COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station

New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 756R

EDGEWORTH EQUILIBRIA

by

C. D. Aliprantis, D. J. Brown, and O. Burkinshaw

September 1985

(Revised)

EDGEWORTH EQUILIBRIA

BY

C. D. ALIPRANTIS[1], D. J. BROWN[2], AND O. BURKINSHAW[1]

The paper studies pure exchange economies with infinite dimensional commodity spaces in the setting of Riesz dual systems. Several new concepts of equilibrium are introduced. An allocation $(x_1,\ldots,x_m)$ is said to be a) an *Edgeworth equilibrium* whenever it belongs to the core of every n-fold replication of the economy; and b) an ε-*Walrasian equilibrium* whenever for each $\varepsilon > 0$ there exists some price $p \neq 0$ with $p\cdot\omega = 1$ (where $\omega = \Sigma\,\omega_i$ is the total endowment) and with $x \succ_i x_i$ implying $p\cdot x \geqslant p\cdot\omega_i - \varepsilon$. The major results of the paper are the following.

THEOREM 1. Edgeworth equilibria exist.
THEOREM 2. An allocation is an Edgeworth equilibrium if and only if it is an ε-Walrasian equilibrium.
THEOREM 3. If preferences are proper, then every Edgeworth equilibrium is a quasiequilibrium.

1. INTRODUCTION

There are three different methods for proving the existence of a Walrasian equilibrium in a finite exchange economy over a finite dimensional commodity space.

The first is the, now standard, fixed point argument in commodity and price space as, say, in Arrow and Debreu [4]. The second is the method of Negishi [9] based on price supporting Pareto optimal allocations. The third is due to Debreu and Scarf [7] where one price supports an allocation that is in the core of every replica of the given economy. Each of these arguments has been extended to demonstrate the existence of Walrasian equilibria in finite exchange economies with infinite dimensional commodity spaces.

---

[1] Research supported in part by NSF grant DMS 83-19594.
[2] Research supported in part by NSF grant SES 83-19611.

The authors are pleased to acknowledge the many conversations that they have enjoyed with their colleagues on the issues addressed in this paper. In particular, they wish to thank Monique Florenzano, Larry Jones, Andreu Mas-Colell, Joe Ostroy, Herb Scarf, and Bill Zame.

Bewley [5] extended the Arrow-Debreu argument to L_∞ by taking the limit of equilibria on subeconomies with finite dimensional commodity spaces, where he applied the Arrow-Debreu existence theorem to each subeconomy. Recently, Mas-Colell [8] demonstrated the existence of Walrasian equilibria in a large class of Banach lattices by using Negishi's argument and imposing a new condition on preferences which he calls properness. The appeal of the Negishi argument is that it is carried out in the space of utilities, and therefore is "independent" of the dimensionality of the commodity space, subject to some compactness assumptions. Peleg and Yaari [10] have extended the Debreu-Scarf argument [7] to $\mathbb{R}_\infty$, the space of real-valued sequences. Again, the "trick" is to recognize that the proof of Debreu-Scarf demonstrates the existence of a core allocation in a finite market game with nontransferable utility, i.e., is "independent" of the dimension of the commodity space.

The Peleg-Yaari model [10] is stimulating in a number of respects. First, the equilibrium notion is not the conventional one, in that equilibrium prices never give finite valuation to every commodity bundle in the commodity space. Consequently, equilibrium price vectors are not in the dual of the commodity space. This is the first paper, with a finite number of agents, which is outside the analytical framework first proposed by Debreu [6], where the commodity space and price space are dual vector spaces. Second, although the positive cone has empty interior with respect to the product topology (the topology used by Peleg-Yaari), the assumptions on preferences are the same as those in the finite dimensional commodity space case. In particular, there is no condition like properness, which is quite surprising given Mas-Colell's work [8]. Moreover, there are no proper preferences on $\mathbb{R}_\infty$ which satisfy the other conditions of monotonicity, continuity and convexity imposed by Peleg and Yaari.

This paper generalizes the results of Peleg and Yaari in the light of the above observations. To extend their results, we introduce several new concepts of equilibrium. An allocation is said to be an *Edgeworth equilibrium* whenever it belongs to the core of every n-fold replica of the economy. An allocation $(x_1,\ldots,x_m)$ is an *ε-Walrasian equilibrium* whenever for each $\varepsilon > 0$ there exists some price $p \neq 0$ with $p\cdot\omega = 1$ (where $\omega = \Sigma\omega_i$ is the total endowment) and with $x \succ_i x_i$ implying $p\cdot x \geq p\cdot\omega_i - \varepsilon$. An allocation is called an *extended Walrasian equilibrium* if it is a Walrasian equilibrium with respect to an extended price system, i.e., with respect to a price system whose values are extended real numbers.

The major results of the paper are the existence of Edgeworth equilibria, the equivalence of our three new equilibrium notions, and the demonstration that under properness every Edgeworth equilibrium is a quasiequilibrium.

Our model, which uses a Riesz dual system to describe the commodity-price duality, specializes to most of the models of exchange in the literature on the existence of Walrasian equilibria.

This paper is divided into five sections. In the fifth section we illustrate our results by considering a number of special models of exchange which have been used to analyze options markets, intertemporal exchange, markets with differentiated commodities, etc. In addition, we present a counterexample to show that one of our topological assumptions (the weak compactness of the order interval $[0,\omega]$) cannot be dropped. That is, in our model, the weak compactness of the order interval $[0,\omega]$ is an essential condition for the existence of Edgeworth equilibria.

2. MATHEMATICAL PRELIMINARIES

To make this paper relatively self-contained, we review in this section some of the basic properties of Riesz spaces (vector lattices) which will be used throughout the paper. We shall follow the notation and terminology of [2] and [3].

A partially ordered vector space E is said to be a *Riesz space* (or a *vector lattice*), whenever for any two elements x and y of E the supremum (least upper bound) and the infimum (greatest lower bound) of the set $\{x,y\}$ exist; notation

$$x \vee y = \sup\{x,y\} \quad \text{and} \quad x \wedge y = \inf\{x,y\}\,.$$

Most of the classical spaces are Riesz spaces. For example, the spaces ℓ_p, $L_p(\mu)$, and $C(\Omega)$ are all Riesz spaces. Throughout the discussion in this section E will denote a Riesz space.

Every element $x \in E$ which satisfies $x \geqslant 0$ is referred to as a positive element. The cone of all positive elements of E will be denoted by E^+, i.e., $E^+ = \{x \in E: x \geqslant 0\}$. For an element $x \in E$, its positive part, negative part, and absolute value are defined by

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \text{ and } |x| = x \vee (-x).$$

If the Riesz space E is a function space over a set T, then for every function $f \in E$ we have

$$f^+(t) = \max\{f(t),0\}, \quad f^-(t) = \max\{-f(t),0\}, \quad \text{and} \quad |f|(t) = \max\{f(t),-f(t)\}$$

for all $t \in T$. A vector subspace F of E is said to be a *Riesz subspace* whenever F is a Riesz space in its own right with the operations of E, i.e., whenever $x,y \in F$ imply that both $x \vee y$ and $x \wedge y$ belong to F. The Riesz space E is said to be *Dedekind complete* whenever every non-empty subset of E that is order bounded from above in E has a supremum (least upper bound). As usual, a set A is said to be order bounded from above whenever there exists some $x \in E$ satisfying $a \leqslant x$ for all $a \in A$.

A subset A of E is said to be a *solid set* whenever $|x| \leqslant |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of E is known as an *ideal*. For an element $\omega \geqslant 0$ there exists a smallest (with respect to inclusion) ideal of E that contains ω. This ideal is called the ideal generated by ω and is precisely the set

$$A_\omega = \{x \in E: \exists\ \lambda > 0 \text{ with } |x| \leqslant \lambda\omega\}.$$

Any ideal of the form A_ω is known as a *principal ideal*. As we shall see, the principal ideals will play a crucial role in our approach.

The ideal generated by $\omega \geqslant 0$ is closely related to the band generated by ω. Recall that the symbolism $x_\alpha \uparrow x$ means that the net $\{x_\alpha\}$ is increasing (i.e., $\alpha \geqslant \beta$ implies $x_\alpha \geqslant x_\beta$) and that x is the least upper bound of $\{x_\alpha\}$. The *band* B_ω generated by ω is the ideal

$$B_\omega = \{x \in E: |x| \wedge n\omega \uparrow |x|\}.$$

Clearly, $A_\omega \subseteq B_\omega$ holds. The disjoint complement of B_ω is the ideal defined by

$$B_\omega^d = \{x \in E: \ |x| \wedge \omega = 0\}.$$

In case E is Dedekind complete, we have $E = B_\omega \oplus B_\omega^d$. For more about ideals and bands see [3, Section 3].

Let $x,y \in E$ satisfy $x \leqslant y$. Then the set

$$[x,y] = \{z \in E: \ x \leqslant z \leqslant y\}$$

is referred to as an *order interval* of E. Subsets of order intervals of E are called order bounded sets. A linear functional $f:E \longrightarrow \mathbb{R}$ is said to be order bounded whenever it carries order bounded subsets of E onto bounded subsets of $\mathbb{R}$. The set of all order bounded linear functionals of E is referred to as the *order dual* of E and is denoted by $E^\sim$, i.e.,

$$E^\sim = \{f:E \longrightarrow \mathbb{R} \mid f \text{ is linear and order bounded}\}.$$

Clearly, $E^\sim$ is a vector space. Remarkably, it is also a Dedekind complete Riesz space. Its lattice operations are given by

$$f \vee g(x) = \sup\{f(y) + g(z): \ y,z \in E^+ \ \text{ and } \ y + z = x\}, \text{ and}$$
$$f \wedge g(x) = \inf\{f(y) + g(z): \ y,z \in E^+ \ \text{ and } \ y + z = x\}$$

for all $x \in E^+$ and all $f,g \in E^\sim$.

A linear functional $f \in E^\sim$ is said to be *order continuous* (or a *normal integral*) whenever $x_\alpha \downarrow 0$ in E implies $f(x_\alpha) \longrightarrow 0$ in $\mathbb{R}$. Similarly, a linear functional $f \in E^\sim$ is said to be *σ-order continuous* (or an *integral*) whenever $x_n \downarrow 0$ implies $f(x_n) \longrightarrow 0$. The order continuous dual $E_n^\sim$ of E is defined by

$$E_n^\sim = \{f \in E^\sim: \ f \text{ is order continuous}\}.$$

Similarly, the σ-order continuous dual $E_c^\sim$ of E is defined by

$$E_c^\sim = \{f \in E^\sim: \ f \text{ is } \sigma\text{-order continuous}\}.$$

Both $E_n^\sim$ and $E_c^\sim$ are bands of the order dual $E^\sim$. Here are some examples of these duals.

1. $(C[0,1])_n^\sim = (C[0,1])_c^\sim = \{0\}$.
2. $(L_p[0,1])_n^\sim = (L_p[0,1])_c^\sim = L_q[0,1]$; $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

A seminorm q on E is said to be a *lattice seminorm* whenever $|x| \leqslant |y|$ in E implies $q(x) \leqslant q(y)$. A locally convex-solid Riesz space (E,τ) is a Riesz space E equipped with a locally convex topology τ that is generated by a family of lattice seminorms. In this case τ is called a *locally convex-solid topology*, and it has a basis at zero consisting of open, convex and solid sets. It is important to notice that the lattice operations (sup, inf, and absolute value) are continuous with respect to any locally convex-solid topology. Special examples of locally convex-solid Riesz spaces are the Banach lattices. A *Banach lattice* is a Riesz space equipped with a lattice norm under which it is a Banach space. The Riesz spaces ℓ_p, $L_p(\mu)$, and $C_b(\Omega)$ are all examples of Banach lattices.

Let (E,τ) be a Hausdorff locally convex-solid Riesz space. The topological dual of (E,τ) will be denoted by E', i.e.,

$$E' = \{f: E \longrightarrow \mathbb{R} \mid f \text{ is linear and } \tau\text{-continuous}\}.$$

It is an important result that E' is an ideal of $E^\sim$. If E is a Banach lattice, then E' and $E^\sim$ coincide, i.e., $E' = E^\sim$ holds. The *absolute weak topology* $|\sigma|(E,E')$ is the locally convex-solid topology on E generated by the family of lattice seminorms $\{\rho_f: f \in E'\}$, where

$$\rho_f(x) = |f|(|x|),\ x \in E.$$

The topology $|\sigma|(E,E')$ is the coarsest locally convex-solid topology on E consistent with the dual system $\langle E,E'\rangle$, i.e., $(E,|\sigma|(E,E'))' = E'$ holds. The locally convex-solid topology on E of uniform convergence on the solid, convex, and $\sigma(E',E)$-compact subsets of E' is the finest locally convex-solid topology on E consistent with $\langle E,E'\rangle$. This topology is called the *absolute Mackey topology* of E and is denoted by $|\tau|(E,E')$. Thus, we have

$$\sigma(E,E') \subseteq |\sigma|(E,E') \subseteq |\tau|(E,E') \subseteq \tau(E,E').$$

In connection with the last relations, it should be noted that:

a) A locally convex topology τ_1 on E is consistent with the dual system $\langle E,E'\rangle$ if and only if $\sigma(E,E') \subseteq \tau_1 \subseteq \tau(E,E')$; and

b) A locally convex-solid topology τ_2 on E is consistent with $\langle E,E'\rangle$ if and only if $|\sigma|(E,E') \subseteq \tau_2 \subseteq |\tau|(E,E')$ holds.

If E is a Banach lattice and E' is its norm dual, then $|\tau|(E,E') = \tau(E,E')$ holds, i.e., the Mackey topology is locally convex-solid. For details, see

[3, Section 11]. Also, it should be noted that if E is Dedekind complete and $E_c^\sim$ separates the points of E, then $\tau(E,E_c^\sim) = |\tau|(E,E_c^\sim)$ holds; see [2, Exercise 4, p. 163].

