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AN EVERYWHERE CONVERGENT SERIES REPRESENTATION OF THE DISTRIBUTION OF HOTELLING's GENERALIZED T_0^2

by

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0. ABSTRACT

A new series representation of the distribution of Hotelling's generalized T_0^2 statistic is found which is everywhere convergent. Earlier results by Constantine which are convergent on the interval $[0,1)$ are also derived from the formulae given here. The new results are made possible by the use of a matrix operator calculus developed by the author.

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1. INTRODUCTION

Let S_1 $(m \times m)$ and S_2 $(m \times m)$ have independent Wishart distributions with n_1, n_2 degrees of freedom respectively and the same population covariance matrix Σ. S_1 may be noncentral and we denote the noncentrality matrix by Ω. The generalized T_0^2 statistic is then defined by [2]

$$T = T_0^2/n_2 = \mathrm{tr}(S_1 S_2^{-1}) .$$

Since its introduction by Lawley [11] and later by Hotelling [9] [10] in connection with wartime problems of multivariate quality control the distribution of this statistic has attracted a good deal of theoretical interest amongst statisticians. A fundamental contribution was made by Constantine [2], who found a zonal polynomial series representation of the distribution of T. However, Constantine's series converge only for $0 \leq T < 1$. In subsequent research, Davis [3] [4] has discovered a differential equation satisfied by the density of T in the null case $(\Omega = 0)$ which has facilitated the numerical computation of percentage points for special cases $(m = 3, 4)$. Pillai and Young [18] have also worked on the problem and found further specialized results for the null distribution. When $m = 2$ Hotelling [10] derived a simple formula for the null distribution which can be written as a Gaussian hypergeometric series which is everywhere convergent in T [2].[1] This formula has been the source of conjectures by Constantine [2], Pillai [17] and others concerning possible general forms of

[1] There appears to be an error in the formula given by Constantine [2] and later by Davis [3]. The correct formula has an additional factor of (1/2). [15] provides a new derivation of this formula.

the density. However, until the present no progress has been made on the analytic derivation of the exact density in the general case. The reader is referred to the articles [16], [17] by Pillai for a detailed review.

The purpose of the present paper is to offer a fresh approach to the problem of the distribution of T. The techniques used here involve a matrix operator calculus which has been developed by the author in other work [13], [14], [15] to solve certain problems of distribution theory primarily of interest in econometrics. However, these methods offer a convenient solution to the apparently long standing problem of a general solution to the distribution of T in both the null and the noncentral cases. It seems likely that these methods will also provide a means by which related problems of multivariate statistical theory, such as the distribution of Pillai's trace, may be resolved.

2. THE NULL DISTRIBUTION OF T

Since T is invariant under the transformations $S_1 \to \Sigma^{-1/2} S_1 \Sigma^{-1/2}$ and $S_2 \to \Sigma^{-1/2} S_2 \Sigma^{-1/2}$ we set the common population covariance matrix $\Sigma = I$. Now let $S_1 = XX'$ where the $m \times n_1$ matrix X is $N_{m,n_1}(M, I_{mn_1})$. We write $T = \mathrm{tr}(S_1 S_2^{-1}) = x'(I_{n_1} \otimes S_2^{-1})x$ where $x = \mathrm{vec}(X)$ and $\mathrm{vec}(\ )$ denotes vectorization by columns. Conditional on S_2, T is distributed as a quadratic form in normal variates. In the null case $M = 0$, $\Omega = MM'/2 = 0$ and we have the density

$$(1)\quad \mathrm{pdf}(T|S_2) = \left[2^{mn_1/2}\Gamma(mn_1/2)\right]^{-1} T^{mn_1/2-1}\left(\det(I\otimes S_2)\right)^{1/2} {}_0F_0\left(-(1/2)I\otimes S_2,\ T\right)$$

where

$$ {}_0F_0\left(-(1/2)I\otimes S_2,\ T\right) = \int_{O(mn_1)} \mathrm{etr}\left(-(1/2)T(I\otimes S_2)HE_{11}H'\right)(\underline{dH}) $$

and where $O(mn_1)$ is the orthogonal group of $mn_1 \times mn_1$ matrices, $E_{11} = e_1e_1'$ where e_1 is the first unit vector and $(\underline{dH})$ represents the normalized Haar measure on $O(mn_1)$.

The unconditional density of T therefore has the following form:

$$ (2)\quad \mathrm{pdf}(T) = \frac{y^{mn_1/2-1}}{2^{m(n_1+n_2)/2}\,\Gamma(mn_1/2)\Gamma_m(n_2/2)} \cdot\int_{S_2>0} \mathrm{etr}\left\{-(1/2)S_2\right\}\left(\det S_2\right)^{(n_1+n_2-m-1)/2} {}_0F_0\left(-(1/2)I\otimes S_2,\ T\right)dS_2\ . $$

We introduce an auxiliary matrix W of dimension $m\times m$ and use ∂W in what follows to denote the matrix differential operator $\partial/\partial W$. With this notation we may write in operator form:

$$ (3)\quad {}_0F_0\left(-(1/2)I\otimes S_2,\ T\right) = \int_{O(mn_1)} \mathrm{etr}\left\{-(1/2)(I\otimes\partial W)HE_{11}H'\right\}\mathrm{etr}(S_2W)\Big|_{W=0}(\underline{dH}) $$

$$ = \int_{O(mn_1)} \mathrm{etr}\left\{-(1/2)(I\otimes\partial W)HE_{11}H'\right\}(\underline{dH}) $$

$$ = {}_0F_0\left(-(1/2)I\otimes\partial W,\ T\right)\mathrm{etr}(S_2W)\Big|_{W=0}\ . $$

The evaluation at $W = 0$ may be performed outside the integral in view of its uniform convergence; and ${}_0F_0\left(-(1/2)I\otimes\partial W,\ T\right)$ may be regarded as a linear operator on a matrix space of complex analytic functions of W .

