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ON THE STOCHASTIC STEADY-STATE BEHAVIOR OF OPTIMAL ASSET ACCUMULATION IN THE PRESENCE OF RANDOM WAGE FLUCTUATIONS AND INCOMPLETE MARKETS

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ABSTRACT

We establish rigorously the existence and properties of the stationary probability distribution which characterizes the accumulation of non-contingent financial claims by a risk averse individual who confronts random wage fluctuations and incomplete insurance markets. We show that there exists a unique, almost-everywhere continuous stationary cummulative distribution function which characterizes the accumulation of non-contingent financial claims in a stochastic steady-state. This distribution is shown to possess a single mass point coinciding with the non-negative, finite borrowing limit faced by the individual. We establish that the stationary distribution which characterizes the asset accumulation of low time preference individuals is at least as large, in the sense of first-degree stochastic dominance, as that of individuals with higher rates of time preference. We prove that, so long as individuals are allowed to borrow in amounts which can be repayed with probability one, additive differences in the probability distribution governing random wage earnings imply inversely proportional additive differences in the stationary probability distributions which govern the accumulation of non-contingent financial claims.

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ON THE STOCHASTIC STEADY-STATE BEHAVIOR
OF OPTIMAL ASSET ACCUMULATION IN THE PRESENCE
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Richard H. Clarida

In this paper, we examine the asset accumulation behavior of a risk averse individual who faces random fluctuations in his wage earnings. The individual is assumed to have access to a capital market in one-period consumption loans, but is unable to purchase state-contingent insurance against adverse earnings disturbances.[1] In such an environment, the risk-averse individual holds an inventory of the non-contingent financial claims which are available so as to self-insure against the random wage fluctuations he confronts.[2] The objective of this paper is to establish rigorously the existence and properties of the stationary probability distribution which characterizes this accumulation of non-contingent financial claims in a stochastic steady-state.

We show that there exists a unique, almost-everywhere continuous stationary cummulative distribution function which characterizes the accumulation of non-contingent financial claims in a stochastic steady-state. This distribution is shown to possess a single mass point coinciding with the non-negative, finite borrowing limit faced by the individual. This latter result and its economic interpretation correspond closely to findings obtained by Lucas (1980) and Clarida (1984) in their inventory-theoretic models of optimal money holdings in the presence of uncertainty and a cash-in-advance constraint. These results extend earlier, important work by Schechtman and Escudero (1977) who first established the existence of a stochastic steady-state in a model of optimal accumulation in the presence of random income fluctuations.

Having established existence, we go on to investigate the influence of time preference and average wage earnings on the accumulation of non-contingent financial claims in a stochastic steady state. The effect of utility function curvature on the stochastic steady state behavior optimal asset holdings is not explored since, as was demonstrated by Hakannson (1970), its influence on optimal consumption and asset accumulation decisions is indeterminate if, as we shall assume, preferences are time-separable.

We first compare the the stochastic steady-state behavior of indiviuals who differ in their pure rates of time preference. We show that the stationary probability distribution which characterizes the asset accumulation of low time preference individuals is at least as large, in the sence of first-degree stochastic dominance, as that of individuals with higher rates of time preference. This is shown to imply that the expected asset holdings of high time preference individuals are not greater than the expected holdings of low time preference individuals. Furthermore, for the special case in which the individual is not allowed to borrow against future earnings, we establish that none of the moments of the stationary distribution governing the asset holdings of high time preference individuals exceeds the corresponding moment of the stationary distribution governing the asset holdings of low time preference individuals.

We next compare the the stochastic steady-state behavior of indiviuals who differ in their average wage earnings. We show that, so long as individuals are allowed to borrow in amounts which can be repayed with probability one, additive differences in the probability distributions governing random wage earnings imply inversely proportional additive differences in the stationary probability distributions which govern the accumulation of non-contingent financial claims. Furthermore, this constant of proportionality is shown to be equal to the inverse of the real rate of interest. These findings of course imply that the individual's expected accumulation of non-contingent financial claims is inversely related to his average wage earnings. They also suggests that the variance and higher central moments of the stationary distribution governing asset accumulation are invariant to differences in average wage earnings.

The plan of the paper is as follows. In Section 1, we formulate and study the solution to the individual's "income fluctuations" problem. We prove that the optimal consumption decision rule is a strictly increasing function of total wealth, a result which sharpens the weak monotonicity property of optimal consumption decisions established by Schechtman and Escudero (1977) and Mendelson and Amihud (1982). In Section 2, we establish the existence of a unique, almost-everywhere continuous stationary cummulative distribution function which characterizes the accumulation of non-contingent financial claims in a stochastic steady-state. We relax two of the sufficient conditions employed by Schechtman and Escudero (1977) in their original existence proof: the restriction against borrowing and the assumption that the wage income received each period is a countable random variable. In Section 3, we investigate the influence of time preference on the stochastic steady-state behavior of optimal asset accumulation, using the result, which we prove, that optimal consumption at a given level of wealth is directly related to the rate of time preference. In Section 4, we investigate the influence of average wage earnings on the stochastic steady-state behavior of optimal asset accumulation, using the result, which we prove, that optimal consumption at a given level of wealth in invariant to average wage earnings so long as borrowing is allowed. In Section 5, we provide some concluding remarks.