A linear topology τ on E is said to be *order continuous* whenever $x_\alpha \downarrow 0$ in E implies $x_\alpha \xrightarrow{\tau} 0$. If τ is a locally convex-solid topology, then τ is order continuous if and only if it is generated by a family of order continuous lattice seminorms. (A lattice seminorm q on E is said to be order continuous if $x_\alpha \downarrow 0$ in E implies $q(x_\alpha) \downarrow 0$.)

We now turn our attention to the important concept of a Riesz dual system which will be the basic framework of our economic model. A *Riesz dual system* $\langle E,E'\rangle$ is a dual system such that:

1. E is a Riesz space;
2. E' is an ideal of the order dual $E^\sim$ separating the points of E; and
3. The duality of the system is the natural one, i.e.,

$$\langle x,x'\rangle = x'(x)$$

holds for all $x \in E$ and all $x' \in E'$.

Here are some examples of Riesz dual systems.

a) $\langle \mathbb{R}^n, \mathbb{R}^n \rangle$;
b) $\langle \ell_p,\ell_q\rangle$, $1 \leq p,q \leq \infty$, $1/p + 1/q = 1$;
c) $\langle L_p(\mu),L_q(\mu)\rangle$, $1 \leq p,q \leq \infty$, $1/p + 1/q = 1$;
d) $\langle c_0,\ell_1\rangle$;
e) $\langle \ell_1,c_0\rangle$;
f) $\langle C_b(\Omega), ca(\Omega)\rangle$;
g) $\langle ca(\Omega),ca'(\Omega)\rangle$;
h) $\langle \mathbb{R}_\infty, \mathbb{R}_\infty' \rangle$.

Note that if (E,τ) is a Hausdorff locally convex-solid Riesz space and E' is the topological dual of (E,τ), then $\langle E,E'\rangle$ is a Riesz dual system. On the other hand, if $\langle E,E'\rangle$ is a Riesz dual system, then the topological dual of $(E,|\sigma|(E,E'))$ is precisely E'; see [3, Thm 11.6, p. 166]. Thus, the Riesz dual systems are associated with locally convex-solid Riesz spaces.

Now let $\langle E,E'\rangle$ be a Riesz dual system. Then the topology $\sigma(E,E')$ on E will be referred to as the *weak topology*. The strong topology $\beta(E',E)$ on E' is always a (Hausdorff) locally convex-solid topology, and so the topological dual

E'' of $(E',\beta(E',E))$ is an ideal of $(E')^\sim$. Identifying each $x \in E$ with the order bounded linear functional on E'

$$x(x') = x'(x),\ x' \in E',$$

we see that E is a Riesz subspace of E'' (and hence of $(E')^\sim$). If E is an ideal of E'', then $\langle E',E \rangle$ is also a Riesz dual system. Thus, we say that a Riesz dual system $\langle E,E' \rangle$ is *symmetric*, whenever E is an ideal of E''. In the preceding list of Riesz dual systems, all but (f) are examples of symmetric Riesz dual systems.

The following theorem characterizes the symmetric Riesz dual systems, and it will be used in our discussion. For a proof see [3, Theorems 11.10 and 11.13].

THEOREM 2.1. *For a Riesz dual system $\langle E,E' \rangle$ the following statements are equivalent.*

1. *$\langle E,E' \rangle$ is a symmetric Riesz dual system, i.e., E is an ideal of $(E')^\sim$.*
2. *The order intervals of E are weakly compact.*
3. *E is Dedekind complete and there exists an order continuous locally convex-solid topology on E consistent with $\langle E,E' \rangle$.*

If $\langle E,E' \rangle$ is a symmetric Riesz dual system, then $E' \subseteq E_n^\sim$ holds. In particular, in this case, $\langle E,E_n^\sim \rangle$ is also a symmetric Riesz dual system.

Finally, we mention one more important property that will be used later. If E is a Dedekind complete Riesz space and $\omega \in E^+$, then the ideal A_ω under the lattice norm

$$\|x\|_\infty = \inf\{\lambda > 0:\ |x| \leqslant \lambda\omega\},\ x \in A_\omega,$$

is an AM-space with unit ω. That is, A_ω is lattice isometric to some $C(X)$ Banach lattice (X Hausdorff and compact) with ω corresponding to the constant function one on X. For details see [3, Section 12].

3. WALRASIAN EQUILIBRIA AND QUASIEQUILIBRIA

We start by reviewing a few facts about preferences and utility functions. Let E be a Riesz space, and let $\succcurlyeq$ be a preference relation (i.e., a complete and transitive relation) on E^+. Then $\succcurlyeq$ is said to be:

1. *weakly convex*, whenever the set $\{x \in E^+ : x \succcurlyeq y\}$ is convex for each $y \in E^+$;
2. *convex*, whenever $x \succ y$ and $0 < \alpha < 1$ imply $\alpha x + (1-\alpha)y \succ y$;
3. *monotone*, whenever $x > y \geqslant 0$ implies $x \succcurlyeq y$;
4. *strongly monotone*, whenever $x > y \geqslant 0$ implies $x \succ y$; and
5. *continuous for some topology* τ, whenever the sets $\{x \in E^+ : x \succcurlyeq y\}$ and $\{x \in E^+ : y \succcurlyeq x\}$ are both τ-closed for each $y \in E^+$.

The following two basic properties will be used later.

a. *A weakly convex preference* $\succcurlyeq$ *is convex if and only if* $x_i \succcurlyeq y$ *for* $i = 1,\ldots,n$ *and* $x_i \succ y$ *for some* i *imply* $\sum_{i=1}^{n} \alpha_i x_i \succ y$ *for each convex combination with positive weights.*

b. *If a preference relation* $\succcurlyeq$ *is weakly convex, strongly monotone, and continuous for some linear topology on* E*, then* $\succcurlyeq$ *is a convex preference.*

To see (b), let $x \succ y$ and let $0 < \alpha < 1$. Then $x > 0$, and since $\lim_{\varepsilon \uparrow 1} \varepsilon x = x$ holds for every linear topology on E, it follows that there exists $0 < \varepsilon < 1$ with $\varepsilon x \succ y$. By the weak convexity, we have $\alpha(\varepsilon x) + (1-\alpha)y \succcurlyeq y$. On the other hand, from $\alpha x + (1-\alpha)y > \alpha(\varepsilon x) + (1-\alpha)y$ and the strong monotonicity of $\succcurlyeq$, we infer that $\alpha x + (1-\alpha)y \succ \alpha(\varepsilon x) + (1-\alpha)y$. Thus, $\alpha x + (1-\alpha)y \succ y$.

A function $u: E^+ \longrightarrow \mathbb{R}$ is said to be a *utility function* for a preference $\succcurlyeq$ whenever $x \succcurlyeq y$ holds if and only if $u(x) \geqslant u(y)$. A utility function u is said to be:

i) *quasi-concave*, whenever for each $x,y \in E^+$ and each $0 < \alpha < 1$ we have

$$u(\alpha x + (1-\alpha)y) \geqslant \min\{u(x),u(y)\};$$

ii) *monotone*, whenever $x > y \geqslant 0$ implies $u(x) \geqslant u(y)$; and

iii) *strongly monotone*, whenever $x > y \geqslant 0$ implies $u(x) > u(y)$.

Note that a utility function is quasi-concave if and only if the preference it represents is weakly convex.

In our approach we shall consider a pure exchange economy whose commodity-price duality is described by a Riesz dual system $\langle E,E'\rangle$. We shall assume the following.

1. The commodity space is E.
2. The prices are the elements of E'. (As usual, if $p \in E'$, then the value $\langle x,p\rangle$ will also be denoted by $p \cdot x$, i.e., $\langle x,p\rangle = p \cdot x$.)

3. There are m consumers indexed by i such that:

a) Each consumer i has an initial endowment $\omega_i > 0$ and has as his consumption set E^+.

b) The total endowment is denoted by ω, i.e., $\omega = \sum_{i=1}^{m} \omega_i$.

c) The preferences of each consumer i can be represented by a strongly monotone quasi-concave utility function $u_i : E^+ \longrightarrow \mathbb{R}^+$ (and hence the preferences are weakly convex, convex, and strongly monotone)[3].

d) There is a locally convex-solid topology τ on E consistent with $\langle E,E' \rangle$ such that each utility function u_i is τ-continuous.

Therefore, for this paper: A *pure exchange economy* $\mathcal{E}$ (or simply an *economy*) consists of a Riesz dual system $\langle E,E' \rangle$ (representing the commodity-price duality), and m consumers. Each consumer i has E^+ as his consumption set, an initial endowment $\omega_i > 0$ and his preferences are represented by a strongly monotone, quasi-concave and τ-continuous utility function u_i.

Recall that an m-tuple $(x_1,\ldots,x_m)$ is said to be an *allocation* whenever $x_i \in E^+$ for each i and $\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \omega_i = \omega$.

DEFINITION 3.1. *An allocation* $(x_1,\ldots,x_m)$ *is said to be:*

a) A Walrasian (or a competitive) equilibrium whenever there exists a price $p \neq 0$ *such that for each* i *the element* x_i *is a maximal element in the budget set* $\mathcal{B}_i(p) = \{x \in E^+ : p \cdot x \leq p \cdot \omega_i\}$, *i.e.,* $x_i \in \mathcal{B}_i(p)$ *and* $x_i \succsim_i x$ *hold for all* $x \in \mathcal{B}_i(p)$.

b) A quasiequilibrium whenever there exists some price $p \neq 0$ *such that* $x \succsim_i x_i$ *in* E^+ *implies* $p \cdot x \geq p \cdot \omega_i$.

Any price vector $p \neq 0$ that satisfies (a) or (b) above is known as a price vector *supporting* the allocation $(x_1,\ldots,x_m)$. Note that every Walrasian equilibrium is a quasiequilibrium.

THEOREM 3.2. *If a price* $p \neq 0$ *supports a quasiequilibrium allocation* $(x_1,\ldots,x_m)$, *then*

1. $p \geq 0$; *and*

2. $p \cdot x_i = p \cdot \omega_i$ *for each* i.

[3] A. Mas-Colell [8] shows that on each order interval every monotone continuous preference can be represented by a utility function. Representation of preferences by utility functions on the order intervals is all we need to prove our results.

PROOF. Observe that $x \succ_i x_i$ implies $p\cdot x_i \geqslant p\cdot\omega_i$. Since $\sum_{i=1}^m x_i = \sum_{i=1}^m \omega_i$, we have $\sum_{i=1}^m p\cdot x_i = \sum_{i=1}^m p\cdot\omega_i$, and so $p\cdot x_i = p\cdot\omega_i$ must hold for each i.

To see that p is a positive linear functional, let $y \geqslant 0$. Then by the monotonicity of $\succ_1$ we have $x_1 + y \succ_1 x_1$, and so $p\cdot x_1 + p\cdot y = p\cdot(x_1+y) \geqslant p\cdot\omega_1$. Now from $p\cdot x_1 = p\cdot\omega_1$, it follows that $p\cdot y \geqslant 0$.■

Recall that a price $0 \leqslant p \in E'$ is said to be *strictly positive* whenever $x > 0$ (i.e., $x \geqslant 0$ and $x \neq 0$) implies $p\cdot x > 0$. A strictly positive price p will be denoted (as usual) by $p \gg 0$.

Under our assumptions, the Walrasian equilibria are characterized as follows.

THEOREM 3.3. *For an allocation $(x_1,\ldots,x_m)$ the following two statements are equivalent.*

1. *The allocation is a Walrasian equilibrium.*
2. *There exists some $p \in E'$ with $p\cdot\omega = 1$ and such that $x \succ_i x_i$ in E^+ for some i implies $p\cdot x \geqslant p\cdot\omega_i$.*

Moreover, if some price p satisfies (2), then

a) $p \gg 0$;

b) $p\cdot x_i = p\cdot\omega_i$ for all i; and

c) p is a supporting price for $(x_1,\ldots,x_m)$.

PROOF. (1) $\Longrightarrow$ (2) Let $p \neq 0$ be a supporting price for $(x_1,\ldots,x_m)$. By Theorem 3.2 we know that $p \geqslant 0$ and that $x \succ_i x_i$ implies $p\cdot x \geqslant p\cdot\omega_i$.

If $p\cdot\omega = 0$, then $x_1 + \omega \in \mathcal{B}_1(p)$, and $x_1 + \omega \succ_1 x_1$ contradicts the maximality property of x_1 in $\mathcal{B}_1(p)$. Now note that $p/p\cdot\omega$ satisfies the properties of (2).

(2) $\Longrightarrow$ (1) Assume that p satisfies (2). By Theorem 3.2 we know that $p \geqslant 0$ and that $p\cdot x_i = p\cdot\omega_i$ holds for all i. The rest of the proof goes by steps.

α) If $p\cdot\omega_i > 0$ holds, then x_i is a maximal element in the budget set $\mathcal{B}_i(p)$.

Let $p\cdot\omega_i > 0$, and assume by way of contradiction that there exists an element $z \in \mathcal{B}_i(p)$ with $z \succ_i x_i$. Clearly, $p\cdot z \leqslant p\cdot\omega_i$ holds. Since the set $\{y \in E^+ : y \succ_i x_i\}$ is τ-open relative to E^+, $z \succ_i x_i$, and $\tau\text{-}\lim_{\varepsilon\uparrow 1} \varepsilon z = z$, there exists some $0 < \varepsilon < 1$ with $\varepsilon z \succ_i x_i$. Therefore, $p\cdot(\varepsilon z) \geqslant p\cdot\omega_i$. On the other hand, we have

$$p\cdot\omega_i > \varepsilon(p\cdot z) = p\cdot(\varepsilon z) \geqslant p\cdot\omega_i,$$

which is impossible. Hence, x_i is a maximal element in $\mathcal{B}_i(p)$.