From (2) and (3) we deduce that

$$(4)\quad \mathrm{pdf}(T) = \frac{y^{mn_1/2-1}}{2^{m(n_1+n_2)/2}\,\Gamma(mn_1/2)\Gamma_m(n_2/2)}$$

$$\cdot\int_{S_2>0}\left[{}_0F_0\left(-(1/2)I\otimes\partial W,\ T\right)\mathrm{etr}\left\{-(1/2)S_2(I-2W)\right\}\right]_{W=0}\left(\det\ S_2\right)^{(n_1+n_2-m-1)/2}dS_2$$

The integral over S_2 in (4) is absolutely and uniformly convergent for all W satisfying $\mathrm{Re}(W) \leq \varepsilon I$ where ε is any positive quantity less than 1/2. We may therefore take both the operator and the evaluation at $W = 0$ outside the integration, yielding:

$$\mathrm{pdf}(T) = \frac{T^{mn_1/2-1}\,\Gamma_m\left((n_1+n_2)/2\right)}{\Gamma(mn_1/2)\Gamma_m(n_2/2)}\cdot{}_0F_0\left(-(1/2)I\otimes\partial W,\ T\right)\det\left(I-2W\right)^{-(n_1+n_2)/2}\Big|_{W=0}$$

$$(5)\qquad = \frac{\Gamma_m\left((n_1+n_2)/2\right)T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)}\cdot{}_0F_0(-I\otimes\partial Z,\ T)\det(I-Z)^{-(n_1+n_2)/2}\Big|_{Z=0}$$

$$(6)\qquad = \frac{\Gamma_m\left((n_1+n_2)/2\right)T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)}\int_{V_{1,mn_1}}\mathrm{etr}\left\{-T(I\otimes\partial Z)hh'\right\}\underline{(dh)}\det(I-Z)^{-(n_1+n_2)/2}\Big|_{Z=0}$$

$$(7)\qquad = \frac{\Gamma_m\left((n_1+n_2)/2\right)T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)}\int_{V_{1,mn_1}}\det(I+TQ)^{-(n_1+n_2)/2}\underline{(dh)}$$

where V_{1,mn_1} denotes the Stiefel manifold $h'h = 1$ where h is $mn_1 \times 1$, $\underline{(dh)}$ denotes the normalized Haar measure on this manifold and

$$(8)\quad Q = \sum_{i=1}^{n_1} h_i h_i'$$

where the h_i are the $m \times 1$ vectors taken from the partition of

$h' = (h_1', h_2', \ldots, h_{n_1}')$ into n_1 component vectors.

(7) is an extremely simple representation of the exact null distribution of T. Moreover, by expanding $\det(I+TQ)^{-(n_1+n_2)/2}$ in its usual zonal polynomial series (valid of course for $0 \le T < 1$) and integrating, we obtain the series discovered by Constantine [2] in 1966. The reader is referred to [15] for this derivation of Constantine's series. An alternative and simpler demonstration proceeds from (6) and uses the easily established identity:

$$(9) \quad C_{(k)}\left(I_{n_1} \otimes \partial Z\right) = \sum_{\kappa} \frac{\left(\frac{n_1}{2}\right)_\kappa}{\left(\frac{1}{2}\right)_k} C_\kappa(\partial Z) \ ;$$

and rule of differentiation:

$$(10) \quad C_\kappa(\partial Z)\det(I-Z)^{-\alpha} = \left(\frac{\alpha}{2}\right)_\kappa C_\kappa\left((I-Z)^{-1}\right)\det(I-Z)^{-\alpha} \ .$$

Using (9) and (10) in (6) we obtain directly the Constantine formula:

$$(11) \quad \mathrm{pdf}(T) = \frac{\Gamma_m\left((n_1+n_2)/2\right)T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} \sum_{k=0}^{\infty} \frac{(-T)^k}{k!\,(mn_1/2)_k} \sum_{\kappa}\left(\frac{n_1+n_2}{2}\right)_\kappa \left(\frac{n_1}{2}\right)_\kappa C_\kappa(I_m)$$

which is convergent for $0 \le T < 1$.

