COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

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COWLES FOUNDATION DISCUSSION PAPER NO. 575

A PATH FOLLOWING PROCEDURE FOR FINDING A POINT IN THE CORE OF A BALANCED N-PERSON GAME

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December 24, 1980

ABSTRACT

A basic theorem in n-person game theory due to Scarf states that a balanced game has a nonempty core. Scarf's proof presents a procedure to find a point in the core of a discrete game, where every coalition disposes of a finite number of alternatives. The proof for a general game follows by passing to the limit.

In this paper we present a procedure which works with the characteristic sets in original form. They no longer need to be approximated. The procedure consists in following a finite sequence of possibily non-linear paths.

The framework adopted for this paper is more general than needed to treat the core problem. This enables us to present a unified approach treating the latter problem as well as related problems in linear complementarity theory and fixed point computation.

A PATH FOLLOWING PROCEDURE FOR FINDING A POINT IN THE CORE OF A BALANCED N-PERSON GAME

by

Ludo Van der Heyden*

I. Introduction

A basic theorem in n-person game theory due to Scarf (1967) states that a balanced game without side payments has a nonempty core. Scarf's well-known proof goes beyond the existence result. It first presents an algorithm for the computation of a point in the core of a balanced, discrete n-person game. A game is discrete when every coalition disposes of a finite number of possibilities or "utility vectors." Scarf's proof continues with the observation that the set of vectors achievable by a coalition, the "characteristic set" of that coalition, can be approximated to any desired degree of accuracy by a finite number of appropriately chosen utility vectors. The proof for the general case then concludes with a passage to the limit in the discrete game.

Scarf's algorithm is a search among a finite number of combinatorial objects called "primitive sets." These objects arise quite naturally

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*I am grateful to Professor B. Curtis Eaves (Stanford University) and Herbert E. Scarf (Yale University) for suggesting the problem to me, and to Professor Carlton E. Lemke (Rensselaer Polytechnic Institute) for extensive comments on an earlier version of the paper.

Financial support by ONR Grant N00014-77-C-0518 is kindly acknowledged.

in the core problem. Aside from the presence of primitive sets, the other distinct feature of the procedure is the application of the Lemke and Howson (1964) argument to guarantee that the search among these objects is finite and terminates with a primitive set associated with a point in the core of the discrete n-person game. The main point of the argument is to show that, unless a solution is reached, a replacement operation leading to a new primitive set can always be carried out, and that during a sequence of such replacement operations a primitive set cannot be visited twice. This prohibits the existence of a cycle of primitive sets which the procedure is caught visiting forever and insures the finiteness of the procedure.

Shapley (1973) has presented a second procedure to compute a point in the core. His is also based on the Lemke-Howson argument, but the combinatorial objects are the more familiar simplices in a triangulation or simplicial subdivision.

Both approaches are combinatorial and solve a general n-person game by first approximating it with a discrete n-person game and then solving the approximate game precisely. If the approximation is good enough, a point in the core of the approximate game will be close to the core of the original game, if not in it already. In this paper, we present a procedure which works with the characteristic sets in their original form; they no longer need to be approximated. The procedure follows a sequence of paths--usually nonlinear--which are the intersection of the boundaries of characteristic sets of certain coalitions. The procedure is finite modulo the existence of a finite procedure for following 1-dimensional paths. This is the nonconstructive part in our proof of Scarf's theorem. It replaces the passage to the limit in the Scarf and Shapley proofs. Instead of solving an approximate problem exactly, we solve the original problem approximately. The approximation arises because we are generally unable to follow a nonlinear path exactly.

It may be worth pointing out that a "continuous" procedure can be obtained by first reinterpreting Scarf's original procedure in terms of path-following and then taking the limit. This limit operation leads to a procedure distinct from the procedure exposited in this paper. The reason we have opted for our approach is that it permits us to present a unifying framework for the core problem, and for two earlier papers on the related topics of fixed point computation and linear complementarity (Van der Heyden, 1979 and 1980).

A general result is presented in Section II. The result is a continuous version of the main theorem in Scarf's monograph (Scarf, 1973, Theorem 4.2.3). Section III contains a proof of our main result and Section IV seeks to illustrate the procedure used in the proof. Section V applies the procedure to the computation of a point in the core of a balanced n-person game. We end with some concluding remarks in Section VI.

II. A General Theorem about Labelled Sets

We are going to state a property of a collection of closed sets in R^n when each set is assigned an n-dimensional vector label. Let $S^1, S^2, \ldots, S^m$ be a collection of closed sets in n-space $(m > n)$, and let a^j be the label of S^j. These labels can be represented as columns of an $n \times m$ matrix $A = [a^1, a^2, \ldots, a^m]$. Given a right-hand side vector b, we consider the linear system

(2.1) $Ay = b \ , \quad y \geq 0 \ ,$

where $y = (y_1, y_2, \ldots, y_m)$ is the vector of weights associated with the sets in our collection. A collection of n sets $S^{j_1}, S^{j_2}, \ldots, S^{j_n}$ is <u>completely labeled</u> if the labels are associated with a solution y for equation (2.1), i.e., if $y_j = 0$ unless $j = j_1, j_2, \ldots, j_n$.

Our interest lies with <u>undominated</u> points. A point is said to be dominated if it lies in the interior of a set S^j $(1 \leq j \leq m)$, denoted int (S^j). The theorem we are about to prove asserts the existence of an undominated point belonging to the boundaries of a completely labeled collection of sets. The boundary of S^j is denoted bdy (S^j), and we use the notation ext(S^j) for the exterior of set S^j.

Naturally, some conditions need to be imposed for the statement to hold. We require four conditions. One involves the labels, a second requires regularity in the intersection of these sets and their boundaries, and the third and fourth conditions involve both the sets and their labels.

(2.2) (Assumption) The linear system $Ay = b$, $y \geq 0$ is nondegenerate and bounded.

The nondegeneracy assumption can be made without loss of generality as a small perturbation of the right-hand side b eliminates degeneracy when it occurs (for more details on linear programming, we refer to Dantzig, 1963). The assumption implies that completely labeled collections contain at least n sets. Given the boundedness condition, a completely labeled collection of $n+1$ sets contains exactly two completely labeled collections of n sets each. The labels of n completely labeled sets form a basis in the matrix A . Introducing the label of an $(n+1)^{st}$ set, we use a standard linear programming pivot step to identify the label which can be replaced in the current basis. The replacement of this label with the incoming label results in a new basis matrix and, hence, in a new completely labeled collection of n sets. This replacement operation is one of the two operations central to our procedure; the other consists in following 1-dimensional paths.

