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Mamoru Kaneko

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Mamoru Kaneko*

Institute of Socio-Economic Planning
University of Tsukuba
Ibaraki-ken 305, Japan

ABSTRACT

This paper provides an axiomatic model based on an extraneous random device generating objective probabilities for the derivation of expected utilities and subjective probabilities. Four basic axioms fully determine a real-valued utility function and a finitely additive subjective probability measure. The restrictions of these axioms to lotteries depending only upon events of the random device yield the von Neumann-Morgenstern axioms.

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1. Introduction

In this paper, we take the position that it is desirable to relate subjective and objective probabilities, and consider the derivation of expected utilities and subjective probabilities based on an extraneous random device with which objective probabilities are generated. Since Savage's pioneering work [13] on the axiomatic approach to expected utilities and subjective probabilities, there have been a great number of articles on subjective probabilities. (See Fishburn [6] for a recent exhaustive survey.) These can be divided into two categories, according to whether or not the existence of an extraneous random device to generate objective probabilities is assumed. Those without the assumption of an extraneous random device, which we call the first category, include Savage [13], Luce and Krantz [11] and many others. The second category (with the assumption of an extraneous random device) includes Anscombe and Aumann [1], Pratt, Raiffa, and Schlaifer [12], Fishburn [3, 4, 5, 7], DeGroot [2], French [8] and also others. It has sometimes been argued that the approach of the first category is advantageous to that of the second in that there is no need to refer to the concept of objective probability. However this can also be thought of as a relative disadvantage: the approach of the second category gives a richer structure than that of the first in that it allows a decision maker to use a random device generating objective probabilities to reflect on his personal subjective probabilities (see French [8, pp. 19-21]). In this paper, we choose the position to relate subjective and objective probabilities.

Since the axioms of the von Neumann-Morgenstern expected utility theory (cf. Herstein and Milnor [9] and Jensen [10]) are quite simple and are well interpreted, it is useful to be able to compare this theory with others of expected utilities and subjective probabilities. The von Neumann-Morgenstern expected utility theory can be regarded as an extreme case in the second category, in that expected utilities are derived based on objective probabilities but subjective probabilities are not incorporated into the theory. Anscombe and Aumann [1], Fishburn [3, 4, 5, 7] and others have developed several models of expected utilities and subjective probabilities assuming directly the von Neumann-Morgenstern theory as a substructure. In these papers, however, a random device generating objective probabilities is treated separately from "real-world" events, and the relationships between the concepts of objective and subjective probabilities are not fully explored. Pratt, Raiffa and Schlaifer [12] provided a model in which an extraneous random device was formulated explicitly as an event space at the same level as that of "real-world" events. However a defect of their model is that it does not have an explicit relation to the von Neumann-Morgenstern model, and their model is not fully elaborated. DeGroot [2, Chapter 6] and French [8] took similar approaches focussing their attentions only on the derivations of subjective probabilities based on an extraneous random device without incorporating utilities into the models. In our model, we have an extraneous random device, which is similar to that of Pratt, Raiffa and Schlaifer [12], DeGroot [2] and French [8], but it has an explicit relation to the von Neumann-Morgenstern theory, and this is developed.

A rough sketch of our model is as follows. The state space is given as the product $X \times Y$ of the set X of "real-world" states and the set Y of outcomes of a random device. The decision maker can use the random device to generate objective probabilities for his decision making. The random device associates events (subsets of Y) with objective probabilities.

Another type of events (subsets of X) are not given probabilities. The lottery space is the set of simple functions from $X \times Y$ to a reward space M . The decision maker's preference relation is defined on the lottery space. We impose, upon the preference relation, four axioms (1) Rationality, (2) Continuity, (3) the Sure-Thing Principle and (4) Objectivity of the Random Device. These axioms determine a unique (up to a positive linear transformation) utility function and a unique finitely additive subjective probability measure. Axiom 4 enables us to compare the other axioms with the von Neumann-Morgenstern axioms, and the restrictions of the other axioms to the lotteries depending only upon events of the random device are equivalent to the von Neumann-Morgenstern axioms. That is, our axiomatization can be regarded as a direct generalization of the von Neumann-Morgenstern utility theory. In particular, we compare Jensen's [8] axiomatization, as one of the most elaborated axiomatization of the von Neumann-Morgenstern theory, with ours. The model and result will be given in Section 2 and the proof of the main theorem will be given in Section 3.

2. Model and Result

We consider a one-person decision problem under uncertainty. The uncertain events can be divided into two classes--events of "real-world" states and events of outcomes of a random device, called, respectively, real-world events and auxiliary events. Any real-world events are not, a priori, given "objective" probabilities in our model, while auxiliary events are assocwated with "objective" probabilities. The auxiliary event space represents the probability space generated by the random device. The decision maker can generate every probability in [0,1] using the random device, independently of real-world events, and can "fix" the random device to reflect his subjective (personal judgmental) probability for each real-world event. By this we mean that the decision maker can set or manipulate the random device so that each possible auxiliary event has the desired probability (see the example following the Theorem).

Formally the whole space is assumed to be the product of the real-world event space and the auxiliary event space. Let (X, B_X) be the real-world event space, where X is the set of real-world states, and B_X an algebra of subsets of X. Let (Y, B_Y, μ) be the auxiliary event space, where Y is the set of all outcomes of the random device, B_Y a σ-algebra of subsets of Y and μ a nonatomic probability measure. Each element S in B_Y is an auxiliary event associated with its objective probability $\mu(S)$. The decision maker can not attach an objective probability to each real world event R in B_X; however, he does have a subjective belief (of the probability) of the occurrence of each event. This subjective belief (probability) will be revealed by his choice behavior. Let B be the minimal algebra including $B_X \times B_Y = \{R \times S : R \in B_X \text{ and } S \in B_Y\}$. The whole event space is given as $(X \times Y, B)$. A pair (r,s) of a state r in X and s in Y fully describes a state of the world. Note that every F in B can be represented as

$$(1) \qquad F = \bigcup_{t=1}^{k} R_t \times S_t \quad \text{for some} \quad R_1 \times S_1, \ldots, R_k \times S_k \in B_X \times B_Y .$$

Remark 1. One may feel that the "real-world" and "auxiliary" events should be attributed to events $R \times Y$ and $X \times S$ instead of R and S ($R \in B_X$ and $S \in B_Y$). However we can regard R and S as the same as $R \times Y$ and $X \times S$, respectively. This should not cause any confusion.

The *space of rewards* is denoted by M, an arbitrary set. A *lottery* is a (simple) function $f : X \times Y \to M$ such that

(2) there is a finite partition $(F_1, \ldots, F_k)$ of $X \times Y$ with (i) $F_t \in B$ for $t = 1, \ldots, k$ and (ii) $f(r,s) = f(r', s')$ for all $(r,s), (r', s') \in F_t$ and $t = 1, \ldots, k$.

