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CHARACTERISTIC FUNCTIONS AND THE TAIL BEHAVIOR OF PROBABILITY DISTRIBUTIONS

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0. ABSTRACT

The theory of Fourier transforms of generalized functions is used to extract general formulae for the tail behavior of a probability distribution from the behavior of its characteristic function in the locality of the origin. The theory is applied to develop asymptotic formula for the tails of the stable distributions.

1. INTRODUCTION

The tail behavior of a probability distribution is known to be closely related to the behavior of the characteristic function of the distribution in the neighborhood of the origin. While certain precise results have been established about this relation (see, for example, [4], [5]), no general formulae which characterize this relationship seem to be available in the literature. The object of this paper is to show that, from a very general representation of the characteristic function in the locality of the origin, it is possible to extract corresponding asymptotic formulae which describe the tail behavior of the

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probability distribution. The formulae derived should be applicable to a wide class of probability distributions and useful in many applications. We illustrate their use in the case of the stable distributions.

2. GENERAL FORMULAE FOR THE TAILS OF A DISTRIBUTION

We let $cf(s)$ be the characteristic function of a real valued random variable. The behavior of $cf(s)$ as $s \to 0$ is assumed to be given by the following asymptotic series

$$(1) \qquad cf(s) \sim e^{i\eta s}\left\{\sum_{m=0}^{M-1} p_m (is)^m + |s|^\mu \sum_{j=0}^{\infty} \sum_{k=0}^{K(j)} \sum_{\ell=0}^{L(j)} q_{jk\ell} |s|^{\nu j} (i\,\mathrm{sgn}(s))^k (\ell n|s|)^\ell \right\}$$

where η , μ , ν , p_m , $q_{jk\ell}$ are real constants and $\mathrm{sgn}(s) = 1, 0, -1$ for $s > 0$, $= 0$, < 0 . In general, we will find in most applications that $\mu \geq M$, $\nu > 0$, $K(j) = 0$ and $L(j) = 0$ or 1 for all j .

The representation (1) is sufficiently general to include a very wide class of distributions and should cover most distributions of practical interest in mathematical statistics. The first component in braces on the right side of (1) is analytic and ensures, when $\mu \geq M$, that integral moments of the distribution will exist to order $M-1$ if this is an even integer and to order $M-2$ if $M-1$ is odd [4]. In cases where M is finite and the distribution does not possess all its moments, the second component of (1) is important in the local behavior of $cf(s)$ in the locality of the origin and is instrumental in determining the form of the tails of the distribution as the following result shows.

<u>THEOREM. If the characteristic function</u> cf(s) <u>is absolutely integrable and can be decomposed into the form</u>

$$cf(s) = cf_1(s) + cf_2(s) + cf_3(s)$$

<u>where</u>

$$cf_1(s) = e^{i\eta s}\sum_{m=0}^{M-1} p_m (is)^m$$

$$cf_2(s) = e^{i\eta s}|s|^{\mu}\sum_{j=0}^{J}\sum_{k=0}^{K(j)}\sum_{\ell=0}^{L(j)} q_{jk\ell}|s|^{\nu j}(i\,\mathrm{sgn}(s))^k(\ln|s|)^{\ell}\ , \quad \mu \geq M\ , \quad \nu > 0$$

$cf_3^{(j)}(s)$ <u>is absolutely integrable for</u> $j = 0, 1, \ldots, N$ <u>and</u> N <u>is the smallest integer</u> $\geq \mu + J\nu + 1$

$cf_3^{(j)}(s) \to 0$ <u>as</u> $s \to \pm\infty$ <u>for</u> $j = 0, 1, \ldots, N$

<u>then the corresponding probability density function</u> pdf(x) <u>has the following asymptotic expansion as</u> $|x| \to \infty$

$$(2) \qquad \mathrm{pdf}(x) = \frac{1}{\pi|x-\eta|^{\mu+1}}\sum_{j=0}^{J}\sum_{k=0}^{K(j)}\left[\sum_{\ell=0}^{L(j)}(q_{jk\ell}\partial^{\ell}/\partial z^{\ell})\Gamma(z+\mu+1)|y|^{-z}\cdot\frac{1}{2}\left\{i^k e^{-\frac{1}{2}i\pi\,\mathrm{sgn}(y)(z+\mu+1)} + (-1)^k e^{\frac{1}{2}i\pi\,\mathrm{sgn}(y)(z+\mu+1)}\right\}\right]_{\substack{z=j\nu\\ y=x-\eta}} + O(|x|^{-N})\ .$$

<u>Proof</u>. We use the theory of asymptotic expansions of Fourier transforms as developed in Lighthill [3] and Jones [2].