1: The "Income Fluctuations" Problem

We consider the following "income fluctuations" problem. An individual with an infinite planning horizon must decide at the beginning of each period how much to consume and how many financial claims to purchase or sell. A financial claim costs one consumption good and entitles its owner to $\rho \equiv 1+r$ consumption goods next period, where r is the constant, non-negative rate of interest on consumption loans. Total resources available for consumption and saving in period t, w_t, are

$$w_t = \rho a_t + \varepsilon_t + \phi; \tag{1}$$

where a_t is the stock of financial claims purchased at the beginning of period t-1, ε_t is the wage received at the beginning of period t (the inelastic supply of labor being normalized to unity), and ϕ is a non-negative, finite limit on borrowing. Consumption of $c_t \leq w_t$ goods in period t yields utility of $\beta^t u(c_t)$ where $\beta \equiv 1/1+\delta$, δ the positive rate of time preference. The individual accumulates financial claims and wealth according to

$$a_{t+1} = \rho a_t + \varepsilon_t - c_t; \tag{2}$$

$$w_{t+1} = \rho(w_t - c_t) + \varepsilon_{t+1} - r\phi. \tag{3}$$

Formally, the individual's problem is

$$\max E \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + a_{t+1} = \rho a_t + \varepsilon_t;$$

$$c_0 + a_1 = \varepsilon_0;$$

$$c_t \geq 0; \; a_t + \phi \geq 0.$$

We shall find it useful to make the following standard assumptions:

(A1) $u(c)$ is a strictly increasing, strictly concave, bounded and and twice continuously differentiable function with strictly convex first derivative;

(A2) The wage received in each period is an independent and identically distributed random variable. Furthermore, there exists finite $\underline{\varepsilon}$ and $\bar{\varepsilon}$ for which the stationary cumulative distribution function of ε, denoted G, satisfies:

$$G(\varepsilon) = 0,\ \varepsilon \leq \underline{\varepsilon} < 0;\quad G(\varepsilon) = 1,\ \varepsilon \geq \bar{\varepsilon} > 0;$$

$$dG(\varepsilon) > 0,\ \text{continuously for all } \varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}];$$

(A3) The limit on borrowing satisfies:

$$0 \leq \phi < \underline{\varepsilon}/r;$$

a restriction which insures solvency with probability one.

Let $v(w)$ be the value of the objective function of an individual who begins the period with resources w and behaves optimally. This function must satisfy:

$$v(w) = \max_{c \leq w} \{\ u(c) + \beta \int v(\rho(w - c) + \varepsilon' - r\phi) dG(\varepsilon')\ \} \tag{4}$$

Denote the optimal consumption and asset accumulation decision rules by c and g respectively. Following closely the arguments presented in Lucas (1978;1980), we now establish the existence and properties of optimal consumption and assset accumulation decisions in the presence of random wage fluctuations.

Theorem 1: By Assumptions (A1), (A2), and (A3), there exists a unique solution to the agent's problem. There is a unique, bounded, continuous, strictly increasing, strictly concave, and once continuously differentiable function v such that

$$v(w) = \max_{c \leq w} \{ u(c) + \beta\int v(\rho(w - c) + \varepsilon' - r\phi)dG(\varepsilon') \} \tag{5}$$

There also exists a unique, continuous, and strictly increasing function c such that

$$v(w) = u(c(w)) + \beta\int v(\rho(w - c(w)) + \varepsilon' - r\phi)dG(\varepsilon'); \tag{6}$$

that is, c is the optimal consumption function in the sense of obtaining maximum expected utility (see Figure 1). Furthermore,

$$v'(w) = u'(c(w)). \tag{7}$$

For all $0 \leq w \leq \hat{w}$, where $\hat{w}$ is the unique solution to

$$u'(\hat{w}) = \beta\rho\int v'(\varepsilon' - r\phi)dG(\varepsilon'); \tag{8}$$

optimal consumption is given by

$$c(w) = w. \tag{9}$$

For all $w > \hat{w}$, the optimal consumption decision rule satisfies $\hat{w} < c(w) < w$ and is uniquely defined by

$$u'(c(w)) = \beta\rho\int v'(\rho(w - c(w)) + \varepsilon' - r\phi)dG(\varepsilon'). \tag{10}$$

Proof: We follow Lucas (1978;1980). Let L be the space of continuous, bounded functions u: $R^+ \to R$ normed by

$$||u|| = \sup_w |u(w)| . \tag{11}$$

Define T as the operator on L such that (5) reads v = Tv. Using Berge (1963), T: L $\to$ L. Using Blackwell (1965), T is a contraction so that Tv = v has a unique solution $v^* \varepsilon$ L and $||T^n u - v^*|| \to 0$ as $n \to \infty$ for all u ε L.

It is easy to verify that T takes nondecreasing, concave functions of w into strictly increasing, strictly concave functions of w. It follows that $v^* = \lim_{n \to \infty} T^n 0$ is nondecreasing and concave, and then, since $v^* = Tv^*$, that these properties hold strictly.

The maximiztion problem (5) involves maximizing a continuous, strictly concave function over a compact, convex set. Hence c(w) is uniquely defined and, from Berge (1963), p. 116, continuous. For all w such that c(w) < w, the differentiability of v follows directly from Lucas (1978) proposition 2. For w such that c(w) = w

$$v(w) = u(w) + \beta\int v(\varepsilon' - r\phi)dG(\varepsilon'); \tag{12}$$

and (7) follows directly. Since the one-sided derivaties agree, the continuous differentiabiltiy property follows. That c is strictly increasing follows from the strict concavity of u and v. The existence of a unique, positive $\hat{w}$ follows from the strict concavity of u and v and the condition, imposed below, that $\beta\rho < 1$. Otherwise we set $\hat{w} = 0$. Q.E.D.

Corollary: There exists a unique, continuous, and non-decreasing function g such that

$$v(w) = u(w - g(w) - \phi) + \beta\int v(\rho(g(w) + \phi) + \varepsilon' - r\phi)dG(\varepsilon'); \tag{13}$$

that is, g is the optimal asset accumulation decision rule (see Figure 2). For all $0 \leq w \leq \hat{w}$, optimal asset accumulation is given by

$$g(w) = -\phi. \tag{14}$$

For all $w > \hat{w}$, g is strictly increasing and is uniquely defined by

$$u'(w - g(w) - \phi) = \beta\rho\int v'(\rho(g(w) + \phi) + \varepsilon' - r\phi)dG(\varepsilon'). \tag{15}$$

Proof: Follows immediately from the properties of c, the strict concavity of v, and the budget constraint $c + g = w - \phi$.