The *set of all lotteries* is denoted by L. A lottery f with a partition $(F_1, \ldots, F_k)$ of $X \times Y$ will be denoted as

$$f = [F_1, \ldots, F_k; m_1, \ldots, m_k] ,$$

where $f(r,s) = m_t$ for all $(r,s) \in F_t$ $(t = 1, \ldots, k)$. A finite partition of $X \times Y$ with condition (2)(i) will simply be called a *partition*. It follows from (1) that every f in L can be represented as

$$f = [R_1 \times S_1, \ldots, R_\ell \times S_\ell; m_1, \ldots, m_\ell] \text{ for some } R_1 \times S_1, \ldots, R_\ell \times S_\ell \in B_X \times B_Y .$$

Identifying $f = [X \times Y; m]$ with m, we can regard the set M as a subset of L.

For $f^1, \ldots, f^k \in L$ and a partition $(E_1, \ldots, E_k)$ of $X \times Y$, a new lottery h, denoted by $\sum_{t=1}^{k} E_t f^t$, is defined by

(3) $h(r,s) = f^t(r,s)$ if $(r,s) \in E_t$ $(t = 1, \ldots, k)$.

The new lottery h agrees with f^t on E_t for $t = 1, \ldots, k$, respectively. Of course, this h belongs to the lottery space L. As will be explained, this operation corresponds to the convex combination operation in the von Neumann-Morgenstern utility theory.

For $E = R \times S \in B_X \times B_Y$ and $f = [R_1 \times S_1, \ldots, R_k \times S_k; m_1, \ldots, m_k]$, we say that E is <u>independent of</u> f iff

(4) $R = X$ or $R_1 = \ldots = R_k = X$;

(5) $\mu(S \cap S_t) = \mu(S)\mu(S_t)$ for all $t = 1, \ldots, k$.

Condition (5) is the standard definition of stochastic independency under a given probability measure μ . Since we don't specify a probability measure on B_X , condition (4) is only possible for stochastic independency.

<u>Remark 2</u>. Every $R \times S \in B_X \times B_Y$ is independent of $m = [X \times Y, m]$ for all $m \in M$. We will use this fact without giving a remark in Section 3.

We now consider the decision maker's choice behavior on the lottery space L . His choice behavior is described by a preference relation $\succsim$ on L . Formally, the preference relation is a binary relation on L . That is, he makes pairwise comparisons on L . Denote the symmetric and nonsymmetric parts of $\succsim$ by $\sim$ and $\succ$, respectively, i.e., $f \sim g \Leftrightarrow f \succsim g$ and $g \succsim f$, and $f \succ g \Leftrightarrow f \succsim g$ and not $f \sim g$.

Let N_X be a subclass of B_X with $X \notin N_X$, and let $N = \{R \times S \in B_X \times B_Y : R \in N_X \text{ or } \mu(S) = 0\}$. An element in N is called a <u>null</u> event.

We make the following axioms on the preference relation $\succsim$.

<u>Axiom 1</u> (<u>Rationality</u>). The preference relation $\succsim$ is a complete preordering on L , i.e., (i) $f \succsim g$ or $g \succsim f$ for all $f, g \in L$ and (ii) $f \succsim g$ and $g \succsim h$ implies $f \succsim h$.

Axiom 2 (Continuity). If $f \succ g \succ h$, then there are S_1 and S_2 in B_Y $(0 < \mu(S_1), \mu(S_2) < 1)$ such that $X \times S_t$ and $X \times S_t^c$ are independent of f and h, respectively, for $t = 1, 2$ and $(X \times S_1)f + (X \times S_1^c)h \succ g$ and $g \succ (X \times S_2)f + (X \times S_2^c)h$.

Axiom 3 (The Sure-Thing Principle). Let f, g, h be lotteries in L. If $R \times S \in B_X \times B_Y - N$ is independent of f and g, then $f \succ g \Longleftrightarrow (R \times S)f + (R \times S)^c h \succ (R \times S)g + (R \times S)^c h$; and if $R \times S \in N$, then $(R \times S)f + (R \times S)^c h \sim (R \times S)g + (R \times S)^c h$.

Axiom 4 (Objectivity of the Random Device). For all $f = [R_1 \times S_1, \ldots, R_k \times S_k; m_1, \ldots, m_k]$ and $g = [R_1 \times S_1^*, \ldots, R_k \times S_k^*; m_1, \ldots, m_k]$, if $\mu(S_t) = \mu(S_t^*)$ for all $t = 1, \ldots, k$, then $f \sim g$.

No explanation is necessary for Axioms 1 and 2. Axiom 3 says that if $R \times S$ is non-null and independent of f, g, then f is preferred to g if and only if $(R \times S)f + (R \times S)^c h$ is preferred to one obtained by replacing f by g; and that if $R \times S$ is a null event, then $(R \times S)f + (R \times S)^c h$ and $(R \times S)g + (R \times S)^c h$ are always indifferent. Axiom 4 simply says that the preference relation depends upon objective probabilities but not the specification of auxiliary events.

Remark 3. (i) The former half of Axiom 3 is equivalent to that if $R \times S \in B_X \times B_Y - N$ is independent of f and g, then $f \succsim g \Longleftrightarrow (R \times S)f + (R \times S)^c h \succsim (R \times S)g + (R \times S)^c h$. (ii) It follows from the latter half of Axiom 3 that if $R \times S \in N$, then $(R \times S)f + (R \times S)^c h \sim h$ for all $f, h \in L$.

The main result of this paper is the following theorem, which will be proved in Section 3.

Theorem. Assume that $\overline{m} \succ \underline{m}$ for some $\overline{m}, \underline{m} \in M$. The preference relation $\succsim$ satisfies Axioms 1 to 4 if and only if there is a real-valued function u on L and a finitely additive probability measure P on B_X such that

(6) for all $f, g \in L$, $f \succsim g$ if and only if $u(f) \geqq u(g)$;

(7) $$u([R_1 \times S_1, \ldots, R_k \times S_k; m_1, \ldots, m_k]) = \sum_{t=1}^{k} P(R_t)\mu(S_t)u(m_t)$$

for all $[R_1 \times S_1, \ldots, R_k \times S_k; m_1, \ldots, m_k] \in L$.

The function u is unique up to a positive linear transformation, and the measure P is uniquely determined by

(8) for all $R \in B_X$, $P(R) = \mu(S)$, where

$$[R \times Y, R^c \times Y; \overline{m}, \underline{m}] \sim [X \times S, X \times S^c; \overline{m}, \underline{m}] .$$

It also holds that

(9) $P(R) = 0$ for all $R \in N_X$.

This theorem claims that under Axioms 1 to 4, the preference relation $\succsim$ is represented by expected utilities in terms of subjective and objective probabilities. The determination of P by (8) is important: The decision maker can find his subjective probability $P(R)$ for each real-world event $R \in B_X$, comparing auxiliary events with it. For example, consider the event R :

"it will rain tomorrow morning."