β) $p \gg 0$ holds, i.e., $x > 0$ implies $p \cdot x > 0$.

Let $x > 0$, and assume by way of contradiction that $p \cdot x = 0$. From $\sum_{i=1}^{m} p \cdot \omega_i = p \cdot \omega = 1$, there exists some i with $p \cdot \omega_i > 0$. By (α), x_i is a maximal element in $\mathcal{B}_i(p)$. On the other hand, by the strong monotonicity of $\succ_i$ we have $x_i + x \succ_i x_i$, and in view of

$$p \cdot (x_i + x) = p \cdot x_i + p \cdot x = p \cdot x_i = p \cdot \omega_i,$$

we see that $x_i + x \in \mathcal{B}_i(p)$, which is a contradiction. Hence, $p \cdot x > 0$ holds, proving that p is a strictly positive functional.

Now to complete the proof note that by (β) we have $p \cdot \omega_i > 0$ for all i, and so by (α) each x_i is a maximal element in the budget set $\mathcal{B}_i(p)$. That is, $(x_1, \ldots, x_m)$ is a Walrasian equilibrium supported by the price vector p.■

It should be noted that if $(x_1, \ldots, x_m)$ is a Walrasian equilibrium supported by a price vector p, then $x \succ_i x_i$ in E^+ implies $p \cdot x > p \cdot \omega_i$.

4. EDGEWORTH EQUILIBRIA

The set of all allocations will be denoted by $\mathcal{A}$, i.e.,

$$\mathcal{A} = \{(x_1, \ldots, x_m) \in E^m \colon x_i \in E^+ \text{ for each } i \text{ and } \sum_{i=1}^{m} x_i = \omega\}.$$

Clearly, the set $\mathcal{A}$ is a non-empty convex set. When $[0,\omega]$ is weakly compact, $\mathcal{A}$ is likewise weakly compact.

THEOREM 4.1. *If the order interval $[0,\omega]$ is weakly compact, then the set $\mathcal{A}$ of all allocations is a non-empty weakly compact convex subset of E^m.*[4]

PROOF. Let $\{(x_1^\alpha, \ldots, x_m^\alpha)\}$ be a net of $\mathcal{A}$. Since $0 \leqslant x_i^\alpha \leqslant \omega$ holds for all i and α and the order interval $[0,\omega]$ is weakly compact, it follows that for each i every subnet of $\{x_i^\alpha\}$ has a weakly convergent subnet. Thus, by passing to an appropriate subnet, we see that $\{(x_1^\alpha, \ldots, x_m^\alpha)\}$ has a weakly convergent subnet to some $(x_1, \ldots, x_m)$ in E^m. Clearly, $x_i \geqslant 0$ holds for all i, and from $\sum_{i=1}^{m} x_i^\alpha = \omega$

[4] It should be noted that the converse is also true for $m \geqslant 2$. That is, for $m \geqslant 2$ the set $\mathcal{A}$ is weakly compact if and only if $[0,\omega]$ is weakly compact.

for all α, we see that $\sum_{i=1}^{m} x_i = \omega$, i.e., $(x_1,\ldots,x_m) \in \mathcal{A}$. Thus, every net of $\mathcal{A}$ has a weakly convergent subnet in $\mathcal{A}$, and so $\mathcal{A}$ is a weakly compact subset of E^m.■

Recall that a coalition S can improve upon an allocation $(x_1,\ldots,x_m)$ whenever there exists an allocation $(y_1,\ldots,y_m)$ with $\sum_{i\in S} y_i = \sum_{i\in S} \omega_i$,

$$y_i \succsim_i x_i \text{ for each } i \in S \text{ and } y_i \succ_i x_i \text{ for some } i \in S.$$

Note that with strongly monotone continuous preferences a coalition S can improve upon an allocation $(x_1,\ldots,x_m)$ if and only if there is another allocation $(z_1,\ldots,z_m)$ satisfying

$$\sum_{i\in S} z_i = \sum_{i\in S} \omega_i \quad \text{and} \quad z_i \succ_i x_i \quad \text{for each } i \in S.$$

An allocation $(x_1,\ldots,x_m)$ that cannot be improved by any coalition is known as a *core allocation*. The set of all core allocations for an economy $\mathcal{E}$ will be denoted by $\mathrm{Core}(\mathcal{E})$. It is well known that every Walrasian allocation is a core allocation.

The next result asserts that core allocations always exist.

THEOREM 4.2. *If the order interval* $[0,\omega]$ *is weakly compact, then the core is a non-empty weakly compact subset of the set of all allocations.*

PROOF. The argument will be done in two steps, and follows the proof of the finite dimensional case due to Debreu and Scarf [7].

STEP I. *The core is non-empty.*

Start by defining the subsets $V(S)$ of $\mathbb{R}^m$ as follows. If S is a coalition of agents, then

$$V(S) = \{(x_1,\ldots,x_m) \in \mathbb{R}^m\colon \exists \text{ an allocation } (y_1,\ldots,y_m) \text{ with } \sum_{i\in S} y_i = \sum_{i\in S}\omega_i \text{ and } x_i \leqslant u_i(y_i) \text{ for all } i \in S\}.$$

The subsets $V(S)$ of $\mathbb{R}^m$ have the following properties.

1) Each $V(S)$ is a non-empty proper closed subset of $\mathbb{R}^m$.

Fix a non-empty set S of agents. Then clearly, $V(S)$ is non-empty. On the other hand, if $(x_1,\ldots,x_m) \in V(S)$, then $x_i \leqslant u_i(\omega)$ holds for all $i \in S$, and so $V(S)$ is a proper subset of $\mathbb{R}^m$.

To see that $V(S)$ is closed, assume that a net $\{(x_1^\alpha,\ldots,x_m^\alpha)\}$ of $V(S)$ satisfies $(x_1^\alpha,\ldots,x_m^\alpha) \longrightarrow (x_1,\ldots,x_m)$ in $\mathbb{R}^m$. For each α pick an allocation $(y_1^\alpha,\ldots,y_m^\alpha)$ with $x_i^\alpha \leqslant u_i(y_i^\alpha)$ for all $i \in S$ and $\sum_{i\in S} y_i^\alpha = \sum_{i\in S} \omega_i$. Since $0 \leqslant y_i^\alpha \leqslant \omega$ holds for all i and all α, and $[0,\omega]$ is weakly compact, we can assume (by passing to appropriate subnets) that $y_i^\alpha \xrightarrow{w} y_i$ holds for all i. Clearly, $(y_1,\ldots,y_m)$ is an allocation and $\sum_{i\in S} y_i = \sum_{i\in S} \omega_i$ holds. Now fix $i \in S$ and let $\varepsilon > 0$. Pick some β with

$$x_i - \varepsilon < x_i^\alpha \quad \text{for all } \alpha \geqslant \beta.$$

Since y_i is in the weak closure of the set $\mathrm{co}\{y_i^\alpha\colon \alpha \geqslant \beta\}$, it follows that y_i is also in the τ-closure of $\mathrm{co}\{y_i^\alpha\colon \alpha \geqslant \beta\}$. Thus, by the τ-continuity of u_i there exists a convex combination $\sum_{j=1}^{k} \lambda_j y_i^{\alpha_j}$ with $\alpha_j \geqslant \beta$ such that

$$u_i\Big(\sum_{j=1}^{k} \lambda_j y_i^{\alpha_j}\Big) < u_i(y_i) + \varepsilon .$$

If γ is an index among $\{\alpha_1,\ldots,\alpha_k\}$ with

$$u_i(y_i^\gamma) = \min\{u_i(y_i^{\alpha_j})\colon j = 1,\ldots,k\},$$

then $\gamma \geqslant \beta$ holds, and by the quasi-concavity of u_i we see that

$$x_i - \varepsilon < x_i^\gamma \leqslant u_i(y_i^\gamma) \leqslant u_i\Big(\sum_{j=1}^{k} \lambda_j y_i^{\alpha_j}\Big) < u_i(y_i) + \varepsilon .$$

Thus, $x_i \leqslant u_i(y_i) + 2\varepsilon$ holds for all $\varepsilon > 0$, from which it follows that $x_i \leqslant u_i(y_i)$ for all $i \in S$. Therefore, $(x_1,\ldots,x_m) \in V(S)$, and so $V(S)$ is also closed.

2) Each $V(S)$ is comprehensive, i.e., $x \in V(S)$ and $y \leqslant x$ imply $y \in V(S)$.

3) If $x \in V(S)$ and $y \in \mathbb{R}^m$ satisfy $x_i = y_i$ for all $i \in S$, then $y \in V(S)$.

4) For each coalition S, the set $V(S) \cap \mathbb{R}^m_+$ is non-empty and bounded relative to $\mathbb{R}^S_+$.

Clearly, $V(S) \cap \mathbb{R}^m_+$ is non-empty, and if $(x_1,\ldots,x_m) \in V(S)$, then $x_i \leqslant u_i(\omega)$ holds for all $i \in S$, proving that $V(S) \cap \mathbb{R}^m_+$ is bounded relative to $\mathbb{R}^S_+$.

5) The exchange game derived from the economy is balanced.

The proof of (5) is Scarf's original proof [11] which we include it here for completeness. Consider a balanced family $\mathcal{B}$ of allocations with weights $\{w_S\colon S \in \mathcal{B}\}$. That is, $\sum_{S\in\mathcal{B}_i} w_S = 1$ holds for all i, where, as usual, $\mathcal{B}_i = \{S \in \mathcal{B}\colon i \in S\}$.

Now let $(x_1,\ldots,x_m) \in \bigcap_{S\in\mathcal{B}} V(S)$. We have to show that $(x_1,\ldots,x_m) \in V(\{1,\ldots,m\})$.

Let $S \in \mathcal{B}$. Since $(x_1,\ldots,x_m) \in V(S)$, there exists an allocation $(x_1^S,\ldots,x_m^S)$ with $\sum_{i\in S} x_i^S = \sum_{i\in S} \omega_i$ and $x_i \leq u_i(x_i^S)$ for all $i \in S$. Now put

$$y_i = \sum_{S\in\mathcal{B}_i} w_S x_i^S,\ i = 1,\ldots,m.$$

Since each y_i is a convex combination, it follows from the quasi-concavity of u_i that $u_i(y_i) \geq x_i$ holds for all i. Moreover, we have

$$\sum_{i=1}^m y_i = \sum_{i=1}^m \sum_{S\in\mathcal{B}_i} w_S x_i^S = \sum_{S\in\mathcal{B}} w_S\Big(\sum_{i\in S} x_i^S\Big) = \sum_{S\in\mathcal{B}} w_S\Big(\sum_{i\in S}\omega_i\Big)$$

$$= \sum_{i=1}^m \omega_i\Big(\sum_{S\in\mathcal{B}_i} w_S\Big) = \sum_{i=1}^m \omega_i = \omega,$$

which proves that $(x_1,\ldots,x_m) \in V(\{1,\ldots,m\})$, as desired.

Next, by Scarf's classical result [12], the exchange game defined from the economy has a non-empty core, and consequently our economy has a non-empty core.

STEP II. *The set of all core allocations is a weakly compact set.*

Denote by C the (non-empty) set of all core allocations. Clearly, C is a subset of the weakly compact set $\mathcal{A}$, and hence the weak closure of C is also weakly compact. To see that C is weakly closed, let $(x_1,\ldots,x_m)$ be an allocation lying in the weak closure of C, and assume by way of contradiction that there exist an allocation $(y_1,\ldots,y_m)$ and a coalition S of consumers with

$$y_i \succ_i x_i \text{ for all } i \in S \text{ and with } \sum_{i\in S} y_i = \sum_{i\in S}\omega_i .$$

For each $i \in S$, the set of allocations

$$U_i = \{(z_1,\ldots,z_m) \in \mathcal{A}:\ z_i \succ_i y_i\}$$

is convex and weakly closed in E^m. Thus, the set $U = \bigcup_{i\in S} U_i$ is weakly closed in E^m, and so its complement U^c is weakly open. From $(x_1,\ldots,x_m) \in U^c$, we see that $U^c \cap C \neq \emptyset$. Pick $(z_1,\ldots,z_m) \in U^c \cap C$. Thus, we have

$$y_i \succ_i z_i \text{ for all } i \in S \text{ and } \sum_{i\in S} y_i = \sum_{i\in S}\omega_i ,$$

which is a contradiction. Hence $(x_1,\ldots,x_m) \in C$, and so C is weakly closed. The proof of the theorem is now complete.■

It is well known that if E is finite dimensional and $\omega \gg 0$, then an allocation is a Walrasian equilibrium if and only if it belongs to the core of every n-fold replica of the economy. One purpose of this paper is to present an infinite dimensional analogue of this result. We start with the following basic definition.

DEFINITION 4.3. *An allocation is said to be an* ***Edgeworth equilibrium*** *whenever it is in the core of every n-fold replica of the economy.*

Observe that in case E is finite dimensional, then every Edgeworth equilibrium is a quasiequilibrium.

The reader should observe that the concept of an Edgeworth equilibrium is "price free." That is, the concept of an Edgeworth equilibrium is an intrinsic property of the commodity space. As a consequence, note that if a Riesz subspace F contains the order interval $[0,\omega]$, then by considering the utility functions restricted to F, it is not difficult to see that an allocation is an Edgeworth equilibrium with respect to E if and only if it is an Edgeworth equilibrium with respect to F.

Do Edgeworth equilibria exist in the infinite dimensional case?

The next theorem presents an affirmative answer to this question.