The existence and basic properties of v, c, and g are established in Schechtman (1976), Schechtman and Escudero (1977), and Mendelson and Amihud (1982). However, the results that v is strictly concave, that c is strictly increasing, and that g is strictly increasing for $w > \hat{w}$ are new. These monotonicity properties of c and g are particularly important when we come to establish the continuity properites of the stationary probability distribution which governs the accumulation of financial claims.

2: <u>Existence of a Stationary Distribution for Optimal Asset Holdings</u>

The function c and the cummulative distribution function G of ε together define a Markov process

$$w_{t+1} = \rho(w_t - c(w_t)) + \varepsilon_{t+1} - r\phi \tag{16}$$

with state space R^+. That is, given an initial distribution for total wealth $F_0(w)$, the distribution $G(\varepsilon)$ and the difference equation (14) together determine the sequence of distributions $F_1(w)$, $F_2(w)$, . . . which prevail at dates t = 1,2, . . . The stationary distribution for total wealth, if it exists, is the limit of this sequence. Schechtman and Escudero (1977) provide the following restrictions on preferences which insure that the wealth accumulation process is bounded:

(A4) The elasticity of $u'(c)$ is uniformly bounded; i.e. there exists a $\bar{c}$ s.t. for all $c > \bar{c}$:

$$-cu''(c)/\,u'(c) \leq M < \infty;$$

(A5) The rate of time preference strictly exceeds the real rate of interest on consumption loans; i.e.

$$0 \leq r < \delta.$$

Schechtman and Escudero (1977) go on to show that (A1), (A4), and (A5) and the additional assumptions that the wage received in each period is a countable random variable and that borrowing is prohibited are sufficient to prove the existence of a limiting distribution for total wealth.

Our first task shall be to relax these latter two assumptions and to prove the existence of a unique, continuous limiting disribution for total wealth under assumptions (A1) through (A5). Armed with these results and the properties of g, we then establish the existence of a unique, almost-everywhere continuous stationary cummulative distribution function which characterizes the accumulation of non-contingent financial claims in a stochastic steady-state.

To say that a sequence $w_0, w_1 \ldots$ is subject to the transition probabilities K means that $K(w,w')$ is the conditional probability of the event $\{w_1 < w'\}$ given that $w_0 = w$. Equation (16) and the definition of G imply that

$$K(w,w') = G(w' - \rho h(w) + r\phi); \tag{17}$$

where $h(w) \equiv w - c(w)$. If the probability distribution of w_0 is F_0, the distribution of w_1 is given by

$$F_1(w') = \int_\Omega F_0\{dw\}K\{w,w'\}. \tag{18}$$

Definition: The distribution F is a stationary distribution for K if TF = F, where the operator T is defined by

$$TF(w') = \int_\Omega F\{dw\}K\{w,w'\}. \tag{19}$$

Feller's (1971) ergodic theorems for Markov chains establish conditions on K and Ω which guarantee the existence of a unique stationary distribution F. Furthermore, F represents the asymptotic distribution of w_n under any initial distribution. That is, the influence of the initial state fades away and the system tends to a steady state governed by the stationary solution. Mendelson and Sobel (1980) have applied the Feller (1971) ergodic theorems to study capital accumulation in the context of a general renewable resource model.

Following closely the arguments used by Danthine and Donaldson (1981) in their examination of the limiting behavior of capital in the Brock - Mirman (1972) stochastic optimal growth model, we now prove the following theorem.

Theorem 2: By Assumption (A1) through (A5), there exists a continuous and increasing stationary distribution function for wealth, denoted $F(w)$, which is the unique solution to the functional equation $TF = F$:

$$F(w') = \int_{\Omega} G(w' - \rho h(w) + r\phi)dF(w). \tag{20}$$

For any F_0,

$$\lim_{n\to\infty} T^n F_0 = F. \tag{21}$$

Furthermore, F possesses a continuous density function which is positive on the compact subset $\Omega \equiv [\underline{\varepsilon} - r\phi, \overline{w}]$ of R^+ where

$$\overline{w} = \min\ [\overline{w}: \rho h(\overline{w}) + \overline{\varepsilon} - r\phi = \overline{w}]. \tag{22}$$

Proof: By assumption (A2), the stochastic kernel

$$K(w,w') \equiv G(w' - \rho h(w) + r\phi) \tag{23}$$

has a continuous density, denoted $k(w,w')$. It follows from Feller (1971), p. 272 that $K(w,w')$ is regular. That is, the family of transforms $\mu_t(\ )$ defined by

$$\mu_t(w') = \int k(w,w')\mu_{t-1}(w)dw; \quad \mu_0 = 0; \tag{24}$$

for μ continuous and bounded is equicontinuous whenever μ_0 is uniformly continuous.

We must show that the compact set $\Omega \equiv [\underline{\varepsilon} - r\phi, \overline{w}]$ is the ergodic set of the Markov process (16) with transition probabilities $K(w,w')$, and that the complement of Ω is the transient set. This will insure that all of the probability weight of the stationary distribution lies in Ω, and that any point of Ω can be reached from any other point in that interval in a finite number of steps.