An alternative formula for the density which is everywhere convergent over $T > 0$ is obtained as follows. We note in fact that

$$\int_{V_{1,mn_1}} \text{etr}\left\{-T(I\otimes\partial Z)hh'\right\}(\underline{dh})$$

$$= {}_1F_1\left(\frac{1}{2}, \frac{mn_1}{2}; -T(I\otimes\partial Z)\right)$$

$$= {}_1F_1\left(\frac{mn_1}{2} - \frac{1}{2}; \frac{mn_1}{2}; T(I\otimes\partial Z)\right)\text{etr}\left\{-T(I\otimes\partial Z)\right\}$$

and

$$\text{etr}\left(-T(I\otimes\partial Z)\right)\det(I-Z)^{-(n_1+n_2)/2} = \text{etr}(-n_1T\partial Z)\det(I-Z)^{-(n_1+n_2)/2}$$

$$= \det[(1+n_1T)I - Z]^{-(n_1+n_2)/2}$$

$$= (1+n_1T)^{-m(n_1+n_2)/2}\det[I-Z/(1+n_1T)]^{-(n_1+n_2)/2}$$

Transforming $Z \to Z/(1+n_1T) = X$ we obtain

(12)
$$\text{pdf}(T) = \frac{\Gamma_m\left((n_1+n_2)/2\right)T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)(1+n_1T)^{m(n_1+n_2)/2}}$$

$$\cdot {}_1F_1\left(\frac{mn_1}{2}-\frac{1}{2}, \frac{mn_1}{2}; \frac{T}{1+n_1T}(I\otimes\partial X)\right)\det(I-X)^{-(n_1+n_2)/2}\Bigg|_{X=0}$$

$$= \frac{T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)(1+n_1T)^{m(n_1+n_2)/2}}\cdot\int_{S>0}\text{etr}(-S)(\det S)^{(n_1+n_2-m-1)/2}$$

$$\cdot {}_1F_1\left(\frac{mn_1-1}{2}; \frac{mn_1}{2}; \frac{T}{1+n_1T}(I\otimes S)\right)ds$$

$$(13) = \frac{T^{mn_1/2-1}(1+n_1T)^{-m(n_1+n_2)/2}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} \sum_{k=0}^{\infty} \frac{\left(T/(1+n_1T)\right)^k}{k!}$$

$$\cdot \sum_{\kappa} \frac{\left(\frac{mn_1-1}{2}\right)_{\kappa}}{\left(\frac{mn_1}{2}\right)_{\kappa}} \int_{S>0} \operatorname{etr}(-S)(\det S)^{(n_1+n_2-m-1)/2} C_{\kappa}(I \otimes S)\,dS \ .$$

We write

$$(14) \quad C_{\kappa}(I_{n_1} \otimes S) = \sum_{\theta} b_{\theta}^{\kappa} C_{\theta}(S)$$

where the summation is over all partitions θ of k into $\leq m$ parts. The b_{θ}^{κ} are constant coefficients in this expansion for which explicit formulae are available in the case $\kappa = (k)$, viz $b_{\theta}^{(k)} = \left(\frac{n_1}{2}\right)_{\theta} \Big/ \left(\frac{1}{2}\right)_k$ (compare (9) above).

Using (14) in (13) we find upon integration:

$$(15) \quad \operatorname{pdf}(T) = \frac{\Gamma_m\left((n_1+n_2)/2\right) T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)(1+n_1T)^{m(n_1+n_2)/2}}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\left(T/(1+n_1T)\right)^k}{k!} \sum_{\theta} \left\{ \sum_{\kappa} b_{\theta}^{\kappa} \left(\frac{mn_1-1}{2}\right)_{\kappa} \Big/ \left(\frac{mn_1}{2}\right)_{\kappa} \right\} \left(\frac{n_1+n_2}{2}\right)_{\theta} C_{\theta}(I_m) \ .$$

To show that (15) is everywhere convergent in T we note first from (12) that the density may also be written as:

$$(16)\quad \mathrm{pdf}(T) = \frac{\Gamma_m\left((n_1+n_2)/2\right) T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)(1+n_1T)^{m(n_1+n_2)/2}}$$

$$\cdot \int_{V_{mn_1-1,mn_1}} \mathrm{etr}\left\{\left(T/(1+n_1T)\right)(I\otimes\partial X)KK'\right\}(\underline{dK})\det(I-X)^{-(n_1+n_2)/2}\Big|_{X=0}$$

where $K'K = I_{mn_1-1}$ and $(\underline{dK})$ denotes the normalized Haar measure on V_{mn_1-1,mn_1}. Now

$$(17)\quad \int_{V_{mn_1-1,mn_1}} \mathrm{etr}\left\{\left(T/(1+n_1T)\right)(I\otimes\partial X)KK'\right\}(\underline{dK})\det(I-X)^{-(n_1+n_2)/2}\Big|_{X=0}$$

$$= \int_{V_{mn_1-1,mn_1}} \mathrm{etr}\left\{\left(T/(1+n_1T)\right)P\partial X\right\}(\underline{dK})\det(I-X)^{-(n_1+n_2)/2}\Big|_{X=0}$$

$$= \int_{V_{mn_1-1,mn_1}} \det\left[I - \left(T/(1+n_1T)\right)P\right]^{-(n_1+n_2)/2}(\underline{dK})$$

where $P = \sum_{i=1}^{n_1} K_iK_i'$ and the K_i are the component $m\times(mn_1-1)$ matrices in the partition $K' = [K_1' \vdots K_2' \vdots \cdots \vdots K_{n_1}']$.

