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AN EVERYWHERE CONVERGENT SERIES REPRESENTATION OF THE DISTRIBUTION OF HOTELLING'S GENERALIZED T_0^2

by

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0. ABSTRACT

A new series representation of the distribution of Hotelling's generalized T_0^2 statistic is obtained. Unlike earlier work, the series representation given here is everywhere convergent. Explicit formulae are given for both the null and the noncentral distributions. Earlier results by Constantine [1], which are convergent on the interval [0,1), are also derived quite simply from our formulae.

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1. INTRODUCTION

Let S_1 $(m \times m)$ and S_2 $(m \times m)$ have independent Wishart distributions with n_1 , n_2 degrees of freedom, respectively, and the same population covariance matrix Σ . S_1 may be noncentral and we denote the noncentrality matrix by Ω . The generalized T_0^2 statistic [1] is then defined by:

$$T = T_0^2/n_2 = \mathrm{tr}(S_1 S_2^{-1}) .$$

Since its introduction by Lawley [12] and later by Hotelling [8, 9] in connection with wartime problems of multivariate quality control, the distribution of this statistic has attracted a good deal of theoretical interest among statisticians. A fundamental contribution was made by Constantine [1], who found a zonal polynomial series representation of the distribution of T . However, Constantine's series converge only for $0 \le T < 1$. In subsequent research, Davis [2] discovered a linear homogeneous differential equation that is satisfied by the density of T in the null case $(\Omega = 0)$. This approach has facilitated the numerical computation of percentage points of the null distribution; and in a series of articles [3, 4, 5] Davis has provided tabulations of the upper 5% and 1% points of the distribution of T for dimensions $m = 3$ through 10. Pillai and Young [19] and Pillai and Sudjana [20] have also worked on the problem and found some specialized results for the case where $m \le 4$ and n_1 is small. Additional contributions have been made by Krishnaiah and Chang [10] and Krishnaiah and Chattopadhyay [11]. Readers are referred to the articles by Pillai [17, 18] for a detailed review of the field.

When $m = 2$ Hotelling [9] derived a very simple formula for the null distribution of T. This formula may be written as a Gaussian hypergeometric series and is everywhere convergent in T [1].[1] Hotelling's formula has been the source of conjectures by Constantine [1], Pillai [17] and others concerning possible general forms of the density. However, until the present, no progress has been made on the analytic derivation of the exact density in the general case even for the null distribution of T.

The purpose of the present paper is to offer a fresh approach to the problem of the distribution of T. We shall give general formulae for the exact density (pdf) of T in both the null and the noncentral case. Unlike earlier work, the series representations that we obtain are everywhere convergent in T. Our results, therefore, provide a solution to the long standing problem of the distribution of T in the general case.

2. THE NULL DISTRIBUTION OF T

Since T is invariant under the transformations $S_1 \to \Sigma^{-1/2} S_1 \Sigma^{-1/2}$ and $S_2 \to \Sigma^{-1/2} S_2 \Sigma^{-1/2}$ we set the common population covariance matrix $\Sigma = I$. Now let $S_1 = XX'$ where the $m \times n_1$ matrix X is $N_{m,n_1}(M, I_{mn_1})$. We write $T = \operatorname{tr}(S_1 S_2^{-1}) = x'(I_{n_1} \otimes S_2^{-1})x$ where $x = \operatorname{vec}(X)$ and $\operatorname{vec}(\ )$ denotes vectorization by columns. Conditional on

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[1] There appears to be a minor error in the expression given by Constantine [1] and later by Davis [2]. The correct formula has an additional factor of (1/2). [15] provides a new derivation of this formula.

S_2 , T is distributed as a quadratic form in normal variates. In the null case $M = 0$, $\Omega = MM'/2 = 0$ and we have the density:

$$\text{(1)} \qquad \text{pdf}(T|S_2) = \frac{T^{mn_1/2-1} |S_2|^{n_1/2}}{(2\pi)^{mn_1/2} \Gamma(mn_1/2)} \, {}_0F_0(-(1/2)(I \otimes S_2), T)$$

where

$$\text{(2)} \qquad {}_0F_0(-(1/2)(I \otimes S_2), T) = \int_V \text{etr}\{-(1/2)T(I \otimes S_2)hh'\}(\underline{dh})$$