The decision maker compares the lottery $[R \times Y, R^c \times Y; \overline{m}, \underline{m}]$ (that he will get the reward $\overline{m}$ (say, \$100) if it rains tomorrow morning or $\underline{m}$ (say, nothing) otherwise) with other lotteries $[X \times S, X \times S^c; \overline{m}, \underline{m}]$ (that he will get \$100 tomorrow if the random device indicates the event S or nothing otherwise). Altering S continuously from a null event $(\mu(S) = 0)$ to bigger auxiliary events, he can find an S which makes him indifferent between those two lotteries $[R \times Y,\ R^c \times Y;\ \overline{m},\ \underline{m}]$ and $[X \times S,\ X \times S^c;\ \overline{m},\ \underline{m}]$.

It is useful to make a comparison between our model and the von Neumann-Morgenstern theory for the consideration of the meanings of our axioms and also for the proof of the main theorem. As one of the most elaborated axiomatizations of expected utilities, we employ Jensen's [10] set of axioms. This comparison will give a better understanding of our axioms and also the von Neumann-Morgenstern model itself.

Let FM be the set of all probability distributions on M with finite supports, i.e., each $\psi \in FM$ is a nonnegative real-valued function on M and has a finite subset A of M such that $\psi(m) > 0$ implies $m \in A$ and $\sum_{m \in A} \psi(m) = 1$. A distribution $\psi \in FM$ can be represented as $[\alpha_1, \ldots, \alpha_k;\ m_1, \ldots, m_k]$, where $\psi(m_t) = \alpha_t$ for all $t = 1, \ldots, k$ and $\sum_{t=1}^{k} \alpha_t = 1$. The set FM can be regarded as a convex subset of the linear space, on reals, of all real-valued functions on M . (A convex combination $\alpha\psi + (1-\alpha)\xi$ corresponds to a compound lottery or a mixture in the expected utility literature.)

Jensen [10] provided axioms for a binary relation $\succsim^*$ on FM to be represented in terms of a real-valued function.

Theorem (Jensen). A binary relation $\succsim^*$ on FM satisfies

(J-1) $\succsim^*$ is a complete preordering on FM ;

(J-2) for all $\psi, \xi, \zeta \in FM$, if $\psi \succ^* \xi \succ^* \zeta$, then there are $\alpha, \beta \in (0,1)$ such that $\alpha\psi + (1-\alpha)\zeta \succ^* \xi$ and $\xi \succ^* \beta\psi + (1-\beta)\zeta$;

(J-3) for all $\psi, \xi, \zeta \in FM$ and $\alpha \in (0,1]$, $\psi \succ^* \xi$ if and only if $\alpha\psi + (1-\alpha)\zeta \succ^* \alpha\xi + (1-\alpha)\zeta$

if and only if there is a real-valued function u^* on FM such that

(10) $\psi \succ^* \xi$ if and only if $u^*(\psi) \geqq u^*(\xi)$;

(11) $u^*(\alpha\psi + (1-\alpha)\xi) = \alpha u^*(\psi) + (1-\alpha)u^*(\xi)$ for all $\psi, \xi \in FM$ and $\alpha \in [0,1]$.

This function u^* is unique up to a positive linear transformation.

Remark 4. Axiom (J-3) in the above theorem was originally (and can be) stated in a little weaker form:

(J-3') for all $\psi, \xi, \zeta \in FM$ and $\alpha \in (0,1)$, if $\psi \succ^* \xi$, then $\alpha\psi + (1-\alpha)\zeta \succ^* \alpha\xi + (1-\alpha)\zeta$.

Here we adopt the stronger form (J-3) for the sake of a comparison with our axiomatization.

Remark 5. It follows from condition (11) that

$$(11') \quad u^*(\psi) = u^*([\alpha_1, \ldots, \alpha_k; m_1, \ldots, m_k]) = \sum_{t=1}^{k} \alpha_t u^*(m_t) \quad \text{for all} \quad \psi = [\alpha_1, \ldots, \alpha_k; m_1, \ldots, m_k] \in FM .$$

Conversely, (11) follows also from (11').

The reader can notice a great similarity between Axioms 1-3 and (J-1) -(J-3). Indeed, if Axioms 1-3 are restricted to the lotteries depending only upon auxiliary events, then they are equivalent to (J-1)-(J-3): Define

$$L_0 = \{f \in L : f = [X \times S_1, \ldots, X \times S_k; m_1, \ldots, m_k] \text{ for some } S_1, \ldots, S_k \in B_Y\} .$$

A lottery f in L_0 depends only upon auxiliary events. The function Φ on L_0 defined by

(12) $\Phi([X \times S_1, \ldots, X \times S_k; m_1, \ldots, m_k]) = [\mu(S_1), \ldots, \mu(S_k); m_1, \ldots, m_k]$

is an onto-mapping to the set FM . The preference $\succsim$ on L with the restriction of Axiom 4 to L_0 and $\succsim^*$ on FM can be connected by the following rule:

(13) for all $f, g \in L_0$, $f \succsim g$ if and only if $\Phi(f) \succsim^* \Phi(g)$.

Then, noting that for all $f = [X \times S_1, \ldots, X \times S_k; m_1, \ldots, m_k]$, $g = [X \times T_1, \ldots, X \times T_\ell; m_1^*, \ldots, m_\ell^*]$ in L_0 and $S \in B_Y$ with $\mu(S) \in (0,1)$,

(14) $\Phi\Big((X \times S)f + (X \times S^c)g\Big) = [\mu(S \cap S_1), \ldots, \mu(S \cap S_k), \mu(S^c \cap T_1), \ldots, \mu(S^c \cap T_\ell); m_1, \ldots, m_k, m_1^*, \ldots, m_\ell^*]$;

(14') if $X \times S$ and $X \times S^c$ are independent of f and g , respectively, then $\Phi\Big((X \times S)f + (X \times S^c)g\Big) = \mu(S)\Phi(f) + (1 - \mu(S))\Phi(g)$,

the restrictions of Axioms 1-3 to L_0 can be translated into Axioms (J-1)-(J-3), and vice versa. Summing these up, we have the following lemma.

Lemma 1. (i) The binary relation $\succsim$ on L_0 satisfies the restriction of Axiom 4 to L_0 if and only if there is a reflective binary relation $\succsim^*$ on FM with condition (13).[1)]

(ii) Assume that $\succsim$ on L_0 and $\succsim^*$ on FM satisfy (13). For $t = 1, 2, 3$, the binary relation $\succsim$ satisfies the restriction of Axiom t to L_0 if and only if the binary relation $\succsim^*$ satisfies Axiom (J-t) .