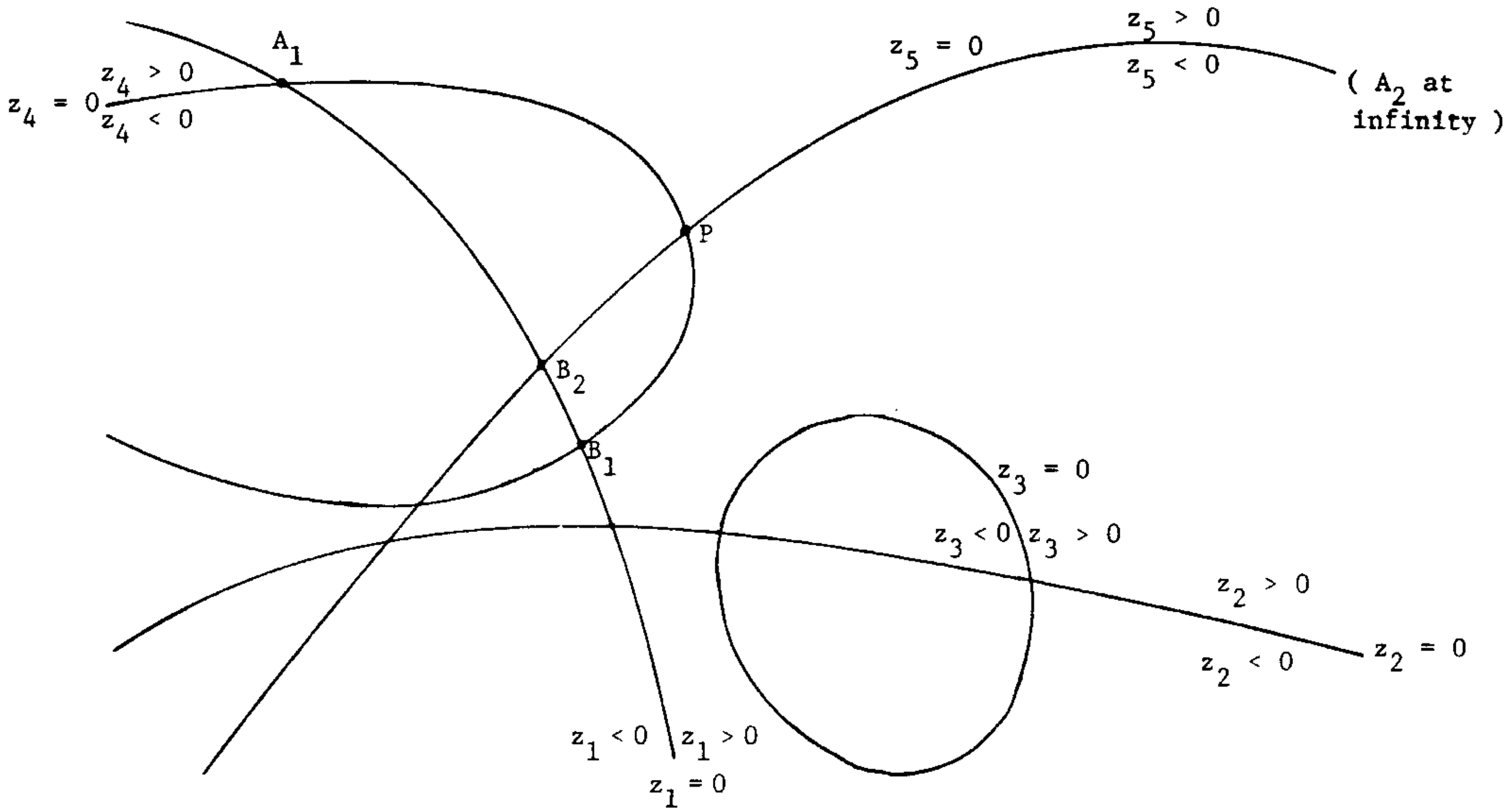
(2.3) (Assumption) The boundaries of the sets $S^1, S^2, \ldots, S^m$ meet <u>nondegenerately</u>:

a. No point belongs to the boundary of more than n sets.

b. Any bounded set contains at most a finite number of points belonging to n boundaries.

c. Consider a point belonging to the intersection of n boundaries, say $x \in \text{bdy}(S^{j_k})$ for $k = 1, 2, \ldots, n$. For each k there exist two connected sets of points belonging to $\text{bdy}(S^{j_h})$ for $1 \leq h \neq k \leq n$ but not to any other $\text{bdy}(S^j)$, $1 \leq j \leq m$. These sets are called <u>arcs</u> and are homeomorphic to an open interval with x as endpoint; one arc lies in $\text{ext}(S^{j_k})$, the other lies in $\text{int}(S^{j_k})$. The second endpoint, if it exists, lies in $\text{bdy}(S^j)$ for some j , $1 \leq j \leq m$. If x is the only endpoint, then the arc is unbounded.

Assumption (2.3) is illustrated in Figure 2.4. It imposes fairly minimal regularity conditions on the sets and their boundaries. A possibly more elegant formulation of assumption (2.3) in terms of manifolds will be given in Section V, but it involves more restrictive differentiability



2.4 (Figure) An example of 5 sets $(m = 5)$ in 2-space $(n = 2)$ whose boundaries meet nondegenerately. The vector z describes the position of a point relative to the sets S^j :

$$z_j(x) \begin{cases} > 0 & \\ = 0 & \text{for} \quad x \in \\ < 0 & \end{cases} \begin{matrix} \text{ext}(S^j) \\ \text{bdy}(S^j) \\ \text{int}(S^j) \end{matrix}$$

Point P belongs to the intersection of $\text{bdy}(S^4)$ and $\text{bdy}(S^5)$. There are four (= 2n) arcs leaving P . Three arcs lead to $\text{bdy}(S^1)$. The arcs and their characterizing inequalities are:

$PA_1 : z_4 = 0 , \quad z_5 > 0 ;$

$PB_1 : z_4 = 0 , \quad z_5 < 0 ;$

$PA_2 : z_5 = 0 , \quad z_4 > 0 ;$

$PB_2 : z_5 = 0 , \quad z_4 < 0 .$

The third arc (PA_2) is unbounded.

conditions and may obfuscate the assumptions central to the procedure. Point x belongs to the intersection of n lines, each representing the intersection of the boundaries of $n-1$ sets. There are 2 ways to leave x along such a line, one in the interior, the other belonging to the exterior of the n^{th} set. In total, there are $2n$ different arcs one can follow to leave x. It is clear from Figure 2.4 that two of these arcs may lead to the same point.

(2.5) (Assumption) The first n sets are given by:

$$S^i = \{x \mid x_i \leq 0\} \quad \text{for} \quad i = 1, 2, \ldots, n .$$

These sets are completely labeled.

This assumption implies a privileged role for the first n sets. It allows the procedure to start at the origin, the only undominated point among these sets. From there, other points will be generated, but they will never be dominated by any of the first n sets, i.e., they will all be points in the nonnegative orthant.