Here V is the Stiefel manifold $\{h \in R^{mn_1} : h'h = 1\}$ and $(\underline{dh})$ represents the normalized invariant measure on V . Using (1) and (2) we deduce the unconditional density of T as follows:

$$\text{pdf}(T) = \frac{T^{mn_1/2-1}}{2^{m(n_1+n_2)/2} \Gamma(mn_1/2)\Gamma_m(n_2/2)} \int_{S_2>0} \int_V$$

$$\cdot \text{etr}\{-(1/2)S_2(I+TQ)\} |S_2|^{(n_1+n_2-m-1)/2} (\underline{dh}) dS_2$$

$$\text{(3)} \qquad = \frac{\Gamma_m((n_1+n_2)/2) T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} \int_V |I+TQ|^{-(n_1+n_2)/2} (\underline{dh})$$

where

$$\text{(4)} \qquad Q = \sum_{i=1}^{n_1} h_i h_i'$$

and h_i $(i = 1, \ldots, n_1)$ are the m-vectors taken from the partition of $h' = (h_1', h_2', \ldots, h_{n_1}')$ into n_1 component vectors.

Formula (3) is an extremely simple representation of the exact null distribution of T. It may be used to derive in a straightforward way the series discovered by Constantine [1] in 1966. We first give the following useful integral:

LEMMA 2.1

$$\int_V C_\kappa(Q)(\underline{dh}) = \frac{(n_1/2)_\kappa C_\kappa(I_m)}{(mn_1/2)_k} . \tag{5}$$

PROOF. We note that:

$$\int_V C_k((I_{n_1} \otimes Z)hh')(\underline{dh}) = C_k(I_{n_1} \otimes Z)/C_k(I_{mn_1})$$

$$= \Sigma_\kappa \frac{(n_1/2)_\kappa C_\kappa(Z)}{(mn_1/2)_k} \tag{6}$$

But the left side is also equal to:

$$\int_V (tr((I \otimes Z)hh'))^k(\underline{dh}) = \int_V (tr(ZQ))^k(\underline{dh})$$

$$= \Sigma_\kappa \frac{C_\kappa(Z)}{C_\kappa(I_m)} \int_V C_\kappa(Q)(\underline{dh}) \tag{7}$$

Equating coefficients of $C_\kappa(Z)$ in (6) and (7) we obtain the stated result. |‾|

To obtain Constantine's [1] series from (3) we now simply expand $|I+TQ|^{-(n_1+n_2)/2}$ in its usual zonal polynomial series (which is valid for $0 \le T < 1$) and integrate over V by using (5). This gives:

(8) $$\mathrm{pdf}(T) = \frac{\Gamma_m((n_1+n_2)/2)T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} \sum_{k=0}^{\infty} \frac{(-T)^k}{k!(mn_1/2)_k} \Sigma_\kappa \left(\frac{n_1+n_2}{2}\right)_\kappa \left(\frac{n_1}{2}\right)_\kappa C_\kappa(I_m)$$

which is convergent for $0 \le T < 1$.

Formula (3) may also be used to obtain an alternative series representation of the density which is everywhere convergent over $T \ge 0$. Given $h \in V$ we introduce an $mn_1 \times (mn_1-1)$ matrix K for which $H = [K,h]$ is orthogonal. We partition K conformably with h as $K' = [K_1', K_2', \ldots, K_{n_1}']$ where the component matrices κ_i are $m \times (mn_1-1)$. Define $P = \Sigma_1^{n_1} K_i K_i'$. Since $K_i K_i' + h_i h_i' = I_m$ $(i = 1, \ldots, n_1)$ we deduce that:

(9) $$P = n_1 I_m - Q$$

and

(10) $$|C_\kappa(P)| \le n_1^k C_\kappa(I_m) \ .$$

We now write

$$|I+TQ|^{-(n_1+n_2)/2} = (1+n_1T)^{-m(n_1+ n_2)/2} |I-(T/(1+n_1T))P|^{-(n_1+n_2)/2}$$

and thus:

(11) $$\mathrm{pdf}(T) = \frac{\Gamma_m((n_1+n_2)/2)T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)(1+n_1T)^{m(n_1+n_2)/2}}$$

$$\cdot \Sigma_{k=0}^{\infty} \frac{(T/(1+n_1T))^k}{k!} \Sigma_\kappa \left(\frac{n_1+n_2}{2}\right)_\kappa \int_V C_\kappa(P)(\underline{dh})$$

The series is everywhere convergent in $T \geq 0$ by majorization in view of (10).