THEOREM 4.4. *If the order interval $[0,\omega]$ is weakly compact, then the set of all Edgeworth equilibria is non-empty and weakly compact.*

PROOF. This proof is due to Debreu and Scarf [7]. Let $\mathcal{E}_n$ be the economy which is the n-fold replica of our original economy $\mathcal{E}$. As usual, an allocation $(x_1,\ldots,x_m)$ of $\mathcal{E}$ will be considered as an allocation on $\mathcal{E}_n$ by assigning the bundle x_i to each consumer of type i. For each n, let

$$C_n = \{(x_1,\ldots,x_m) \in \mathcal{A}\colon\ (x_1,\ldots,x_m) \in \mathrm{Core}(\mathcal{E}_n)\}.$$

The sets C_n have the following properties.

1. Each C_n is non-empty.

Note first that in $\mathcal{E}_n$ the consumers' characteristics satisfy all assumptions of Theorem 4.2, and hence $\mathrm{Core}(\mathcal{E}_n) \neq \emptyset$. Let $(x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},\ldots,x_{m1},\ldots,x_{mn})$ be a core allocation for $\mathcal{E}_n$. Then we claim that

$$x_{ij} \sim_i x_{ik} \quad \text{for } j,k=1,\ldots,n \text{ and } i=1,\ldots,m,$$

i.e., no consumer prefers his bundle to that of another consumer of the same type.

To see this, note first that (by rearranging the consumers of each type), we can suppose that $x_{ij} \succsim_i x_{i1}$ holds for all i and j. Put

$$y_{i1} = \frac{1}{n}\sum_{j=1}^{n} x_{ij}\,, \quad i=1,\ldots,m.$$

Then $\sum_{i=1}^{m} y_{i1} = \omega$, and by the weak convexity we have $y_{i1} \succsim_i x_{i1}$ for each i. Now assume by way of contradiction that there exists some k $(1 \leqslant k \leqslant m)$ and some r $(1 \leqslant r \leqslant n)$ with $x_{kr} \succ_k x_{k1}$. The latter, in view of the convexity of $\succsim_k$, implies $y_{k1} \succ_k x_{k1}$. Now if each consumer $(i,1)$ gets the bundle y_{i1}, then the coalition $\{(i,1)\colon i=1,\ldots,m\}$ blocks the original core allocation, which is impossible. This contradiction establishes the validity of our claim.

Next, note that by the quasi-concavity of the utility functions we have $y_{i1} \succsim_i x_{ij}$ for $j=1,\ldots,n$. An easy argument now shows that $(y_{11},y_{21},\ldots,y_{m1}) \in C_n$, and thus C_n is non-empty.

2. Each C_n is weakly closed.

Let the net $\{(x_1^\alpha,\ldots,x_m^\alpha)\}$ of C_n satisfy $(x_1^\alpha,\ldots,x_m^\alpha) \xrightarrow{w} (x_1,\ldots,x_m)$ in E^m. By Theorem 4.2 we know that $\mathrm{Core}(\mathcal{E}_n)$ is a weakly compact subset of E^{mn}, and so if we consider each $(x_1^\alpha,\ldots,x_m^\alpha)$ in $\mathrm{Core}(\mathcal{E}_n)$, it follows easily that $(x_1,\ldots,x_m)$ must be also in $\mathrm{Core}(\mathcal{E}_n)$. That is, $(x_1,\ldots,x_m) \in C_n$, and so C_n is a weakly closed set.

3. For each n we have $C_{n+1} \subseteq C_n$.

Since $\mathcal{A}$ is a weakly compact set and the sequence $\{C_n\}$ has the finite intersection property, it follows that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$. To complete the proof, note that the set of Edgeworth equilibrium allocations is precisely $\bigcap_{n=1}^{\infty} C_n$. ■

The next lemma regarding locally convex-solid Riesz spaces will be needed later.

LEMMA 4.5. *Let (E,τ) be a locally convex-solid Riesz space, and let two nets $\{x_\alpha\}$ and $\{y_\alpha\}$ satisfy $0 \leqslant x_\alpha \leqslant y_\alpha + x$ for all α and some $x \in E^+$. If $y_\alpha \xrightarrow{\tau} 0$ and $[0,x]$ is weakly compact, then the net $\{x_\alpha\}$ has a weakly convergent subnet.*

PROOF. Since $y_\alpha \xrightarrow{\tau} 0$ implies $y_\alpha^+ \xrightarrow{\tau} 0$, replacing $\{y_\alpha\}$ by $\{y_\alpha^+\}$, we can assume that $y_\alpha \geqslant 0$ holds for all α. By the Riesz decomposition property we can write $x_\alpha = w_\alpha + v_\alpha$ with $0 \leqslant w_\alpha \leqslant y_\alpha$ and $0 \leqslant v_\alpha \leqslant x$ for all α. Then $w_\alpha \xrightarrow{\tau} 0$, and since $[0,x]$ is weakly compact, we see that $\{v_\alpha\}$ has a weakly convergent subnet. Therefore, $\{x_\alpha\}$ has a weakly convergent subnet.■

Let $(x_1,\ldots,x_m)$ be an allocation. Following Peleg and Yaari [10], we define

$$F_i = \{x \in E^+ : x \succ_i x_i\}, \text{ and}$$

$$G_i = F_i - \omega_i = \{x \in E : x + \omega_i \succ_i x_i\}.$$

Clearly, F_i and G_i are convex and weakly closed sets. We denote by G the convex hull of $\bigcup_{i=1}^m G_i$, i.e.,

$$G = co(\bigcup_{i=1}^m G_i) = \{\sum_{i=1}^m \lambda_i y_i : \lambda_i \geqslant 0,\ \sum_{i=1}^m \lambda_i = 1 \text{ and } y_i \in G_i\}.$$

A basic property of the convex set G is described in the next result. This property was proved first by Peleg and Yaari for the $\mathbb{R}_\infty$ case; see Lemmas 4.1-4.3 in [10].

THEOREM 4.6. *If $(x_1,\ldots,x_m)$ is an Edgeworth equilibrium and $[0,\omega]$ is weakly compact, then for each $\varepsilon > 0$ we have $0 \notin \varepsilon\omega + \overline{G} = \overline{\varepsilon\omega + G}$ (where the bar denotes weak closure).*

PROOF. Let $\varepsilon > 0$ be fixed, and assume by way of contradiction that $0 \in \varepsilon\omega + \overline{G}$.

Since G is convex, we see that zero belongs to the τ-closure of $\varepsilon\omega + G$. Thus, there exists a net $\{y_\alpha\}$ of G with $\varepsilon\omega + y_\alpha \xrightarrow{\tau} 0$. Write $y_\alpha = \sum_{i=1}^m \lambda_i^\alpha y_i^\alpha$ with $y_i^\alpha \in G_i$, $\lambda_i^\alpha \geqslant 0$ $(i = 1,\ldots,m)$ and $\sum_{i=1}^m \lambda_i^\alpha = 1$. By passing to appropriate subnets, we can assume without loss of generality that $\lambda_i^\alpha \xrightarrow[\alpha]{} \lambda_i \geqslant 0$ holds in $\mathbb{R}$ for each i. Clearly, $\sum_{i=1}^m \lambda_i = 1$ holds. Put $S = \{j : \lambda_j > 0\}$, and note that $S \neq \emptyset$. Also, we can suppose that there exists some $\lambda > 0$ satisfying $\lambda_j^\alpha > \lambda$ for $j \in S$ and all α.

Let $v_i^\alpha = y_i^\alpha + \omega_i$. Then clearly $v_i^\alpha \in F_i$, and so $v_i^\alpha \geqslant 0$. We claim that for each $j \in S$ the net $\{v_j^\alpha\}$ has a weakly convergent subnet in E. To see this, fix $j \in S$ and note that

$$0 \leqslant \lambda_j^\alpha v_j^\alpha \leqslant \sum_{i=1}^m \lambda_i^\alpha v_i^\alpha = \sum_{i=1}^m \lambda_i^\alpha (y_i^\alpha + \omega_i) = \sum_{i=1}^m \lambda_i^\alpha y_i^\alpha + \sum_{i=1}^m \lambda_i^\alpha \omega_i$$

$$= y_\alpha + \sum_{i=1}^m \lambda_i^\alpha \omega_i \leqslant y_\alpha + \omega \leqslant y_\alpha + \varepsilon\omega + \omega\ .$$

Hence,

$$0 \leqslant v_j^\alpha \leqslant \frac{1}{\lambda_j^\alpha}(y_\alpha + \varepsilon\omega) + \frac{1}{\lambda_j^\alpha}\omega \leqslant \frac{1}{\lambda_j^\alpha}(y_\alpha + \varepsilon\omega) + \frac{1}{\lambda}\omega$$

holds in E, and so by Lemma 4.5 the net $\{v_j^\alpha\}$ has a weakly convergent subnet. Therefore, by passing to appropriate subnets, we can assume that $v_j^\alpha \xrightarrow{w} v_j$ holds in E for each $j \in S$. Since each F_j is weakly closed, we infer that $v_j \in F_j$ (i.e., $v_j \succcurlyeq_j x_j$) for each $j \in S$. From

$$y_\alpha = \sum_{i=1}^m \lambda_i^\alpha(v_i^\alpha - \omega_i) \geqslant \sum_{j\in S} \lambda_j^\alpha v_j^\alpha - \sum_{i=1}^m \lambda_i^\alpha \omega_i \xrightarrow{w} \sum_{j\in S} \lambda_j v_j - \sum_{j\in S} \lambda_j \omega_j$$

(and the fact that E^+ is weakly closed) we infer that

$$-\varepsilon\omega = \text{w-lim}\, y_\alpha \geqslant \sum_{j\in S} \lambda_j v_j - \sum_{j\in S} \lambda_j \omega_j .$$

Therefore,

$$\sum_{j\in S} \lambda_j(v_j + \varepsilon\omega) \leqslant \sum_{j\in S} \lambda_j \omega_j . \qquad (\star)$$

Now if n is a positive integer denote by n_j the smallest integer greater or equal than $n\lambda_j$ (i.e., $0 \leqslant n_j - n\lambda_j \leqslant 1$). Since $\lim_{n\to\infty} n\lambda_j/n_j = 1$ and $v_j + \varepsilon\omega \succ_j x_j$ for each $j \in S$, we can choose (by the continuity) n large enough so that

$$z_j = \frac{n\lambda_j}{n_j}(v_j + \varepsilon\omega) \succ_j x_j,\ j \in S . \qquad (\star\star)$$

Taking into account $(\star)$, we get

$$\sum_{j\in S} n_j z_j = \sum_{j\in S} n\lambda_j(v_j + \varepsilon\omega) \leqslant \sum_{j\in S} n\lambda_j \omega_j \leqslant \sum_{j\in S} n_j \omega_j .$$

The preceding inequality, coupled with $(\star\star)$, shows that $(x_1,\ldots,x_m)$ can be blocked by an allocation in some replication of the economy, which is impossible. Hence $0 \notin \varepsilon\omega + \bar{G}$ must hold, as desired.■

In order to characterize an Edgeworth equilibrium we need to introduce a weaker notion of Walrasian equilibrium.

DEFINITION 4.7. *An allocation* $(x_1,\dots,x_m)$ *is said to be an* ε-*Walrasian equilibrium whenever for each* $\varepsilon > 0$ *there exists some price* $p \neq 0$ *(depending upon* ε*) such that:*

1. $p\cdot\omega = 1$; *and*
2. *If* $x \in E^+$ *satisfies* $x \succ_i x_i$ *for some* i, *then* $p\cdot x \geqslant p\cdot\omega_i - \varepsilon$.

Observe that if a price p satisfies $p\cdot x \geqslant p\cdot\omega_i - \varepsilon$ whenever $x \succ_i x_i$, then $p \geqslant 0$ holds. Indeed, if $y \geqslant 0$, then $x_1 + \delta^{-1}y \succ_1 x_1$ holds for all $\delta > 0$, and so $p\cdot x_1 + \delta^{-1}p\cdot y \geqslant p\cdot\omega_1 - \varepsilon$ for all $\delta > 0$. Thus, $p\cdot y \geqslant \delta[p\cdot(\omega_1 - x_1) - \varepsilon]$ holds for all $\delta > 0$, from which it follows that $p\cdot y \geqslant 0$. In particular, note that the prices p in Definition 4.7 are positive prices.

Clearly, every Walrasian equilibrium is an ε-Walrasian equilibrium. On the other hand, as we shall see next, every Edgeworth equilibrium is an ε-Walrasian equilibrium.

THEOREM 4.8. *If* $[0,\omega]$ *is weakly compact, then every Edgeworth equilibrium is an* ε-*Walrasian equilibrium.*

PROOF. Let $(x_1,\dots,x_m)$ be an Edgeworth equilibrium and let $\varepsilon > 0$. Then, by Theorem 4.6, $0 \notin \varepsilon\omega + \bar{G}$. Since $\varepsilon\omega + \bar{G}$ is a weakly closed convex set, it follows from the classical separation theorem (see, for example, [3, Thm 9.12, p. 136]) that there exists some $p \in E'$ satisfying

$$p\cdot(y + \varepsilon\omega) > 0 \quad \text{for all} \quad y \in G. \qquad (\star)$$

Clearly, $p \geqslant 0$ and we claim that $p\cdot\omega > 0$. Indeed, from $x_i - \omega_i \in G_i$, we see that $0 = \sum_{i=1}^{m} \frac{1}{m}(x_i - \omega_i) \in G$, and so $p\cdot\omega = \frac{1}{\varepsilon}p\cdot(0 + \varepsilon\omega) > 0$.