(2.6) (Assumption) Let $S^{j_1}, S^{j_2}, \ldots, S^{j_{n-1}}$ be sets whose boundaries intersect along $L = \bigcap_{i=1}^{n-1} \mathrm{bdy}(S^{j_i})$. If $S^{j_1}, S^{j_2}, \ldots, S^{j_n}$ form a completely labeled collection and $j_n = \max(j_1, j_2, \ldots, j_n)$, then the intersection $L \cap S^{j_n}$ is bounded from above.

We will soon see that, when considered jointly with assumption 2.5, assumption 2.6 bounds the region in which a solution can be found. It is clearly satisfied when the intersection of n completely labeled sets is bounded from above. The latter condition is very natural for the core problem.

It may be worthwhile to point out that the principal assumptions are the boundedness of the linear system (2.2), the upper-boundedness condition related to a completely labeled collection of sets (2.6), and the privileged role of the first n sets (2.5). The nondegenerate intersection property (2.3) imposes minimal regularity conditions on the sets and their boundaries. We will show that in the discrete game, for example, they can always be satisfied.

Having developed terminology and having presented our assumptions, we now state the theorem we are about to prove.

(2.7) (Theorem) Consider a collection of closed sets $S^1, S^2, \ldots, S^m$ $(m > n)$ whose boundaries meet nondegenerately and whose labels are associated with a nondegenerate and bounded linear system. If these sets and their labels satisfy assumptions 2.5 and 2.6, then there exists an undominated point belonging to n completely labeled sets.

The point whose existence is the object of Theorem 2.7 will be called a solution. Since it is undominated it lies in the boundary of any set to which it belongs.

III. A Proof for Theorem 2.7

Our proof starts with the observation that assumption 2.5 causes the theorem to be trivially satisfied when considering only the first n sets in our collection. Their boundaries intersect at the origin and there is no other set to dominate the origin. Starting at the origin, we will verify the statement in theorem 2.7 for an increasingly larger collection of sets. The subproblem which finds an undominated point belonging to n completely labeled sets among $S^1, S^2, \ldots, S^h$, and thus verifies the statement in theorem 2.7 for the first h sets, is called the <u>h-problem</u>. Solving the m-problem proves theorem 2.7. Notice that the procedure is very dependent on the indexing of the sets, though the indexing enters the statement of the theorem only marginally through assumption 2.6. The indices determine the order in which the various sets will be considered by the procedure.

Our procedure follows a special type of arc in a subproblem. Sets in a subproblem meet nondegenerately so that subproblems have their own collection of arcs, some of which may coincide with arcs in another subproblem.

(3.1) (Definition) A <u>special arc</u> is associated with n sets $S^{g_1}, S^{g_2}, \ldots, S^{g_n}$ and meets the following conditions with $g_n = \max(g_1, g_2, \ldots, g_n)$:

a. the special arc is an arc of the g_n-problem;

b. points on the arc belong to the boundaries of $S^{g_1}, S^{g_2}, \ldots, S^{g_{n-1}}$;

c. the first set which dominates points on the arc is S^{g_n} ;

d. $S^{g_1}, S^{g_2}, \ldots, S^{g_n}$ are completely labeled.

Following such an arc we reach one of its endpoints, if the arc is bounded. We now argue that the endpoints of these arcs are of one of two types. Moving along the special arc of definition 3.1, we are attempting to solve the g_n-problem since S^{g_n} is the first set in the sequence which dominates points on the arc. The endpoint of the arc is a candidate for solving the g_n-problem, and also the m-problem. If the endpoint belongs to bdy(S^{g_n}) , we have solved the g_n-problem. If no set in the m-problem dominates the solution for the g_n-problem, it is a solution (for the m-problem) as well. If not, we have found a solution for the g_n-problem which is not a solution for a subproblem of size larger than g_n . The procedure will then be said to be at a <u>position of type 1</u>. Instead of first meeting bdy(S^{g_n}) , we could meet another boundary, say bdy(S^g) with $g<g_n$. Continuation along the same line makes S^g the first set to dominate points along the line and brings us into a different subproblem. To avoid this, the procedure generally follows a different arc. If the sets $S^{g_1}, S^{g_2}, \ldots, S^{g_{n-1}}, S^g$ are completely labeledwe have solved the h-problem where $h = \max(g, g_1, g_2, \ldots, g_{n-1}) < g_n$. The procedure has reached another position of type 1. If $S^{g_1}, S^{g_2}, \ldots, S^{g_{n-1}}, S^g$ are not completely labeled, then the procedure has reached a different type of position, called <u>a position of type 2</u>. We have established that solutions, positions of type 1, and positions of type 2 are the only endpoints special arcs can have. As a way of summary, we formally define positions of type 1 and 2.

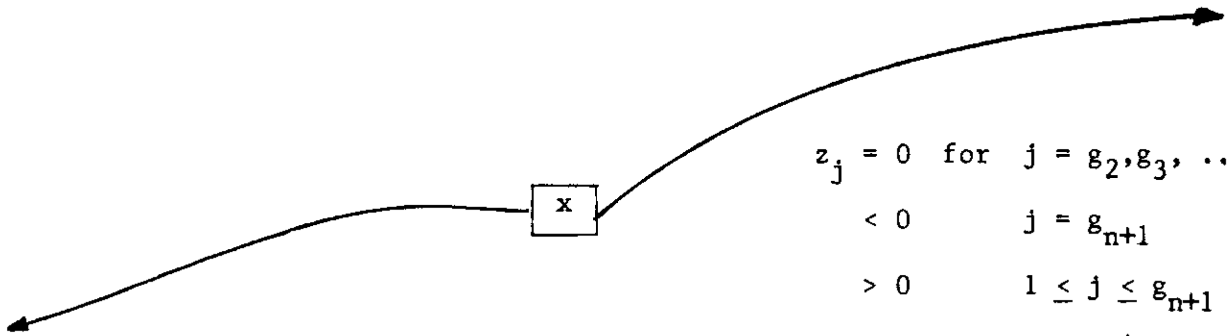
(3.2) (Definition) A position of type 1 is a point belonging to the boundaries of n completely labeled sets $S^{g_1}, S^{g_2}, \ldots, S^{g_n}$, undominated in the h-problem, where $h = \max(g_1, g_2, \ldots, g_n)$, but dominated by a set $S^{g_{n+1}}$ where $m \geq g_{n+1} > h$.

(3.3) (Definition) A position of type 2 is a point belonging to the boundaries of n sets $S^{g_1}, S^{g_2}, \ldots, S^{g_n}$ and undominated in the h-problem, where $h = \max(g_1, g_2, \ldots, g_n)$. Consider the first set which dominates the point, say $S^{g_{n+1}}$ $(m \geq g_{n+1} > h)$. The collection $S^{g_1}, S^{g_2}, \ldots, S^{g_n}$ is not completely labeled, but becomes completely labeled by adding $S^{g_{n+1}}$.