LEMMA 2.2.

$$(12) \qquad \int_V C_\kappa(P)(\underline{dh}) = \left\{ \Sigma_{t=0}^{k} \frac{(-1)^t n_1^{k-t}}{(mn_1/2)_t} \Sigma_\tau a_{\kappa,\tau} (n_1/2)_\tau \right\} C_\kappa(I_m)$$

where the $a_{\kappa,\tau}$ are Constantine's coefficients given in [1].

PROOF. We use the binomial expansion [1]:

$$C_\kappa(P) = n_1^k C_\kappa(I - (1/n_1)Q)$$
$$= n_1^k \{\Sigma_{t=0}^{k}(-1/n_1)^t \Sigma_\tau a_{\kappa,\tau} C_\tau(Q)/C_\tau(I_m)\} C_\kappa(I_m)$$

and the result follows by integration from (5). |_|

We deduce the following explicit series representation of the density of T :

$$(13 \; \mathrm{pdf}(T) = \frac{\Gamma_m((n_1+n_2)/2) T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)(1+n_1 T)^{m(n_1+n_2)/2}}$$
$$\cdot \Sigma_{k=0}^{\infty} \frac{(T/(1+n_1 T))^k}{k!} \Sigma_\kappa \left(\frac{n_1+n_2}{2}\right)_\kappa \left\{ \Sigma_{t=0}^{k} \frac{(-1)^t n_1^{k-t}}{(mn_1/2)_t} \Sigma_\tau a_{\kappa,\tau} (n_1/2)_\tau \right\} C_\kappa(I_m)$$

which, like (11), is everywhere convergent in $T \geq 0$.

3. THE NONCENTRAL DISTRIBUTION OF T

Since $T = \mathrm{tr}(XX'S_2^{-1})$ we start with the joint density of (X, S_2) :

$$\left[2^{m(n_1+n_2)/2}\pi^{mn_1/2}\Gamma_m(n_2/2)\right]^{-1}\mathrm{etr}\{-(1/2)(X-M)(X-M)'\}\mathrm{etr}\{-(1/2)S_2\}|S_2|^{(n_2-m-1)/2}$$

$$(14) \quad = \left[2^{m(n_1+n_2)/2}\pi^{mn_1/2}\Gamma_m(n_2/2)\right]^{-1}\mathrm{etr}(-\Omega)\mathrm{etr}\{-(1/2)XX'\}\mathrm{etr}(-XM')$$

$$\cdot\mathrm{etr}\{-(1/2)S_2\}|S_2|^{(n_2-m-1)/2}$$

T is invariant under the simultaneous transformations $X \to HXJ$, $S_2 \to HS_2H'$ where $H \in O(m)$ and $J \in O(n_1)$. Hence, making these substitutions in (14) and integrating over the (normalized) orthogonal groups we have:

$$(15) \quad \left[2^{m(n_1+n_2)/2}\pi^{mn_1/2}\Gamma_m(n_2/2)\right]^{-1}\mathrm{etr}(-\Omega)\mathrm{etr}\{-(1/2)XX'\}$$

$$\cdot{}_0F_1^{(m)}\left(\frac{n_1}{2};\ \frac{1}{2}XX',\ \Omega\right)\mathrm{etr}\{-(1/2)S_2\}|S_2|^{(n_2-m-1)/2} \quad .$$

We now transform $X \to S_2^{-1/2}X = Y$ in (15), giving:

$$\left[2^{m(n_1+n_2)/2}\pi^{mn_1/2}\Gamma_m(n_2/2)\right]^{-1}\mathrm{etr}(-\Omega)\mathrm{etr}\{-(1/2)S_2YY'\}$$

$$\cdot{}_0F_1^{(m)}\left(\frac{n_1}{2};\ \frac{1}{2}S_2YY',\ \Omega\right)\mathrm{etr}\{-(1/2)S_2\}|S_2|^{(n_1+n_2-m-1)/2} \quad .$$