We first show that, once the process (16) has entered Ω, there is zero probability that it will depart from it (see Figure 3). It follows immediately from Schechtman and Escudero (1977) Theorem 3.3, p. 158 that there exists a $\tilde{w} < \infty$ such that

$$\rho(w - c(w)) + \bar{\varepsilon} - r\phi \leq \tilde{w}, \quad w \geq \tilde{w}. \tag{25}$$

This implies that there exists at least one $\bar{w}$ which solves (22) and that

$$\rho(w - c(w)) + \bar{\varepsilon} - r\phi \leq \bar{w}, \quad w \leq \bar{w}. \tag{26}$$

Observe that for $w > \hat{w}$, $u'(c(w)) < \beta\rho u'(c(\rho h(w) + \underline{\varepsilon} - r\phi))$ which implies that $w > \rho h(w) + \underline{\varepsilon} - r\phi$. It follows that

$$\underline{\varepsilon} - r\phi \leq \rho h(w) + \varepsilon' - r\phi \leq \bar{w}. \tag{27}$$

That is, for W not in Ω and $w \varepsilon \Omega$

$$k(w,W) = dG(W - \rho h(w) + r\phi) = 0. \tag{28}$$

We next show that there is no transient subset of Ω. This follows from the fact that, for all $w \varepsilon \Omega$, $\rho h(w) + \bar{\varepsilon} - r\phi > w$, $\rho h(w) + \underline{\varepsilon} - r\phi < w$, and that c and dG are continuous. Hence, for all $w \varepsilon \Omega$, there exists an interval of positive length surrounding w, $l(w)$, such that $k(w,w') > 0$. These intervals cover Ω.

Finally, we show that any interval disjoint from Ω is a transient set. There are two types of intervals to consider, depicted as $[\bar{w},w_1]$ and $[w_1,w_2]$ in Figure 3. The interval $[\bar{w},w_1]$ is characterized by the fact that, for all ε' $\rho h(w) + \varepsilon' - r\phi < w$. The wealth accumulation process will clearly leave such intervals in a finite number of steps with probability one. That is, there exists an N such that, for all $n > N$,

$$T^n[\rho h(w) + \varepsilon' - r\phi] < \bar{w} \tag{29}$$

for all possible realization of ε'. With regards to intervals such as $[w_1,w_2]$ there exists an $\underline{\varepsilon} < \hat{\varepsilon} < \bar{\varepsilon}$ such that, for all w in $[w_1,w_2]$ and $\underline{\varepsilon} < \varepsilon' < \hat{\varepsilon}$ $\rho h(w) + \varepsilon' - r\phi < w$. Thus, there exists an N such that, for $n > N$,

$$T^n[\rho h(w) + \varepsilon' - r\phi] < w_1, \text{ provided } \varepsilon' < \hat{\varepsilon}. \tag{30}$$

The probability of this occurrence is at least ζ^N where

$$\zeta = \int_{\underline{\varepsilon}}^{\hat{\varepsilon}} dG(\varepsilon) > 0. \tag{31}$$

Thus, with probability of at least ζ^N, the process leaves $[w_1,w_2]$ never to return. In general, the expected number of visits to $[w_1,w_2]$ is less than

$$\sum_{j=1}^{\infty}(1 - \zeta^N)^j < \infty. \tag{32}$$

Hence, $[w_1,w_2]$ is transient (probability of one of leaving, zero probability of re-entering). Exactly the same reasoning applies to intervals to the left of Ω. Q.E.D.

Corollary: Let assumptions (A1) through (A5) hold. Then there exists a unique, almost-everywhere continuous, increasing stationary distribution, denoted $X(a')$, which characterizes the accumulation of non-contingent financial assets. The support of $X(a')$ is the compact interval $[-\phi, g(\overline{w})]$ of R. X has a single mass point at $a' = -\phi$ such that

$$X(-\phi) \equiv \text{Prob}(a' = -\phi) = F(\hat{w}). \tag{33}$$

For $a' > -\phi$, $X(a')$ is defined by

$$X(a') \equiv \text{Prob}(a \leq a') = F(g^{-1}(a')). \tag{34}$$

Proof: From the corollary to Theorem 1, $g(w) = -\phi$ for $0 \leq w \leq \hat{w}$ which implies (21). $g(w)$ is continuous and strictly increasing over the compact interval $[\hat{w}, \overline{w}]$ so that the inverse $g^{-1}(a')$ exists and is right continuous at $a' = -\phi$ which implies (22) and the right continuity of X at $-\phi$. Q.E.D.

The shape of the probability distribution which characterizes the holdings of non-contingent financial claims is shown in Figure 4. There is a mass point at the finite borrowing limit $-\phi$ and then a smooth distribution on $(-\phi, g(\overline{w})]$. The intuition is straightforward. The individual's available line of credit $\phi + a$ is an inventory which is held to self-insure against adverse wage disturbances. Since the rate of time preference is positive, the optimal inventory policy cannot be bounded away from zero (cf. Lucas (1980) p.140 and Clarida (1984) p.21). This property of the stationary distribution which governs the accumulation of non-contingent financial claims corresponds closely to the findings of Lucas (1980) and Clarida (1984) in their inventory-theoretic models of optimal money holdings in the presence of uncertainty.

3: The Role of Time Preference

In this section, we compare the accumulation behavior of individuals who differ in their pure rates of time preference. Individuals with low rates of time preference can be thought of as facing more rapid fluctuations in their random wage earnings (Bewley (1977;1980)). We first determine the influence of time preference on optimal consumption decisions at a given level of wealth. We next study how differences in consumption functions induced by differences in time preference influence the accumulation of assets over time. We show that differences in transition probabilities $K(w,w') = G[w' - \rho(w - g(w;\delta)) + r\phi]$ induced by differences in time preference have specific implications for the stationary distributions governing optimal asset holdings uniquely associated with them. Well known results due to Hadar and Russell (1971) allow us to determine the link between time preference and the moments of the stationary distribution which governs the accumulation of non-contingent claims in a stochastic steady-state.

We begin by showing that optimal consumption at a given level of wealth is directly related to time preference.

Theorem 4: Consider two otherwise identical individuals who differ only in their pure rates of time preference. The agent with the lower rate of time preference consumes no more at each level of wealth than does the agent with the higher rate of time preference:

$$c(w;\delta_2) = c(w;\delta_1) = w, \quad \text{for } w \leq \hat{w}(\delta_2); \tag{35}$$

$$c(w;\delta_2) < c(w;\delta_1) \leq w, \quad \text{for } w > \hat{w}(\delta_2); \tag{36}$$

where $\delta_2 < \delta_1$.