Replacing p by $p/p\cdot\omega$, we can assume that $(\star)$ holds for a price $p \geqslant 0$ with $p\cdot\omega = 1$. Finally, we show that $(x_1,\dots,x_m)$ is an ε-Walrasian equilibrium. To this end, let $y \in E^+$ satisfy $y \succ_i x_i$ for some i. Then $y - \omega_i \in G_i \subseteq G$, and so from $(\star)$ we get $p\cdot(y - \omega_i + \varepsilon\omega) > 0$. This implies

$$p\cdot y \geqslant p\cdot\omega_i - \varepsilon,$$

and the proof of the theorem is finished.■

Now let us take a closer look at the ideal generated by ω. Recall that

$$A_\omega = \{x \in E: \exists\ \lambda > 0 \text{ with } |x| \leq \lambda\omega\}.$$

On A_ω a lattice norm is defined by the formula

$$\|x\|_\infty = \inf\{\lambda > 0: |x| \leq \lambda\omega\},\ x \in A_\omega .$$

Thus, A_ω with the topology generated by the $\|\cdot\|_\infty$-norm is a locally convex-solid Riesz space. In case A_ω is a Dedekind complete Riesz space[5], $(A_\omega, \|\cdot\|_\infty)$ is in fact a Banach lattice[6]. On the other hand, for each $x \in A_\omega$ we have

$$|x| \leq \|x\|_\infty \cdot \omega . \qquad (\star\star)$$

This implies that the $\|\cdot\|_\infty$-closed unit ball of A_ω coincides with the order interval $[-\omega, \omega]$. Therefore, if A'_ω is the norm dual of $(A_\omega, \|\cdot\|_\infty)$, then $A'_\omega = A^\sim_\omega$ holds. In particular, the restrictions of the functionals of E' to A_ω belong to A'_ω.

If q is a lattice seminorm on E, then from $(\star\star)$ it follows that $q(x) \leq q(\omega)\|x\|_\infty$ holds for all $x \in A_\omega$. In other words, every lattice seminorm of E is $\|\cdot\|_\infty$-continuous on A_ω. In particular, each utility function is $\|\cdot\|_\infty$-continuous on A_ω. Therefore, the utility functions define an economy on the Riesz dual system $\langle A_\omega, A'_\omega \rangle$ with the original endowments.

Summarizing the above discussion: *The Riesz dual system $\langle A_\omega, A'_\omega \rangle$ with the same preferences and endowments defines a new economy.*

The importance of the economy with respect to the Riesz dual system $\langle A_\omega, A'_\omega \rangle$ will become apparent in the next theorems.

THEOREM 4.9. *If $(x_1, \ldots, x_m)$ is an ε-Walrasian equilibrium for an economy, then there exists a strictly positive functional p on A_ω such that:*

1. $p \cdot \omega = 1$ *; and*
2. $0 \leq y \in A_\omega$ *and* $y \succ_i x_i$ *imply* $p \cdot y \geq p \cdot \omega_i$ *.*

In other words, every ε-Walrasian equilibrium is a Walrasian equilibrium with respect to the economy whose preferences and endowments are the same and with a commodity-price duality defined by the Riesz dual system $\langle A_\omega, A'_\omega \rangle$.

---

[5] If either E is Dedekind complete or $[0, \omega]$ is weakly compact, then A_ω is a a Dedekind complete Riesz space.

[6] In general, the ideal A_ω is a Banach lattice under the $\|\cdot\|_\infty$-norm if and only if A_ω is a uniformly complete Riesz space; see [3, Exercise 12, p. 198].

PROOF. Let $(x_1,\ldots,x_m)$ be an ε-Walrasian equilibrium. For each n pick some $0 \leqslant p_n \in E'$ with $p_n \cdot \omega = 1$ and such that

$$y \succ_i x_i \ \text{ in } \ E^+ \ \text{ implies } \ p_n \cdot y \geqslant p_n \cdot \omega_i - \frac{1}{n}\,. \qquad (\star)$$

The condition $p_n \cdot \omega = 1$ tells us that each p_n restricted to $(A_\omega, \|\cdot\|_\infty)$ has norm one. Let p be $\sigma(A'_\omega, A_\omega)$-accumulation point of $\{p_n\}$ (where each p_n is now considered restricted to A_ω). Then $p \cdot \omega = 1$, and from $(\star)$ we see that

$$0 \leqslant y \in A_\omega \quad \text{and} \quad y \succ_i x_i \quad \text{imply} \quad p \cdot y \geqslant p \cdot \omega_i .$$

By Theorem 3.3, it follows that $(x_1,\ldots,x_m)$ is a Walrasian equilibrium for the economy whose preferences and endowments are the original ones and whose Riesz dual system is $\langle A_\omega, A'_\omega \rangle$. ■

The next result describes two basic properties of any price that supports a quasiequilibrium on A_ω.

THEOREM 4.10. *Let $(x_1,\ldots,x_m)$ be a quasiequilibrium allocation on A_ω supported by a price $0 \leqslant p \in A'_\omega$. Then:*

1. *p is τ-continuous on $[0,\omega]$; and*
2. *If $[0,\omega]$ is weakly compact, then p is order continuous on A_ω, i.e., $v_\alpha \downarrow 0$ in A_ω implies $p(v_\alpha) \downarrow 0$ in $\mathbb{R}$.*

PROOF. (1) Assume that a net $\{y_\alpha\}$ satisfies $0 \leqslant y_\alpha \leqslant x_i$ and $y_\alpha \xrightarrow{\tau} 0$. Fix $\varepsilon > 0$. From $x_i - y_\alpha + \varepsilon\omega \xrightarrow{\tau} x_i + \varepsilon\omega \succ_i x_i$, we see that there exists some α_0 such that $x_i - y_\alpha + \varepsilon\omega \succ_i x_i$ holds for all $\alpha \geqslant \alpha_0$. Therefore,

$$p(x_i) - p(y_\alpha) + \varepsilon p(\omega) \geqslant p(\omega_i) = p(x_i)$$

holds for all $\alpha \geqslant \alpha_0$, and so $0 \leqslant p(y_\alpha) \leqslant \varepsilon p(\omega)$ for all $\alpha \geqslant \alpha_0$, i.e., $\lim p(y_\alpha) = 0$.

Next, suppose that $0 \leqslant y_\alpha \leqslant \omega = \sum_{i=1}^{m} x_i$ satisfies $y_\alpha \xrightarrow{\tau} 0$. By the Riesz decomposition property we can write $y_\alpha = y_\alpha^1 + \cdots + y_\alpha^m$ with $0 \leqslant y_\alpha^i \leqslant x_i$. From $0 \leqslant y_\alpha^i \leqslant y_\alpha$, we see that $y_\alpha^i \xrightarrow{\tau} 0$ for each i. Thus,

$$\lim p(y_\alpha) = \lim \sum_{i=1}^{m} p(y_\alpha^i) = 0.$$

Now let $\{y_\alpha\} \subseteq [0,\omega]$ satisfy $y_\alpha \xrightarrow{\tau} y$. Then $(y_\alpha - y)^+ \xrightarrow{\tau} 0$ and $(y_\alpha - y)^- \xrightarrow{\tau} 0$ hold, and so

$$p(y_\alpha) - p(y) = p(y_\alpha - y) = p[(y_\alpha - y)^+] - p[(y_\alpha - y)^-] \longrightarrow 0\,.$$

(2) Let $v_\alpha \downarrow 0$ hold in A_ω. Without loss of generality we can assume that $0 \leq v_\alpha \leq \omega$ holds for all α. Since $[0,\omega]$ is weakly compact, the net $\{v_\alpha\}$ has a weakly convergent subnet, and from $v_\alpha \downarrow 0$, it follows that $v_\alpha \xrightarrow{w} 0$. The latter implies $v_\alpha \xrightarrow{\tau} 0$; see [2, Thm 9.8, p. 63]. Finally, by part (1) we see that $p(v_\alpha) \downarrow 0$ holds in $\mathbb{R}$. ■

For our next discussion we shall use some of the properties of the weak closure $\overline{A_\omega}$ of A_ω. Recall that the band generated by ω is the ideal given by

$$B_\omega = \{x \in E: \ |x| \wedge n\omega \uparrow |x|\}.$$

Clearly, $A_\omega \subseteq B_\omega$. Since every band is τ-closed [2, Thm 5.6, p. 35], it follows that $\overline{A_\omega} \subseteq B_\omega$. Of course, $\overline{A_\omega}$ is also the τ-closure of A_ω, which implies that $\overline{A_\omega}$ is also an ideal. The ideal $\overline{A_\omega}$ is the set

$$\overline{A_\omega} = \{x \in E: \ |x| \wedge n\omega \xrightarrow{\tau} |x|\} .$$

If τ is order continuous, then clearly $\overline{A_\omega} = B_\omega$. On the other hand, in general $\overline{A_\omega}$ does not coincide with B_ω. For instance, in $\langle \ell_\infty, \ell_\infty' \rangle$ if $\omega = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, then $\overline{A_\omega} \subseteq c_0 \neq \ell_\infty = B_\omega$ and the sup norm is order continuous on $\overline{A_\omega}$. Two basic properties dealing with the weak compactness of the order interval $[0,\omega]$ are the following.

a. *If* $[0,\omega]$ *is weakly compact, then* τ *is order continuous on* $\overline{A_\omega}$.

b. *If* E *is Dedekind complete and* τ *is order continuous on* $\overline{A_\omega}$*, then* $[0,\omega]$ *is weakly compact.*

To see (a), assume that $[0,\omega]$ is weakly compact. Let $x_\alpha \downarrow 0$ in $\overline{A_\omega}$, and let V and W be two solid τ-neighborhoods of zero satisfying $W + W \subseteq V$. We can assume that there exists some $0 \leq x \in \overline{A_\omega}$ with $0 \leq x_\alpha \leq x$ for all α. Pick some $0 \leq y \in A_\omega$ with $x - y \in W$, and note that $[0,y]$ is weakly compact. Since $x_\alpha \wedge y \downarrow 0$ holds in $[0,y]$, it follows that $x_\alpha \wedge y \xrightarrow{w} 0$, and so $x_\alpha \wedge y \xrightarrow{\tau} 0$, see [2, Thm 9.8, p. 63]. Pick some α_0 with $x_\alpha \wedge y \in W$ for all $\alpha \geq \alpha_0$. If $\alpha \geq \alpha_0$, then we have

$$0 \leq x_\alpha = (x_\alpha - y)^+ + x_\alpha \wedge y \leq (x - y)^+ + x_\alpha \wedge y \in W + W \subseteq V,$$

and so $x_\alpha \in V$ for all $\alpha \geq \alpha_0$. Therefore, $x_\alpha \xrightarrow{\tau} 0$ so that τ is order continuous on $\overline{A_\omega}$.

To see (b), let E be Dedekind complete and let τ be order continuous on $\overline{A_\omega}$. Let $(\overline{A_\omega})'$ denote $(\overline{A_\omega}, \tau)'$ and consider $\overline{A_\omega}$ equipped with the topology τ.

The Dedekind completeness of E implies that $\overline{A_\omega}$ (as an ideal of E) is Dedekind complete, and so by Theorem 2.1 the Riesz dual system $\langle \overline{A_\omega}, (\overline{A_\omega})' \rangle$ is symmetric. In particular, (by Theorem 2.1 again) $[0,\omega]$ is $\sigma(\overline{A_\omega}, (\overline{A_\omega})')$-compact. Since $(\overline{A_\omega})'$ consists precisely of the restrictions of the functionals of E' to A_ω, it follows that $[0,\omega]$ is $\sigma(E,E')$-compact.

To continue our discussion we need to introduce the concept of an extended Walrasian equilibrium.

DEFINITION 4.11. *An allocation* $(x_1,\ldots,x_m)$ *is said to be an extended Walrasian equilibrium whenever there exists a function* $\pi\colon(\overline{A_\omega})^+ \longrightarrow [0,\infty]$ *(called an extended price supporting the allocation) such that:*

1. $\pi(\omega) = 1$;
2. π *is additive, i.e.,* $\pi(x+y) = \pi(x) + \pi(y)$ *holds for all* $x,y \in (\overline{A_\omega})^+$; *and*
3. $y \succ_i x_i$ *in* $(\overline{A_\omega})^+$ *implies* $\pi(y) \geqslant \pi(\omega_i)$.

Let $(x_1,\ldots,x_m)$ be an allocation supported by an extended price π. Then we have the following properties.

1. π *is monotone, i.e.,* $0 \leqslant y \leqslant x$ *implies* $\pi(y) \leqslant \pi(x)$.

   Indeed, if $0 \leqslant y \leqslant x$ holds, then from $x = (x-y) + y$ and the additivity of π, we see that

$$\pi(x) = \pi(x-y) + \pi(y) \geqslant \pi(y).$$

2. π *is finite on* A_ω^+, *i.e.,* $\pi(x) < \infty$ *holds for all* $x \in A_\omega^+$.

   If $x \in A_\omega^+$, then pick some positive integer n with $0 \leqslant x \leqslant n\omega$, and note that by (1) and the additivity of π we have

$$\pi(x) \leqslant \pi(n\omega) = \pi(\omega + \cdots + \omega) = \pi(\omega) + \cdots + \pi(\omega) = n\pi(\omega) = n < \infty .$$

3. π *defines a positive linear functional on* A_ω.

   This follows from a classical result of L. V. Kantorovič; see [3, Thm 1.7, p. 7]. The formula defining π on A_ω is given by

$$\pi(x) = \pi(x^+) - \pi(x^-),\ x \in A_\omega.$$

The extended Walrasian equilibria and the Walrasian equilibria on the ideal A_ω are related as follows.

THEOREM 4.12. *An allocation* $(x_1,\ldots,x_m)$ *is a Walrasian equilibrium on* A_ω *if and only if it is an extended Walrasian equilibrium.*

Moreover, in this case, if $0 \leqslant p \in A'_\omega$ *supports the allocation, then the formula*

$$\pi(x) = \sup\{p(x \wedge n\omega):\ n = 1,2,\ldots\},\ x \in (\overline{A_\omega})^+,$$

defines an extended price supporting the allocation.