We write $y = \mathrm{vec}(Y) = hT^{1/2}$ where $h \in V$ and then

$T = y'y = x'(I \otimes S_2^{-1})x = \mathrm{tr}(XX'S_2^{-1})$. The measure transforms according to $dy = (1/2)T^{mn_1/2-1} dT(dh)$ where (dh) represents the invariant measure on V . We deduce that:

$$\mathrm{pdf}(T) = \left[2^{m(n_1+n_2)/2}\pi^{mn_1/2}\Gamma_m(n_2/2)\right]^{-1} \mathrm{etr}(-\Omega)T^{mn_1/2-1}$$

$$\cdot\int_V\int_{S_2>0} \mathrm{etr}\left\{-\left(\frac{1}{2}\right)(I+TQ)S_2\right\} {}_1F_1^{(m)}\left(\frac{n_1}{2};\ \frac{T}{2}S_2Q,\ \Omega\right)|S_2|^{(n_1+n_2-m-1)/2} dS_2(dh)$$

$$(16)\quad = \left[2^{m(n_1+n_2)/2}\pi^{mn_1/2}\Gamma(mn_1/2)\right]^{-1} \mathrm{etr}(-\Omega)T^{mn_1/2-1} \int_V\int_{S_2>0} \mathrm{etr}\left\{-\left(\frac{1}{2}\right)(I+TQ)S_2\right\}$$

$$\cdot {}_0F_1^{(m)}\left(\frac{n_1}{2};\ \frac{T}{2}S_2Q,\ \Omega\right)|S_2|^{(n_1+n_2-m-1)/2} dS_2(\underline{dh})$$

where Q is given by (4).

Performing the integration over $S_2 > 0$ in (16) we obtain:

$$(17)\quad \mathrm{pdf}(T) = \frac{\Gamma_m((n_1+n_2)/2)\mathrm{etr}(-\Omega)T^{mn_1/2-1}}{\Gamma(mn_1/2)\Gamma_m(n_2/2)}$$

$$\cdot\int_V {}_1F_1^{(m)}\left(\frac{n_1+n_2}{2},\ \frac{n_1}{2};\ T(I+TQ)^{-1}Q,\ \Omega\right)|I+TQ|^{-(n_1+n_2)/2}(\underline{dh})$$

which generalizes (3) to the noncentral case.

Constantine's [1] series for the noncentral case may be deduced quite simply from (17). We use the easily established expansion:

$$(18)\qquad |I-Z|^{-a}\,{}_1F_1^{(m)}\left(a,\ \gamma+p;\ B,\ -Z(I-Z)\right)^{-1} = \sum_{k=0}^{\infty}\sum_\kappa \frac{(a)_\kappa L_\kappa^\gamma(B)C_\kappa(Z)}{(\gamma+p)_\kappa k!C_\kappa(I_m)}$$

where $B > 0$, $\|Z\| < 1$, $\gamma > -1$, $p = (m+1)/2$ and $L_\kappa^\gamma(\ )$ denotes Constantine's generalized Laguerre polynomial of matrix argument. (18) was given by Muirhead in [13] (exercise 7.20, p. 290), although his result as stated contains an error in that his exponent for $|I-Z|$ should read "$-a$" as given above.

Now let $Z = -TQ$ for $0 \le T < 1$, $B = \Omega$, $a = (n_1+n_2)/2$, and $\gamma+p = n_1/2$. We find from (17) and (18) the series:

$$\text{pdf}(T) = \{\Gamma_m((n_1+n_2)/2)/\Gamma(mn_1/2)\Gamma_m(n_2/2)\}\text{etr}(-\Omega)T^{mn_1/2-1}$$

$$\cdot \sum_{k=0}^{\infty} \frac{(-T)^k}{k!} \Sigma_\kappa \frac{\left(\frac{n_1+n_2}{2}\right)_\kappa L_\kappa^\gamma(\Omega)}{\left(\frac{n_1}{2}\right)_\kappa C_\kappa(I_m)} \cdot \int_V C_\kappa(Q)(\underline{dh})$$

which is valid for $0 \le T < 1$. Using Lemma 2.1 we have immediately:

(19)
$$\text{pdf}(T) = \{\Gamma_m((n_1+n_2)/2)/\Gamma(mn_1/2)\Gamma_m(n_2/2)\}\text{etr}(-\Omega)T^{mn_1/2-1}$$

$$\cdot \sum_{k=0}^{\infty} \frac{(-T)^k}{k!\left(\frac{mn_1}{2}\right)_k} \Sigma_\kappa \left(\frac{n_1+n_2}{2}\right)_\kappa L_\kappa^\gamma(\Omega) \ , \quad \gamma = (n_1-m-1)/2$$

This is the series given by Constantine in [1] for the noncentral case when $0 \le T < 1$.