The "constructive" procedure--it is constructive modulo the existence of a constructive procedure for following arcs--derives easily from the incidence properties of special arcs and positions. This incidence is summarized in Figure 3.4 and is now explained.

It is easier to first consider a position of type 2. Such a position is associated with a point x belonging to the boundaries of n sets S^{g_i} for $i = 1, 2, \ldots, n$. These sets are not completely labeled, but become completely labeled when joining the label of $S^{g_{n+1}}$, which is the first set in the collection $S^1, S^2, \ldots, S^m$ to dominate x. Any special arc incident to x must be an arc in the g_{n+1}-problem. From standard linear programming theory, we know that there are two completely labeled collections of n sets among $S^{g_1}, S^{g_2}, \ldots, S^{g_{n+1}}$. Assume that these sets are obtained by deleting either S^{g_1} or S^{g_2}. It is then clear that there are exactly two special arcs incident to x, one associated with the sets $S^{g_2}, S^{g_3}, \ldots, S^{g_{n+1}}$, the other with $S^{g_1}, S^{g_3}, S^{g_4}, \ldots, S^{g_{n+1}}$. The first arc belongs to $\text{ext}(S^{g_1})$, the second arc belongs to $\text{ext}(S^{g_2})$.

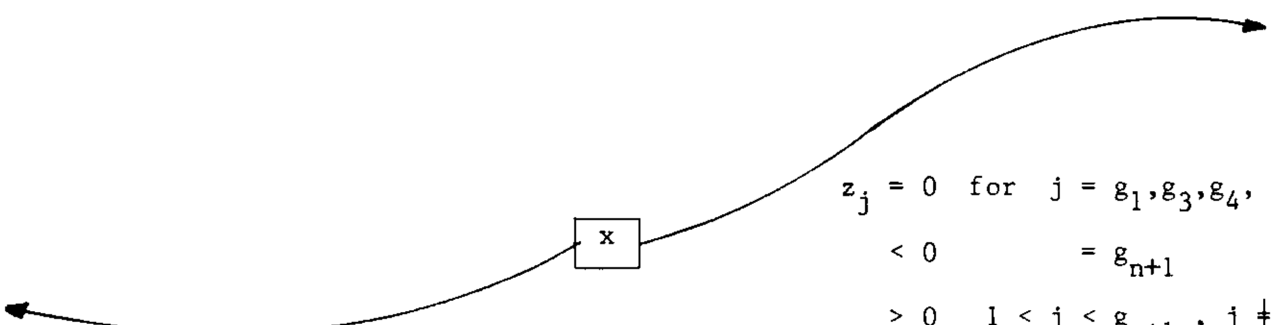
Assume now that x is associated with a position of type 1. The sets $S^{g_1}, S^{g_2}, \ldots, S^{g_n}$ form a completely labeled set, and $S^{g_{n+1}}$ is the first set to dominate x. Again from standard linear programming theory, we know that there is exactly one other completely labeled set among $S^{g_1}, S^{g_2}, \ldots, S^{g_{n+1}}$. Assume this set is obtained by deleting S^{g_1}. This identifies a special arc incident to x, namely the arc in the g_{n+1}-problem associated with the sets $S^{g_2}, S^{g_3}, \ldots, S^{g_n}$ and lying in $\text{ext}(S^{g_1})$. This is the only arc which does not belong to the interior of any of the sets S^{g_i} $(i = 1, 2, \ldots, n)$. The arc which goes into



$$z_j = 0 \quad \text{for} \quad j = g_2, g_3, \ldots, g_n$$
$$< 0 \qquad j = g_{n+1}$$
$$> 0 \qquad 1 \le j \le g_{n+1}, \; j \neq g_2, g_3, \ldots, g_{n+1}$$

$$z_j = 0 \quad \text{for} \quad j = g_1, g_2, \ldots, g_{n-1}$$
$$< 0 \qquad j = g_n$$
$$> 0 \qquad 1 \le j \le g_n, \; j \neq g_1, g_2, \ldots, g_n$$

a. If x represents a position of type 1, then $S^{g_1}, S^{g_2}, \ldots, S^{g_n}$ are completely labeled. We assume that $S^{g_2}, S^{g_3}, \ldots, S^{g_{n+1}}$ also form a completely labeled collection of sets.



$$z_j = 0 \quad \text{for} \quad j = g_1, g_3, g_4, \ldots, g_n$$
$$< 0 \qquad = g_{n+1}$$
$$> 0 \qquad 1 \le j \le g_{n+1}, \; j \neq g_1, g_3, g_4, \ldots, g_{n+1}$$

$$z_j = 0 \quad \text{for} \quad j = g_2, g_3, \ldots, g_n$$
$$< 0 \qquad = g_{n+1}$$
$$> 0 \qquad 1 \le j \le g_{n+1}, \; j \neq g_2, g_3, \ldots, g_{n+1}$$

b. If x represents a position of type 2, then $S^{g_1}, S^{g_2}, \ldots, S^{g_n}$ are not completely labeled, but $S^{g_1}, S^{g_2}, \ldots, S^{g_{n+1}}$ are. The latter collection contains two completely labeled sets of n members, which we assume are obtained by deleting S^{g_1} and S^{g_2} respectively.

Figure 3.4

3.4 (Figure) The incidence properties of special arcs and positions. A position is associated with a point x belonging to the intersection of n sets $S^{g_1}, S^{g_2}, \ldots, S^{g_n}$. Let $g_n = \max(g_1, g_2, \ldots, g_n)$. The first set among $S^1, S^2, \ldots, S^m$ to dominate x is $S^{g_{n+1}}$ $(g_{n+1} > g_n)$. The vector z , as defined in 2.4, describes the position of a point relative to the sets $S^1, S^2, \ldots, S^m$. Point x's position relative to the sets of the g_{n+1}-problem is given by

$$\begin{aligned} z_j(x) &= 0 \quad \text{for} \quad j = g_1, g_2, \ldots, g_n \\ &< 0 \qquad\quad j = g_{n+1} \\ &> 0 \quad 1 \le j \le g_{n+1}, \ j \neq g_1, g_2, \ldots, g_{n+1} . \end{aligned}$$

$int(S^{g_i})$ for some i , $1 \leq i \leq n$, can be a special arc only if $g_i = \max(g_1, g_2, \ldots, g_n)$. This identifies the second special arc incident to x .