It follows from Jensen's Theorem and Lemma 1 that if the preference relation $\succsim$ on L satisfies Axioms 1-4, and the function u^* given in Jensen's Theorem and (13) satisfies

1) A binary relation O^* on a set A is called reflexive iff $a O^* a$ for all $a \in A$.

(15) for all $f, g \in L_0$, $f \succsim g$ if and only if $u^*(\Phi(f)) \geqq u^*(\Phi(g))$;

(16) for all $f = [X \times S_1, \ldots, X \times S_k; m_1, \ldots, m_k] \in L_0$,

$$u^*(\Phi(f)) = \sum_{t=1}^{k} \mu(S_t)u^*(m_t) .$$

In the proof of the main theorem, this utility function u^* will be extended to the whole space L .

We now have to evaluate Axioms 1-4. Axiom 1 is a general condition for measurement. Axiom 2 is continuity, which is usually necessary for mathematical arguments including infinite constituents. Therefore these two axioms are innocuous. Axiom 4 simply requires that the choice behavior depends only upon objective probabilities but not upon specifications of auxiliary events. This axiom represents the assumption of the existence of a random device with which the decision maker generates objective probabilities for his decision making. Under Axiom 4, the restrictions of Axioms 1-3 to L_0 are the translations of (J-1)-(J-3) to our context. Axiom (J-3) is also usually called the Sure-Thing Principle, and can be regarded as the most important condition for the von Neumann-Morgenstern utility theory. This axiom requires that the decision maker's choice be made by evaluating outcomes of lotteries and by taking objective probabilities into account. However, Axiom 3 on the whole lottery space L is not just a translation of the purer form (J-3) of the Sure-Thing Principle. It also ascribes the (real-world) state-independency of the utility function u in the main theorem. It follows from Axiom 3 (Remark 3) that

(*) if $R \times Y$ is independent of f and g , then

$$f \sim g \Rightarrow (R \times Y)f + (R \times Y)^c h \sim (R \times Y)g + (R \times Y)^c h .$$

That is, the utility evaluations of lotteries $(R \times Y)f + (R \times Y)^c h$ and $(R \times Y)g + (R \times Y)^c h$ do not depend upon the relations between the event R and f, g but only upon the "subjective probability" of the event R.

To make this point clear, consider the following example. Let R be the event: "it will rain tomorrow morning," and let

$\underline{m}$: nothing will be given tomorrow morning;

$\overline{m}$: an umbrella will be given tomorrow morning;

m^* : a parasol will be given tomorrow morning.

Assume that the decision maker is indifferent between $\overline{m} = [X \times Y; \overline{m}]$ and $m^* = [X \times Y, m^*]$, that is, he is indifferent between getting an umbrella and doing a parasol tomorrow morning, if there is no contingency in getting one of them. Then consider the choice between $[R \times Y, R^c \times Y, \overline{m}, \underline{m}]$ and $[R \times Y, R^c \times Y, m^*, \underline{m}]$. The lottery $[R \times Y, R^c \times Y, \overline{m}, m]$ ($[R \times Y, R^c \times Y; m^*, \underline{m}]$) is that if it rains tomorrow morning, then the decision maker will get an umbrella (a parasol, respectively). In this case, it is natural to think the decision maker prefers $[R \times Y, R^c \times Y, \overline{m}, \underline{m}]$ to the other; it is normal to prefer an umbrella to a parasol if it rains. In this example, condition (*) implies

$$(*) \qquad [X \times Y; \overline{m}] \sim [X \times Y; m^*] \text{ implies } (R \times Y)[X \times Y; \overline{m}] + (R^c \times Y)[X \times Y; \underline{m}]$$
$$\sim (R \times Y)[X \times Y; m^*] + (R^c \times Y)[X \times Y; \underline{m}] .$$

This condition is violated as was explained above.

The state-independency of the utility function is restrictive and the most crucial point of Axiom 3. If one wishes to generalize our model to incorporate state-dependency, then he should attack Axiom 3.

Finally, we give a remark on a condition for the probability measure to be countably additive.

Remark 6. Assume that B_X is a σ-algebra. Then if we add the following axiom

Axiom 5. Let $f \in L$ and let $\{[R^\nu \times Y, R^{\nu c} \times Y; m_1, m_2]\}$ be a sequence in L such that $R^1 \supset R^2 \supset \ldots$ and $[R^\nu \times Y, R^{\nu c} \times Y; m_1, m_2] \succsim f$ for all ν. Then $[R \times Y, R^c \times Y; m_1, m_2] \succsim f$, where $R = \bigcap_\nu R^\nu$.

then the probability measure P is countably additive. This can be proved in the standard way (e.g. French [8, p. 25]).

3. Proof of the Theorem

Since sufficiency can easily be verified, we omit the proof of sufficiency. Throughout this section, we assume that the prefrence relation $\succsim$ satisfies Axioms 1-4, and then we will prove that there is a real-valued fucntion u on L and a finitely additive probability measure P on B_X with properties (6) and (7).

As was mentioned just after Lemma 1, we can define a real-valued function u on L_0 by

$$\text{(17)}\quad u([X\times S_1,\ldots,X\times S_k;m_1,\ldots,m_k]) = u^*(\Phi([X\times S_1,\ldots,X\times S_k;m_1,\ldots,m_k])$$
$$= u^*([\mu(S_1),\ldots,\mu(S_k);m_1,\ldots,m_k]) \text{ for all } [X\times S_1,\ldots,X\times S_k;m_1,\ldots,m_k]\in L_0,$$

where u* is the real-valued function given in Jensen's Theorem and (13). It follows from (15) and (16) that

(15') for all $f,g \in L_0$, $f \succsim g$ if and only if $u(f) \geqq u(g)$;

(16') for all $f = [X\times S_1,\ldots,X\times S_k;m_1,\ldots,m_k] \in L_0$, $u(f) = \sum_{t=1}^{k} \mu(S_t)u(m_t)$.

We will extend this function u to the whole space L.

Lemma 2. Let R×S be an arbitrary element in $B_X\times B_Y$, and let m_1,m_2 be arbitrary elements in M. Then

(i-1) $m_1 \succsim m_2 \Rightarrow m_1 \succsim [R\times S,(R\times S)^c;m_1,m_2] \succsim m_2$;

(i-2) $m_1 \precsim m_2 \Rightarrow m_1 \precsim [R\times S,(R\times S)^c;m_1,m_2] \precsim m_2$;

(ii) there is a $\hat{T} \in B_Y$ such that $[R\times S,(R\times S)^c;m_1,m_2] \sim [X\times\hat{T},X\times\hat{T}^c;m_1,m_2]$.

Proof. If R×S is a null event, then $[R\times S,(R\times S)^c;m_1,m_2] \sim m_2 = [X\times\emptyset,X\times Y;m_1,m_2]$ by Remark 3, i.e., (i) and (ii) hold. In the following, we assume that R×S is a non-null event, and consider only the case where $m_1 \succsim m_2$.