To obtain an everywhere convergent series we proceed as follows. Using (9) we write $I+TQ = (1+n_1T)I - TP$ and (16) becomes:

$$\left[2^{m(n_1+n_2)/2}\Gamma(mn_1/2)\Gamma_m(n_2/2)\right]^{-1} \mathrm{etr}(-\Omega)T^{mn_1/2-1} \int_V\int_{S_2>0}$$

$$\mathrm{etr}\{-(1/2)(1+n_1T)S_2\}\mathrm{etr}\{(1/2)TS_2P\}{}_0F_1^{(m)}\left(\frac{n_1}{2};\ \frac{T}{2}S_2Q,\ \Omega\right)|S_2|^{(n_1+n_2-m-1)/2} dS_2(\underline{dh})$$

This expression is invariant under the simultaneous transformations ($Q \to L'QL$, $P \to L'PL$) where $L \epsilon O(m)$. Thus, transforming $S_2 \to LS_2L'$ and integrating over the normalized orthogonal group we obtain:

$$\text{(20)}\quad \mathrm{pdf}(T) = \left[2^{m(n_1+n_2)/2}\Gamma(mn_1/2)\Gamma_m(n_2/2)\right]^{-1} \mathrm{etr}(-\Omega)T^{mn_1/2-1}$$

$$\cdot\int_V\int_{S_2>0} \mathrm{etr}\{-(1/2)(1+n_1T)S_2\}|S_2|^{(n_1+n_2-m-1)/2}$$

$$\cdot \sum_{k,\ell=0}^{\infty} \sum_{\kappa,\lambda} \frac{(T/2)^{k+\ell}C_\lambda(\Omega)}{k!\ell!(n_1/2)_\lambda C_\lambda(I_m)} \sum_{\phi\epsilon\kappa\cdot\lambda} \frac{\theta_\phi^{\kappa,\lambda}C_\phi(S_2)C_\phi^{\kappa,\lambda}(P,Q)}{C_\phi(I_m)}\ dS_2(\underline{dh})$$

$$\text{(21)}\quad = \frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right)\mathrm{etr}(-\Omega)}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} \frac{T^{mn_1/2-1}}{(1+n_1T)^{m(n_1+n_2)/2}} \Sigma_{k,\ell=0}^{\infty} \frac{(T/(1+n_1T))^{k+\ell}}{k!\ell!}$$

$$\cdot \sum_{\kappa,\lambda} \frac{C_\lambda(\Omega)}{(n_1/2)_\lambda C_\lambda(I_m)} \sum_{\phi\epsilon\kappa\cdot\lambda} \theta_\phi^{\kappa,\lambda}\left(\frac{n_1+n_2}{2}\right)_\phi \int_V C_\phi^{\kappa,\lambda}(P,Q)(\underline{dh})$$

In the above formula $C_\phi^{\kappa,\lambda}$ is an invariant polynomial in the elements of its two matrix arguments. These polynomials and the constants $\theta_\phi^{\kappa,\lambda} = C_\phi^{\kappa,\lambda}(I_m,\ I_m)/C_\phi(I_m)$ were introduced by Davis [6, 7]. In (20) and (21) ϕ is a partition of the integer $f = k+\ell$ into $\le m$ parts κ is a partition of k into $\le m$ parts and λ is a partition of ℓ into $\le m$

parts. The notation $\emptyset \epsilon \kappa\cdot\lambda$, which is defined in [6], relates the three different partitions in the summation.