We have thus shown that every position is incident to two special arcs. The procedure we will apply for the proof of theorem 2.7 should now become clear. The procedure consists in following special arcs. The special arcs in the procedure are all bounded so that an endpoint of the special arc is always reached. If this endpoint is a solution, the proof of theorem 2.7 is complete. If not, a new position is reached, which the procedure leaves by following the other arc incident to the position. To insure that the procedure terminates with a solution, we need to show that leaving a position along a special arc we always reach another position or a solution, and next that we never return to the same position twice. To verify the first statement we only need to show that every special arc followed by the procedure is bounded, since we already know that an endpoint is either a position or a solution.

The starting point for the procedure is the origin, which is a position of type 1 associated with the first n sets. Unless the origin is a solution, there are two special arcs incident to it. One of them coincides with a negative coordinate axis:

$$(3.5) \qquad x_i \begin{cases} = 0 & \text{for } i = 1, 2, \ldots, n-1 , \\ < 0 & \phantom{\text{for }} = n . \end{cases}$$

Note that this is the only unbounded special arc. This follows from two facts. First, points on a special arc belong to n completely labeled sets, and hence are bounded from above (assumption 2.5). Next, points on a special arc different from arc 3.5 do not belong to $int(S_i)$ for $i = 1, 2, \ldots, n$, which bounds these from below by the origin. Hence, all special arcs used by the procedure are bounded and have two endpoints.

We already showed that when an endpoint of a special arc is neither a position of type 1 nor one of type 2, then it is a solution and the procedure terminates. At a position, the procedure proceeds by following the other special arc incident to the position. The familiar Lemke-Howson argument applies to show that the procedure never returns to a position visited earlier. We repeat the argument for it is both clever and brief. Let $(x^i | i \geq 0)$ be the sequence of positions visited by the algorithm, and let x^i be the first position visited twice, i.e., $x^k = x^i$ with $i < k$. If $i \geq 1$, x^i has two neighboring positions, x^{i-1} and x^{i+1} , which cannot coincide with x^k since this would make x^{i-1} or x^{i+1} the first position visited twice. But then there are three arcs incident at x^i , contradicting the incidence properties summarized in Figure 2.4. The only possibility left is $i = 0$. There is, however, only one bounded arc incident at x^0 , which implies that $x^{k-1} = x^1$. This contradicts the fact that x^0 is the first position visited twice.

Having shown that every special arc, and hence its endpoints, lie in a bounded set, we invoke assumption 2.3.b to conclude that there are at most a finite number of positions. Since we also know that no position is visited twice, the procedure must terminate at a solution. This completes our proof of theorem 2.7.

IV. Algorithmic Statement and Illustration of the Procedure

The procedure used in the proof of theorem 2.7 can easily be stated algorithmically as consisting of successive iterations of four steps. In describing these steps, we use the vector $z(x)$ to indicate the position of point x relative to the sets $\{S^j\}$. The notation $z(x)$ was first introduced in 2.4. The index set J_0 is used to identify the sets whose boundaries a position (of type 1 or type 2) belongs to, while jdom is the index of the first set in the sequence $\{S^j\}$ that dominates the position.

Step 0. Initialize $x = 0$ so that $z_j(x) = 0$ for $j = 1, 2, \ldots, n$. Set $J_0 = \{1, 2, \ldots, n\}$. The feasible basis matrix B consists of the labels of the first n sets: $B = [a^1, a^2, \ldots, a^n]$. Proceed to step 1.

Step 1. If $z(x) \geq 0$ then x is a solution and the procedure terminates. Otherwise x is a position of type 1. Determine the first set S^g which dominates x ($z_j(x) \geq 0$ for $1 \leq j \leq g-1$, $z_j(x) < 0$ for $j = g$). Set both jdom = g and jin = g . Proceed to step 2.

Step 2. Pivot label a^{jin} into current basis matrix B . Say a^{jin} replaces a^{jout} . If jout = jdom , go to step 4. Otherwise, set $J_0 = J_0 - \{jout\}$ and proceed to step 3.

Step 3. Follow the special arc

$$\begin{aligned} z_j(x) &= 0 \quad \text{for } j \in J_0 , \\ &< 0 \quad \text{for } j = \text{jdom} , \\ &> 0 \quad \text{for } j = 1, 2, \ldots, \text{jdom} , \quad j \notin J_0 \cup \{\text{jdom}\} , \end{aligned}$$

until endpoint x' is reached. Say $z^g(x') = 0$ with $g \notin J_0$. Set jin = g and $J_0 = J_0 \cup \{jin\}$. If jin = jdom , proceed to step 1.

If jin ≠ jdom , proceed to step 2.

Step 4. Set $\text{jdom} = \max(j \mid j \in J_0)$ and $J_0 = J_0 - \{\text{jdom}\}$. Proceed to step 3.

Steps 1 and 4 correspond to changes in the size of the subproblem being considered. Step 1 is a progression to a subproblem of larger size, while step 4 is a regression to a subproblem of smaller size. The former step corresponds to a movement along the special arc appearing to the right of the position in figure 3.4.a, while step 4 corresponds to a movement along the special arc appearing to the left of the position. The following example exhibits an instance of nonmonotonicity in the sizes of the successive subproblems considered by the procedure.