PROOF. Assume that $(x_1,\ldots,x_m)$ is a Walrasian equilibrium on A_ω supported by $0 \leqslant p \in A'_\omega$ (where, of course, $p\cdot\omega = 1$). Consider the formula defining $\pi(x)$ and note that $\pi(x) = p(x)$ holds for all $x \in A^+_\omega$, and so $\pi(\omega) = p\cdot\omega = 1$.

To see that π is additive, let $x,y \in (\overline{A_\omega})^+$. From $(x+y)\wedge n\omega \leqslant x \wedge n\omega + y \wedge n\omega$, we see that

$$p((x+y)\wedge n\omega) \leqslant p(x\wedge n\omega) + p(y\wedge n\omega) \leqslant \pi(x) + \pi(y),$$

and so

$$\pi(x+y) \leqslant \pi(x) + \pi(y).$$

On the other hand, the inequality $x\wedge n\omega + y\wedge m\omega \leqslant (x+y)\wedge(n+m)\omega$ implies $p(x\wedge n\omega) + p(y\wedge m\omega) \leqslant p[(x+y)\wedge(n+m)\omega] \leqslant \pi(x+y)$, from which it follows that

$$\pi(x) + \pi(y) \leqslant \pi(x+y).$$

Hence $\pi(x+y) = \pi(x) + \pi(y)$ holds.

Now let $y \succ_i x_i$ in $(\overline{A_\omega})^+$, and let $\varepsilon > 0$. Then $y + \varepsilon\omega \succ_i x_i$ holds. Since $y\wedge n\omega + \varepsilon\omega \xrightarrow{\tau} y + \varepsilon\omega$, there exists some k with $y\wedge k\omega + \varepsilon\omega \succ_i x_i$. It follows that

$$\pi(y) + \varepsilon \geqslant p(y\wedge k\omega) + \varepsilon = p(y\wedge k\omega + \varepsilon\omega) \geqslant p(\omega_i) = \pi(\omega_i)$$

holds. Since $\varepsilon > 0$ is arbitrary, we see that $\pi(y) \geqslant \pi(\omega_i)$.

Conversely, assume that the formula for $\pi(x)$ is an extended price supporting the allocation $(x_1,\ldots,x_m)$. Then π defines (by the formula $\pi(x) = \pi(x^+) - \pi(x^-)$) a finite positive linear functional on A_ω with $\pi(\omega) = 1$ and such that

$$x \succ_i x_i \text{ in } A^+_\omega \text{ implies } \pi(x) \geqslant \pi(\omega_i).$$

Since preferences are $\|\cdot\|_\infty$-continuous on A_ω, it follows from Theorem 3.3 that $(x_1,\ldots,x_m)$ is a Walrasian equilibrium on A_ω supported by π (considered restricted to A_ω).■

Our next major result shows that the notions of Edgeworth equilibrium, ε-Walrasian equilibrium and extended Walrasian equilibrium coincide.

THEOREM 4.13. *If* $[0,\omega]$ *is weakly compact, then for an allocation* $(x_1,\ldots,x_m)$ *the following statements are equivalent.*

1. *The allocation is an Edgeworth equilibrium.*
2. *The allocation is an ε-Walrasian equilibrium.*
3. *The allocation is a Walrasian equilibrium for an economy with the same preferences and endowments and with the commodity-price duality defined by the Riesz dual system* $\langle A_\omega, A_\omega' \rangle$.
4. *The allocation is an extended Walrasian equilibrium.*

PROOF. (1) $\Longrightarrow$ (2) This is Theorem 4.8.

(2) $\Longrightarrow$ (3) This follows from Theorem 4.9.

(3) $\Longrightarrow$ (4) This follows immediately from Theorem 4.12.

(4) $\Longrightarrow$ (1) Let $\pi\colon(\overline{A_\omega})^+ \longrightarrow [0,\infty]$ be an extended price that supports the allocation $(x_1,\ldots,x_m)$. From $\pi(\omega)=1$ and the monotonicity of π, we see that π is finite on A_ω^+, and hence it extends uniquely to a positive linear functional on A_ω; see [3, Thm 1.7, p. 7].

Now by Theorem 3.3 the allocation $(x_1,\ldots,x_m)$ is a Walrasian equilibrium for an economy whose Riesz dual system is $\langle A_\omega, A_\omega' \rangle$ and has the same preferences and endowments. It follows that $(x_1,\ldots,x_m)$ is in the core of every n-fold replica of the economy with Riesz dual system $\langle A_\omega, A_\omega' \rangle$. Since the allocations in E coincide with those in A_ω, the latter shows that $(x_1,\ldots,x_m)$ is in the core of every n-fold replica of the economy with Riesz dual system $\langle E,E' \rangle$. That is, $(x_1,\ldots,x_m)$ is an Edgeworth equilibrium.■

If the Riesz dual system $\langle E,E' \rangle$ is symmetric, then all order intervals of E are weakly compact, and so, in this case, our results hold true for any initial endowment ω. On the other hand, for an arbitrary Riesz dual system $\langle E,E' \rangle$, for some $\omega \in E^+$ the order intervals $[0,\omega]$ are weakly compact and for some they are not weakly compact. For example, in the Riesz dual system $\langle \ell_\infty, \ell_\infty' \rangle$ (ℓ_∞' the norm dual of ℓ_∞) if $\omega=(\alpha_1,\alpha_2,\ldots)$ satisfies $\lim\alpha_n=0$, then $[0,\omega]$ is weakly compact (in fact norm compact), while if $\omega=(1,1,1,\ldots)$, then $[0,\omega]$ is not weakly compact. As an example of the other extreme, the Riesz dual system $\langle L_\infty, L_\infty' \rangle$ over $[0,1]$ (L_∞' the norm dual of L_∞) does not have any weakly compact order intervals. (Reason: Let $0<\omega\in L_\infty$. Pick a measurable set A of positive measure and some $\varepsilon>0$ with $\omega \geqslant \varepsilon\chi_A$, and then select a sequence $\{A_n\}$ of measurable subsets of A of positive measure with $A_n \downarrow \emptyset$. If $v_n = \varepsilon\chi_{A_n} \in [0,\omega]$, then $v_n \downarrow 0$ and $\|v_n\|_\infty = \varepsilon$ for each n. In view of [2, Thm 9.8, p. 63], we see that $\{v_n\}$ does not have any weakly convergent subsequence, and hence $[0,\omega]$ is not weakly compact.)

If $[0,\omega]$ is not weakly compact, then there is no guarantee that Edgeworth equilibria exist. However, in this case we have the following companion of Theorem 4.13.

THEOREM 4.14. *For an allocation* $(x_1,\ldots,x_m)$ *the following statements are equivalent.*

1. *The allocation is an Edgeworth equilibrium.*
2. *The allocation is an Edgeworth equilibrium with respect to* $\langle A_\omega, A'_\omega \rangle$.
3. *The allocation is an* ε*-Walrasian equilibrium with respect to* $\langle A_\omega, A'_\omega \rangle$.
4. *The allocation is a Walrasian equilibrium with respect to* $\langle A_\omega, A'_\omega \rangle$.
5. *The allocation is an extended Walrasian equilibrium.*

PROOF. The implications $(1) \Longrightarrow (2)$ and $(3) \Longrightarrow (4) \Longrightarrow (5) \Longrightarrow (1)$ have already be proven.

$(2) \Longrightarrow (3)$ Let $\varepsilon > 0$ be fixed. Put $F_i = \{x \in A_\omega^+ : x \succ_i x_i\}$, $G_i = F_i - \omega_i$, and let G be the convex hull of $\bigcup_{i=1}^{m} G_i$. Pick some n with $n\omega > x_1$ and note that by the strong monotonicity we have $n\omega \succ_1 x_1$. Since $n\omega$ is an interior point of A_ω^+ and F_1 is open relative to A_ω^+, we see that F_1 has $\|\cdot\|_\infty$-interior points. From $\varepsilon\omega + F_1 - \omega_1 \subseteq \varepsilon\omega + G$, we infer that $\varepsilon\omega + G$ has $\|\cdot\|_\infty$-interior points.

Since $(x_1,\ldots,x_m)$ is an Edgeworth equilibrium, it follows from [7] that $0 \notin \varepsilon\omega + G$, and so by the separation theorem (see, for example, [3, Thm 9.9, p. 135]) there exists a non-zero price $p \in A'_\omega$ satisfying $p\cdot(\varepsilon\omega+g) \geqslant 0$ for all $g \in G$. Clearly, $p \geqslant 0$. On the other hand, $p\cdot\omega = 0$ implies $p = 0$, which is a contradiction. Thus $p\cdot\omega > 0$, and so replacing p by $p/p\cdot\omega$ we can assume that $p\cdot\omega = 1$. Now if $x \succ_i x_i$ holds in A^+_ω, then $x - \omega_i \in G$, and so $p\cdot(\varepsilon\omega + x - \omega_i) \geqslant 0$, from which it follows that

$$p\cdot x \geqslant p\cdot\omega_i - \varepsilon .$$

Therefore, $(x_1,\ldots,x_m)$ is an ε-Walrasian equilibrium with respect to the Riesz dual system $\langle A_\omega, A'_\omega\rangle$.■

In order to prove the existence of quasiequilibria on E, we need to impose an additional continuity condition on preferences. This condition, known as properness, was introduced by A. Mas-Colell in [8].

DEFINITION 4.15. *Let $\langle E,E'\rangle$ be a Riesz dual system, let $\succ$ be a preference relation, and let τ be a locally convex-solid topology on E consistent with $\langle E,E'\rangle$.*

1. *The preference relation $\succ$ is said to be τ-proper at $x \in E^+$ whenever there exist a τ-neighborhood V of zero and some $v > 0$ such that*

$$x - \alpha v + z \succ x \text{ in } E^+ \text{ with } \alpha > 0 \text{ imply } z \notin \alpha V.$$

2. *The preference relation $\succ$ is said to be τ-proper whenever $\succ$ is τ-proper at every point $x \in E^+$ and V and $v > 0$ can be chosen independently of x, i.e., whenever there exist a τ-neighborhood V of zero and some $v > 0$ such that*

$$x - \alpha v + z \succ x \text{ in } E^+ \text{ with } \alpha > 0 \text{ imply } z \notin \alpha V.$$

If $\langle E,E'\rangle$ is the Riesz dual system for an economy, then by saying that the preferences are proper we shall mean that they are τ-proper for the consistent locally convex-solid topology τ on E for which all utility functions are τ-continuous.

The main result of A. Mas-Colell in [8] asserts that if the preferences are proper, then quasiequilibria exist. We shall prove a much stronger result; namely, that every Edgeworth equilibrium is a quasiequilibrium.

THEOREM 4.16. *If preferences are proper, then every Edgeworth equilibrium is a quasiequilibrium.*

In particular, if the order interval $[0,\omega]$ *is weakly compact, then the economy has quasiequilibria.*

PROOF. Let $(x_1,\ldots,x_m)$ be an Edgeworth equilibrium, and let G be the (non-empty) convex set associated with $(x_1,\ldots,x_m)$ as defined before Theorem 4.6. The proof below is an adaptation of A. Mas-Colell's proof of Proposition 7.1 in [8].

By properness, for each i there exists a convex, solid, open τ-neighborhood V_i of zero and some $v_i > 0$ such that $x - \alpha v_i + z \succ_i x$ in E^+ with $\alpha > 0$ imply $z \notin \alpha V_i$. Put $V = \bigcap_{i=1}^{m} V_i$ and $v = v_1 + \cdots + v_m$.

Next, consider the set

$$\Gamma = \{\alpha w\colon\ \alpha > 0 \ \text{ and } \ w \in E \ \text{ satisfies } \ w + v \in \tfrac{1}{2}V\},$$

and note that Γ is a non-empty, convex, open cone of E. We claim that $\Gamma \cap G = \emptyset$.

To see this, assume by way of contradiction that $\Gamma \cap G \neq \emptyset$. Let $\theta \in \Gamma \cap G$. Write

$$\theta = \sum_{i=1}^{m} \lambda_i(z_i - \omega_i),\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1,\ z_i \geq 0 \ \text{ and } \ z_i \succ_i x_i;\ \text{and}$$

$$\theta = \alpha w \ \text{ for some } \ \alpha > 0 \ \text{ and some } \ w \in E \ \text{ with } \ w + v \in \tfrac{1}{2}V.$$

Since for each i there exists a sequence $\{r_n^i\}$ of strictly positive rationals with $r_n^i \downarrow_n \lambda_i$, we can choose rational numbers n_i/n (n_i and n positive integers) with

$$\Big[\frac{1}{\alpha} \sum_{i=1}^{m} \frac{n_i}{n}(z_i - \omega_i) - w\Big] + (w+v) \in \frac{1}{2}V + \frac{1}{2}V = V\,.$$

Hence,

$$\sum_{i=1}^{m} n_i z_i - \sum_{i=1}^{m} n_i\omega_i + \alpha n v \in \alpha n V\,. \qquad (\star)$$

Next, put

$$y = \sum_{i=1}^{m} n_i\omega_i - \alpha n v \quad \text{ and } \quad z = \sum_{i=1}^{m} n_i z_i \geq 0\,.$$

From $z - y = z + \alpha n v - \sum_{i=1}^{m} n_i\omega_i \leq z + \alpha n v$, it follows that

$$(y-z)^- = (z-y)^+ \leq z + \alpha n v = \sum_{i=1}^{m} (n_i z_i + \alpha n v_i)\,.$$

Thus, by the Riesz decomposition property there exist $w_i \in E^+$ $(i = 1,\ldots,m)$ with

$0 \leqslant w_i \leqslant n_i z_i + \alpha n v_i$ and $\sum_{i=1}^{m} w_i = (y-z)^-$. Now let

$$y_i = z_i + \frac{\alpha n}{n_i} v_i - \frac{w_i}{n_i} = \frac{1}{n_i}(n_i z_i + \alpha n v_i - w_i) \geqslant 0,$$

and note that $y_i \succ_i z_i$ holds. Indeed, if this is not true, then we must have $z_i = y_i - \frac{\alpha n}{n_i} v_i + \frac{w_i}{n_i} \succcurlyeq_i y_i$, which (in view of the properness) implies $\frac{w_i}{n_i} \notin \frac{\alpha n}{n_i} V$, or $w_i \notin \alpha n V$. On the other hand, from

$$0 \leqslant w_i \leqslant (y-z)^- \leqslant |y-z| = \Big|\sum_{i=1}^{m} n_i(z_i - \omega_i) + \alpha n v\Big|$$

and ($\star$), we see that $w_i \in \alpha n V$, which contradicts $w_i \notin \alpha n V$. Thus, $y_i \succ_i z_i$ holds for each i.