Proof: From Theorem 1 we know that a necessary and sufficient characterization of the optimal consumption decision rule is given by

$$u'(c(w;\delta)) = \beta\rho\int v'(\rho(w - c(w;\delta)) + \varepsilon' - r\phi;\delta)dG(\varepsilon'), \tag{37}$$

for $w > \hat{w}(\delta)$; otherwise $c(w) = w$. We must show that $v'(w;\delta_2) > v'(w;\delta_1)$. We work recursively with the fact that v can be viewed as the limit of a sequence of valuation problems. The agent's problem in the next to last period is given by

$$v(w;\delta;1) = \max_{c \leq w} \{ u(c) + \beta Eu[\rho(w - c) + \varepsilon' - r\phi] \} \tag{38}$$

Define the maximand as $\xi(w;c;\delta;1)$. Then, for all c $\xi_w(w;c;\delta_2;1) \geq \xi_w(w;c;\delta_1;1)$. Since $v(w;\delta;1) = \xi(w;\tilde{c};\delta;1)$ for some $\tilde{c}$, we have

$$v'(w;\delta_2;1) \geq v'(w;\delta_1;1). \tag{39}$$

Proceeding recursively, we can show that

$$v'(w;\delta_2;i) \geq v'(w;\delta_1;i); \tag{40}$$

provided $v'(w;\delta_2;i-1) \geq v'(w;\delta_1;i-1)$. It follows directly from Mendelson and Amihud (1982) Lemma p. 696 that $v'(w;i)$ converges to $v'(w)$ uniformly on every closed interval so long as $u'(0) < \infty$. It follows that

$$v'(w;\delta_2) \geq v'(w;\delta_1). \tag{41}$$

From (37) and (41), we obtain, for $w > \hat{w}(\delta_2)$,

$$u'(c(w;\delta_1)) < \beta_2\rho\int v'(\rho(w - c(w;\delta_1)) + \varepsilon' - r\phi;\delta_2)dG(\varepsilon'). \tag{42}$$

From the strict concavity of u and v, we know that consumption must be decreased to restore the equality between the marginal utility of consumption today with the expected marginal valuation of wealth tommorow. Furthermore, (42) and the definition of $\hat{w}$ imply $\hat{w}(\delta_2) < \hat{w}(\delta_1)$. Q.E.D.

Corollary: The agent with the lower rate of time preference accumulates no less at a given level of wealth than the agent with the higher rate of time preference:

$$g(w;\delta_2) = g(w;\delta_1) = -\phi, \quad \text{for } w \leq \hat{w}(\delta_2); \tag{43}$$

$$g(w;\delta_2) > g(w;\delta_1), \quad \text{for } w > \hat{w}(\delta_2). \tag{44}$$

Proof: Follows directly from the theorem and the budget constraint $c + g = w - \phi$.

The influence of time preference on the accumulation of non-contingent financial claims is now examined. We show that the stationary distribution which governs the asset accumulation of low time preference individuals is at least as large, in the sense of first-degree stochastic dominance, as that of individuals with higher rates of time preference.

Theorem 5: Consider two otherwise identical individuals who differ only in their rates of time preference. If $\delta_2 < \delta_1$, then $X(a';\delta_2)$ is at least as large, in the sense of first-degree stochastic dominance, as $X(a';\delta_1)$. That is,

$$X(a';\delta_2) \leq X(a';\delta_1). \tag{45}$$

Proof: From Theorem 2 we know that

$$F(w';\delta_1) = \int G(w' - \rho h(w;\delta_1) + r\phi)F(w;\delta_1). \tag{46}$$

Associated with $\delta_2 < \delta_1$ is the function $h(w;\delta_2)$ which, from Theorem 4 and the definition of h is greater than $h(w;\delta_1)$ so long as $w > \hat{w}(\delta_2)$. Define the mapping T_2 as

$$T_2F(w) = \int G(w' - \rho h(w;\delta_2) + r\phi))dF(w). \tag{47}$$

From Theorem 2, we know that

$$\lim_{n\to\infty} T_2^n F = F(w;\delta_2) \tag{48}$$

Since G is a strictly increasing function, we know that

$$G(w' - \rho h(w;\delta_2) + r\phi) \leq G(w' - \rho h(w;\delta_1) + r\phi) \tag{49}$$

with the inequality holding strictly for some $w \in [\underline{\varepsilon} - r\phi, \overline{w}(\delta_2)]$ if $\hat{w}(\delta_2) < \overline{\varepsilon} - r\phi$. This implies that $T_2F(w;\delta_1) \leq F(w;\delta_1)$. It follows that the sequence $\{T_2^n F(w')\}$ is non-increasing and none of its elements are larger than $F(w';\delta_1)$. Thus, $F(w';\delta_2) \leq F(w';\delta_1)$.

Consider now the relationship between $X(a';\delta_2)$ and $X(a';\delta_1)$. There are three cases to examine. If $\overline{\varepsilon} - r\phi \leq \hat{w}(\delta_2)$, then $\overline{w}(\delta_i) = \overline{\varepsilon} - r\phi$ and

$$X(-\phi;\delta_2) = X(-\phi;\delta_1) = 1. \tag{50}$$

If $\hat{w}(\delta_2) < \overline{\varepsilon} - r\phi \leq \hat{w}(\delta_1)$, then $X(a';\delta_1) = 1$ for all $a' \geq -\phi$, but

$$X(a';\delta_2) = F(g^{-1}(a';\delta_2);\delta_2)) < 1 \tag{51}$$

for $a' < g(\overline{w}(\delta_2))$. If $\hat{w}(\delta_1) > \overline{\varepsilon} - r\phi$, we have:

$$X(a';\delta_2) = F(g^{-1}(a';\delta_2);\delta_2)) < F(g^{-1}(a';\delta_2);\delta_2)) = X(a';\delta_1); \qquad (52)$$

which follows directly from the properties of $g = w - c - \phi$ and the fact that, $\overline{w}(\delta_2) > \overline{w}(\delta_1)$. Q.E.D.