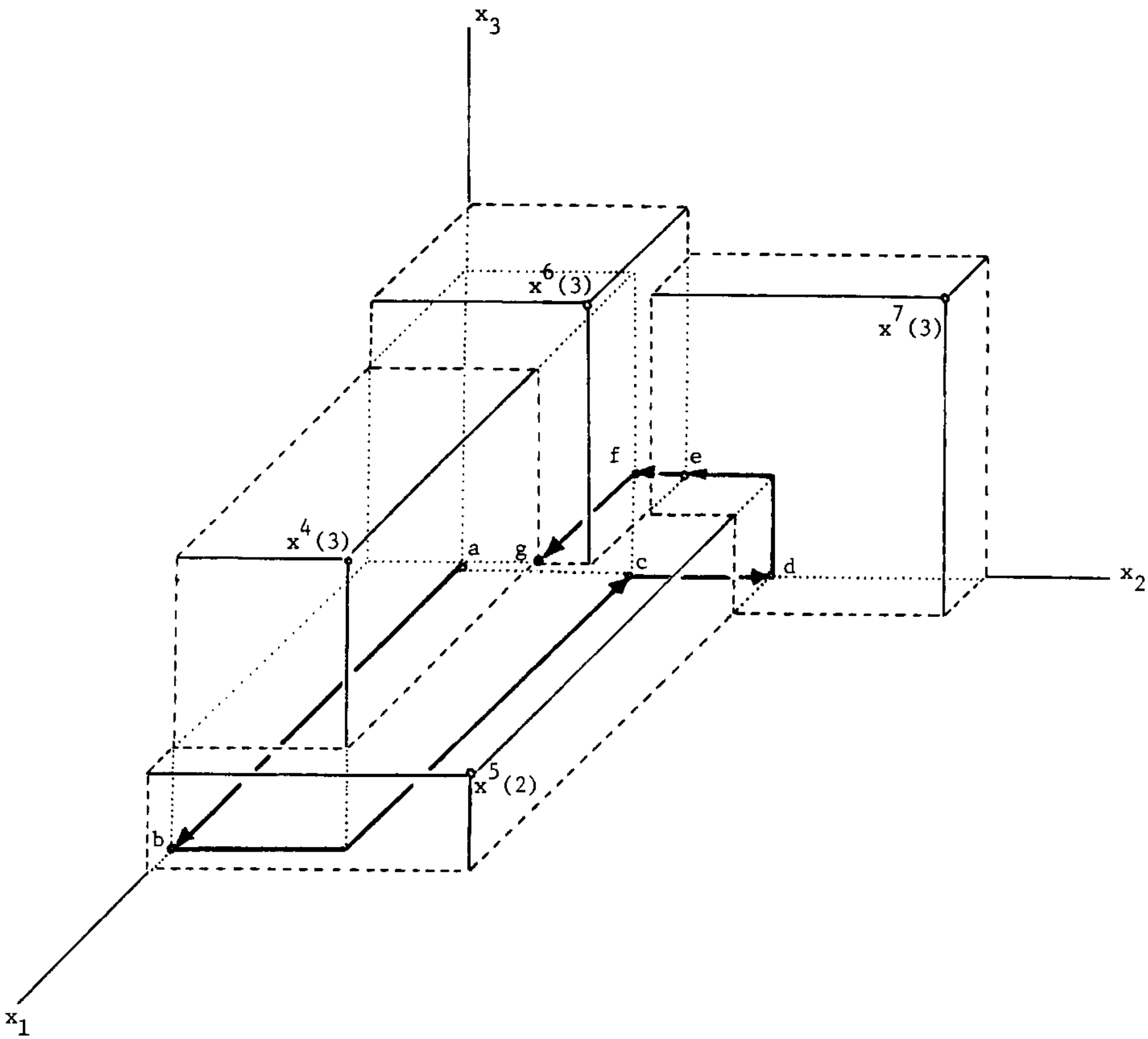
To illustrate the procedure, we present a 3-dimensional example where the sets and their labels take a particularly simple form (figure 4.1). The sets S^1, S^2, S^3 are determined by assumption 2.6, while S^4, S^5, S^6, S^7 are of the form:

$$S^j = \{x \mid x \leq x^j\} \quad \text{for} \quad j = 4, 5, 6, 7 .$$

The labels are, again for simplicity, assumed to be unit vectors and b is a positive vector. To simplify, we assign the <u>scalar</u> label i $(1 \leq i \leq 3)$ to S^j if its vector label is the i^{th} unit vector. Scalar labels are indicated in parentheses in figure 4.1. Three sets are completely labeled if they jointly bear labels 1, 2, 3.

Applied to the problem of figure 4.1 the procedure follows a broken line and generates the following positions:

Position	Type	Sets whose boundaries intersect at the position (with labels)	First set to dominate the position (with label)	Special arc leading to subsequent position (notation introduced in Figure 2.4)
a	1	$S^1(1)$, $S^2(2)$, $S^3(3)$	$S^4(1)$	$z_i = 0$ for $i = 2, 3$ > 0 for $i = 1$ < 0 for $i = 4$
b	1	$S^4(1)$, $S^2(2)$, $S^3(3)$	$S^5(2)$	$z_i = 0$ for $i = 3, 4$ > 0 for $i = 1, 2$ < 0 for $i = 5$
c	2	$S^4(1)$, $S^1(1)$, $S^3(3)$	$S^5(2)$	$z_i = 0$ for $i = 1, 3$ > 0 for $i = 2, 4$ < 0 for $i = 5$
d	1	$S^5(2)$, $S^1(1)$, $S^3(3)$	$S^7(3)$	$z_i = 0$ for $i = 1, 5$ > 0 for $i = 2, 3, 4, 6$ < 0 for $i = 7$
e	1	$S^5(2)$, $S^1(1)$, $S^6(3)$	$S^7(3)$	$z_i = 0$ for $i = 1, 5$ > 0 for $i = 2, 3, 4$ < 0 for $i = 6$
f	2	$S^5(2)$, $S^1(1)$, $S^4(1)$	$S^6(3)$	$z_i = 0$ for $i = 4, 5$ > 0 for $i = 1, 2, 3$ < 0 for $i = 6$
g	(solution)	$S^5(2)$, $S^6(3)$, $S^4(1)$		



4.1 (Figure) The path of our procedure consists of a finite union of broken lines and line segments. The point a is associated with the starting position, the point g with a solution. The points b , d , and e represent the positions of type 1 visited by the procedure before reaching a solution, while c and f represent positions of type 2.

V. An Application: Finding a Point in the Core of a Balanced n-Person Game

Consider an n-person game in characteristic form. Let $N = \{1, 2, \ldots, n\}$ be the set of players and let $\mathcal{C}$ ($\mathcal{C}_0$) be the set of (proper) coalitions, namely the collection of all nonempty (and proper) subsets of N. A coalition C is associated with an n-dimensional set of achievable utility vectors $V(C)$ which satisfies the following conditions:

(5.1) a. $V(C)$ is a nonempty closed set in R^n; if $x \in V(C)$ and $y_i = x_i$ for all $i \in C$ then $y \in V(C)$;

b. if $x \in V(C)$ and $y \leq x$ then $y \in V(C)$;

c. $V(\{i\}) = \{x | x_i \leq 0\}$ for $i \in N$;

d. $V(N)$ is bounded from above.

These are standard conditions in game theory and need not be discussed here (see e.g., Scarf (1973)). An important solution concept in game theory is the notion of the core. A utility vector x belongs to the core if it is a feasible allocation for the grand coalition and if no coalition has an incentive to depart from it. The mathematical form of these conditions is that the core consists of all allocations in the set $V(N) - \bigcup_{C \in \mathcal{C}} \text{int}(V(C))$.

A fundamental result in cooperative game theory due to Scarf (1967) asserts that every balanced game has a nonempty core. To define balancedness for a game, we first introduce the notion of a balanced collection of coalitions. The <u>collection</u> $\{B\}$ <u>of coalitions</u> is said to be <u>balanced</u> if there exist nonnegative weights $\{y_B\}$ verifying

$$\sum_{\{B: i \in B\}} y_B = 1 \quad \text{for each} \quad i \in N .$$

A <u>game</u> is <u>balanced</u> if, for every balanced collection $\{B\}$,

(5.2) $\bigcap_{\{B\}} V(B) \subset V(N)$.

We now show how our path-following procedure applies to determine a point in the core of a balanced game.