From $y_i \succ_i z_i \succcurlyeq_i x_i$, and

$$\sum_{i=1}^{m} n_i y_i = \sum_{i=1}^{m} n_i z_i + \alpha n \sum_{i=1}^{m} v_i - \sum_{i=1}^{m} w_i = z + \alpha n v - (y-z)^-$$

$$\leqslant z + \alpha n v + y - z = y + \alpha n v = \sum_{i=1}^{m} n_i \omega_i ,$$

we see that there exists an allocation on some replication of the economy which blocks $(x_1,\ldots,x_m)$ on the same replication. However, the latter is a contradiction, and hence $\Gamma \cap G = \emptyset$.

Finally, since Γ is open, it follows from the separation theorems (see, for example, [3, Thm 9.10, p. 136]) that there exists some $p \neq 0$ with $p \cdot g \geqslant p \cdot w$ for all $g \in G$ and all $w \in \Gamma$. Since $w \in \Gamma$ implies $\alpha w \in \Gamma$ for all $\alpha > 0$, we see that $p \cdot g \geqslant 0$ holds for all $g \in G$. Thus, if $x \succcurlyeq_i x_i$, then $x - \omega_i \in G$, and so $p \cdot (x - \omega_i) = p \cdot x - p \cdot \omega_i \geqslant 0$ implies $p \cdot x \geqslant p \cdot \omega_i$. This shows that $(x_1,\ldots,x_m)$ is a quasiequilibrium, and the proof of the theorem is finished.∎

A. Mas-Colell [8] observed that if the positive cone has interior points, then every strongly monotone continuous preference is proper. In particular, his observation holds true for the $\mathbb{R}^n$ spaces. With this in mind, our next major result can be viewed as an infinite dimensional analogue of the classical theorem of Debreu and Scarf [7]. (Recall that an element $w \in E^+$ is said to be *strictly positive*, in symbols $w \gg 0$, whenever $p \in E'$ and $p > 0$ imply $p \cdot w > 0$.)

THEOREM 4.17. *If preferences are proper and* $\omega \gg 0$*, then an allocation is an Edgeworth equilibrium if and only if it is a Walrasian equilibrium.*

In particular, in this case, if $[0,\omega]$ *is also weakly compact, then the economy has Walrasian equilibria.*

PROOF. Let $(x_1,\ldots,x_m)$ be an Edgeworth equilibrium. By Theorem 4.16 there exists some price $q \neq 0$ such that $x \succ_i x_i$ in E^+ implies $q\cdot x \geq q\cdot\omega_i$. Clearly, $q \geq 0$ holds. Since $\omega \gg 0$, it is easy to see that $q\cdot\omega > 0$. Therefore, the price $p = q/q\cdot\omega$ satisfies statement (2) of Theorem 3.3, and consequently the allocation $(x_1,\ldots,x_m)$ is a Walrasian equilibrium.■

If ω is not a strictly positive element, then the preceding result takes the following form.

THEOREM 4.18. *If preferences are proper on* $\overline{A_\omega}$*, then an allocation is an Edgeworth equilibrium if and only if it is a Walrasian equilibrium with respect to an economy whose Riesz dual system is* $\langle \overline{A_\omega}, (\overline{A_\omega})' \rangle$.

In particular, in this case, if $[0,\omega]$ *is also weakly compact, then the economy has Walrasian equilibria on* $\overline{A_\omega}$.

5. EXAMPLES

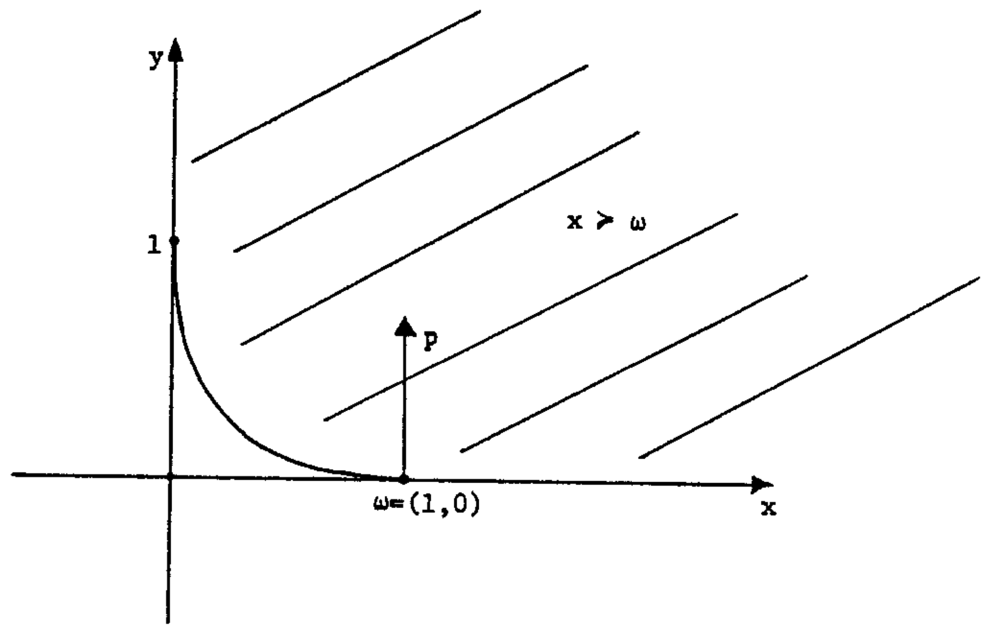
In this section, we illustrate our results by considering several examples of exchange economies that have appeared in the literature of general equilibrium theory.

Example 5.1. This is a modified version of an example due to A. Mas-Colell [8]. Start by observing that in case the economy has only a single consumer, then the set of all allocations has a single element (the initial endowment of the consumer).

Suppose that there are only two goods (hence the commodity space $E = \mathbb{R}^2$) and that we have one consumer with initial endowment $\omega = (1,0)$ and utility function

$$u(x,y) = \sqrt{x} + \sqrt{y}\,.$$

Clearly, u is a strongly monotone, continuous, and strongly quasi-concave function on $\mathbb{R}^2_+$. The following diagram illustrates the situation.



The Edgeworth equilibrium here is the initial endowment $\omega = (1,0)$. Since the preference is proper, ω must be (by Theorem 4.16) also a quasiequilibrium. Observe that this is indeed the case and that the price vector $p = (0,1)$ supports ω. On the other hand, it is easy to see that there is no Walrasian equilibrium. (Note that $(1,0)$ is the only possible Walrasian equilibrium.)

By Theorem 4.13 the initial endowment $(1,0)$ is a Walrasian equilibrium on the ideal generated by $\omega = (1,0)$. Here

$$A_\omega = B_\omega = \{(x,0):\ x \in \mathbb{R}\},$$

and so $A'_\omega = \{(p_1,0):\ p_1 \in \mathbb{R}\}$. It follows that the price $p = (1,0)$ supports the Walrasian equilibrium $(1,0)$ on A_ω. Note that in A_ω the budget set for $p = (1,0)$ is $[0,1]$, while in $\mathbb{R}^2$ it is the positive x-axis.■

Since Bewley's seminal paper [5] on the existence of Walrasian equilibrium on L_∞, economists have considered the existence question in several other infinite dimensional spaces serving as models for differentiated commodities, option markets, etc. Most of these models will be described in the next examples, where we shall simply list the relevant commodity-price dualities and relate them to our analysis.

Example 5.2. We consider the symmetric Riesz dual system $\langle L_\infty(\mu), L_1(\mu)\rangle$ over a σ-finite measure space, and assume that the utility functions are strongly monotone, quasi-concave and $\tau(L_\infty, L_1)$-continuous, and that the total endowment ω is uniformly bounded away from zero. In this case the Mackey topology is a locally convex-solid topology.

In this model, the assumption on ω implies that $A_\omega = L_\infty$. Thus, by Theorems 4.4 and 4.13 there exists a Walrasian equilibrium for the economy whose Riesz dual system is $\langle A_\omega, A'_\omega\rangle = \langle L_\infty, L'_\infty\rangle$. Let $0 < p \in L'_\infty$ be a price supporting this equilibrium. By Theorem 4.10 the price p is order continuous on L_∞. Since the order continuous dual of L_∞ is L_1 (i.e., $(L_\infty)^\sim_n = L_1$), we infer that $p \in L_1$; see [3, Thm 14.12, p. 226]. In other words, our original economy has a Walrasian equilibrium.■

Example 5.3. Consider the Riesz dual system $\langle L_1, L_\infty\rangle$ over a σ-finite measure space, and note that since the norm topology on L_1 is order continuous the Riesz dual system is symmetric. Assume that $\omega \gg 0$ and that the preferences are strongly monotone, quasi-concave, proper and norm continuous.

Thus, $\overline{A_\omega} = B_\omega = L_1$ holds, and so by Theorem 4.17 the economy has a Walrasian equilibrium.■

Example 5.4. Consider the symmetric Riesz dual system $\langle L_2(\mu), L_2(\mu)\rangle$, $\omega \gg 0$ and preferences to be strongly monotone, quasi-concave and norm continuous. Here $\overline{A_\omega} = B_\omega = L_2$, and therefore if preferences are proper, then by Theorem 4.17 this economy has a Walrasian equilibrium.

A similar result holds true for the symmetric Riesz dual system $\langle L_p(\mu), L_q(\mu)\rangle$; $1 < p, q < \infty$, $1/p + 1/q = 1$.■

Example 5.5. Consider the Riesz dual system $\langle ca(\Omega), ca'(\Omega)\rangle$, where $ca'(\Omega)$ is the norm dual of $ca(\Omega)$ equipped with the variational norm (Ω Hausdorff and compact). The variational norm is order continuous and hence $\langle ca(\Omega), ca'(\Omega)\rangle$ is a symmetric Riesz dual system. In general, $ca(\Omega)$ does not have strictly positive elements, and therefore we may only assume $\omega > 0$ (and hence $\overline{A_\omega} \neq ca(\Omega)$).

By Theorem 4.4 Edgeworth equilibria always exist. Moreover, if preferences are proper on A_ω, then an Edgeworth equilibrium is a Walrasian equilibrium with respect to $\langle \overline{A_\omega}, (\overline{A_\omega})' \rangle$ supported by some $0 \ll p \in (\overline{A_\omega})'$ (Theorem 4.18) and a quasiequilibrium with respect to some $0 < q \in ca'(\Omega)$ (Theorem 4.16).■

Example 5.6. Let $\mathbb{R}_\infty$ be the Dedekind complete Riesz space of all real valued sequences equipped with the product topology τ. Then τ is an order continuous locally convex-solid topology, and so $\langle \mathbb{R}_\infty, \mathbb{R}'_\infty \rangle$ is a symmetric Riesz dual system. We assume that preferences are strongly monotone, quasi-concave and τ-continuous. The total endowment ω is assumed to be strictly positive (and hence $\overline{A_\omega} = \mathbb{R}_\infty$).

Since $\mathbb{R}_\infty$ admits no strictly positive linear functional, it follows from Theorem 4.17 that there are no strongly monotone, quasi-concave, τ-continuous utility functions on $\mathbb{R}_\infty$ which are proper. That is, on this commodity space we can only hope for Walrasian equilibria on A_ω, which by Theorems 4.4 and 4.13 exist. Theorem 4.12 guarantees the existence of extended Walrasian equilibria on $\overline{A_\omega} = \mathbb{R}_\infty$. These extended Walrasian equilibria coincide with the equilibrium notion of Peleg and Yaari [10].■

The last example shows that if $[0,\omega]$ is not weakly compact, then there is no guarantee that Edgeworth equilibria exist. As a matter of fact, as we shall see next, if $[0,\omega]$ is not weakly compact, then the economy may even have an empty core.

Example 5.7. For our discussion below L_p $(1 \leqslant p \leqslant \infty)$ will denote $L_p[0,1]$.

Our economy will have two consumers. Its Riesz dual system will be either $\langle C[0,1], ca[0,1] \rangle$ or $\langle L_p, L_q \rangle$ $(1 \leqslant p,q \leqslant \infty;\ 1/p + 1/q = 1)$. The consumers' initial endowments are $\omega_1 = \omega_2 = 1$ (= the constant function one), and their utility functions are given by the formulas

$$u_1(x) = \int_0^{\frac{1}{2}} \sqrt{x(t)}\, dt + \frac{1}{2}\int_{\frac{1}{2}}^1 \sqrt{x(t)}\, dt,$$

$$u_2(x) = \frac{1}{2}\int_0^{\frac{1}{2}} \sqrt{x(t)}\, dt + \int_{\frac{1}{2}}^1 \sqrt{x(t)}\, dt .$$

Clearly, the utility functions are strongly monotone and strongly concave. The order interval $[0,\omega]$ is $\sigma(L_p,L_q)$-compact but it is not $\sigma(C[0,1],ca[0,1])$-compact.

The main property we would like to isolate here is the following: *The economy with Riesz dual system $\langle C[0,1],ca[0,1]\rangle$ has an empty core.*

The proof of this claim will follow from the discussion below.