Writing $P = n_1 I_m - Q$ as before we now use the binomial expansion given by Davis [6, equation (6.6)]:

$$C_{\emptyset}^{\kappa,\lambda}(n_1 I - Q,\ Q) = n_1^k C_{\emptyset}^{\kappa,\lambda}(I - (1/n\ )Q,\ Q)$$

$$= n_1^k (\Sigma_{r=0}^{k} \sum_{\rho,\tau\epsilon\rho\cdot\lambda} b_{\rho,\lambda;\tau}^{\kappa,\lambda;\emptyset} C_{\tau}^{\rho,\lambda}(-(1/n_1)Q,\ Q)/C_{\tau}(I)) C_{\emptyset}(I)$$

$$(22)\quad = n_1^k (\Sigma_{r=0}^{k} \sum_{\rho,\tau\epsilon\rho\cdot\lambda} b_{\rho,\lambda;\tau}^{\kappa,\lambda;\emptyset} (-1/n_1)^{\tau} \theta_{\tau}^{\rho,\lambda} C_{\tau}(Q)/C_{\tau}(I)) C_{\emptyset}(I) \ .$$

In this summation ρ and τ are partitions of the integers r and $r+\ell$, respectively, into $\le m$ parts and the $b_{\rho,\lambda;\tau}^{\kappa,\lambda;\emptyset}$ are constants introduced in [6].

Using (22) and Lemma 2.1 in (21) we deduce the following series representation of the density of T :

$$(23)\quad \mathrm{pdf}(T) = \frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right) \mathrm{etr}(-\Omega)}{\Gamma\left(\frac{mn_1}{2}\right)\Gamma_m\left(\frac{n_2}{2}\right)} \frac{T^{mn_1/2-1}}{(1+n_1 T)^{m(n_1+n_2)/2}}$$

$$\cdot\Sigma_{k,\ell=0}^{\infty} \frac{(T/(1+n_1 T))^{k+\ell}}{k!\,\ell!} \sum_{\kappa,\lambda} \frac{C_{\lambda}(\Omega)}{(n_1/2)_{\lambda} C_{\lambda}(I_m)} \sum_{\emptyset\epsilon\kappa\cdot\lambda} \theta_{\emptyset}^{\kappa,\lambda} \left(\frac{n_1+n_2}{2}\right)_{\emptyset}$$

$$\cdot\left\{\Sigma_{r=0}^{k} \frac{(-1)^r n_1^{k-r}}{(mn_1/2)_{r+\ell}} \sum_{\rho,\tau\epsilon\rho\cdot\lambda} b_{\rho,\lambda;\tau}^{\kappa,\lambda;\emptyset} \theta_{\tau}^{\rho,\lambda} \left(\frac{n_1}{2}\right)_{\tau}\right\} C_{\emptyset}(I_m)$$

When $\Omega = 0$ the series in ℓ terminates at $\ell = 0$ and (23) reduces to the null density given in (13).

Like (13), the series (23) is everywhere convergent in $T \geq 0$. To see this it is simplest to work with the equivalent series (21). Noting that $P \leq n_1 I_m$ we find that (21) is majorized by the series:

$$(24)\quad \frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right)\operatorname{etr}(-\Omega)}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} \frac{T^{mn_1/2-1}}{(1+n_1T)^{m(n_1+n_2)/2}} \Sigma_{k,\ell=0}^{\infty} \frac{(n_1T/(1+n_1T))^k(T/(1+n_1T))^\ell}{k!\ell!}$$

$$\cdot \Sigma_{\kappa,\lambda} \frac{C_\lambda(\Omega)}{(n_1/2)_\lambda C_\lambda(I_m)} \sum_{\phi\in\kappa\cdot\lambda} \theta_\phi^{\kappa,\lambda}\left(\frac{n_1+n_2}{2}\right)_\phi \int_V C_\phi^{\kappa,\lambda}(I_m,\ Q)(\underline{dh})$$

Using

$$C_\phi^{\kappa,\lambda}(I_m,\ Q) = \{\theta_\phi^{\kappa,\lambda} C_\phi(I_m)/C_\lambda(I)\}C_\lambda(Q)$$