We start with applying our procedure to the solution of a <u>discrete</u> n-person game. In such game, each characteristic set contains a finite number of possibilities

$$V(C) = \bigcup_{k\varepsilon K_C} V_k(C)$$

where $C \varepsilon \mathcal{C}$, $K_C = \{1, 2, \ldots, k_C\}$, and $V_k(C) = \{x | x_C \leq x_C^k\}$. The notation used to describe the latter set is $x_C = (x_i | i \varepsilon C)$. These sets are the product of $R^{|N-C|}$ with a set in $R^{|C|}$ which is similar to the ones appearing in figure 4.1. Note that $k_{\{i\}} = 1$ and $x_{\{i\}}^1 = 0$ for $i \varepsilon N$.

The collection of sets $\{V_k(C) | C \varepsilon \mathcal{C}_0, k \varepsilon K_C\}$ are the sets $\{S_1, S_2, \ldots, S_m\}$ on which the procedure operates. In this application $m = \sum_{C\varepsilon\mathcal{C}_0} k_C$. We recall that the procedure depends on the ordering (or indexing) of the sets. The ordering is arbitrary except for the requirement that the first n sets coincide with the 1-person characteristic sets: $S_{\{i\}} = V_1(\{i\}) = V(\{i\})$ for $i = 1, 2, \ldots, n$. The label of set $S_j = V_k(C)$ with $C \varepsilon \mathcal{C}_0$, $k \varepsilon K_C$ is the indicator of coalition C :

$$a_{ij} = 1 \quad \text{if} \quad i \varepsilon C ,$$
$$= 0 \quad \text{otherwise.}$$

We set $b = (1, 1, \ldots, 1)$. Given the resulting linear system $Ay = b$,

$y \geq 0$, a collection of sets is completely labeled if the corresponding coalitions form a balanced collection.

Let us now verify the assumptions needed to apply the path-following procedure. The linear system is clearly bounded and can be made nondegenerate by perturbing the i^{th} component of b $(b_i = 1)$ to $b_i = 1 + \varepsilon^i$, where ε is a small, positive number (Dantzig, 1963). This verifies assumption 2.2. Assumption 2.5 is met by virtue of statement 5.1.c. Assumption 2.6 follows directly from the balancedness property (5.2) and from the upper-boundedness of $V(N)$ (5.1.d). This leaves us with assumption 2.3.

One can verify that the nondegeneracy assumption is met when the sequence of i^{th} coordinates $(x_i^k | i \varepsilon C,\ k \varepsilon K_C,\ C \varepsilon \mathcal{C}_0)$ comprises entries which are all different. This assumption can be made without any loss in generality since a small perturbation of some of the coordinates eliminates ties in the sequence should they occur. Alternatively, the lexicographic ordering can be used to break ties as in Scarf (1973). We argue that if the latter condition holds, then statement 2.3.a is true. If $x \varepsilon \text{bdy}(S^j)$ where $S^j = V_k(C)$ for $C \varepsilon \mathcal{C}_0$, $k \varepsilon K_C$, then at least one coordinate of x_C is equal to a coordinate of x_C^k and the other coordinates are less than or equal to the corresponding coordinates of x_C^k . Since all i^{th} coordinates in the sequence $(x_i^k | i \varepsilon C,\ k \varepsilon K_C,\ C \varepsilon \mathcal{C}_0)$ differ, the requirement that x belongs to n different boundaries determines the n coordinates of x . Hence, x cannot belong to more than n boundaries. Note that $x \varepsilon \text{bdy}(V_{k_i}(C_i))$ for $i = 1, 2, \ldots, n$ implies that

$$x = \min(x_{C_i}^{k_i} | i = 1, 2, \ldots, n)$$

where $\min(x, y, \ldots)$ denotes the vector whose i^{th} component is $\min(x_i, y_i, \ldots)$. This verifies 2.3.b. The argument needed to verify statement 2.3.c can be found in Van der Heyden (1979).

The discrete case then meets all the necessary assumptions for the application of the procedure. A path consisting of broken and straight line segments--similar to that appearing in figure 4.1--leads to an undominated point x belonging to a balanced collection of proper coalitions, hence to $V(N)$. . There exists an index $k \in K_N$ such that $x \in V_k(N)$ and x_N^k undominated in $V(N)$. The point x_N^k belongs to the core of the discrete game.

The reader familiar with Scarf's original procedure will notice a difference with the procedure exposited here. Our path is interior to at least one characteristic set until a solution is found, while no point of the path generated by Scarf's procedure is ever interior to any characteristic set. For more details, we refer to Van der Heyden (1979).

Having described the solution to the discrete case, we now turn to a continuous version of the problem. One purpose of the remainder of this section is to indicate that additional differentiability assumptions--which as shown in our treatment of the discrete case are not intrinsic to the procedure--permit us to restate assumption 2.3 in terms of manifolds. The reader may find the latter formulation more elegant.

In a continuous version of the problem, the characteristic sets are assumed sums of sets

$$V_k(C) = \{x \mid g_C^k(x_C) \leq 0\}$$

where $g_C^k(\cdot)$ is a scalar function with domain $R^{|C|}$. One way to meet nondegeneracy assumption 2.3 in the continuous case is to impose smoothness conditions on the functions $g_C^k(\cdot)$. If $g_C^k(\cdot)$ is a smooth function, i.e., a function with continuous partial derivatives of all orders, then the set $V_k(C)$ is an n-dimensional manifold with boundary given by

$$bdy(V_k(C)) = \{x | g_C^k(x_C) = 0\}$$

(for more details on manifolds we refer, e.g., to Guillemin and Pollack, 1974). In case 0 is not a regular value of the mapping, the above statement cannot be made. However, the statement is generic in that small perturbations of 0 to $0+\varepsilon$, ε being an arbitrary small number, yields regular values. The boundary $bdy(V_k(C))$ itself is an (n-1)-dimensional manifold without boundary. Nondegeneracy assumption 2.3 will be met if the boundary manifolds meet transversally. Again this is a minor requirement because transversality, just as regularity, is a generic property Having described the manifolds in terms of inequalities, the transversality condition takes the familiar form that the gradients associated with the functions $g_C^k(.)$ vanishing at x are linearly independent.
The procedure can then be applied and determines a point x undominated by any proper coalition, and belonging to $V(N)$ if the game is balanced. Any point $x' \geq x$ and in $V(N)$ belongs to the core of the game. The procedure is illustrated in figure 5.3.