1. *The utility functions are continuous for both the $\|\cdot\|_1$-norm and the Mackey topology $\tau(L_\infty,L_1)$. In particular, they are continuous for the sup norm and the L_p-norms.*

   Let a net $\{x_\alpha\}\subseteq L_\infty^+$ satisfy $x_\alpha \xrightarrow{\tau(L_\infty,L_1)} x$. Since $[-1,1]$ is a convex, circled and $\sigma(L_1,L_\infty)$-compact subset of L_1, it follows that

$$V=[-1,1]^{\circ}=\{x\in L_\infty\colon \left|\int_0^1 x(t)y(t)dt\right|\leqslant 1 \text{ for all } y\in[-1,1]\}$$
$$=\{x\in L_\infty\colon \int_0^1 |x(t)|dt\leqslant 1\}$$

is a $\tau(L_\infty,L_1)$-neighborhood of zero. From this, we see that

$$\|x_\alpha - x\|_1 = \int_0^1 |x_\alpha(t)-x(t)|dt \longrightarrow 0\,.$$

   Now assume that $\|x_\alpha - x\|_1 \longrightarrow 0$ holds in L_1^+. If $f=\chi_{[0,\frac12]}+\frac{1}{2}\chi_{(\frac12,1]}$, then we have

$$|u_1(x_\alpha)-u_1(x)|\leqslant \int_0^{\frac12}\left|\sqrt{x_\alpha(t)}-\sqrt{x(t)}\right|dt+\frac{1}{2}\int_{\frac12}^1\left|\sqrt{x_\alpha(t)}-\sqrt{x(t)}\right|dt$$
$$=\int_0^1 f(t)\left|\sqrt{x_\alpha(t)}-\sqrt{x(t)}\right|dt$$
$$\leqslant \int_0^1 f(t)\sqrt{|x_\alpha(t)-x(t)|}\,dt$$
$$\leqslant \sqrt{\frac{5}{8}}\cdot\left(\int_0^1 |x_\alpha(t)-x(t)|dt\right)^{\frac12}\longrightarrow 0\,,$$

where the last inequality holds by virtue of Hölder's inequality. Therefore, $u_1(x_\alpha)\longrightarrow u_1(x)$, and similarly $u_2(x_\alpha)\longrightarrow u_2(x)$.

2. *Let (x_1,x_2) be an allocation with respect to L_1 satisfying $x_1>0$ and $x_2>0$. Then there exist two constants $0\leqslant a\leqslant 2$ and $0\leqslant b\leqslant 2$ with $a\neq b$ such that the allocation (x_1^*,x_2^*), given by*

$$x_1^* = a\chi_{[0,\frac12]}+b\chi_{(\frac12,1]} \quad \text{and}$$
$$x_2^* = (2-a)\chi_{[0,\frac12]}+(2-b)\chi_{(\frac12,1]}\,,$$

   satisfies $x_1^*\succ_1 x_1$ and $x_2^*\succ_2 x_2$.

To see this, let (x_1, x_2) be an allocation with respect to L_1 with $x_1 > 0$ and $x_2 > 0$. Put $a = 2\int_0^{1/2} x_1(t)dt$, $b = 2\int_{1/2}^1 x_1(t)dt$, and note that $0 \leqslant a \leqslant 2$ and $0 \leqslant b \leqslant 2$ hold. Let x_1^* and x_2^* be defined as above. Now using Hölder's inequality, we see that

$$\int_0^{1/2} \sqrt{x_1(t)}\, dt \leqslant \frac{1}{\sqrt{2}} \left[\int_0^{1/2} x_1(t)dt\right]^{1/2} = \frac{\sqrt{a}}{2}, \quad \text{and} \quad \int_{1/2}^1 \sqrt{x_1(t)}\, dt \leqslant \frac{\sqrt{b}}{2}.$$

Thus,

$$u_1(x_1) = \int_0^{1/2} \sqrt{x_1(t)}\, dt + \frac{1}{2}\int_{1/2}^1 \sqrt{x_1(t)}\, dt \leqslant \frac{\sqrt{a}}{2} + \frac{\sqrt{b}}{4} = u_1(x_1^*),$$

i.e., $x_1^* \succcurlyeq_1 x_1$. Similarly,

$$\begin{aligned}
u_2(x_2) &= \frac{1}{2}\int_0^{1/2} \sqrt{2 - x_1(t)}\, dt + \int_{1/2}^1 \sqrt{2 - x_1(t)}\, dt \\
&\leqslant \frac{1}{2\sqrt{2}} \left(\int_0^{1/2} [2 - x_1(t)]dt\right)^{1/2} + \frac{1}{\sqrt{2}} \left(\int_{1/2}^1 [2 - x_1(t)]\right)^{1/2} \\
&= \frac{1}{2\sqrt{2}}\left(1 - \frac{a}{2}\right)^{1/2} + \frac{1}{\sqrt{2}}\left(1 - \frac{b}{2}\right)^{1/2} \\
&= \frac{1}{4}\sqrt{2 - a} + \frac{1}{2}\sqrt{2 - b} \\
&= u_2(x_2^*),
\end{aligned}$$

and so $x_2^* \succcurlyeq_2 x_2$.

Next, we must verify that we can choose a and b with $a \neq b$. To this end, assume that $a = b$. In this case we have $x_1^* = a$ and $x_2^* = 2 - a$. From $x_1^* \succcurlyeq_1 x_1 > 0$ and $x_2^* \succcurlyeq_2 x_2 > 0$, we see that $0 < a < 2$. By the symmetry of the situation, we can also assume that $0 < a \leqslant 1$. Then we claim that the allocation (y_1, y_2), given by

$$y_1 = \frac{3}{2} a \chi_{[0,1/2]} + \frac{1}{2} a \chi_{(1/2,1]} \quad \text{and}$$

$$y_2 = \left(2 - \frac{3}{2}a\right)\chi_{[0,1/2]} + \left(2 - \frac{1}{2}a\right)\chi_{(1/2,1]},$$

satisfies $y_1 \succ_1 x_1^*$ and $y_2 \succ_2 x_2^*$ (and hence $y_1 \succ_1 x_1$ and $y_2 \succ_2 x_2$). Indeed, note first that

$$u_1(y_1) = \frac{1}{2}\sqrt{\frac{3}{2}a} + \frac{1}{4}\sqrt{\frac{a}{2}} = \frac{2\sqrt{3} + 1}{4\sqrt{2}}\sqrt{a} > \frac{3}{4}\sqrt{a} = u_1(x_1^*).$$

On the other hand, we have

$$u_2(y_2) = \frac{1}{4}\sqrt{2 - \frac{3}{2}a} + \frac{1}{2}\sqrt{2 - \frac{1}{2}a} = \frac{1}{4\sqrt{2}}(\sqrt{4-3a} + 2\sqrt{4-a}),$$

and an easy calculation shows that $u_2(y_2) > \frac{3}{4}\sqrt{2-a} = u_2(x_2^*)$. The proof of part 2 is now complete.

3. *Assume that (x_1,x_2) is an allocation with $0 < x_i \in C[0,1]$ $(i=1,2)$. Then there exists an allocation (y_1,y_2) with $y_i \in C[0,1]$ $(i=1,2)$ satisfying $y_i \succ_i x_i$ for $i=1,2$. In other words, the economy with Riesz dual system $\langle C[0,1], ca[0,1]\rangle$ has no core allocations.*

To see this, let (x_1,x_2) be an allocation with $0 < x_i \in C[0,1]$ $(i=1,2)$. By part 2 there exist two constants $0 \leqslant a \leqslant 2$ and $0 \leqslant b \leqslant 2$ with $a \neq b$ such that the allocation (x_1^*,x_2^*), given by

$$x_1^* = a\chi_{[0,\frac{1}{2}]} + b\chi_{(\frac{1}{2},1]} \quad \text{and}$$

$$x_2^* = (2-a)\chi_{[0,\frac{1}{2}]} + (2-b)\chi_{(\frac{1}{2},1]},$$

satisfies $x_1^* \succ_1 x_1$ and $x_2^* \succ_2 x_2$. Since x_1^* and x_2^* are not continuous functions, we see that $x_1^* \neq x_1$ and $x_2^* \neq x_2$. Thus, by the strong concavity of the utility functions, we infer that the allocation (θ_1,θ_2) given by $\theta_1 = \frac{1}{2}(x_1 + x_1^*)$ and $\theta_2 = \frac{1}{2}(x_2 + x_2^*)$ satisfies $\theta_1 \succ_1 x_1$ and $\theta_2 \succ_2 x_2$.

Now since $C[0,1]$ is $\|\cdot\|_1$-dense in L_1, there exists a sequence $\{z_n\} \subseteq C[0,1]$ satisfying $0 \leqslant z_n \leqslant 2$ for all n and $\lim \|\theta_1 - z_n\|_1 = 0$ (and, of course, $\lim \|\theta_2 - (2 - z_n)\|_1 = 0$). By virtue of the $\|\cdot\|_1$-continuity of the utility functions (part 1), there exists some n so that the allocation $(z_n, 2 - z_n)$ satisfies $z_n \succ_1 x_1$ and $2 - z_n \succ_2 x_2$. This completes the proof of part 3.

We saw before (in Example 5.2) that the economy with Riesz dual system $\langle L_\infty, L_1\rangle$ has Walrasian equilibria. Part 2 tells us that these Walrasian equilibria must be of a very special type. Next, we exhibit a Walrasian equilibrium for the economy with Riesz dual system $\langle L_p, L_q\rangle$.

4. *The allocation (x_1,x_2), given by*

$$x_1 = \frac{8}{5}\chi_{[0,\frac{1}{2}]} + \frac{2}{5}\chi_{(\frac{1}{2},1]} \quad \textit{and}$$

$$x_2 = \frac{2}{5}\chi_{[0,\frac{1}{2}]} + \frac{8}{5}\chi_{(\frac{1}{2},1]},$$

is a Walrasian equilibrium for an economy whose Riesz dual system is $\langle L_p, L_q \rangle$*;* $1 \leq p,q \leq \infty$*,* $1/p + 1/q = 1$*. Moreover, in this case, the Lebesgue integral is a price that supports the allocation* (x_1, x_2)*.*

To see this, note first that $u_1(x_1) = u_2(x_2) = \sqrt{\frac{5}{8}}$. Also, if $f = \chi_{[0,\frac{1}{2}]} + \frac{1}{2}\chi_{(\frac{1}{2},1]}$, then we have $u_1(x) = \int_0^1 f(t)\sqrt{x(t)}\,dt$. Therefore, if $x \succ_1 x_1$, then by using Hölder's inequality, we see that

$$\begin{aligned}\sqrt{\frac{5}{8}} = u_1(x_1) \leq u_1(x) &= \int_0^1 f(t)\sqrt{x(t)}\,dt \\ &\leq \left(\int_0^1 [f(t)]^2 dt\right)^{\frac{1}{2}} \cdot \left(\int_0^1 x(t)dt\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{5}{8}} \cdot \left(\int_0^1 x(t)dt\right)^{\frac{1}{2}},\end{aligned}$$

and so $\left(\int_0^1 x(t)dt\right)^{\frac{1}{2}} \geq 1$, i.e.,

$$\int_0^1 x(t)dt \geq 1 = \int_0^1 \omega_1(t)dt .$$

Similarly, $x \succ_2 x_2$ implies $\int_0^1 x(t)dt \geq \int_0^1 \omega_2(t)dt$. In other words, the Lebesgue integral is a price that supports (x_1, x_2), and so (x_1, x_2) is a Walrasian equilibrium.∎

Finally, we mention a model of considerable importance for which our existence theorems do not apply. This is an economy whose Riesz dual system is $\langle C_b(\Omega), ca(\Omega) \rangle$. Example 5.7 shows that such an economy may not have an Edgeworth equilibrium. The Riesz dual system $\langle C_b(\Omega), ca(\Omega) \rangle$ is *not* in general symmetric and in general no order interval of $C_b(\Omega)$ is weakly compact. For this case, the only existence theorem known to us is that of Aliprantis and Brown [1] where demand functions are taken as primitive.

REFERENCES

1. C. D. ALIPRANTIS AND D. J. BROWN, Equilibria in markets with a Riesz space of commodities, *J. Math. Economics* 11(1983), 189-207.
2. C. D. ALIPRANTIS AND O. BURKINSHAW, *Locally Solid Riesz Spaces*, Academic Press, New York & London, 1978.
3. C. D. ALIPRANTIS AND O. BURKINSHAW, *Positive Operators*, Academic Press, New York & London, 1985.
4. K. J. ARROW AND G. DEBREU, Existence of an equilibrium for a competitive economy, *Econometrica* 22(1954), 265-290.
5. T. F. BEWLEY, Existence of equilibria in economies with infinitely many commodities, *J. Economic Theory* 4(1972), 514-540.
6. G. DEBREU, Valuation equilibrium and Pareto optimum, *Proc. Nat. Acad. Sci. U.S.A.* 40(1954), 558-592.
7. G. DEBREU AND H. SCARF, A limit theorem on the core of an economy, *International Economic Review* 4(1963), 235-246.
8. A. MAS-COLELL, The price equilibrium problem in Banach lattices, *Econometrica*, forthcoming.
9. T. NEGISHI, Welfare economics and existence of an equilibrium for a competitive economy, *Metroeconomica* 12(1960), 92-97.
10. B. PELEG AND M. E. YAARI, Markets with countably many commodities, *International Economic Review* 11(1970), 369-377.
11. H. SCARF, The core of an n person game, *Econometrica* 35(1967), 50-69.

C. D. Aliprantis and O. Burkinshaw
DEPARTMENT OF MATHEMATICAL SCIENCES
INDIANA UNIVERSITY-PURDUE UNIVERSITY AT INDIANAPOLIS
INDIANAPOLIS, IN 46223

AND

D. J. Brown
DEPARTMENT OF ECONOMICS
YALE UNIVERSITY
NEW HAVEN, CT 06520