We note that if h is an $mn_1\times 1$ vector chosen so that $H'H = I_{mn_1}$ where $H = [K \vdots h]$ and that if h is partitioned as $h' = (h_1', h_2', \ldots, h_{n_1}')$ into n_1 component m-vectors h_i then

$$(18)\quad P = \sum_{i=1}^{n_1} K_iK_i' = n_1I_m - \sum_{i=1}^{n_1} h_ih_i'$$

and

$$(19)\quad |C_\kappa(P)| \le n_1^k C_\kappa(I_m)\ .$$

It follows that the series expansion

$$(20)\quad \int_{V_{mn_1-1,mn_1}} \det\left[I - \left(T/(1+n_1T)\right)P\right]^{-(n_1+n_2)/2} (\underline{dK})$$

$$= \sum_{k=0}^{\infty} \frac{\left(T/(1+n_1T)\right)^k}{k!} \sum_{\kappa}\left(\frac{n_1+n_2}{2}\right)_{\kappa} \int_{V_{mn_1-1,mn_1}} C_{\kappa}(P)\,(\underline{dK})$$

is convergent for all $T > 0$ since, in view of (19), it is majorized by the series

$$\sum_{k=0}^{\infty} \frac{\left(n_1T/(1+n_1T)\right)^k}{k!} \sum_{\kappa}\left(\frac{n_1+n_2}{2}\right)_{\kappa} C_{\kappa}(I_m)$$

which is everywhere convergent in $T > 0$. Moreover, since the series (20) is everywhere convergent in T so is the series representation of the density given by (15).

Two useful integral formulae are implied by the above results. First from (7) and (11) we have:

$$(21)\quad \int_{V_{1,mn_1}} C_{\kappa}(Q)\,(\underline{dh}) = \left(\frac{n_1}{2}\right)_{\kappa} C_{\kappa}(I_m) \Big/ \left(\frac{mn_1}{2}\right)_k$$

where $Q = \sum_{i=1}^{n_1} h_i h_i'$ is $m \times m$ and $h' = (h_1', h_2', \ldots, h_{n_1}')$. Second from (15) and (20) we have:

$$(22)\quad \int_{V_{mn_1-1,mn_1}} C_\kappa(P)(\underline{dK}) = \left\{\sum_{\kappa'} b_\kappa^{\kappa'} \left(\frac{mn_1-1}{2}\right)_{\kappa'} \left(\frac{mn_1}{2}\right)_{\kappa'}\right\} C_\kappa(I_m)$$

where P is defined in (18) and the summation over κ' is over all partitions of k into $\leq mn_1-1$ parts.

4. THE NONCENTRAL DISTRIBUTION OF T

Since $T = tr(XX'S_2^{-1})$ we start with the joint density of (X, S_2) :

$$\left[2^{m(n_1+n_2)/2}\pi^{mn_1/2}\Gamma_m(n_2/2)\right]^{-1} etr\{-(1/2)(X-M)(X-M)'\}etr\{-(1/2)S_2\}(\det S_2)^{(n_2-m-1)/2}$$

$$(23) = \left[2^{m(n_1+n_2)/2}\pi^{mn_1/2}\Gamma_m(n_2/2)\right]^{-1} etr(-\Omega)etr\{-(1/2)XX'\}etr(-XM')$$

$$\cdot etr\{-(1/2)S_2\}(\det S_2)^{(n_2-m-1)/2}$$

T is invariant under the simultaneous transformations $X \to HXK$, $S_2 \to HS_2H'$ where $H \in O(m)$ and $K \in O(n_1)$. Hence, making these substitutions in (23) and integrating over the (normalized) orthogonal groups we have:

$$(24)\quad \left[2^{m(n_1+n_2)/2}\pi^{mn_1/2}\Gamma_m(n_2/2)\right]^{-1} etr(-\Omega)etr\{-(1/2)XX'\}$$

$$\cdot {}_0F_1^{(m)}\left(\frac{n_1}{2}; \frac{1}{2}XX', \Omega\right)etr\{-(1/2)S_2\}(\det S_2)^{(n_2-m-1)/2} .$$

Transform $X \to S_2^{-1/2}X = Y$ in (24) giving

$$\left[2^{m(n_1+n_2)/2}\pi^{mn_1/2}\Gamma_m(n_2/2)\right]^{-1} etr(-\Omega)etr\{-(1/2)S_2YY'\}$$

$$\cdot {}_0F_1^{(m)}\left(\frac{n_1}{2}; \frac{1}{2}S_2YY', \Omega\right)etr\{-(1/2)S_2\}(\det S_2)^{(n_1+n_2-m-1)/2} .$$

We now write

$$y = \text{vec}(Y) = hT^{1/2}$$

where $T = y'y = x'(I \otimes S_2^{-1})x = \text{tr}(XX'S_2^{-1})$ and $h \in V_{1,mn_1}$. The measure changes according to $dy = (1/2)T^{mn_1/2-1}\, dT(dh)$ and we deduce:

$$\text{pdf}(T) = \left[2^{m(n_1+n_2)/2+1}\pi^{mn_1/2}\Gamma_m(n_2/2)\right]^{-1} \text{etr}(-\Omega)T^{mn_1/2-1}$$

$$\cdot\int_{V_{1,mn_1}}\int_{S_2>0} \text{etr}\left\{-\left(\frac{1}{2}\right)(I+TAA')S_2\right\}{}_0F_1^{(m)}\left(\frac{n_1}{2};\ \frac{T}{2}S_2AA',\ \Omega\right)(\det S_2)^{(n_1+n_2-m-1)/2}\, dS_2(dh)$$

$$(25) = \left[2^{m(n_1+n_2)/2}\Gamma(mn_1/2)\Gamma_m(n_2/2)\right]^{-1} \text{etr}(-\Omega)T^{mn_1/2-1}\int_{V_{1,mn_1}}\int_{S_2>0} \text{etr}\left\{-\left(\frac{1}{2}\right)(I+TAA')S_2\right\}$$

$$\cdot {}_0F_1^{(m)}\left(\frac{n_1}{2};\ \frac{T}{2}S_2AA',\ \Omega\right)(\det S_2)^{(n_1+n_2-m-1)/2}\, dS_2\underline{(dh)}$$

where $A = [h_1 \vdots \cdots \vdots h_{n_1}]$ the $m \times n_1$ matrix formed from the n_1 component m-vectors in the partition of $h' = (h_1', \ldots, h_{n_1}')$.