(i): It follows from Axiom 3 (Remark 3) that

$$[R\times S,(R\times S)^c;m_1,m_2] = (R\times S)m_1+(R\times S)^c m_2 \succsim (R\times S)m_2+(R\times S)^c m_2 = m_2 .$$

Further it holds that

$$[R\times S,(R\times S)^c;m_1,m_2] = (R\times S)m_1+(R^c\times S)m_2+(R\times S^c)m_2+(R^c\times S^c)m_2$$

$$= (R^c\times S)m_2+(R^c\times S)^c[(R\times S)m_1+(R^c\times S)m_2+(R\times S^c)m_2+(R^c\times S^c)m_2]$$

$$\precsim (R^c\times S)m_1+(R^c\times S)^c[(R\times S)m_1+(R^c\times S)m_2+(R\times S^c)m_2+(R^c\times S^c)m_2] \quad \text{(by Axiom 3)}$$

$$= (R\times S^c)m_2+(R\times S^c)^c[(R\times S)m_1+(R^c\times S)m_1+(R\times S^c)m_2+(R^c\times S^c)m_2]$$

$$\precsim (R\times S^c)m_1+(R\times S^c)^c[(R\times S)m_1+(R^c\times S)m_1+(R\times S^c)m_2+(R^c\times S^c)m_2] \quad \text{(by Axiom 3)}$$

$$= (R^c\times S^c)m_2+(R^c\times S^c)^c[(R\times S)m_1+(R^c\times S)m_1+(R\times S^c)m_1+(R^c\times S^c)m_2]$$

$$\precsim (R^c\times S^c)m_1+(R^c\times S^c)^c[(R\times S)m_1+(R^c\times S)m_1+(R\times S^c)m_1+(R^c\times S^c)m_2] \quad \text{(by Axiom 3)}$$

$$= m_1 .$$

Thus we have $m_1 \succsim [R\times S,(R\times S)^c;m_1,m_2] \succsim m_2$.

(ii): Denote $[R\times S,(R\times S)^c;m_1,m_2]$ by f. If $m_1 \sim f$ or $m_2 \sim f$, then it suffices to put $\hat{T} = Y$ or $\hat{T} = \emptyset$, respectively. Consider the case where $m_1 \succ f \succ m_2$. Since $u([X\times T,X\times T^c;m_1,m_2]) = \mu(T)u(m_1)+(1-\mu(T))u(m_2)$ by (16'), the sets $\{\mu(T): [X\times T,X\times T^c;m_1,m_2] \succ f\}$ and $\{\mu(T) : f \succ [X\times T,X\times T^c;m_1,m_2]\}$ are disjoint intervals of [0,1]. If $\{\mu(T): [X\times T,X\times T^c;m_1,m_2] \succ f\}$ is a closed interval, say $[\alpha,1]$, then by Axiom 4, $[X\times\tilde{T},X\times\tilde{T}^c;m_1,m_2] \succ f$ for all $\tilde{T}$ with $\mu(\tilde{T}) = \alpha$. Then it follows from Axiom 2 that there is a $W \in B_X$ ($\mu(W) < 1$) such that $X\times W$ is independent of $[X\times\tilde{T},X\times\tilde{T}^c;m_1,m_2]$ and $(X\times W)[X\times\tilde{T},X\times\tilde{T}^c;m_1,m_2]+(X\times W^c)m_2 \succ f$. However $(X\times W)[X\times\tilde{T},(X\times\tilde{T})^c;m_1,m_2]+(X\times W^c)m_2 = [X\times(W\cap\tilde{T}),X\times(W\cap\tilde{T})^c;m_1,m_2]$ and $\mu(W\cap\tilde{T}) = \mu(W)\mu(\tilde{T}) < \alpha$, which is a contradiction. We can similarly proved that the set $\{\mu(T): f \succ [X\times T,X\times T^c;m_1,m_2]\}$ is also an open interval of [0,1]. Let $\hat{\alpha}$ be an element in [0,1] but in neither (of the) intervals, and let $\hat{T}$ be an element in B_X with $\mu(\hat{T}) = \hat{\alpha}$. Then $[X\times\hat{T},X\times\hat{T}^c;m_1,m_2] \sim f$. Q.E.D.

<u>Lemma 3.</u> Assume that $m_1 \succ m_2$ and $m_3 \succ m_4$. Then $[R\times Y, R^c\times Y; m_1, m_2] \sim [X\times S, X\times S^c; m_1, m_2]$ implies $[R\times Y, R^c\times Y; m_3, m_4] \sim [X\times S, X\times S^c; m_3, m_4]$.

<u>Proof</u>. Let $m_1 \succ m_2$. Consider the case where $m_1 \succsim m_3 \succsim m_2$. Then $u(m_3) = \alpha u(m_1) + (1-\alpha)u(m_2)$ for some $\alpha \in [0,1]$. We can find a $\hat{T} \in B_X$ such that $\mu(\hat{T}) = \alpha$ and $\mu(\hat{T} \cap S) = \mu(\hat{T})\mu(S)$. Then it follows from Axiom 3 that

$$(X\times\hat{T})[R\times Y, R^c\times Y; m_1, m_2] + (X\times\hat{T}^c)m_2 \sim (X\times\hat{T})[X\times S, X\times S^c; m_1, m_2] + (X\times\hat{T}^c)m_2 ,$$

that is,

$$(18) \quad [R\times\hat{T}, (R\times\hat{T})^c; m_1, m_2] \sim [X\times(\hat{T}\cap S), X\times(\hat{T}\cap S)^c; m_1, m_2] .$$

Since $m_3 \sim [X\times\hat{T}, X\times\hat{T}^c; m_1, m_2]$ by the definition of $\hat{T}$, we have

$$\begin{aligned}
& [R\times Y, R^c\times Y; m_3, m_2] = (R\times Y)m_3 + (R^c\times Y)m_2 \\
\sim\ & (R\times Y)[X\times\hat{T}, X\times\hat{T}^c; m_1, m_2] + (R^c\times Y)m_2 && \text{(by Axiom 3)} \\
=\ & [R\times\hat{T}, (R\times\hat{T})^c; m_1, m_2] \sim [X\times(\hat{T}\cap S), X\times(\hat{T}\cap S)^c; m_1, m_2] && \text{(by (18))} \\
=\ & (X\times S)[X\times\hat{T}, X\times\hat{T}^c; m_1, m_2] + (X\times S^c)m_2 \sim (X\times S)m_3 + (X\times S^c)m_2 && \text{(by Axiom 3)} \\
=\ & [X\times S, X\times S^c; m_3, m_2].
\end{aligned}$$

Thus we have shown that

(19) if $m_1 \succsim m_3 \succsim m_2$ and if $[R\times Y, R^c\times Y; m_1, m_2] \sim [X\times S, X\times S^c; m_1, m_2]$, then $[R\times Y, R^c\times Y; m_3, m_2] \sim [X\times S, X\times S^c; m_3, m_2]$.

