics and existence of an equilibrium for a competitive economy, *Metroeconomica* 12(1960), 92-97.
19. H. NIKAIDO, *Convex Structures and Economic Theory*, Academic Press, New York and London, 1968.
20. J. OSTROY, On the existence of Walrasian equilibria in large square economies, *J. Math. Economics* 13(1984), 143-164.

21. B. PELEG AND M. E. YAARI, Markets with countably many commodities, *International Economic Review* 11(1970), 369-377.

22. S. F. RICHARD AND W. R. ZAME, Proper preferences and quasi-concave utility functions, Preprint, November 1985.

23. H. H. SCHAEFER, *Banach Lattices and Positive Operators*, Springer-Verlag, New York & Berlin, 1974.

24. H. SCARF, The core of an n-person game, *Econometrica* 35(1967), 50-69.

25. N. C. YANNELIS, The core of an economy without ordered preferences, Center for Economic Research, University of Minnesota, Discussion Paper No. 214, 1985.

26. N. C. YANNELIS AND W. R. ZAME, Equilibria in Banach lattices without ordered preferences, *J. Math. Economics*, forthcoming.

27. W. R. ZAME, Equilibria in production economies with an infinite-dimensional commodity space, Institute for Mathematics and its Applications-University of Minnesota, Preprint # 127, 1985.

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