The notation $ft_i(x)$ is used to denote the inverse Fourier transform of $cf_i(s)$. Since the functions $cf_i(s)$ for $i = 1, 2$ are not absolutely integrable, the $ft_i(x)$ cannot be defined in the usual way but do exist as generalized functions. In particular, the $cf_i(s)$ can first be defined as generalized functions since there exists a $G > 0$ for which $(1+s^2)^{-G} cf_i(s)$ is absolutely integrable; the $ft_i(x)$ are then defined as the generalized functions obtained as the inverse Fourier transforms of the generalized functions $cf_i(s)$ [3].

Starting with $cf_1(s)$, we now write

$$cf_1(s) = \lim_{t\to 0+} e^{i\eta s} \sum_{m=0}^{M-1} p_m e^{-|s|t} (is)^m$$

and then, by definition,

$$ft_1(x) = \lim_{t\to 0+} \left\{ \frac{1}{2\pi} \sum_{m=0}^{M-1} p_m \int_{-\infty}^{\infty} e^{-isx} e^{i\eta s} e^{-|s|t} (is)^m ds \right\}$$

$$= \frac{1}{2\pi} \sum_{m=0}^{M-1} p_m (-1)^m \lim_{t\to 0+} \left\{ \frac{d^m}{dx^m} \int_{-\infty}^{\infty} e^{-i(x-\eta)s} e^{-|s|t} ds \right\}$$

$$= \frac{1}{2\pi} \sum_{m=0}^{M-1} p_m (-1)^m \lim_{t\to 0+} \left\{ \frac{d^m}{dx^m} \left[\int_0^{\infty} e^{[-i(x-\eta)-t]s} ds + \int_0^{\infty} e^{[i(x-\eta)-t]s} ds \right] \right\}$$

$$= \frac{1}{2\pi} \sum_{m=0}^{M-1} p_m (-1)^m \delta^{(m)} (x-\eta) \tag{3}$$

where $\delta(y)$ is the Dirac delta function and $\delta^{(m)}(y)$ is its m^{th} derivative. We deduce the asymptotic behavior of $ft_1(x)$ as $|x| \to \infty$ immediately from (3) as

$$ft_1(x) = O(|x|^{-k}) \tag{4}$$

for any value of $k > 0$.

The second component of $cf(s)$ is

$$cf_2(s) = e^{i\eta s}|s|^{\mu} \sum_{j=0}^{J} \sum_{k=0}^{K(j)} \sum_{\ell=0}^{L(j)} q_{ij\ell}|s|^{\nu j}(isgn(s))^k(\ell n|s|)^{\ell}$$

$$= e^{i\eta s} \sum_{j=0}^{J} \sum_{k=0}^{K(j)} \left[\sum_{\ell=0}^{L(j)} (q_{jk\ell}\partial^{\ell}/\partial z^{\ell})|s|^{z+\mu}(isgn(s))^k\right]_{z=j\nu}$$

$$= \lim_{t\to 0+} e^{i\eta s} \sum_{j=0}^{J} \sum_{k=0}^{K(j)} \left[\sum_{\ell=0}^{L(j)} (q_{jk\ell}\partial\ell/\partial z^{\ell})|s|^{z+\mu}(isgn(s))^k e^{-|s|t}\right]_{z=j\nu}$$

On inversion, we obtain

$$ft_2(x) = \lim_{t\to 0+}\left\{\frac{1}{2\pi}\sum_{j=0}^{J} \sum_{k=0}^{K(j)} \left[\sum_{\ell=0}^{L(j)} (q_{jk\ell}\partial^{\ell}/\partial z^{\ell})\int_{-\infty}^{\infty} e^{-isx+i\eta s-|s|t}|s|^{z+\mu}(isgn(s))^k ds\right]_{z=j\nu}\right.$$

$$= \lim_{t\to 0+}\left\{\frac{1}{2\pi}\sum_{j=0}^{J} \sum_{k=0}^{K(j)} \left[\sum_{\ell=0}^{L(j)} (q_{jk\ell}\partial^{\ell}/\partial z^{\ell})\left\{i^k\int_{0}^{\infty} e^{-(iy+t)s}s^{z+\mu}ds\right.\right.\right.$$

$$\left.\left.\left. + (-i)^k\int_{0}^{\infty} e^{-(-iy+t)s}s^{z+\mu}ds\right\}\right]_{\substack{z=j\nu\\ y=x-\eta}}\right\}$$

$$= \frac{1}{2\pi}\sum_{j=0}^{J} \sum_{k=0}^{K(j)} \left[\sum_{\ell=0}^{L(j)} (q_{jk\ell}\partial^{\ell}/\partial z^{\ell})(\lim_{t\to 0+} \Gamma(z+\mu+1)\{i^k(t+iy)^{-z-\mu-1}\right.$$