Finally, we examine the relationship between time preference and the moments of the probability distribution which characterizes the accumulation of non-contingent financial claims. We show that expected asset holdings of high time preference individuals do not exceed the expected asset holdings of low time preference individuals. Furthermore, for the special case in which borrowing against future earnings is not permitted, we show that none of the moments of the stationary distribution governing the asset holdings of high time preference individuals exceeds the corresponding moment of the stationary distribution governing the asset holdings of low time preference individuals.

Corollary: The expected asset holdings of high time preference individuals do not exceed the expected asset holdings of low time preference individuals:

$$E(a';\delta_2) > E(a';\delta_1), \quad \delta_2 < \delta_1. \qquad (52)$$

For the special case in which $\phi = 0$, none of the moments of the stationary distribution governing the asset holdings of high time preference individuals exceeds the corresponding moment of the stationary distribution governing the asset holdings of low time preference individuals.

$$E(a';\delta_2)^i > E(a';\delta_1)^i \qquad i = 1,2, \ldots \qquad (53)$$

Proof: Follows directly from Theorem 5 and Hadar and Russell (1971) Corollary 1, p. 291.

4: The Influence of Average Wage Earnings

In this section we compare the accumulation behavior of individuals who differ only in their average wage earnings. We shall assume that borrowing is allowed in amounts which can be repayed with probability one. It follows that additive differences in probability distributions governing the random earnings of two otherwise identical individuals imply directly proportional differences in the borrowing capacity available to these individuals. We shall now explore the stochastic-steady state implications of this simple fact.

We begin by showing that c, the optimal consumption function, is invariant to additive differnces in the probability distribution governing random wages.

Theorem 6: Consider two otherwise identical individuals whose wage earnings are governed by the probability distributions $G(\varepsilon_t)$ and $G(\varepsilon_t;\psi)$ respectively, $\psi > 0$, such that $G(\varepsilon_t;\psi) = G(\varepsilon_t + \psi)$. The two individuals have identical consumption functions. That is,

$$c(w) = c(w;\psi). \tag{54}$$

Proof: From Theorem 1 we know that a necessary and sufficient characterization of optimal consumption is given by

$$u'(c(w)) = \beta\rho\int v'(\rho(w - c(w)) + \varepsilon' - r\phi)dG(\varepsilon'). \tag{55}$$

for $w > \hat{w}$, $c = w$ otherwise. The theorem follows directly from the assumption that $\phi(\psi) = \phi + \psi/r$ and the definition of $G(\varepsilon;\psi)$. Together these imply that the random variables $\varepsilon' - r\phi$ and $\varepsilon' + \psi - r\phi(\psi)$ are identical. Q.E.D.

Corollary: At any given level of wealth, the agent with the higer average wage earnings purchases fewer non-contingent assets. Furthermore,

$$g(w;\psi) - g(w) = -\psi/r. \tag{56}$$

Proof: Follows directly from the theorem and the budget constraint $c + g = w - \phi(\psi)$.

These results make sense. So long as the individual with the higher average wage earnings is allowed to borrow against them - to a limit of ψ/r which can be repayed with probability one - the stationary statistical decision problems faced by him and another individual with lower average wage earnings are identical except for intitial conditions. Thus, the optimal consumption functions coincide. However, it is the case that the individual with the higher average wage earnings does consume more initially, accomplishing this by borrowing against his greater human capital.

The next theorem establishes the stochastic steady-state implications of these results.

Theorem 7: Consider two otherwise identical individuals whose wage earnings are governed by the probability distributions $G(\varepsilon_t)$ and $G(\varepsilon_t;\psi)$ respectively. The stationary probability distributions which governs these individiuals' accumulation of non-contingent financial claims differ by the location parameter ψ/r. In particular

$$X(a';\psi) = X(a' + \psi/r). \tag{57}$$

Proof: The stationary probability distribution which governs total wealth is the unique solution to the functional equation

$$F(w') = \int G(w' - \rho h(w) + r\phi) dF(w). \tag{58}$$

The solution to equation (58) depends only on G and h, which by Theorem 6, is independent of ψ. Using the fact that $g(w;\psi) = h(w) - \phi(\psi)$, we obtain,

$$X(a') = F[h^{-1}(a' + \phi)]; \tag{59}$$

$$X(a';\psi) = F[h^{-1}(a' + \phi + \psi/r)]. \tag{60}$$

If follows directly that

$$X(a';\psi) = X(a' + \psi/r); \quad \text{Q.E.D.} \tag{61}$$

Corollary 1: Differences in average wage earnings are, on average, offset by the accumulation of non-contingent financial claims:

$$E[a;\phi] - E[a;\phi(\psi)] = \psi/r. \tag{62}$$

Corollary 2: The central moments of the stationary probability distribution which governs the accumulation of non-contingent financial claims are invariant to differences in average wages:

$$E[a - E(a;\phi);\phi]^j = E[a - E(a;\phi(\psi));\phi(\psi)]^j \tag{63}$$

for $j = 1, 2, \ldots$

5: Concluding Remarks

In this paper, we have established rigorously the existence and properties of the stationary probability distribution which characterizes the accumulation of non-contingent financial claims by a risk averse individual who confronts random wage fluctuation and incomplete insurance markets. We showed that there exists a unique, almost-everywhere continuous stationary cummulative distribution function which characterizes the accumulation of non-contingent financial claims in a stochastic steady-state. This distribution was shown to possess a single mass point coinciding with the non-negative, finite borrowing limit faced by the individual. We went on to investigate the influence of time preference and average wage earnings on the accumulation of non-contingent financial claims in a stochastic steady-state. We established that the stationary probability distribution which characterizes the asset accumulation of low time preference individuals is at least as large, in the sense of first-degree stochastic dominance, as that of individuals with higher rates of time preference. We proved that, so long as individuals are allowed to borrow in amounts which can be repayed with probability one, additive differences in the probability distributions governing random wage earnings imply inversely proportional additive differences in the stationary probability distributions which govern the accumulation of non-contingent financial claims.