We can also prove that

(19') if $m_1 \succsim m_4 \succsim m_2$ and if $[R\times Y, R^c\times Y; m_1, m_2] \sim [X\times S, X\times S^c; m_1, m_2]$, then $[R\times Y, R^c\times Y; m_1, m_4] \sim [X\times S, X\times S^c; m_1, m_4]$.

In the case where $m_1 \succsim m_3 \succ m_4 \succsim m_2$, we have the assertion of the Lemma, using (19) and (19').

Next suppose that $m_1 \succsim m_3 \succ m_2 \succ m_4$. From Lemma 2, there is a $\hat{T} \varepsilon B_Y$ such that $[R\times Y, R^c\times Y; m_3, m_4] \sim [X\times\hat{T}, X\times\hat{T}^c; m_3, m_4]$. Then it follows from (19') that

$$[R\times Y, R^c\times Y; m_3, m_2] \sim [X\times\hat{T}, X\times\hat{T}^c; m_3, m_2]$$

Since $[R\times Y, R^c\times Y; m_3, m_2] \sim [X\times S, X\times S^c; m_3, m_2]$ by (19), we have $\mu(S) = \mu(\hat{T})$ by (16'). Axiom 4 together with this implies

$$[R\times Y, R^c\times Y; m_3, m_4] \sim [X\times\hat{T}, X\times\hat{T}^c; m_3, m_4] \sim [X\times S, X\times S^c; m_3, m_4] .$$

Consider the case where $m_1 \succ m_2 \succsim m_3 \succ m_4$. In the same way as the above, we have

$$[R\times Y, R^c\times Y; m_1, m_2] \sim [X\times S, X\times S^c; m_1, m_2] \text{ implies } [R\times Y, R^c\times Y; m_1, m_4] \sim [X\times S, X\times S^c; m_1, m_4].$$

Then it follows from (19) that $[R\times Y, R^c\times Y; m_3, m_4] \sim [X\times S, X\times S^c; m_3, m_4]$.

The same argument can be applied to the cases where $m_3 \succ m_4 \succsim m_1 \succ m_2$ and $m_3 \succ m_1 \succ m_4$. Q.E.D.

<u>Lemma 4.</u> If $[R\times Y, R^c\times Y; m_1, m_2] \sim [X\times S, X\times S^c; m_1, m_2]$, then $[R\times Y, R^c\times Y; m_2, m_1] \sim [X\times S, X\times S^c; m_2, m_1]$.

<u>Proof.</u> If $m_1 \sim m_2$, then, from Lemma 2, $m_1 \sim [R\times Y, R^c\times Y; m_1, m_2] \sim m_2$ and $m_1 \sim [X\times S, X\times S^c; m_1, m_2] \sim m_2$, i.e., $[R\times Y, R^c\times Y; m_1, m_2] \sim [X\times S, X\times S^c; m_1, m_2]$.

Assume that $m_1 \succ m_2$ or $m_2 \succ m_1$. From Lemma 2, there is a $\hat{T} \varepsilon B_Y$ such that

$$[R\times Y, R^c\times Y; m_2, m_1] \sim [X\times\hat{T}, X\times\hat{T}^c; m_2, m_1] .$$

We can find a $W \in B_Y$ so that $\mu(W) = \frac{1}{2}$ and W is independent of S and $\hat{T}$. Then it follows from Axiom 3 that

$$(20) \quad (X\times W)[R\times Y, R^c\times Y; m_1, m_2] + (X\times W^c)[R\times Y, R^c\times Y; m_2, m_1] \sim (X\times W)[X\times S, X\times S^c; m_1, m_2] + (X\times W^c)[X\times \hat{T}, X\times \hat{T}^c; m_2, m_1].$$

The first lottery of (20) can be written as

$$[R\times W, R^c\times W, R\times W^c, R^c\times W^c; m_1, m_2, m_2, m_1].$$

Since $\mu(W) = \mu(W^c) = \frac{1}{2}$, this lottery is, by Axiom 4, indifferent to

$$[R\times W, R^c\times W^c, R\times W^c, R^c\times W\ ; m_1, m_2, m_2, m_1] = [X\times W, X\times W^c; m_1, m_2].$$

Then it follows from (20) that

$$[X\times W, X\times W^c; m_1, m_2] \sim (X\times W)[X\times S, X\times S^c; m_1, m_2] + (X\times W^c)[X\times \hat{T}, X\times \hat{T}^c; m_2, m_1]$$
$$= [X\times (W\cap S), X\times (W\cap S^c), X\times (W^c\cap \hat{T}), X\times (W^c\cap \hat{T}^c); m_1, m_2, m_2, m_1].$$

This togeter with (16') implies that $\mu(W) = \mu((W\cap S)\cup(W^c\cap \hat{T}^c)) = \frac{1}{2}$. That is, $\frac{1}{2}\mu(S) + \frac{1}{2}\mu(\hat{T}^c) = \frac{1}{2}$, which is equivalent to $\mu(S) = \mu(\hat{T})$. Therefore it follows from Axiom 4 that

$$[R\times Y, R^c\times Y; m_2, m_1] \sim [X\times \hat{T}, X\times \hat{T}^c, m_2, m_1] \sim [X\times S, X\times S^c; m_2, m_1].$$

Q.E.D.

Let m_1, m_2 be arbitrary elements in M with $m_1 \succ m_2$ or $m_2 \succ m_1$. For each $R \in B_X$, define P(R) by

$$(21) \quad P(R) = \mu(\hat{T}), \text{ where } [R\times Y, R^c\times Y; m_1, m_2] \sim [X\times \hat{T}, X\times \hat{T}^c; m_1, m_2].$$

Lemmas 3 and 4 ensure that this definition of $P : B_X \to [0,1]$ is possible independently of the choice of m_1, m_2 ($m_1 \succ m_2$ or $m_2 \succ m_1$). Lemma 2 ensures the existence of such a $\hat{T}$, and the uniqueness of $\mu(\hat{T})$ follows (16').

Lemma 5. Let $m_1 \succ m_2$ or $m_2 \succ m_1$. Then $[R\times S, (R\times S)^c; m_1, m_2] \sim [X\times T, X\times T^c; m_1, m_2]$ if and only if $\mu(T) = P(R)\mu(S)$.