Performing the integration over $S_2 > 0$ in (25) we obtain

$$(26)\quad \text{pdf}(T) = \left(\Gamma_m\left((n_1+n_2)/2\right)/\Gamma(mn_1/2)\Gamma_m(n_2/2)\right)\text{etr}(-\Omega)T^{mn_1/2-1}$$

$$\cdot\int_{V_{1,mn_1}} {}_1F_1^{(m)}\left(\frac{n_1+n_2}{2},\ \frac{n_1}{2};\ T(I+TAA')^{-1}AA',\ \Omega\right)\det(I+TAA')^{-(n_1+n_2)/2}\underline{(dh)}\ .$$

Once again this is a very convenient general form for the density. Constantine's [2] series for the noncentral case may be deduced quite simply from (26). We use the easily established expansion:

$$(27)\quad \det(I-Z)^{-a}\,{}_1F_1^{(m)}\left(a,\ \gamma+p;\ B,\ -Z(I-Z)^{-1}\right) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a)_\kappa L_\kappa^{\gamma}(B)C_\kappa(Z)}{(\gamma+p)_\kappa k!C_\kappa(I_m)}$$

where $B > 0$, $\|Z\| < 1$, $\gamma > -1$, $p = (m+1)/2$ and $L_\kappa^{\gamma}(\ )$ denotes Constantine's generalized Laguerre polynomial of matrix argument. (27) was given by Muirhead in [12] (exercise 7.20, p. 290) although his result is in error in that his exponent for $\det(I-Z)$ should read "-a" as given above.

Now let $Z = -TAA'$ for $0 \le T < 1$, $B = \Omega$, $a = (n_1+n_2)/2$, and $\gamma+p = n_1/2$. We find from (26) and (27) the series:

$$\mathrm{pdf}(T) = \left(\Gamma_m\left((n_1+n_2)/2\right)/\Gamma(mn_1/2)\Gamma_m(n_2/2)\right)\mathrm{etr}(-\Omega)T^{mn_1/2-1}$$
$$\cdot\sum_{k=0}^{\infty}\frac{(-T)^k}{k!}\sum_{\kappa}\frac{\left(\frac{n_1+n_2}{2}\right)_\kappa L_\kappa^{\gamma}(\Omega)}{\left(\frac{n_1}{2}\right)_\kappa C_\kappa(I_m)}\cdot\int_{V_{1,mn_1}} C_\kappa(AA')(\underline{dh})$$

which is valid for $0 \le T < 1$. Using (21) we have immediately

$$(28)\quad \mathrm{pdf}(T) = \left(\Gamma_m\left((n_1+n_2)/2\right)/\Gamma(mn_1/2)\Gamma_m(n_2/2)\right)\mathrm{etr}(-\Omega)T^{mn_1/2-1}$$
$$\cdot\sum_{k=0}^{\infty}\frac{(-T)^k}{k!\left(\frac{mn_1}{2}\right)_k}\sum_{\kappa}\left(\frac{n_1+n_2}{2}\right)_\kappa L_\kappa^{\gamma}(\Omega)\ ,\quad \gamma = (n_1-m-1)/2$$

the series obtained by Constantine in [2] for the noncentral case by a different method.

To obtain an everywhere convergent series representation of the density of T we use the following generalized operator representation of the ${}_1F_1$ function in (26):

$$(29)\quad {}_1F_1^{(m)}\left(\frac{n_1+n_2}{2}, \frac{n_1}{2}; TAA'(I+TAA')^{-1}, \Omega\right)$$

$$= {}_1F_1^{(m)}\left(\frac{n_1+n_2}{2}, \frac{n_1}{2}; \partial W(I+\partial W)^{-1}, \Omega\right)\operatorname{etr}(TAA'W)\Big|_{W=0} .$$

In (29) W is an $m \times m$ matrix of auxiliary variables, ∂W denotes the matrix operator $\partial/\partial W$ and the ${}_1F_1$ function is an absolutely convergent power series of zonal polynomials in the operator $\zeta_W = \partial W(I+\partial W)^{-1}$. The zonal polynomial $C_\kappa(\zeta_W)$ may be interpreted as a Fourier integral operator defined by:

$$(30)\quad C_\kappa(\zeta_W)f(W) = \frac{2^{m(m-1)/2}\Gamma_m\left(\frac{m+1}{2}, \kappa\right)}{(2\pi i)^{m(m+1)/2}} \cdot\int_{\operatorname{Re}(Z)>0}\left\{\operatorname{etr}(\zeta_W Z)f(W)\right\}(\det Z)^{-(m+1)/2}C_\kappa(Z^{-1})dZ$$

provided the integral converges. In (30) $f(W)$ is a complex analytic function of W , $\Gamma_m(t,\kappa)$ is the coefficient defined by Constantine [1] and the integration is taken over $Z = X + iY$ with fixed $X > 0$ and Y ranging over all real symmetric matrices. In the present case we have

$$C_\kappa(\zeta_W)\operatorname{etr}(SW) = \frac{2^{m(m-1)/2}\Gamma_m\left(\frac{m+1}{2}, \kappa\right)\operatorname{etr}(SW)}{(2\pi i)^{m(m+1)/2}} \cdot\int_{\operatorname{Re}(Z)>0}\operatorname{etr}\{S(I+S)^{-1}Z\}(\det Z)^{-(m+1)/2}C_\kappa(Z^{-1})dZ$$

which converges for all $S > 0$.