$$\left. + (-i)^k(t-iy)^{-z-\mu-1})\right]_{\substack{z=j\nu\\ y=x-\eta}}$$

$$= \frac{1}{2\pi}\sum_{j=0}^{J} \sum_{k=0}^{K(j)} \left[\sum_{\ell=0}^{L(j)} (q_{jk\ell}\partial\ell/\partial z^{\ell})\Gamma(z+\mu+1)|y|^{-z-\mu-1}\left\{i^k e^{-\frac{1}{2}i\pi sgn(y)(z+\mu+1)}\right.\right.$$

$$\left.\left. + (-i)^k e^{\frac{1}{2}i\pi sgn(y)(z+\mu+1)}\right\}\right]_{\substack{z=j\nu\\ y=x-\eta}}$$

$$(5) \qquad = \frac{1}{\pi|x-\eta|^{\mu+1}} \sum_{j=0}^{J} \sum_{k=0}^{K(j)} \left[\sum_{\ell=0}^{L(j)} (q_{jk\ell}\partial^{\ell}/\partial z^{\ell})\Gamma(z+\mu+1)|y|^{-z} \frac{1}{2}\left\{ i^k e^{-\frac{1}{2}i\pi \operatorname{sgn}(y)(z+\mu+1)} + (-i)^k e^{\frac{1}{2}i\pi \operatorname{sgn}(y)(z+\mu+1)} \right\} \right]_{\substack{z=j\nu \\ y=x-\eta}} .$$

The third component of the characteristic function is $cf_3(s)$. Now $ft_3(x)$ is the inverse Fourier transform of $cf_3(s)$ so that $(-ix)^N ft_3(x)$ has the inverse Fourier transform $cf_3^{(N)}(s)$. By assumption, $cf_3^{(N)}(s)$ is absolutely integrable over $(-\infty,\infty)$ and it follows from the Riemann Lebesgue lemma that $(-ix)^N ft_3(x) = O(1)$ as $|x| \to \infty$. This last result together with (4) and (5) implies that as $|x| \to \infty$

$$\begin{aligned} pdf(x) &= ft_1(x) + ft_2(x) + ft_3(x) \\ &= \frac{1}{\pi|x-\eta|^{\mu+1}} \sum_{j=0}^{J} \sum_{k=0}^{K(j)} \left[\sum_{\ell=0}^{L(j)} (q_{jk\ell}\partial^{\ell}/\partial z^{\ell})\Gamma(z+\mu+1)|y|^{-z} \right. \\ &\quad \left. \cdot\frac{1}{2}\left\{ i^k e^{-\frac{1}{2}i\pi \operatorname{sgn}(y)(z+\mu+1)} + (-i)^k e^{\frac{1}{2}i\pi \operatorname{sgn}(y)(z+\mu+1)} \right\} \right]_{\substack{z=j\nu \\ y=x-\eta}} + O(|x|^{-N}) . \end{aligned}$$

□

3. APPLICATIONS

To illustrate the use of the theorem, we first take the simple example of the Cauchy distribution with $cf(s) = e^{-|s|}$. In this case

$$cf_1(s) = 1 ,$$

$$cf_2(s) = |s| \sum_{j=0}^{J} \frac{(-1)^{j+1}}{(j+1)!}|s|^j ,$$

$$cf_3(s) = \frac{|s|^{J+2}}{(J+2)!}e^{-\theta|s|} \,, \quad 0 < \theta < 1$$

and we deduce from (2) by setting $\mu = 1$, $\eta = 0$, $\nu = 1$, $K(j) = 0$ and $L(j) = 0$ for all j that

$$\mathrm{pdf}(x) = \frac{1}{\pi x^2} \sum_{j=0}^{J} \frac{(-1)^{j+1}\Gamma(j+2)\cos\left\{\frac{1}{2}\pi(j+2)\right\}}{(j+1)!}|x|^{-j} + o(|x|^{-J-2})$$

$$(6) \qquad = \frac{1}{\pi x^2} \sum_{n=0}^{N} (-1)^n (x^2)^{-n} + o((x^2)^{-(N+1)})$$

where $2n = j$ and $2N = J$. The expansion (6) can be verified directly from the probability density $\mathrm{pdf}(x) = [\pi(1+x^2)]^{-1}$ itself.