Proof. Necessity: Let W be an element in B_Y such that W is independent of S and $\mu(W) = P(R)$. Then we have, by the definition of P,

$$(22) \quad [R\times Y, R^c\times Y; m_1, m_2] \sim [X\times W, X\times W^c; m_1, m_2].$$

It follows that

$$[R\times S, (R\times S)^c; m_1, m_2] = (X\times S)[R\times Y, R^c\times Y; m_1, m_2] + (X\times S^c)m_2$$

$$\sim (X\times S)[X\times W, X\times W^c; m_1, m_2] + (X\times S^c)m_2 \qquad \text{(by Axiom 3 and (22))}$$

$$= [X\times(S\cap W), X\times(S\cap W)^c; m_1, m_2] .$$

Therefore $[X\times(S\cap W), X\times(S\cap W)^c; m_1, m_2] \sim [X\times T, X\times T^c; m_1, m_2]$. Since $m_1 \succ m_2$ or $m_2 \succ m_1$, we have $\mu(T) = \mu(S\cap W) = \mu(S)\mu(W) = \mu(S)P(R)$.

Sufficiency: From Lemma 2, there is a $\hat{T} \in B_Y$ such that $[R\times S, (R\times S)^c; m_1, m_2] \sim [X\times\hat{T}, X\times\hat{T}^c; m_1, m_2]$. In the proof of necessity, we have shown that if $[R\times S, (R\times S)^c; m_1, m_2] \sim [X\times\hat{T}, X\times\hat{T}^c; m_1, m_2]$, then $\mu(\hat{T}) = P(R)\mu(S)$. Axiom 4 states that $\hat{T}$ can be replaced by any $T \in B_Y$ with the same objective probability $\mu(\hat{T})$. That is, $[R\times S, (R\times S); m_1, m_2] \sim [X\times T, X\times T^c; m_1, m_2]$ for all T with $\mu(T) = P(R)\mu(S)$. Q.E.D.

Remark 7. The sufficiency of Lemma 5 is true without the assumption that $m_1 \succ m_2$ or $m_2 \succ m_1$. Indeed, if $m_1 \sim m_2$, then $m_1 \sim [X\times T, X\times T^c; m_1, m_2] \sim m_2$ and $m_1 \sim [R\times S, (R\times S)^c; m_1, m_2] \sim m_2$ by Lemma 2.

Lemma 6. Let $(R_1\times S_1,\ldots,R_k\times S_k)$ and $(X\times T_1,\ldots,X\times T_k)$ be partitions of $X\times Y$, respectively. Then, for all $m_1,\ldots,m_k \in M$,

(i) if $P(R_t)\mu(S_t) \leqq \mu(T_t)$ for all $t = 1,\ldots,k$, then $[R_1\times S_1,\ldots,R_k\times S_k;m_1,\ldots,m_k] \precsim [X\times T_1,\ldots,X\times T_k;m_1,\ldots,m_k]$;

(ii) if $P(R_t)\mu(S_t) \geqq \mu(T_t)$ for all $t = 1,\ldots,k$, then $[R_1\times S_1,\ldots,R_k\times S_k;m_1,\ldots,m_k] \succsim [X\times T_1,\ldots,X\times T_k;m_1,\ldots,m_k]$.

Proof. It sufficies to prove (i). Let $\underline{m}$ be an element in M with $m_t \succsim \underline{m}$ for all $t = 1,\ldots,k$. Let $\hat{T}_t$ be an element in B_Y with $\mu(\hat{T}_t) = P(R_t)\mu(S_t)$ for $t = 1, \ldots,k$. Then we have, from Lemma 5 and Remark 7,

$$[R_t\times S_t,(R_t\times S_t)^c;m_t,\underline{m}] \sim [X\times\hat{T}_t,X\times\hat{T}_t^c;m_t,\underline{m}] \quad \text{for all } t = 1,\ldots,k.$$

Since $\mu(T_t) \geqq \mu(\hat{T}_t) = P(R_t)\mu(S_t)$ and $m_t \succsim \underline{m}$ for all $t = 1,\ldots,k$, it follows from (16') that

(23) $[X\times T_t,X\times T_t^c;m_t,\underline{m}] \succsim [R_t\times S_t,(R_t\times S_t)^c;m_t,\underline{m}]$ for all $t = 1,\ldots,k$.

Let $(X\times S_1^*,\ldots,X\times S_j^*)$ be the corsest partition of $X\times Y$ satisfying

$$S_t\cap S_{t'}^* \neq \phi \text{ implies } S_t\supset S_{t'}^* .$$

Let $(X\times U_1,\ldots,X\times U_i)$ be the corsest common refinement of $(X\times S_1^*,\ldots,X\times S_j^*)$ and $(X\times T_1,\ldots,X\times T_k)$. Let $(X\times W_1,\ldots,X\times W_k)$ be a partition of $X\times Y$ such that $\mu(W_t\cap U_\ell) = \mu(U_\ell)/k$ for all $\ell = 1,\ldots,i$. Then it can be verified that

(24) $\mu(W_t) = 1/k$ for all $t = 1,\ldots,k$;

(25) W_t is independent of S_ℓ and $T_{\ell'}$ for all ℓ,ℓ' $(1 \leqq \ell,\ell' \leqq k)$.

Denote $[R_t\times S_t,(R_t\times S_t)^c;m_t,\underline{m}]$ and $[X\times T_t,X\times T_t^c;m_t,\underline{m}]$ by f_t and g_t ($t = 1,\ldots,k$) respectively. Now consider the lottery $(X\times W_1)f_1+(X\times W_1^c)\sum_{t=1}^{k}(X\times W_t)f_t$. Since

W_1 is independent of S_1 and T_1, we have, by Axiom 3,

$$(X\times W_1)f_1+(X\times W_1^c)\sum_{t=1}^{k}(X\times W_t)f_t \precsim (X\times W_1)g_1+(X\times W_1^c)\sum_{t=1}^{k}(X\times W_t)f_t$$
$$= (X\times W_2)f_2+(X\times W_2^c)[(X\times W_1)g_1+\sum_{t=1}^{k}(X\times W_t)f_t] .$$

Since W_2 is independent of S_2 and T_2 , we have, again by Axiom 3,

$$(X\times W_2)f_2+(X\times W_2^c)[(X\times W_1)g_1+\sum_{t=2}^{k}(X\times W_t)f_t]$$
$$\precsim (X\times W_2)g_2+(X\times W_2^c)[(X\times W_1)g_1+\sum_{t=2}^{k}(X\times W_t)f_t]$$
$$= (X\times W_3)f_3+(X\times W_3^c)[(X\times W_1)g_1+(X\times W_2)g_2+\sum_{t=3}^{k}(X\times W_t)f_t].$$

Repeating the same argument, we have, for $\ell = 1,\ldots,k$,

$$(26) \quad \sum_{t=1}^{k}(X\times W_t)f_t \precsim \sum_{t=1}^{\ell}(X\times W_t)g_t+\sum_{t=\ell+1}^{k}(X\times W_t)f_t \precsim \sum_{t=1}^{k}(X\times W_t)g_t .$$