The calculus underlying (30) has been explored by the author in other work [14] in the context of possibly complex and fractional operators such as $(\det \partial W)^\mu$. In particular, matrix operators such as $(\det \partial W)^\mu$ are developed in [14] as a generalization of the scalar Weyl calculus. The only other reference to matrix methods such as these that the author has been able to find is Garding [7]. In [7] Garding works with symmetric matrices W and a matrix generalization of the Riemann Liouville integral definition of a fractional operator. Both here and in my other work [13], [14] I have found that the Weyl definition of a fractional operator provides a basis for matrix generalizations which appears to be more suited to problems of multivariate statistical theory. One of the reasons for this is that Weyl operators applied to elementary functions (such as etr( ) and det( )) yield elementary functions, whereas Riemann Liouville operators applied to the same elementary functions no longer in general yield elementary functions. This gives the Weyl operator a significant advantage in the simplification of multivariate integrals. The fact that, in general, Weyl operators require stricter convergence criteria because the domain of integration is unbounded (whereas it is bounded for the Riemann Liouville operator) has not been found to be a limitation in this work so far. In large part this is due to the fact that the operators are applied to simple functions of the type etr(SW) and the domain of definition of S usually assures convergence. The reader is referred to [14] for a detailed development.

From (26) and (29) we obtain

$$(31)\quad \mathrm{pdf}(T) = \left[\Gamma_m\!\left((n_1+n_2)/2\right)/\Gamma(mn_1/2)\Gamma_m(n_2/2)\right]\mathrm{etr}(-\Omega)T^{mn_1/2-1}$$
$$\cdot\left[{}_1F_1^{(m)}\!\left(\frac{n_1+n_2}{2},\ \frac{n_1}{2};\ \zeta_W,\ \Omega\right)\int_{V_{1,mn_1}} \mathrm{etr}(TAA'W)\det(I+TAA')^{-(n_1+n_2)/2}\,\underline{(dh)}\right]_{W=0} .$$

We select an $mn_1 \times mn_1-1$ matrix K such that $H'H = I_{mn_1}$ where $H = [h \vdots K]$. Partitioning K as $K' = [K_1' \vdots K_2' \vdots \cdots \vdots K_{n_1}']$ where the component matrices K_i are $m \times mn_1-1$, we have

$$(32)\quad Q = AA' = \sum_{i=1}^{n_1} h_i h_i' = n_1 I_m - \sum_{i=1}^{n_1} K_i K_i' = n_1 I_m - P$$

(compare (18) above). Then

$$(33)\quad \det(I+TAA')^{-(n_1+n_2)/2} = (1+n_1T)^{-m(n_1+n_2)/2}\det\Big(I - (T/(1+n_1T))P\Big)^{-(n_1+n_2)/2}$$

$$= (1+n_1T)^{-m(n_1+n_2)/2}\operatorname{etr}\Big\{\Big(T/(1+n_1T)\Big)P\partial V\Big\}\det(I-V)^{-(n_1+n_2)/2}\Big|_{V=0}$$

which, in view of (10) and (19), has a convergent series representation for all $T \geq 0$.

From (31), (32) and (33) we have

$$(34)\quad \operatorname{pdf}(T) = \Big[\Gamma_m\Big((n_1+n_2)/2\Big)/\Gamma(mn_1/2)\Gamma_m(n_2/2)\Big]\operatorname{etr}(-\Omega)T^{mn_1/2-1}(1+n_1T)^{-m(n_1+n_2)/2}$$

$$\cdot\Big[{}_1F_1^{(m)}\Big(\frac{n_1+n_2}{2},\frac{n_1}{2};\zeta_W,\Omega\Big)\int_{O(mn_1)}\operatorname{etr}(TQW)\operatorname{etr}\Big\{\Big(T/(1+n_1T)\Big)P\partial V\Big\}(\underline{dH})\det(I-V)^{-(n_1+n_2)/2}\Big]_{\substack{V=0\\W=0}}$$

where ∂V denotes the matrix operator $\partial/\partial V$ and V is an $m \times m$ matrix of auxiliary variables. Now

$$(35)\quad \int_{O(mn_1)}\operatorname{etr}(TQW)\operatorname{etr}\Big\{\Big(T/(1+n_1T)\Big)P\partial V\Big\}(\underline{dH})$$

$$= \int_{O(mn_1)}\operatorname{etr}\Big\{T(I_{n_1}\otimes W)HE_{11}H'\Big\}\operatorname{etr}\Big\{\Big(T/(1+n_1T)\Big)(I_{n_1}\otimes\partial V)HE_{22}H'\Big\}(\underline{dH})$$

where $E_{11} = e_1 e_1'$ with e_1 the first unit vector and $E_{22} = I_{mn_1} - E_{11}$.
(35) may be expanded as an absolutely convergent power series in the form [5]

$$(36)\quad \sum_{k,\ell=0}^{\infty} \sum_{\kappa,\lambda;\varphi\in\kappa\cdot\lambda} C_{\varphi}^{\kappa,\lambda}\left(TI_{n_1}\otimes W, (T/(1+n_1T))I_{n_1}\otimes\partial V\right) C_{\varphi}^{\kappa,\lambda}(E_{11},E_{22})/k!\,\ell!\,C_{\varphi}(I_{mn_1}) \ .$$

In this expression $C_{\varphi}^{\kappa,\lambda}$ is an invariant polynomial in the elements of its two argument matrices. Such polynomials were introduced by Davis [5], [6] to extend the zonal polynomials. κ , λ and φ are partitions of k , ℓ and $f = k+\ell$ respectively into $\leq mn_1$ parts and the notation $\varphi \in \kappa\cdot\lambda$ which is defined in [5] relates the sets of partitions in the summation.