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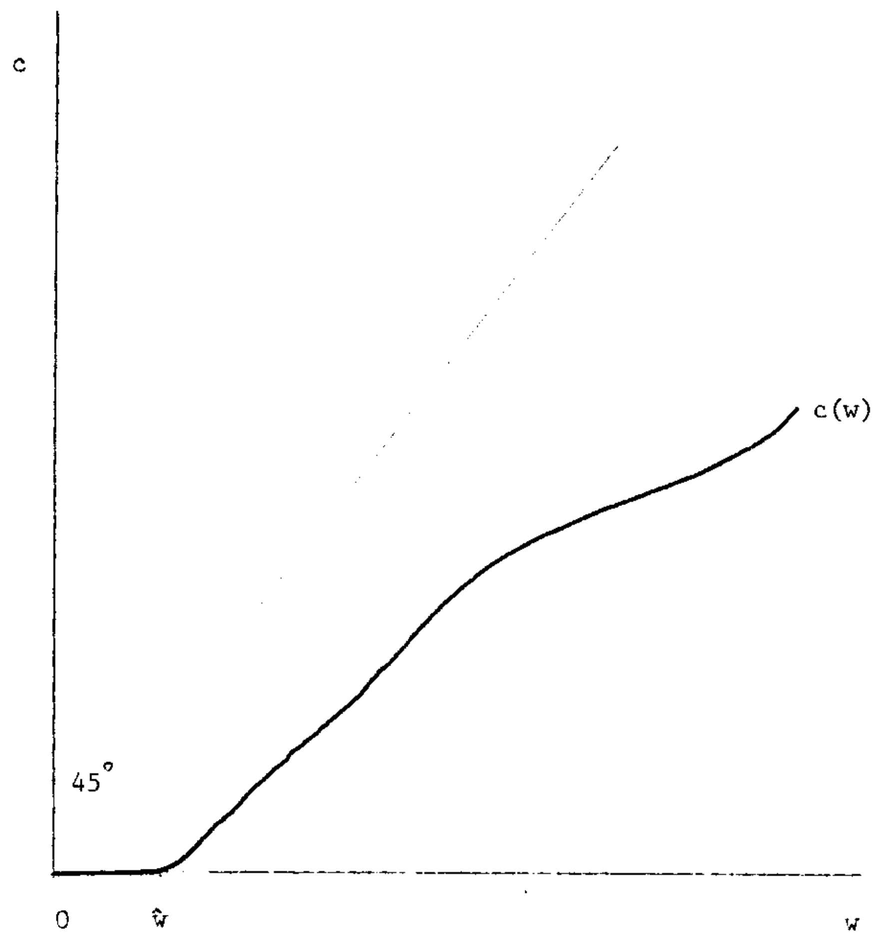
<u>NOTES</u>

(1) One may think of insurance as being impossible because assymetric information about the individual's circumstances offers an incentive to cheat.

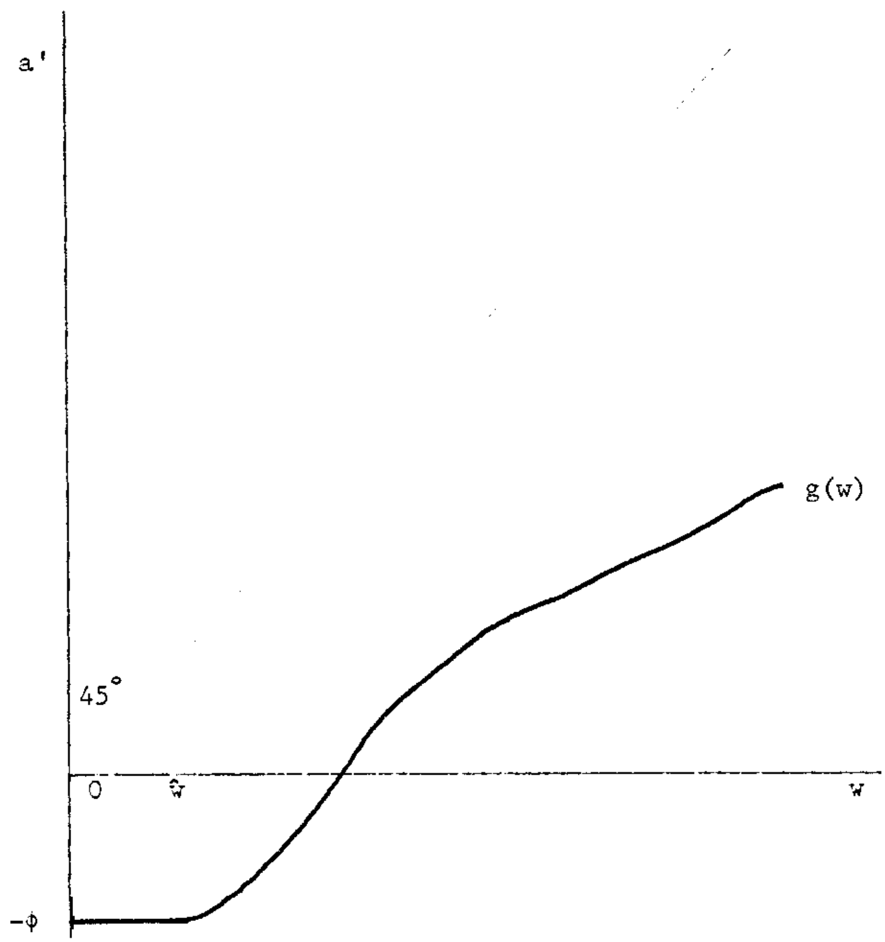
(2) Bewley (1977;1980) has studied the role of non-contingent assets in an intertemporal, general equilibrium model.