[7, equation (5.2)] and Lemma 2.1 we write (24) as follows:

$$\frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right)\operatorname{etr}(-\Omega)}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} \frac{T^{mn_1/2-1}}{(1+n_1T)^{m(n_1+n_2)/2}} \Sigma_{k,\ell=0}^{\infty} \frac{(n_1T/(1+n_1T))^k(T/(1+n_1T))^\ell}{k!\ell!(mn_1/2)_\ell}$$

$$\Sigma_{\kappa,\lambda} \frac{C_\lambda(\Omega)}{C_\lambda(I_m)} \sum_{\phi\in\kappa\cdot\lambda} (\theta_\phi^{\kappa,\lambda})^2 \left(\frac{n_1+n_2}{2}\right)_\phi C_\phi(I_m)$$

$$= \frac{\operatorname{etr}(-\Omega)}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} \frac{T^{mn_1/2-1}}{(1+n_1T)^{m(n_1+n_2)/2}} \Sigma_{k,\ell=0}^{\infty} \frac{(n_1T/(1+n_1T))^k(T/(1+n_1T))^\ell}{k!\ell!(mn_1/2)_\ell}$$

$$\Sigma_{\kappa,\lambda} \frac{C_\lambda(\Omega)}{C_\lambda(I_m)} \sum_{\phi\in\kappa\cdot\lambda} (\theta_\phi^{\kappa,\lambda})^2 \int_{S>0} \operatorname{etr}(-S)|S|^{(n_1+n_2-m-1)/2} C_\phi(S)\,dS$$

$$= \frac{\mathrm{etr}(-\Omega)}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} \frac{T^{mn_1/2-1}}{(1+n_1T)^{m(n_1+n_2)/2}} \Sigma_{k,\ell=0}^{\infty} \frac{(n_1T/(1+n_1T))^k (T/(1+n_1T))^{\ell}}{k!\ell!(mn_1/2)_{\ell}}$$

$$\Sigma_{\kappa,\lambda} \frac{C_\lambda(\Omega)}{C_\lambda(I_m)} \int_{S>0} \mathrm{etr}(-S)|S|^{(n_1+n_2-m-1)/2} C_\kappa(S)C_\lambda(S)dS$$

$$= \frac{\mathrm{etr}(-\Omega)}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} \frac{T^{mn_1/2-1}}{(1+n_1T)^{m(n_1+n_2)/2}} \Sigma_{k,\ell=0}^{\infty} \frac{(T/(1+n_1T))^{\ell}}{\ell!(mn_1/2)_{\ell}}$$

$$\Sigma_\lambda \frac{C_\lambda(\Omega)}{C_\lambda(I)} \int_{S>0} \mathrm{etr}\{-1(1+n_1T)S\}|S|^{(n_1+n_2-m-1)/2} C_\lambda(S)dS$$

$$= \frac{\Gamma_m((n_1+n_2)/2)\mathrm{etr}(-\Omega)}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} T^{mn_1/2-1} \Sigma_{\ell=0}^{\infty} \frac{T^{\ell}}{\ell!(mn_1/2)_{\ell}} \Sigma_\lambda C_\lambda(\Omega)\left(\frac{n_1+n_2}{2}\right)_\lambda$$

Since $(mn_1/2)_\ell \geq (n_1/2)_\lambda$ for all m the final series above is majorized by

$$\frac{\Gamma_m((n_1+n_2)/2)\mathrm{etr}(-\Omega)}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} T^{mn_1/2-1} \Sigma_{\ell=0}^{\infty} \frac{T^{\ell}}{\ell!} \Sigma_\lambda \frac{\left(\frac{n_1+n_2}{2}\right)_\lambda}{\left(\frac{n_1}{2}\right)_\lambda} C_\lambda(\Omega)$$

$$= \frac{\Gamma_m((n_1+n_2)/2)\mathrm{etr}(-\Omega)}{\Gamma(mn_1/2)\Gamma_m(n_2/2)} T^{mn_1/2-1} {}_1F_1\left(\frac{n_1+n_2}{2}, \frac{n_1}{2}; T\Omega\right)$$

which is convergent for all $T \geq 0$. It follows that the series representation of the density given by (23) is everywhere convergent in $T \geq 0$.

4. CONCLUSION

This paper provides a mathematical solution to the long standing analytic problem of the exact distribution of Hotelling's generalized T_0^2 statistic. The formulae presented here are primarily useful for analytic purposes in that they extend and unify existing distributional results.

The T_0^2 statistic is a special case of the Wald statistic for testing general linear restrictions on the coefficients in the multivariate linear model. The exact distribution of the latter statistic has recently been obtained by the author in [16] using operator methods. Methods similar to those of [16] may also be used to treat the distribution of the T_0^2 statistic. Such an approach was adopted in the first version of this paper [14] and was the original stimulus for the present investigation.

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