From (34), (35) and (36) we deduce the following general formula for the density of T :

$$(37)\quad \mathrm{pdf}(T) = \left[\Gamma_m\left((n_1+n_2)/2\right)/\Gamma(mn_1/2)\Gamma_m(n_2/2)\right] \mathrm{etr}(-\Omega)\, T^{mn_1/2-1}(1+n_1T)^{-m(n_1+n_2)/2}$$
$$\cdot\Bigg[{}_1F_1^{(m)}\left(\frac{n_1+n_2}{2}, \frac{n_1}{2}; \zeta_W, \Omega\right) \sum_{k,\ell=0}^{\infty} \sum_{\kappa,\lambda;\varphi\in\kappa\cdot\lambda} C_{\varphi}^{\kappa,\lambda}\left(TI_{n_1}\otimes W, (T/(1+n_1T))I_{n_1}\otimes\partial V\right)$$
$$\cdot C_{\varphi}^{\kappa,\lambda}(E_{11},E_{22})/k!\,\ell!\,C_{\varphi}(I_{mn_1}) \det(I-V)^{-(n_1+n_2)/2}\Bigg]_{\substack{V=0\\W=0}} \ .$$

This series is everywhere convergent in $T \geq 0$ and thereby extends the Constantine series given in (28). Some further reduction of (37) may be possible using expansions for invariant polynomials of tensor products of matrices (such as $I\otimes W$ and $I\otimes\partial V$) into sums of polynomials in the component matrices (W and ∂V); but since the formulae for the coefficients in these expansions are generally unknown we prefer to leave (37) in its present form. Note, however, that it is easy to deduce the null distribution (15) from

(37) by setting $\Omega = 0$ and using (14) and the fact that

$$C_\lambda(I_{mn_1-1})/C_\lambda(I_{mn_1}) = \left((mn_1-1)/2\right)_\lambda/(mn_1/2)_\lambda .$$

4. AN ALTERNATE APPROACH TO CONSTANTINE'S SERIES

There is an interesting alternate approach to (28) which makes use of the polynomials introduced by Hayakawa [8]. We note that the density (25) remains invariant if we replace the matrix A in (25) by H_1A for any orthogonal $H_1 \in O(m)$. Moreover, (25) may be written in the form

$$(38)\quad \left[2^{m(n_1+n_2)/2}\Gamma(mn_1/2)\Gamma_m(n_2/2)\right]^{-1} \operatorname{etr}(-\Omega)T^{mn_1/2-1}$$
$$\cdot\int_{V_{1,mn_1}}\int_{S_2>0}\int_{O(m)}\int_{O(n_1)} \operatorname{etr}\left\{-(1/2)S_2\right\}\operatorname{etr}\left\{-(1/2)A'H_1'S_2H_1A\right\}$$
$$\cdot\operatorname{etr}\left\{T^{1/2}A'H_1S_2^{1/2}MH_2\right\}(\underline{dH_1})(\underline{dH_2})(\det S_2)^{(n_1+n_2-m-1)/2} dS_2(\underline{dh}) .$$

But from Theorem 7 of Hayakawa [8]

$$(39)\quad \int_{O(m)}\int_{O(n_1)} \operatorname{etr}\left\{-(T/2)A'H_1'S_2H_1A + T^{1/2}A'H_1S_2^{1/2}MH_2\right\}(\underline{dH_1})(\underline{dH_2})$$
$$= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{P_\kappa(2^{-1/2}M',\ 2^{-1}TS_2)C_\kappa(A'A)}{k!\left(\frac{m}{2}\right)_\kappa C_\kappa(I_{n_1})}$$

where the polynomial $P_\kappa(\ ,\ )$ is given by (see (34) of [8]):

$$(40)\quad \operatorname{etr}(-\Omega)P_\kappa(2^{-1/2}M',\ 2^{-1}TS_2)$$

$$= \frac{(-1)^k}{\pi^{mn_1/2}} \int_U \operatorname{etr}(-2^{1/2}iMU)\operatorname{etr}(-UU')C_\kappa\Big((T/2)US_2U'\Big)dU$$

where U is a real $n_1 \times m$ matrix .