As our second example, we consider the stable distributions, whose characteristic functions take the form [1]

$$(7) \qquad cf(s) = \exp\left\{i\gamma s - c|s|^{\alpha}\left[\exp - \tfrac{1}{2}i\pi K(\alpha)\beta s/|s|\right]\right\}$$

for $\alpha \neq 1$ and

$$(8) \qquad cf(s) = \exp\left\{i\gamma s - c|s|\left(1 - i\beta\frac{2}{\pi}\frac{s}{|s|}\ell n|s|\right)\right\}$$

when $\alpha = 1$ and where $K(\alpha) = 1 - |1-\alpha|$. It is clear that both (7) and (8) fall within the class of characteristic functions with local expansions at the origin of the form given by (1). As discussed in [1] we may, without loss of generality, restrict ourselves to the case in which $\gamma = 0$, $c = 1$, $x \geq 0$. The behavior of (7) as $s \to 0$ is then governed by the series

$$cf(s) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^j |s|^{j\alpha}}{j!k!} \left(-\frac{1}{2} j\pi K(\alpha)\beta\right)^k (i\,\mathrm{sgn}(s))^k .$$

Setting $\mu = 0$, $\eta = 0$, $\nu = \alpha$ in (1) we then obtain directly from (2) the following asymptotic series for the probability density as $x \to \infty$

$$\mathrm{pdf}(x) \sim \frac{1}{\pi x} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^j \left[-\frac{1}{2} j\pi K(\alpha)\beta\right]^k}{j!k!} \left[\Gamma(z+1) y^{-z} \frac{1}{2}\left\{ i^k e^{-\frac{1}{2} i\pi\,\mathrm{sgn}(y)(z+1)} + (-i)^k e^{\frac{1}{2} i\pi\,\mathrm{sgn}(y)(z+1)} \right\}\right]_{\substack{z=j\alpha \\ y=x}}$$

$$= \frac{1}{\pi x} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \Gamma(j\alpha+1) x^{-j\alpha} \frac{1}{2}\left\{ e^{-\frac{1}{2} ij\pi K(\alpha)\beta} e^{-\frac{1}{2} i\pi\,\mathrm{sgn}(x)(j\alpha+1)} + e^{\frac{1}{2} ij\pi K(\alpha)\beta} e^{\frac{1}{2} i\pi\,\mathrm{sgn}(x)(j\alpha+1)} \right\}$$

$$= \frac{1}{\pi x} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \Gamma(j\alpha+1) x^{-j\alpha} \left\{ \frac{i}{2}(e^{iz} - e^{-iz}) \right\}_{z = \frac{1}{2} j\pi K(\alpha)\beta + \frac{1}{2}\pi j\alpha}$$

$$= \frac{1}{\pi x} \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j!} \Gamma(j\alpha+1) \sin\left\{ \frac{1}{2}\pi j(\alpha + K(\alpha)\beta) \right\} x^{-j\alpha} .$$

This formula has been obtained by contour integration in the separate cases $\alpha < 1$ and $\alpha > 1$ by other authors (see [1], pp. 54-56).

When $\alpha = 1$ we have

$$cf(s) = \exp\left\{-|s|\left[1 - \beta\frac{2}{\pi}\ln|s|(i\,\mathrm{sgn}(s))\right]\right\}$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{j+k}\binom{j}{k}\left(\frac{2\beta}{\pi}\right)^k}{j!}|s|^j(\ln|s|)^k(i\,\mathrm{sgn}(s))^k$$

which is also of the general form (1). We set $K(j) = j$, $\ell = k$, $\eta = 0$, $\mu = 0$ and $\nu = 1$ in (2) and we find the corresponding asymptotic series for the probability density as $x \to \infty$

$$\mathrm{pdf}(x) \sim \frac{1}{\pi x}\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}\frac{(-1)^j(-1)^k\binom{j}{k}\left(\frac{2\beta}{\pi}\right)^k}{j!}\,\partial^k/\partial z^k\,\Gamma(z+1)y^{-z}\right.$$

$$\left.\cdot\frac{1}{2}\left\{i^k e^{-\frac{1}{2}i\pi\,\mathrm{sgn}(y)(z+1)} + (-i)^k e^{\frac{1}{2}i\pi\,\mathrm{sgn}(y)(z+1)}\right\}\right]_{\substack{z=j\\y=x}}$$

$$= \frac{1}{\pi x}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left[\frac{1}{2}\left\{\left(1 - \frac{2i\beta}{\pi}\frac{\partial}{\partial z}\right)^j\Gamma(z+1)y^{-z}e^{-\frac{1}{2}i\pi(z+1)}\right.\right.$$

$$\left.\left.+\left(1 + \frac{2i\beta}{\pi}\frac{\partial}{\partial z}\right)^j\Gamma(z+1)y^{-z}e^{\frac{1}{2}i\pi(z+1)}\right\}\right]_{\substack{z=j\\y=x}}$$

which has been given in alternative integral form in [1].

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