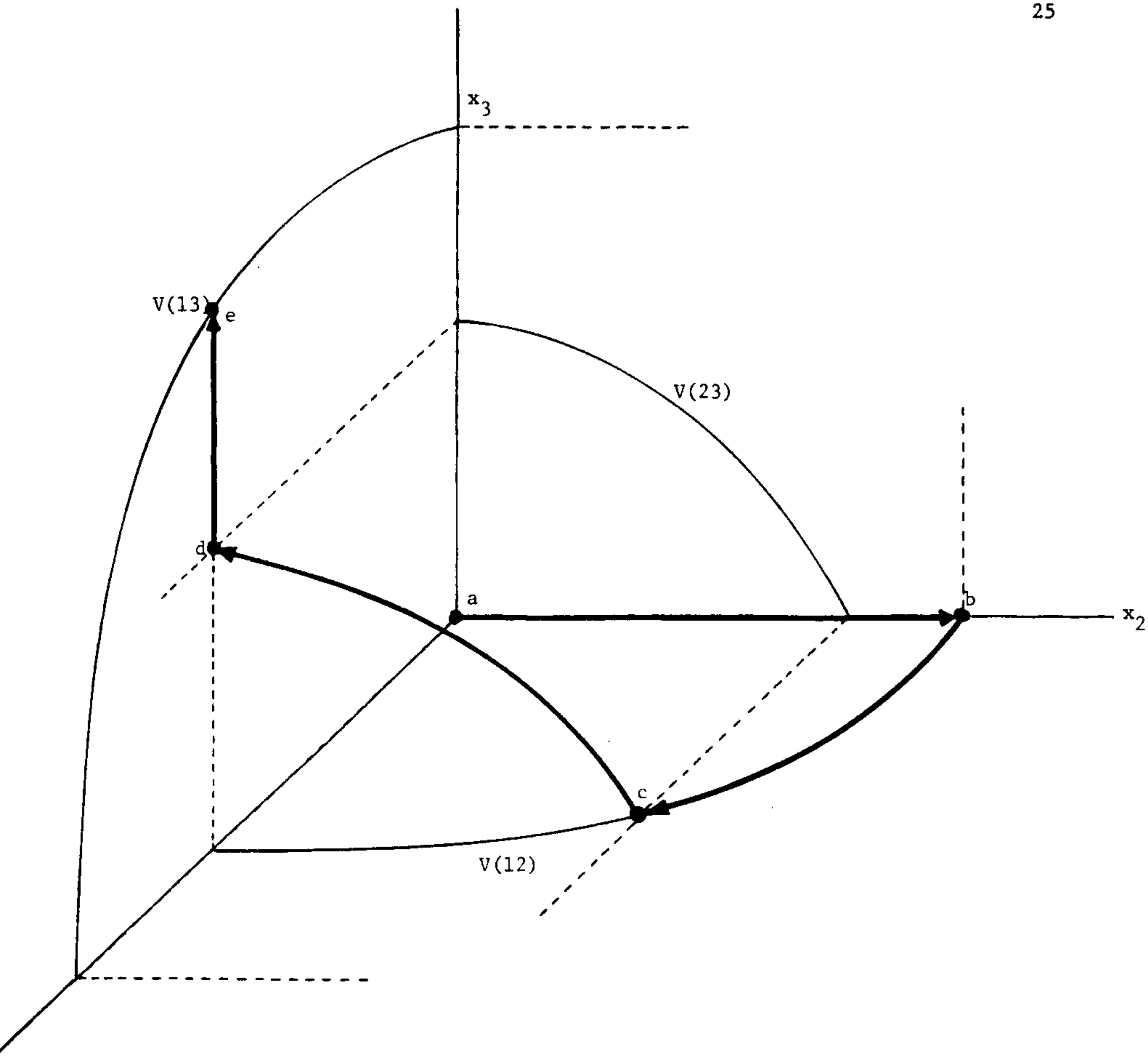


Figure 5.3. The path following procedure applied to the solution of a 3-person game ($n = 3$, $m = 6$, $k_i = 1$ for all i). The set S^i ($i = 1, 2, 3$) is the i^{th} coordinate plane. The other sets are $S^4 = V(12)$, $S^5 = V(23)$, $S^6 = V(13)$. Point a represents the starting position. Point b is the only other position of type 1. Points c and d represent positions of type 2, point e is associated with a solution.

VI. Concluding Remarks

This paper has exposited a path-following procedure for proving a general theorem (2.7) "constructively." The quotes are due to the fact that the method assumes the existence of a constructive procedure to follow 1-dimensional paths. Although the statement of the theorem does not, in any major way, take account of the indexing of the sets, the procedure does. The indices determine the order in which the various sets are considered by the procedure. Different orderings may lead to the discovery of multiple solutions.

The procedure links subproblems of different sizes and starts with a solution for one subproblem when attempting to solve a subproblem of a different size. The subproblems are linked so as to apply the Lemke-Howson argument and avoid cycling. This idea can be traced back to Shapley (1973). An interesting feature of the procedure is its occasional regression to a subproblem of smaller size considered earlier in the computation. This nonmonotonic behavior occurs whenever in solving a given subproblem, a new solution is found for a subproblem of smaller size, one already solved earlier. Several regressions may follow each other, but the Lemke-Howson argument ensures that eventually the procedure returns to a forward mode.

Nonmonotonicity is directly related to uniqueness of solutions, as multiple solutions for a subproblem are found every time the procedure regresses to a subproblem solved earlier. Index theoretic arguments, similar to the ones appearing in Van der Heyden (1979), can identify conditions sufficient for monotonicity and uniqueness (see also Scarf and Eaves, 1976).

A second distinct feature of the procedure is its interpretation as a refinement procedure. Starting with a solution for the n-problem the procedure generates--sometimes in nonmonotonic fashion--one or several solutions for subproblems of size larger than n. Each solution can be interpreted as an approximate solution for the main problem (the m-problem). The accuracy of this approximation increases with the size of the subproblem. Once a subproblem is solved, the problem is refined through the introduction of one or more sets, the last of which is the first set in the sequence to dominate the solution for the former subproblem. The procedure then uses the solution for the old subproblem as its starting point for the solution of the new subproblem. Note that until the h-problem is solved for the first time, the sets S_{h+1}, S_{h+2}, ... need not be specified. In certain applications, the solution for the h-problem will be a satisfactory approximate solution for the main problem and further computations will be unnecessary.

We have illustrated the procedure by applying it to determine a point in the core of an n-person game. We have done this both for the discrete game, where every coalition has the choice over a finite set of alternatives, and for the case where every coalition is characterized by a continuous set of alternatives. There is a natural link between both. The procedure for the continuous game can be seen to be the limit of the procedure for a discrete version of the game when the number of discrete alternatives used to approximate each continuous characteristic set becomes increasingly large.

There are other problems to which the procedure can be applied. One is the computation of fixed points of certain mappings (Van der Heyden, 1979); the other is the linear complementarity problem, which includes

2-person nonzero sum games (Van der Heyden, 1980). Both papers present algorithms which can be seen to be applications of the general procedure outlined in this paper. The reader may wish to refer to the first paper for a discussion of the relationship between the procedure developed here and other existing fixed point procedures.

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