Since $C_\kappa(A'A) = C_\kappa(AA') = C_\kappa(Q)$ and $(n_1/2)_\kappa C_\kappa(I_m) = (m/2)_\kappa C(I_{n_1})$ it follows from (38), (39) and (21) that

$$(41)\quad \operatorname{pdf}(T) = \left[2^{m(n_1+n_2)/2}\Gamma(mn_1/2)\Gamma_m(n_2/2)\right]^{-1}\operatorname{etr}(-\Omega)T^{mn_1/2-1}$$

$$\cdot\int_{S_2>0}\operatorname{etr}\Big\{-(1/2)S_2\Big\}(\det S_2)^{(n_1+n_2-m-1)/2}\sum_{k=0}^{\infty}\frac{(T/2)^k}{k!(mn_1/2)_k}\sum_\kappa P_\kappa(2^{-1/2}M',S_2)dS_2 \ .$$

Note that if we had been working with the conditional distribution $\operatorname{pdf}(T|S_2)$ instead of the marginal density we would at this point in the argument have obtained Hayakawa's Theorem 10 in [8].

For $0 \leq T < 1$ it is readily verified that the series in (41) may be integrated term by term. To perform the integration we note that

$$(41)\quad \int_{S_2>0}\operatorname{etr}\Big\{-(1/2)S_2\Big\}(\det S_2)^{(n_1+n_2-m-1)/2}P_\kappa(2^{-1/2}M',S_2)dS_2$$

$$= \Gamma_m\Big((n_1+n_2)/2\Big)P_\kappa(2^{-1/2}M',\partial W)\det\Big((1/2)I-W\Big)^{-(n_1+n_2)/2}\Bigg|_{W=0}$$

where W is a matrix of auxiliary variables. Moreover, using (40) we can write (41) as

$$(42)\quad \frac{(-1)^k\Gamma_m\left((n_1+n_2)/2\right)\mathrm{etr}(\Omega)}{\pi^{mn_1/2}2^{-m(n_1+n_2)/2}}\int_U \mathrm{etr}(-2^{1/2}iMU)\mathrm{etr}(-UU')$$

$$\cdot C_\kappa(U\partial WU')dU\ \det(I-2W)^{-(n_1+n_2)/2}\Big|_{W=0}\ .$$

Changing variables according to $Z = 2(U'U)^{-1/2}W(U'U)^{-1/2}$ the operator changes as $2\partial Z = (U'U)^{1/2}\partial W(U'U)^{1/2}$ and (42) becomes

$$(43)\quad \frac{(-2)^k 2^{m(n_1+n_2)/2}\Gamma_m\left((n_1+n_2)/2\right)\mathrm{etr}(\Omega)}{\pi^{mn_1/2}}$$

$$\cdot\left[C_\kappa(\partial Z)\int_U \mathrm{etr}(-2^{1/2}iMU)\mathrm{etr}(-UU')\det(I-ZU'U)^{-(n_1+n_2)/2}dU\right]_{Z=0}\ .$$

We now write $U = VR^{1/2}$ where $R = U'U$ and $V'V = I_m$. The measure changes according to $dU = 2^{-m}(\det R)^{(n_1-m-1)/2}dR(dV)$ and we find after integration of V over V_{m,n_1} in (43)

$$(44)\quad \frac{(-2)^k 2^{m(n_1+n_2)/2}\mathrm{etr}(\Omega)\Gamma_m\left((n_1+n_2)/2\right)}{\Gamma_m(n_1/2)}$$

$$\cdot\left[C_\kappa(\partial Z)\int_{R>0}\mathrm{etr}(-R)\det(I-ZR)^{-(n_1+n_2)/2}(\det R)^{(n_1-m-1)/2}{}_0F_1\left(\frac{n_1}{2};\ -R\Omega\right)dR\right]_{Z=0}\ .$$

Summing over partitions κ of k into $\leq m$ parts as in (41) we have

$$(45)\quad \sum_\kappa C_\kappa(\partial Z)\det(I-ZR)^{-(n_1+n_2)/2}\Big|_{Z=0} = (\mathrm{tr}\ \partial Z)^k\det(I-ZR)^{-(n_1+n_2)/2}\Big|_{Z=0}$$

$$= \sum_\kappa\left(\frac{n_1+n_2}{2}\right)_\kappa C_\kappa(R)\ .$$

We also note that

$$(46)\quad \int_{R>0} \operatorname{etr}(-R)(\det R)^{(n_1-m-1)/2} C_\kappa(R)\, {}_0F_1\left(\frac{n_1}{2}; -R\Omega\right) dR$$

$$= \Gamma_m\left(\frac{n_1}{2}\right) \operatorname{etr}(-\Omega) L_\kappa^\gamma(\Omega) \quad , \quad \gamma = (n_1-m-1)/2$$

using Constantine's definition of $L_\kappa^\gamma(\Omega)$ in [2].

Substitution of (43)-(46) in (41) now yields the Constantine series:

$$\operatorname{pdf}(T) = \left[\Gamma_m\left((n_1+n_2)/2\right) / \Gamma(mn_1/2)\Gamma_m(n_2/2)\right] \operatorname{etr}(-\Omega) T^{mn_1/2-1}$$

$$\cdot \sum_{k=0}^{\infty} \frac{(-T)^k}{k!(mn_1/2)k} \sum_\kappa \left(\frac{n_1+n_2}{2}\right)_\kappa L_\kappa^\gamma(\Omega) \quad , \quad 0 \le T < 1 \; .$$

5. CONCLUSION

This paper provides a mathematical solution to the long standing problem of the distribution of Hotelling's trace, T_0^2 . The formulae presented here are primarily useful for analytic purposes in that they extend and unify existing distributional results. Moreover, it is hoped that the method of derivation will itself find application to a variety of other unsolved problems in multivariate distribution theory. Some indication of the possible range of application is given by the author's other work in [13], [14] and [15].

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