The lottery $\sum_{t=1}^{k}(X\times W_t)f_t$ can be written as

$$[R_1\times(W_1\cap S_1),\ldots,R_k\times(W_k\cap S_k),\bigcup_{t\neq 1}R_t\times(W_1\cap S_t),\ldots,\bigcup_{t\neq 1}R_t\times(W_k\cap S_t);$$
$$m_1,\ldots,m_k,\underline{m},\ldots,\underline{m}]$$

$$= [R_1\times(W_1\cap S_1),\ldots,R_k\times(W_k\cap S_k),R_1\times(W_1^c\cap S_1),\ldots,R_k\times(W_k^c\cap S_k);$$
$$m_1,\ldots,m_k,\underline{m},\ldots,\underline{m}].$$

Since $\mu(W_t\cap S_t) = \mu(S_t)/k = \mu(W_1\cap S_t)$ and $\mu(W_t^c\cap S_t) = \mu(W_t^c)\mu(S_t) = \frac{k-1}{k}\mu(S_t)$ $= \mu(W_1^c\cap S_t)$ for all $t = 1,\ldots,k$ by (24) and (26), we have, by Axiom 4,

$$[R_1\times(W_1\cap S_1),\ldots,R_k\times(W_k\cap S_k),R_1\times(W_1^c\cap S_1),\ldots,R_k\times(W_k^c\cap S_k);m_1,\ldots,m_k,\underline{m},\ldots,\underline{m}]$$
$$\sim [R_1\times(W_1\cap S_1),\ldots,R_k\times(W_1\cap S_k),R_1\times(W_1^c\cap S_1),\ldots,R_k\times(W_1^c\cap S_k);m_1,\ldots,m_k,\underline{m},\ldots,\underline{m}]$$

$$= (X\times W_1)[R_1\times S_1,\ldots,R_k\times S_k;m_1,\ldots,m_k] +(X\times W_1^c)\underline{m} .$$

In the same way, we can prove that

$$\sum_{t=1}^{k} (X\times W_t)g_t \sim (X\times W_1)[X\times T_1,\ldots,X\times T_k;m_1,\ldots,m_k]+(X\times W_1^c)\underline{m} .$$

Then it follows from (26) that

$$(X\times W_1)[R_1\times S_1,\ldots,R_k\times S_k;m_1,\ldots,m_k]+(X\times W_1^c)\underline{m}$$

$$\precsim (X\times W_1)[X\times T_1,\ldots,X\times T_k;m_1,\ldots,m_k]+(X\times W_1^c)\underline{m} .$$

Then it follows from Axiom 3 that $[R_1\times S_1,\ldots,R_k\times S_k;m_1,\ldots,m_k] \precsim [X\times T_1,\ldots,X\times T_k;$ $m_1,\ldots,m_k]$. Q.E.D.

<u>Lemma 7.</u> For all partition $(R_1\times S_1,\ldots,R_k\times S_k)$ of $X\times Y$, $\sum_{t=1}^{k} P(R_t)\mu(S_t) = 1$.

<u>Proof.</u> Suppose that $\sum_{t=1}^{k} P(R_t)\mu(S_t) < 1$. Then there is a partition $(X\times\hat{T}_1,\ldots,$ $X\times\hat{T}_k)$ of $X\times Y$ such that $\mu(\hat{T}_t) > P(R_t)\mu(S_t)$ for all $t = 1,\ldots,k$. Then Lemma 6 states that $[X\times\hat{T}_1,\ldots,X\times\hat{T}_k;m_1,\ldots,m_k] \succsim [R_1\times S_1,\ldots,R_k\times S_k;m_1,\ldots,m_k]$ for all $m_1,\ldots,m_k \in M$.

Let $m_1 \prec m_2 = \ldots = m_k$. Then the above claim states that $[X\times\hat{T}_1,X\times\hat{T}_1^c;m_1,m_2]$ $= [X\times\hat{T}_1,X\times\hat{T}_2,\ldots,X\times\hat{T}_k;m_1,m_2,\ldots,m_2] \succsim [R_1\times S_1,\ldots,R_k\times S_k;m_1,m_2,\ldots,m_2] =$ $[R_1\times S_1,(R_1\times S_1)^c;m_1,m_2]$. However Lemma 5 states that if $\mu(W_1) = P(R_1)\mu(S_1)$, then $[X\times W_t,X\times W_t^c;m_1,m_2] \sim [R_1\times S_1,(R_1\times S_1)^c;m_1,m_2]$. Since $\mu(\hat{T}_1) > P(R_1)\mu(S_1)$ $= \mu(W_1)$ and $m_2 \succ m_1$, we have

$$[X\times W_1,X\times W_1^c;m_1,m_2] \succ [X\times\hat{T}_1,X\times\hat{T}_1^c;m_1,m_2] \succsim [R_1\times S_1,(R_1\times S_1)^c;m_1,m_2] ,$$

which is a contradiction. Therefore we have $\sum_{t=1}^{k} P(R_t)\mu(S_t) \geq 1$.

In the same way, we can prove that $\sum_{t=1}^{k} P(R_t)\mu(S_t) \leq 1$. Q.E.D.

<u>Lemma 8.</u> Let $(R_1 \times S_1, \ldots, R_k \times S_k)$ and $(X \times T_1, \ldots, X \times T_k)$ be partitions of $X \times Y$ with $\mu(T_t) = P(R_t)\mu(S_t)$ for all $t = 1, \ldots, k$. Then

$$(27) \quad [R_1 \times S_1, \ldots, R_k \times S_k; m_1, \ldots, m_k] \sim [X \times T_1, \ldots, X \times T_k; m_1, \ldots, m_k]$$

$$\text{for all } m_1, \ldots, m_k \in M.$$

<u>Proof.</u> This lemma follows immediately Lemma 6.

Extend the utility function u defined by (17) to L by

$$(28) \quad u([R_1 \times S_1, \ldots, R_k \times S_k; m_1, \ldots, m_k]) = \sum_{t=1}^{k} P(R_t)\mu(S_t)u(m_t)$$

$$\text{for all } [R_1 \times S_1, \ldots, R_k \times S_k; m_1, \ldots, m_k] \in L.$$

Lemma 8, (15') and (16') imply condition (6) of the main Theorem. The finite additivity of the measure P follows Lemma 7. Indeed, for arbitrary $R_1, R_2 \in B_X$ with $R_1 \cap R_2 = \emptyset$, $P(R_1)+P(R_2)+P((R_1 \cup R_2)^c) = 1$ and $P(R_1 \cup R_2)+P((R_1 \cup R_2)^c) = 1$ by Lemma 7, because $(R_1 \times Y, R_2 \times Y, (R_1 \cup R_2)^c \times Y)$ and $((R_1 \cup R_2) \times Y, (R_1 \cup R_2)^c \times Y)$ are partitions of $X \times Y$. Therefore we have $P(R_1)+P(R_2) = P(R_1 \cup R_2)$.

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