Figure 1



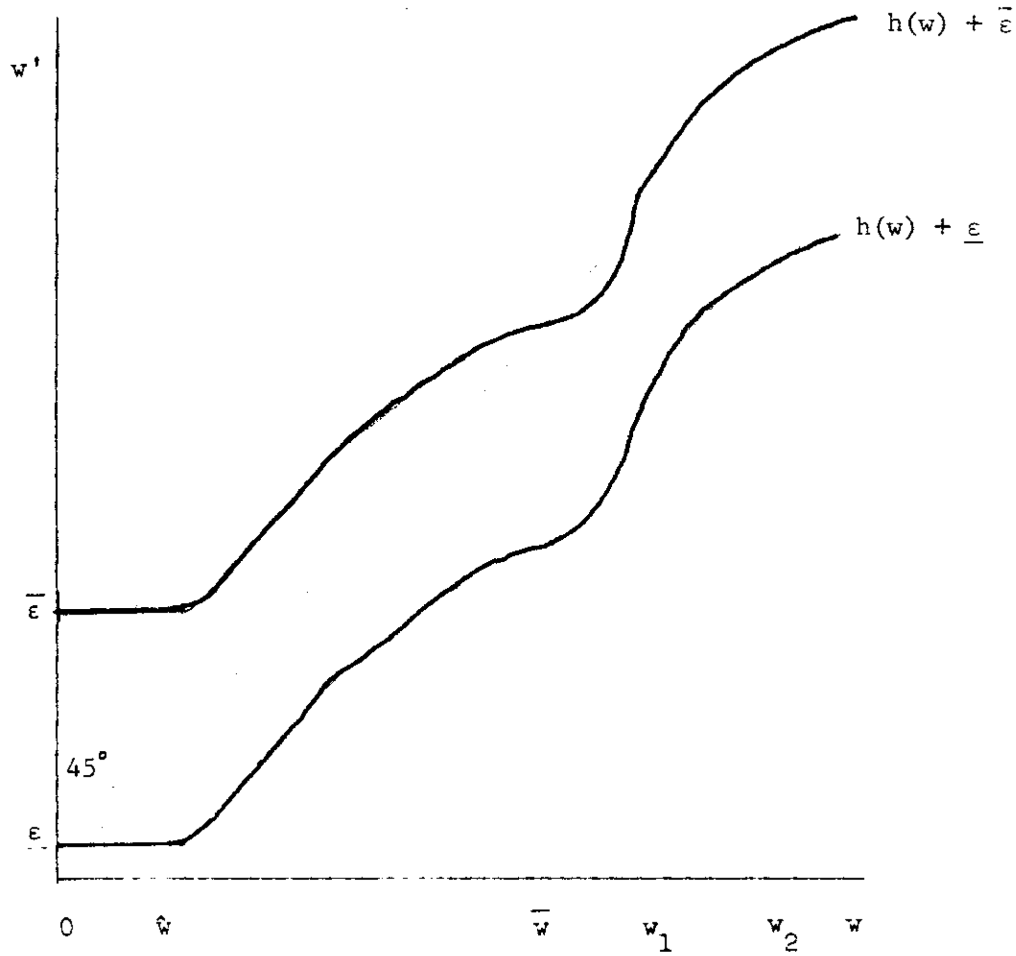
The optimal consumption decision rule c(w).

Figure 2



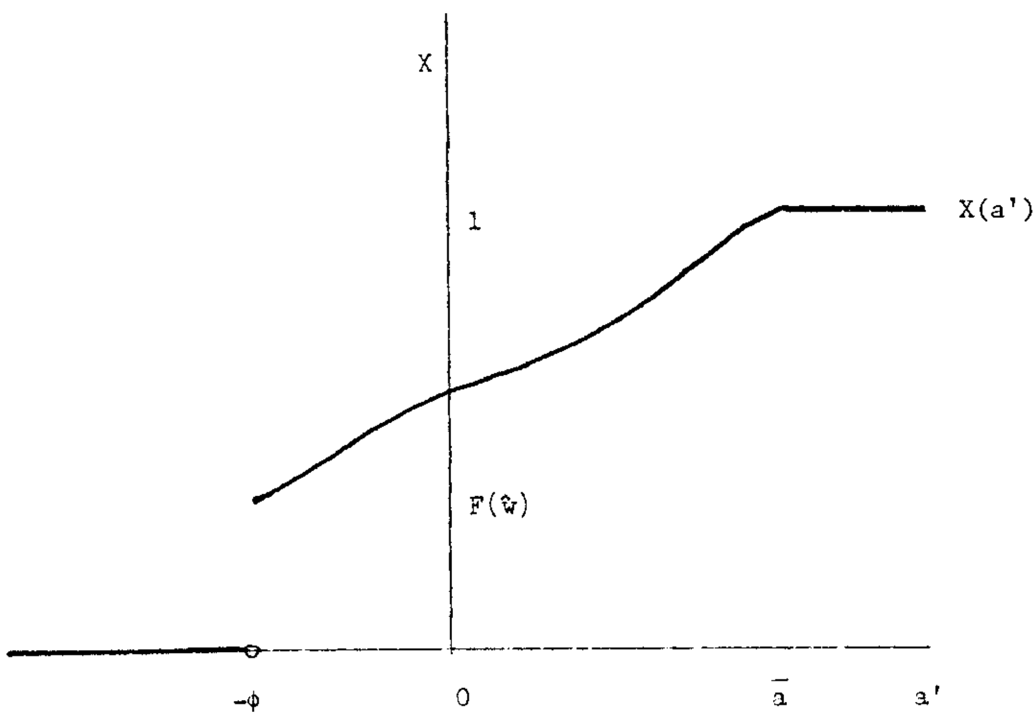
The optimal asset accumulation decision rule g(w).

Figure 3



The ergodic set $\Omega = [\underline{\varepsilon}, \bar{w}]$.

Figure 4



The limiting distribution for optimal asset holdings X(a').